Renormalization-group analysis of metamagnetic tricritical behavior

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A renormalization-group treatment of the tricitical behavior of a d-dimensional metamagnet in a magnetic field is presented. For small $\epsilon = 4 - d > 0$, the tricritical behavior is described by competitions between pairs of Gaussian-like and Ising-like fixed points. Despite the increased number of independent interacting fields, we find that the metamagnetic tricritical exponents maintain their classical values for $d > 3$, with logarithmic corrections in three dimensions. The conclusions thus agree with those of Riedel and Wegner for an intrinsically simpler model appropriate to He^3-He^4 mixtures. The effect of an ordering field is analyzed, and Ising exponents are found on the "wings" of the tricritical point.

I. INTRODUCTION

An ideal metamagnetic crystal is made up of identical plane layers of spins with uniaxially anisotropic ferromagnetic coupling between spins within each layer but relatively weak *anti*ferromagnetic coupling between adjacent layers. In zero external field, a metamagnet behaves as an anisotropic (Ising-like) antiferromagnet. The corresponding antiferromagnetic transition remains continuous (i.e., of critical character) in a small enough external field H. However, for sufficiently large fields, the transition becomes first order. The change-over from continuous to first-order transition occurs at the tricritical point,[1] (H_t, T_t). Considerable interest attaches to the behavior of a metamagnet in the vicinity of the tricritical point and, in particular, one would like to know the values of the corresponding tricritical exponents.[1]

Riedel and Wegner[2,3] have discussed a single-component spin model which provides a reasonable description of He^3-He^4 mixtures in which a tricritical point is also observed. Using a renormalization-group approach[4,5] they conclude that for spatial dimensionalities exceeding $d = 3$ the tricritical point of their model is fully classical, i.e., in accord with a Landau-type phenomenological theory with exponents[1] $\alpha_t = \frac{1}{2}$, $\beta_t = \frac{1}{4}$, $\gamma_t = 1$, $\delta_t = 5$, etc. At $d = 3$ they find the exponents retain their classical values but with logarithmic corrections to the critical behavior of most quantities. (For $d < 3$ nonclassical tricritical exponents appear[6] but we shall not be concerned with this region.)

The Hamiltonian needed to describe a metamagnetic system is appreciably more complicated than that considered by Riedel and Wegner. In the first place there is the spatial anisotropy implied by the layer structure. In the second place, in the presence of an external field H (which is vital to the observation of a metamagnetic tricritical point) there are essentially two nonequivalent classes of spins (since H acts oppositely on ferromagnetic layers oriented in opposite senses). In particular, if a staggered ordering field $H^\dagger$, which is oriented "up" on alternate layers and "down" on the interleaving layers, is introduced there is no question as to the inequivalence of the two sets of layers. This indicates that a proper renormalization-group analysis of a metamagnet, which is the aim of the present work, should allow for two independent local spin fields, or, equivalently, consider a two-component (or $n = 2$) spin field. To omit this precaution may lead one to overlook ordering effects associated with the sublattice difference by averaging them out in the initial renormalization stages. The proper treatment might well lead to *nonclassical* tricritical exponents for $d \geq 3$ in contradistinction to the results of Riedel and Wegner. Indeed, the two-component Baxter-like model treated by Wilson and Fisher[4] resembles the system we will analyze here and does exhibit nonclassical tricritical behavior for $d < 4$ (with Ising, or $n = 1$, exponents).

In the Sec. II we discuss the Hamiltonian of a metamagnetic system in a form suitable for a renormalization-group analysis. In particular we consider a d-dimensional system with identical, ferromagnetic $(d-1)$-dimensional layers coupled in the remaining dth direction by antiferromagnetic interactions. As usual,[4,5] continuous, Ising-like $(n = 1)$ spins with s^4 and s^6 weighting factors are utilized. However, the distinctions between adjacent sublattices is retained. By iteration of the renormalization-group transformation for small $\epsilon = 4 - d > 0$, with suitably chosen, distinct rescaling factors for the different field components, it is shown that many terms in the full Hamiltonian become thermodynamically irrelevant. After a sufficient number of iterations, these variables may thus be neglected and a simple picture emerges. Contrary to the expectations aroused by the Baxter-like model, however, the tricritical behavior is described (in zero staggered field) by a competition only between pairs of Ising-like and Gaussian fixed points. The calculations then indicate that the tri-

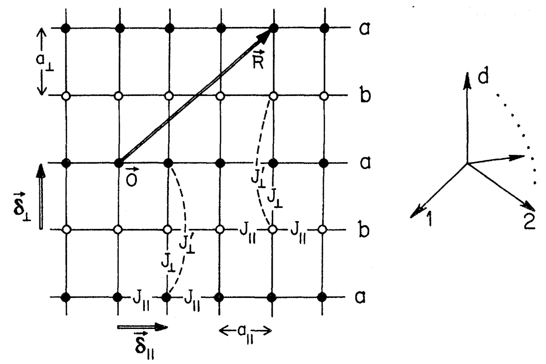

FIG. 1. Lattice structure of a d-dimensional layered metamagnet.

critical exponents retain their classical values down to $d=3$; no new nonclassical tricritical behavior is found! Indeed, an explicit partial trace over the renormalized Hamiltonian leads to a reduced Hamiltonian of just the form considered by Riedel and Wegner.

It is straightforward to extend the analysis to include the staggered field. This then shows that the critical points remaining on the "wings" of the full phase diagram[1] have Ising-like character; this result is, perhaps, not surprising but has not previously been established.

II. METAMAGNETIC HAMILTONIAN

We consider a system made up of continuous, Ising-like or single-component ($n=1$) spins each associated with a spin weighting factor[4,5]

$$e^{-w(s)}=\exp(-\tfrac{1}{2}s^2-f_4s^4-f_6s^6-\cdots)\,. \qquad (2.1)$$

The spins are located on the N sites, $\vec{R}$, of a d-dimensional lattice. Our interest centers on the case of a tetragonal lattice which is composed of $(d-1)$-dimensional layers. It is convenient to write $d-1=d'$. In each layer we suppose sites are separated by nearest-neighbor vectors $\vec{\delta}_\parallel$ of length $|\vec{\delta}_\parallel|=a_\parallel$ (see the diagram in Fig. 1). Corresponding spins in adjacent layers are taken to be separated by an orthogonal nearest-neighbor vector $\vec{\delta}_\perp$ of length $|\vec{\delta}_\perp|=a_\perp$. All sites are equivalent under translations generated by $\vec{\delta}_\perp$ and the $\vec{\delta}_\parallel$. However, for the reasons explained in Sec. I, we will label the layers alternately a and b (see Fig. 1); the sublattice of a sites will be denoted A, and of b sites, B.

We are interested in a system in which the coupling between spins in the same sublattice (in particular within each layer) is predominantly ferromagnetic, while the coupling between sublattices (in particular between adjacent layers) is predominantly antiferromagnetic. On such a system we wish to impose a uniform field H which, for small values, should merely shift the zero-field antiferromagnetic transition. However, when H is large enough, it is expected to produce tricritical behavior and then a first-order transition as the temperature T varies. The appropriate ordering field (at least for small H) is the staggered magnetic field $H^\dagger$, which acts equally but oppositely on the two sublattices A and B. For theoretical convenience, however, we transform this system into one in which all the interactions are predominantly ferromagnetic, by changing the sign of each spin $s(\vec{R})$ lying on one sublattice. This does not, of course, alter the partition function or free energy. The magnetic part of the Hamiltonian may then be written

$$\mathcal{H}_{\rm int}=-\frac{1}{2}\sum_{\vec{R},\vec{R}'}J(\vec{R}-\vec{R}')s(\vec{R})s(\vec{R}')$$
$$-H\sum_{\vec{R}}e^{i\vec{k}_0\cdot\vec{R}}s(\vec{R})-H^\dagger\sum_{\vec{R}}s(\vec{R}), \qquad (2.2)$$

where we take the origin of $\vec{R}$ on an a site, while

$$\vec{k}_0=\pi\vec{\delta}_\perp/a_\perp^2 \qquad (2.3)$$

is the reciprocal vector corresponding to the displacement between the two sublattices. The coupling $J(\vec{R}-\vec{R}')$ may now be taken as (predominantly) positive, i.e., ferromagnetic. The total effective Hamiltonian is

$$\overline{\mathcal{H}}\{s(\vec{R})\}=-\mathcal{H}_{\rm int}/k_BT-\sum_{\vec{R}}w[s(\vec{R})]\,, \qquad (2.4)$$

and the trace operation required to calculate the partition function $Z(T,H,H^\dagger)$, from $e^{\overline{\mathcal{H}}}$ is simply integration over each $s(\vec{R})$ from $-\infty$ to ∞.

It is evident from (2.2) that when $H^\dagger$ and H are nonzero, the spins on the two sublattices are inequivalent. Thus we now implement a scheme designed to retain the distinction, in accordance with the philosophy explained in Sec. I. To this end we define sublattice delta functions by

$$\Delta_a(\vec{R})=1,\quad \Delta_b(\vec{R})=0 \quad \text{if } \vec{R}\subset A,$$
$$\Delta_a(\vec{R})=0,\quad \Delta_b(\vec{R})=1 \quad \text{if } \vec{R}\subset B, \qquad (2.5)$$

and write

$$s(\vec{R})=s_a(\vec{R})\Delta_a(\vec{R})+s_b(\vec{R})\Delta_b(\vec{R})\,. \qquad (2.6)$$

This definition of the new spin variable $s_a(\vec{R})$ leaves it undefined when $\vec{R}\subset B$; for convenience we then assume that $s_a(\vec{R})$ vanishes identically—likewise for $s_b(\vec{R})$ when $\vec{R}\subset A$. The total Hamiltonian (2.4) then becomes

$$\mathcal{H}=\frac{1}{2}\sum_{\vec{R},\vec{R}'}\{K_{aa}(\vec{R}-\vec{R}')[s_a(\vec{R})s_a(\vec{R}')+s_b(\vec{R})s_b(\vec{R}')]+K_{ab}(\vec{R}-\vec{R}')[s_a(\vec{R})s_b(\vec{R}')+s_b(\vec{R})s_a(\vec{R}')]\}$$
$$+L\sum_{\vec{R}}[s_a(\vec{R})-s_b(\vec{R})]+L^{\dagger}\sum_{\vec{R}}[s_a(\vec{R})+s_b(\vec{R})]-\frac{1}{2}\sum_{\vec{R}}\{[s_a(\vec{R})]^2+[s_b(\vec{R})]^2\}-f_4\sum_{\vec{R}}\{[s_a(\vec{R})]^4+[s_b(\vec{R})]^4\},\quad(2.7)$$

where $L^{\dagger}=H^{\dagger}/k_BT$, $L=H/k_BT$, and

$$K_{aa}(\vec{R}-\vec{R}')=K_{bb}(\vec{R}-\vec{R}')=J(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_a(\vec{R}')/k_BT,\quad(2.8)$$

$$K_{ab}(\vec{R}-\vec{R}')=J_{ab}(\vec{R}-\vec{R}')/k_BT$$
$$=J(\vec{R}-\vec{R}')\Delta_a(\vec{R})\Delta_b(\vec{R}')/k_BT.\quad(2.9)$$

The equality of K_{aa} and K_{bb} stems from the translational invariance of the Hamiltonian when $H^{\dagger}=H=0$. For simplicity, terms of order higher than s^4, arising from the weighting function in (2.1), have not been indicated. The significance of these terms will be discussed later.

In order to diagonalize the quadratic part of the Hamiltonian (2.7) we define the transformed spin variables $\hat{s}_+(\vec{q})$ and $\hat{s}_-(\vec{q})$ by

$$s_{\pm}(\vec{q})=\frac{1}{2}\sum_{\vec{R}}e^{i\vec{q}\cdot\vec{R}}[s_a(\vec{R})\Delta_a(\vec{R})\pm s_b(\vec{R})\Delta_b(\vec{R})],\quad(2.10)$$

where $\vec{q}$ runs over a half-size Brillouin zone of $N_a=\frac{1}{2}N$ points (N being the total number of spins) specified by

$$|\vec{q}_{\parallel}|\le\pi/a_{\parallel},\quad|\vec{q}_{\perp}|\le\pi/2a_{\perp}.\quad(2.11)$$

The wave-vector components $\vec{q}_{\parallel}$ are parallel to the layers, while $\vec{q}_{\perp}$ is directed perpendicular to the layers (i.e., parallel to $\vec{\delta}_{\perp}$). The inverse transformation to (2.10) is

$$s_a(\vec{R})=N_a^{-1}\sum_{\vec{q}}e^{-i\vec{q}\cdot\vec{R}}[\hat{s}_+(\vec{q})+\hat{s}_-(\vec{q})],\qquad\vec{R}\subset A,$$
$$s_b(\vec{R})=N_a^{-1}\sum_{\vec{q}}e^{-i\vec{q}\cdot\vec{R}}[\hat{s}_+(\vec{q})-\hat{s}_-(\vec{q})],\qquad\vec{R}\subset B,\quad(2.12)$$

so that the Hamiltonian can be written as

$$\mathcal{H}=-N_a^{-1}\sum_{\vec{q}}\{[1-\hat{K}_+(\vec{q})]\hat{s}_+(\vec{q})\hat{s}_+(-\vec{q})+[1-\hat{K}_-(\vec{q})]\hat{s}_-(\vec{q})\hat{s}_-(-\vec{q})\}+2L^{\dagger}\hat{s}_+(\vec{0})+2L\hat{s}_-(\vec{0})$$
$$-2f_4N_a^{-3}\sum_{\vec{q}}\sum_{\vec{q}'}\sum_{\vec{q}''}[\hat{s}_+(\vec{q})\hat{s}_+(\vec{q}')\hat{s}_+(\vec{q}'')\hat{s}_+(-\vec{q}-\vec{q}'-\vec{q}'')+6\hat{s}_+(\vec{q})\hat{s}_+(\vec{q}')\hat{s}_-(\vec{q}'')\hat{s}_-(-\vec{q}-\vec{q}'-\vec{q}'')$$
$$+\hat{s}_-(\vec{q})\hat{s}_-(\vec{q}')\hat{s}_-(\vec{q}'')\hat{s}_-(-\vec{q}-\vec{q}'-\vec{q}'')].\quad(2.13)$$

Note that H and $H^{\dagger}$ ($\propto L, L^{\dagger}$) now couple only to zero-momentum spin components. The transformed interactions are defined by

$$\hat{K}_{\pm}(\vec{q})=\hat{K}_{aa}(\vec{q})\pm\hat{K}_{ab}(\vec{q}),\quad(2.14)$$

where

$$\hat{K}_{aa}(\vec{q})=\sum_{\vec{R}}e^{i\vec{q}\cdot\vec{R}}K_{aa}(\vec{R}),$$
$$\hat{K}_{ab}(\vec{q})=\sum_{\vec{R}}e^{i\vec{q}\cdot\vec{R}}K_{ab}(\vec{R}).\quad(2.15)$$

It is not hard to show that

$$\hat{K}_+(\vec{q})=\hat{J}(\vec{q})/k_BT=(k_BT)^{-1}\sum_{\vec{R}}e^{i\vec{q}\cdot\vec{R}}J(R),\quad(2.16)$$

where $J(\vec{R})$ is the original spin-spin coupling in $\mathcal{H}_{\text{int}}$. In addition, one has

$$k_BT\hat{K}_-(\vec{q})=\hat{J}(\vec{q})-2\hat{J}_{ab}(\vec{q}),\quad(2.17)$$

where $\hat{J}_{ab}(\vec{q})$ is defined in analogy to (2.16) but in terms of $J_{ab}(\vec{R})$ [see (2.9)].

For interactions of finite range we may now make a low-momentum expansion of the quadratic terms in (2.18) as

$$1-\hat{K}_+(\vec{q})=(T-T_0)/T+(j^{\parallel}a_{\parallel}^2/k_BT)q_{\parallel}^2+(j^{\perp}a_{\perp}^2/k_BT)q_{\perp}^2+\cdots,\quad(2.18)$$

$$1-\hat{K}_-(\vec{q})=(T-T_-)/T+(j_-^{\parallel}a_{\parallel}^2/k_BT)q_{\parallel}^2+(j_-^{\perp}a_{\perp}^2/k_BT)q_{\perp}^2+\cdots,\quad(2.19)$$

where

$$k_BT_0=\hat{J}(\vec{0})\quad\text{and}\quad k_BT_-=\hat{J}(\vec{0})-2\hat{J}_{ab}(\vec{0}).\quad(2.20)$$

Thus T_0 is the usual mean-field critical temperature; but note that for ferromagnetic interplanar coupling one has $\hat{J}_{ab}(\vec{0})>0$ so that $T_0>T_-$.

It is instructive at this point to specialize to the case, indicated in Fig. 1, where there are only nearest-neighbor interactions of strength $J_{\parallel}$ in the layers but first- and second-neighbor interactions of strengths $J_{\perp}$ and $J_{\perp}'$, respectively, perpendicular to the layers. One finds

$$\hat{J}(\vec{0})=2(d'J_{\parallel}+J_{\perp}+J_{\perp}'),\quad\hat{J}_{ab}(\vec{0})=2J_{\perp}\quad(2.21)$$

and

$$j^{\parallel}=J_{\parallel}, \quad j^{\perp}=J_{\perp}+4J'_{\perp}, \tag{2.22}$$

$$j^{\parallel}_{-}=J_{\parallel}, \quad j^{\perp}_{-}=-J_{\perp}+4J'_{\perp}. \tag{2.23}$$

From this it is clear that we may expect the coefficients $j^{\parallel}$, $j^{\perp}$, and $j^{\parallel}_{-}$ to be positive (as usual). However, the coefficient $j^{\perp}_{-}$ entering in (2.19) may be of either sign. While the possibility of a negative sign is, at first sight, surprising, it does not violate the stability of the Hamiltonian with respect to $\hat{s}_{-}(\vec{q})$ fluctuations since there is a finite momentum cutoff. Furthermore, the coefficient $j^{\perp}_{-}$ will be shown in Sec. IV to correspond to a strongly irrelevant variable.

Finally, in order to put the Hamiltonian in a form convenient for renormalization group calculations, we rescale the spins by writing

$$\hat{s}_{+}(\vec{q})=(k_B T/j^{\parallel}a_{\parallel}^2)^{1/2}\sigma_{1,\vec{q}}, \quad \hat{s}_{-}(\vec{q})=(k_B T/j^{\parallel}_{-}a_{\parallel}^2)^{1/2}\sigma_{2,\vec{q}} . \tag{2.24}$$

The Hamiltonian then becomes

$$\overline{\mathcal{H}}=-\frac{1}{2}\int_{\vec{q}}(r_1+q_{\parallel}^2+\kappa_1 q_{\perp}^2)\sigma_{1,\vec{q}}\sigma_{1,-\vec{q}}-\frac{1}{2}\int_{\vec{q}}(r_2+q_{\parallel}^2+\kappa_2 q_{\perp}^2)\sigma_{2,\vec{q}}\sigma_{2,-\vec{q}}+h_1\sigma_{1,\vec{0}}+h_2\sigma_{2,\vec{0}}$$
$$-\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}(u_{11}\sigma_{1,\vec{q}}\sigma_{1,\vec{q}'}\sigma_{1,\vec{q}''}\sigma_{1,-\vec{q}-\vec{q}'-\vec{q}''}+2u_{12}\sigma_{1,\vec{q}}\sigma_{1,\vec{q}'}\sigma_{2,\vec{q}''}\sigma_{2,-\vec{q}-\vec{q}'-\vec{q}''}+u_{22}\sigma_{2,\vec{q}}\sigma_{2,\vec{q}'}\sigma_{2,\vec{q}''}\sigma_{2,-\vec{q}-\vec{q}'-\vec{q}''}), \tag{2.25}$$

where $\int_{\vec{q}}$ here denotes $a_{\parallel}^{-d}N_2^{-1}\sum_{\vec{q}}$, while

$$r_1=k_B(T-T_0)/j^{\parallel}a_{\parallel}^2, \quad r_2=k_B(T-T_{-})/j^{\parallel}_{-}a_{\parallel}^2, \tag{2.26}$$

$$\kappa_1=j^{\perp}a_{\perp}^2/j^{\parallel}a_{\parallel}^2, \quad \kappa_2=j^{\perp}_{-}a_{\perp}^2/j^{\parallel}_{-}a_{\parallel}^2, \tag{2.27}$$

$$h_1=\sqrt{2}\,H^{\dagger}/(k_B T j^{\parallel}a_{\parallel}^{d+2})^{1/2}, \quad h_2=\sqrt{2}\,H/(k_B T j^{\parallel}_{-}a_{\parallel}^{d+2})^{1/2}, \tag{2.28}$$

$$u_{11}=\tfrac{1}{2}a_{\parallel}^{d-4}f_4(k_B T/j^{\parallel})^2, \quad u_{12}=\tfrac{3}{2}a_{\parallel}^{d-4}f_4(k_B T)^2(j^{\parallel}j^{\parallel}_{-})^{-1}, \quad u_{22}=\tfrac{1}{2}a_{\parallel}^{d-4}f_4(k_B T/j^{\parallel}_{-})^2 . \tag{2.29}$$

The final form (2.25) of the Hamiltonian resembles the anisotropic XY (or $n=2$) model treated by Fisher and Pfeuty[7] and by Wegner.[8] However, they discussed only the case of vanishing fields $h_1, h_2 \to 0$. Note that h_1 now describes the staggered or ordering field of the original metamagnet, while h_2 is proportional to the original uniform field H which is expected to bring about the tricritical behavior. Note also that if the system exhibits metamagnetic ordering tendencies so that $\hat{J}_{ab}(\vec{0})>0$, then the inequality $r_1<r_2$ follows (since, as explained before, $T_0>T_{-}$). Hence, as discussed by Fisher and Pfeuty,[7] the system will cross over rapidly to Ising-like ($n=1$) behavior when $h_1=h_2=0$. For our purposes, however, it will be essential to investigate effects of the terms $h_1\sigma_1$ and $h_2\sigma_2$ for finite h_1 and h_2. Indeed, if we did this with neglect of the momentum squared (or gradient) terms in $\overline{\mathcal{H}}$, we would just maximize $\overline{\mathcal{H}}$ which would amount to a classical or Landau-type treatment of the problem. This in turn would lead to a phase diagram with a tricritical point exhibiting classical exponent values. The renormalization group treatment, of course, allows for the momentum dependence (i.e., the gradient terms) and determines which variables are relevant and which irrelevant. Dimensionality-dependent corrections (if any) to the classical exponent values are expected to emerge from the calculation.

III. RENORMALIZATION-GROUP TRANSFORMATION

Before undertaking a renormalization of the Hamiltonian (2.25) we make two transformations. First, in order to remove the spatial inhomogeneity represented by the coefficient κ_1 in (2.25), it is convenient to rescale the momentum variable in the direction perpendicular to the layers according to $q_{\perp}\to\kappa_1^{-1/2}q_{\perp}$. To keep the resulting coefficient of q^2 equal to unity, the spins $\sigma_{1,\vec{q}}$ and $\sigma_{2,\vec{q}}$ must likewise be rescaled by a factor $\kappa_1^{1/4}$. The effect is then to replace κ_2 in (2.25) by

$$\kappa=\kappa_2/\kappa_1. \tag{3.1}$$

As a matter of fact, the parameter κ_1 corresponds to a *marginal* operator which can change the symmetry of the critical scattering from "spherical" to "ellipsoidal." This can, indeed, already be seen in the exact solutions of the anisotropic square-lattice Ising model.[9] However, we are not here interested in this effect, so we will, via the rescaling device, ignore it. Furthermore, in the interests of simplicity we will write subsequent expressions as if $\kappa_1\equiv 1$ (which is, of course, perfectly possible).

The second transformation is to make a shift of the spin variables $\sigma_{1,\vec{q}}$ and $\sigma_{2,\vec{q}}$ so as to eliminate the linear, or external, field terms in the Hamiltonian. This amounts to defining new spin variables which are chosen to vanish at the "classical" minimum of the Hamiltonian. Thus on putting

$$\sigma_{1,\vec{q}} = \tilde{\sigma}_{1,\vec{q}} + N_a M_1 \delta_{\vec{0},\vec{q}},$$
$$\sigma_{2,\vec{q}} = \tilde{\sigma}_{2,\vec{q}} + N_a M_2 \delta_{\vec{0},\vec{q}}, \tag{3.2}$$

and neglecting a spin-independent term, the Hamiltonian becomes

$$\begin{aligned}\overline{\mathcal{H}} = &-\frac{1}{2}\int_{\vec{q}}[(r_{11}+q^2)\tilde{\sigma}_1\tilde{\sigma}_1 + 2r_{12}\tilde{\sigma}_1\tilde{\sigma}_2 + (r_{22}+q_{\parallel}^2+\kappa q_{\perp}^2)\tilde{\sigma}_2\tilde{\sigma}_2] \\ &-\int_{\vec{q}}\int_{\vec{q}'}(w_1\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_2 + w_2\tilde{\sigma}_2\tilde{\sigma}_2\tilde{\sigma}_2 \\ &\qquad + w_3\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_2 + w_4\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1) \\ &-\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}(u_{11}\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1 \\ &\qquad + 2u_{12}\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_2 + u_{22}\tilde{\sigma}_2\tilde{\sigma}_2\tilde{\sigma}_2\tilde{\sigma}_2),\end{aligned} \tag{3.3}$$

where the momentum-conserving subscripts $\vec{q}$, $-\vec{q}$, $\vec{q}'$, $-\vec{q}-\vec{q}'$, etc., have been omitted in the interests of clarity. Note that in the three-spin terms w_1 and w_2 label the terms even in $\tilde{\sigma}_{1,q}$, while w_3 and w_4 label those odd in $\tilde{\sigma}_{1,q}$. The elimination of the terms linear in $\tilde{\sigma}_{1,q}$ and $\tilde{\sigma}_{2,q}$ is subject to choosing M_1 and M_2 as the unique roots of

$$\begin{aligned}4u_{12}M_1M_2^2 + 4u_{11}M_1^3 + M_1r_1 &= h_1, \\ 4u_{12}M_1^2M_2 + 4u_{22}M_2^3 + M_2r_2 &= h_2,\end{aligned} \tag{3.4}$$

which go linearly to zero with h_1 and h_2, respectively. The new interaction parameters are then given by

$$\begin{aligned}r_{11} &= r_1 + 12u_{11}M_1^2 + 4u_{12}M_2^2, \quad r_{12} = 8u_{12}M_1M_2, \\ r_{22} &= r_2 + 12u_{22}M_2^2 + 4u_{12}M_1^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}w_1 &= 4u_{12}M_2, \quad w_2 = 4u_{22}M_2, \\ w_3 &= 4u_{12}M_1, \quad w_4 = 4u_{11}M_1.\end{aligned} \tag{3.6}$$

These various relations would, of course, be altered if we had explicitly carried along the six-spin terms in the original spin weighting function (2.1) instead of dropping them in (2.7). However, it is clear that we could, by making a shift, still obtain a Hamiltonian of the form (3.3) except that five-spin and six-spin terms would enter. However, we expect these to be thermodynamically irrelevant.

The thermodynamic limit may be taken by letting $N_a = \frac{1}{2}N \to \infty$ whereupon the sums $\int_{\vec{q}}$ become the integrals $(2\pi)^{-d}\int d^dq$. These momentum integrals run over the "rectangular" Brillouin zone (2.11) (rescaled as explained above). This zone may be retained formally for most of the renormalization-group arguments developed below although for technical reasons it may be useful in explicit calculations to approximate it by a "spherical" zone $|\vec{q}| \le \pi/a$, with some suitable mean lattice spacing a. We will do this where convenient.

The tricritical point is expected to be in the plane $h_1 = 0$ (i.e., $H^\dagger = 0$) where $M_1 = 0$. The only field then acting on the system is h_2 which corresponds to the original uniform field H acting on the metamagnet, which is the situation of experimental interest. Accordingly, we consider this case first. Many terms in (3.3) now vanish; in particular we have $r_{12} = w_3 = w_4 = 0$ so that the only parameters are r_{11}, r_{22}, κ, w_1, w_2, and the u_{ij}. Furthermore, any fifth-order terms like $\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1$, $\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_2$, etc., which are odd in $\tilde{\sigma}_1$, must also vanish identically.

To generate the renormalization-group recursion relations, we now assume, as usual, that the nonquadratic parts of the Hamiltonian are small and calculate by perturbation theory.[4,5] A new, renormalized Hamiltonian $\overline{\mathcal{H}}'$ is generated from $\overline{\mathcal{H}}$ by choosing a rescaling factor $b > 1$ and integrating out all spin variables $\tilde{\sigma}_{1,\vec{q}}$ and $\tilde{\sigma}_{2,\vec{q}}$ of momentum such that $b\vec{q}$ lies outside the original Brillouin zone. We will indicate wave vectors in this outer momentum shell by a superscript >. A rescaling of momentum space by the factor b, and of the spins by factors $\hat{c}_1$ and $\hat{c}_2$ which, in contrast to previous work we will allow to be *distinct*, results in a Hamiltonian of the same form as $\overline{\mathcal{H}}$ (allowing for higher-order terms). The recursion relations give the values of the renormalized interaction parameters.

The perturbation theory involves the two distinct inverse Feynman propagators

$$\begin{aligned}G_1^{-1}(\vec{q}, r_{11}, e_1) &= r_{11} + e_1q^2, \\ G_2^{-1}(\vec{q}, r_{22}, e_2, \kappa) &= r_{22} + e_2(q_{\parallel}^2 + \kappa q_{\perp}^2),\end{aligned} \tag{3.7}$$

in which, for reasons to be explained, we have allowed for variable amplitudes of the q^2 terms. Typical graphs which arise from the w_1, w_2, and u_{ij} vertices are shown in Fig. 2 where the propagator G_1 is indicated by a solid line, while G_2 is denoted by a broken line.

As a result of integrating over spins in the outer momentum shell, terms linear in $\tilde{\sigma}_{2,\vec{q}}$ are regenerated spontaneously. The graphs responsible for this in leading order are shown in Figs. 2(a) and (b). To retain zero h_2', we accordingly make a secondary shift of the $\tilde{\sigma}_{2,\vec{q}}$ variable after each iteration of the renormalization-group transformation chosen to eliminate this new linear term. Thus the recursion relations we will now quote include the effects of the momentum shell integration, the spin rescaling, and the secondary shift.

IV. RECURSION RELATIONS AND FIXED POINTS

We now analyze, in detail, the recursion relations for the case discussed above, of zero ordering field. It will become apparent that the parameters u_{22}, u_{12}, and w_2 are strongly irrelevant, going to zero rapidly as the renormalization process progresses. Consequently, we will here suppress the

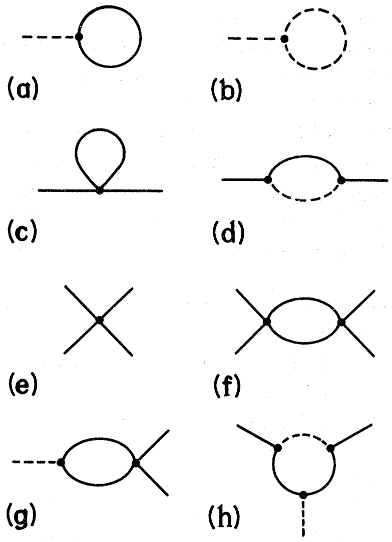

FIG. 2. Some of the graphs involved in the perturbative calculation of the recursion relations. Solid lines denote the propagator $G_1(\vec{q})$; broken lines, $G_2(\vec{q})$.

precise dependence of the recursion relations on these variables. The full recursion relations to leading order are given for reference in Appendix A. The essential features of the recursion relations to this order are given by

$$r_{11}' = \hat{c}_1^2 b^{-d}[r_{11} + 12A_{10}u_{11} - 4A_{11}w_1^2 - 2A_{10}w_1^2/r_{22} + O(u_{12}, w_1w_2)], \tag{4.1}$$

$$r_{22}' = \hat{c}_2^2 b^{-d}[r_{22} - 2A_{20}w_1^2 + O(u_{12}, u_{22}, w_2^2, w_1w_2)], \tag{4.2}$$

$$e_1' = \hat{c}_1^2 b^{-d-2} e_1 + O(w_1^2,\ w_1w_2), \tag{4.3}$$

$$e_2' = \hat{c}_2^2 b^{-d-2} e_2 + O(w_2^2, w_1w_2),$$

$$\kappa' = \kappa + O(u_{12}, u_{22}, w_2^2, w_1w_2), \tag{4.4}$$

$$w_1' = \hat{c}_1^2\hat{c}_2 b^{-2d}[w_1 - 12A_{20}w_1u_{11} + 4A_{21}w_1^3 + O(w_1u_{12}, w_2u_{12}, w_1w_2^2)], \tag{4.5}$$

$$w_2' = \hat{c}_2^3 b^{-2d}[w_2 + \tfrac{4}{3}A_{30}w_1^3 + O(w_2u_{22}, w_1u_{12}, w_2^3, w_1u_{22})], \tag{4.6}$$

and

$$u_{11}' = \hat{c}_1^4 b^{-3d}[u_{11} - 36A_{20}u_{11}^2 + 24A_{21}u_{11}w_1^2 - 4A_{22}w_1^4 + O(u_{12}^2, u_{12}w_1^2)], \tag{4.7}$$

$$u_{12}' = \hat{c}_1^2\hat{c}_2^2\, b^{-3d}[u_{12} + 24A_{30}u_{11}w_1^2 - 8A_{31}w_1^4 + O(u_{12}u_{11}, u_{12}^2, u_{12}w_1^2, u_{22}w_1^2,\ w_1^2w_2^2)], \tag{4.8}$$

$$u_{22} = \hat{c}_2^4 b^{-3d}[u_{22} - 2A_{40}w_1^4 + O(u_{22}^2, u_{12}^2, u_{12}w_1^2, u_{22}w_2^2, w_2^4)], \tag{4.9}$$

where

$$A_{lm} = A_{lm}^d(r_{11}, r_{22}, e_1, e_2, \kappa; b)$$

$$= \int_{\vec{q}}^{>} [G_1(\vec{q})]^l [G_2(\vec{q})]^m, \tag{4.10}$$

in which the superscript > indicates integration over the outer d-dimensional shell as explained previously. The factor $1/r_{22}$ in the coefficient of w_1^2 in (4.1) arises from the secondary shift of the spin variables (see Appendix A).

For vanishing w_1 and w_2 it is known that the recursion relations have fixed points with $u = O(\epsilon)$ and $r = O(\epsilon)$.[4,5] Inspection of the full relations indicates that the influence of w_1 and w_2 becomes felt when they are of order $\sqrt{\epsilon}$. This justifies the orders of the terms retained in (4.1) to (4.9).

Normally the spin rescaling factors $\hat{c}_1$ and $\hat{c}_2$ are chosen to keep the coefficients of q^2, namely e_1 and e_2, constant and equal to unity.[4,5,7,8] If this rescaling is used so that $r_2 > r_1 = O(\epsilon)$ as discussed above, then, r_{22} diverges when r_{11} is at criticality. In addition the propagator G_2 approaches zero. Effectively this means that one loses control over the $\tilde{\sigma}_2$ spin variables. To avoid this we rescale so as to keep r_{22} fixed in place of e_2 while keeping $e_1 = 1$ as usual. Consequently we choose

$$\hat{c}_1 = b^{3-\epsilon/2}[1 + O(\epsilon^2)], \tag{4.11}$$

$$\hat{c}_2 = b^{2-\epsilon/2}[1 + A_{20}w_1^2/r_{22} + O(\epsilon^2)]. \tag{4.12}$$

To justify the order indicated for the error terms here we must check that the contribution from the w^2 graph in Fig. 2(d) is independent of q^2 in order ϵ, since this is the leading correction in (4.3). This is done in Appendix B.

Under the spin rescalings (4.11) and (4.12) most of the terms in $\bar{\mathcal{H}}$ become strongly irrelevant. In particular taking $u_{ij} = O(\epsilon)$ and $w_i = O(\sqrt{\epsilon})$ the recursion relations for these strongly irrelevant variables become

$$e_2' = b^{-2}[e_2 + O(\epsilon^2)], \tag{4.13}$$

$$w_2' = b^{-2+\epsilon/2}[w_2 + O(e^{3/2})], \tag{4.14}$$

$$u_{22} = b^{-4+\epsilon}[u_{22} + O(\epsilon^2)], \quad u_{12} = b^{-2+\epsilon}[u_{12} + O(\epsilon^2)]. \tag{4.15}$$

We see, in particular, that the q dependence of the $\tilde{\sigma}_2$ propagator drops out so that

$$G_2(\vec{q}) \to 1/r_{22}. \tag{4.16}$$

It follows from these relations that after a sufficient number of iterations l, the original Hamiltonian (3.3) (with $r_{12} = w_3 = w_4 = 0$) becomes renormalized to the form

$$\bar{\mathcal{H}}_l \approx -\frac{1}{2}\int_{\vec{q}} (r_{11} + q^2)\sigma_1\sigma_1 - \tfrac{1}{2} r_{22}\int_{\vec{q}} \tilde{\sigma}_2\tilde{\sigma}_2 - w_1 \int_{\vec{q}}\int_{\vec{q}''} \sigma_1\sigma_1\tilde{\sigma}_2 - u_{11}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} \sigma_1\sigma_1\sigma_1\sigma_1, \tag{4.17}$$

where we have dropped the tildes from the σ_1 variables since these are unshifted when $h_1 = 0$, and where, as before, we omit the momentum conserving subscripts. As explained, r_{22} is held fixed in this Hamiltonian, so that only r_{11}, w_1, and u_{11} change under further iteration. Truncated at order ϵ their recursion relations are then

$$r'_{11} = b^2(r_{11} + 12A_{10}u_{11} - 6A_{10}w_1^2/r_{22}), \quad (4.18)$$

$$w'_1 = b^{\epsilon/2}(w_1 - 12A_{20}w_1u_{11} + 5A_{20}w_1^3/r_{22}), \quad (4.19)$$

$$u'_{11} = b^{\epsilon}(u_{11} - 36A_{20}u_{11}^2 + 24A_{20}u_{11}w_1^2/r_{22} - 4A_{20}w_1^4/r_{22}^2). \quad (4.20)$$

In deriving these relations we have set $d=4$ in all the integrals A_{lm} and used (4.16) in (4.10) to obtain $A_{lm} = A_{l0}/r_{22}^m$. To simplify further we define

$$x = w_1^2/r_{22} \sim M_2^2 \sim H^2, \quad (4.21)$$

where the last parts of the formula serve as a reminder that w_1 vanishes if the original field H vanishes, and has an initial value proportional to H. The recursion relations may then be written

$$r'_{11} = b^2[r_{11} + 12A_{10}(r_{11})u_{11} - 6A_{10}(r_{11})\,x], \quad (4.22)$$

and

$$\delta u_{11} = u_{11}(\epsilon \ln b - 36A_{20}u_{11} + 24A_{20}x) - 4A_{20}x^2, \quad (4.23)$$

$$\delta x = x(\epsilon \ln b - 24A_{20}u_{11} + 10A_{20}x). \quad (4.24)$$

These relations have a structure quite similar to those for the Baxter-like anisotropic XY model analyzed by Wilson and Fisher.[4] The last pair of relations determine the fixed points; the corresponding Hamiltonian flows in the (u_{11}, x) subspace are sketched in Fig. 3. To order ϵ the fixed points are found to be

$$\begin{aligned}
&\text{(a)}\ u_{11}^* = 0, && x^* = 0 && \text{(Gaussian)},\\
&\text{(b)}\ u_{11}^* = \tfrac{1}{36}\bar{\epsilon}, && x^* = 0 && \text{(Ising-like)},\\
&\text{(c)}\ u_{11}^* = \tfrac{1}{9}\bar{\epsilon}, && x^* = \tfrac{1}{6}\bar{\epsilon} && \text{(Ising-like)},\\
&\text{(d)}\ u_{11}^* = \tfrac{1}{4}\bar{\epsilon}, && x^* = \tfrac{1}{2}\bar{\epsilon} && \text{(Gaussian-like)},
\end{aligned} \quad (4.25)$$

where

$$\bar{\epsilon} = \epsilon \ln b / A_{20}^4(r_{11} = 0,\ e_i = 1) = \tilde{K}\epsilon, \quad (4.26)$$

in which $\tilde{K}$ is a constant independent of b but dependent on the shape of the original Brillouin zone. The corresponding eigenvalues $\Lambda_i = b^{\lambda_i}$ and eigenvectors $\vec{y}_i$ of the linearized recursion relations

$$\begin{bmatrix} u'_{11} - u_{11}^* \\ x' - x^* \end{bmatrix} = L_b \begin{bmatrix} u_{11} - u_{11}^* \\ x - x^* \end{bmatrix} \quad (4.27)$$

are given in Table I.

The fixed-point values of r^* follow from (4.21) and thence, by linearization in the standard way,[4,5] the critical exponent ν may be found to order ϵ (see Table I). By (4.11) it follows that $\eta = O(\epsilon^2)$.[5]

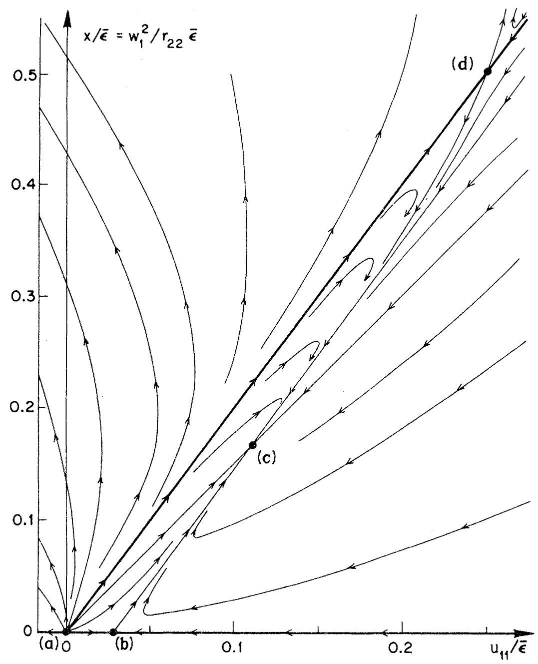

FIG. 3. Fixed points and Hamiltonian flows in the (u_{11}, x) subspace with $x = w_1^2/r_{22}$.

The terms Gaussian, Gaussian-like, and Ising-like, describing the fixed points above, indicate that the exponents found (at least to order ϵ) are the same as those of the normal Gaussian or Ising-like Hamiltonians.[4,5]

To interpret these results we assume initially that $u_{11} > 0$ (as follows from the definitions). For zero uniform field, H, one has $x \equiv 0$; the fixed point (b) is then stable [in the (u_{11}, x) plane] and controls the critical behavior, which is Ising-like ($n=1$), as expected. For small field H and, hence, small initial x there is a crossover to the fixed point (c). This fixed point is stable whenever

$$x = w_1^2/r_{22} < \tfrac{1}{2}u_{11}, \quad (4.28)$$

i.e., for small enough fields H. At first sight a crossover for small fields is surprising since the critical behavior is still expected to be Ising-like. However, since the exponents for fixed points (c) are still of Ising character this expectation is confirmed despite the change of fixed point. (In case our original spins had $n=2$ or more components, a genuine crossover would occur here.[10])

On the borderline $x = \frac{1}{2}u_{11}$, which determines the tricritical field H_t, the fixed point (d) is stable and determines the *tricritical* behavior. For slightly smaller initial fields H the behavior crosses over to the Ising-like fixed point (c); But for larger fields, $x > \frac{1}{2}u_{11}$, there is a "critical run away"[4] leading to negative values of u_{11}. In this circum-

TABLE I. Fixed points, eigenvalues λ_1, and corresponding eigenvectors $\vec{y}_i$ for the recursion relations (4.22) to (4.23). Also given is the exponent ν to $O(\epsilon)$. Note that $\bar{\epsilon}$ is defined in (4.26) and $\eta = O(\epsilon^2)$ at all fixed points.

Fixed points	u_{11}^*	x^*	r^*	λ_1	$\vec{y}_1$	λ_2	$\vec{y}_2$	ν
(a)	0	0	0	$\bar{\epsilon}$	(1, 0)	$\bar{\epsilon}$	(0, 1)	$\frac{1}{2}$
(b)	$\frac{1}{36}\bar{\epsilon}$	0	$\frac{-A_{10}^4(0)\bar{\epsilon}}{3(b^2-1)}$	$-\bar{\epsilon}$	(2, 1)	$\frac{1}{3}\bar{\epsilon}$	(0, 1)	$\frac{1}{2}+\frac{1}{12}\epsilon$
(c)	$\frac{1}{9}$	$\frac{1}{6}\bar{\epsilon}$	$\frac{-A_{10}^4(0)\bar{\epsilon}}{3(b^2-1)}$	$-\frac{1}{3}\bar{\epsilon}$	(1, 2)	$-\bar{\epsilon}$	(2, 3)	$\frac{1}{2}+\frac{1}{12}\epsilon$
(d)	$\frac{1}{4}\bar{\epsilon}$	$\frac{1}{2}\bar{\epsilon}$	0	$-\bar{\epsilon}$	(1, 2)	$\bar{\epsilon}$	(1, 3)	$\frac{1}{2}$

stance sixth-order terms in σ_1 are required to stabilize the Hamiltonian and a new rescaling is needed to keep the mean magnitude of σ_1 finite. This then corresponds to the existence of a first-order transition since the sign of the mean value of σ_1 will be fixed by an infinitesimal ordering field h_1 ($\propto H^\dagger$). The exponents appropriate to the fixed point (d) are the tricritical exponents. But as already explained these are, for small ϵ, just those of a Gaussian fixed point, i.e., in accord with Landau theory.

To this point we have discussed the new fixed points only to leading order. However, it is, in fact, not hard to show that the fixed points (c) and (d) must be Ising-like and Gaussian-like to *all* orders in ϵ. Thus the renormalized Hamiltonian (4.17) is only quadratic in the spin variable $\tilde{\sigma}_2$ and there is no corresponding momentum dependence. Accordingly, in the corresponding partition function we can explicitly integrate over all the $\tilde{\sigma}_{2,\vec{q}}$ variables. Neglecting spin-independent terms this leads to a reduced Hamiltonian

$$\bar{\mathcal{H}}_{\text{red}} = -\frac{1}{2}\int_{\vec{q}}(r_{11}+q^2)\,\sigma_1\,\sigma_1 - \tilde{u}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_1\,\sigma_1\,\sigma_1\,\sigma_1, \quad (4.29)$$

with

$$\tilde{u} = u_{11} - w_1^2/2r_{22}, \quad (4.30)$$

which is independent of the original $\tilde{\sigma}_{2,\vec{q}}$ spins. If the six-spin terms in σ_1 had been carried along they would still appear here; they are specifically needed for stability if $w_1^2 > 2r_{22}u_{11}$ when $\tilde{u}$ becomes negative. For $\tilde{u} > 0$, $\bar{\mathcal{H}}_{\text{red}}$ is a standard Ising-like Hamiltonian. If $\tilde{u}$ can approach zero it becomes precisely the tricritical Hamiltonian treated by Riedel and Wegner.[2,3] Standard analysis shows that the tricritical fixed point remains Gaussian down to three dimensions where logarithmic dependences on H and T appear in the critical behavior.

It is now evident that the crucial feature in our whole calculation was the strong irrelevancy of the parameters e_2, w_2, u_{12}, and u_{22}, which followed from (4.13) to (4.15). About the Gaussian fixed point, ($r_{ii}^* = 0$, $w_i^* = 0$, $u_{ij}^* = 0$) these parameters all remain irrelevant for $\epsilon < 2$ (or $d > 2$). Thus we are justified in integrating out the $\tilde{\sigma}_{2,\vec{q}}$ spin variables and using the reduced Hamiltonian (4.29), down to three dimensions. Hence, we may take over the analysis of Riedel and Wegner for the present model and conclude that the tricritical behavior will be classical except for logarithmic corrections in three dimensions.

V. EFFECTS OF THE ORDERING FIELD

We now discuss the full Hamiltonian (3.3) which includes the ordering field $h_1 \sim H^\dagger$. The procedure will follow that used in Sec. IV. Recursion relations are calculated recursively assumming u_{ij}, w_i, and r_{12} are small. One might include r_{12} in a more general two-component propagator $G_{ij}(\vec{q})$ but this proves unnecessary. As before, the spin rescalings are chosen to keep e_1 and r_{22} fixed. Secondary shifts of both $\tilde{\sigma}_{1,\vec{q}}$ and $\tilde{\sigma}_{2,\vec{q}}$ are now needed at each stage of iteration. The parameters e_2, w_2, u_{12}, and u_{22} are found to be irrelevant as before, with (4.13) to (4.15) still applying. In addition we find

$$w_3' = b^{-1+\epsilon/2}\,[w_3 + O(\epsilon^{3/2})], \quad (5.1)$$

so that w_3 is also strongly irrelevant for small ϵ. After sufficient iterations, therefore, the renormalized Hamiltonian becomes

$$\begin{aligned}\bar{\mathcal{H}}_l \approx &-\frac{1}{2}\int_{\vec{q}}[(r_{11}+q^2)\,\tilde{\sigma}_1\,\tilde{\sigma}_1 + 2r_{12}\,\tilde{\sigma}_1\,\tilde{\sigma}_2 + r_{22}\,\tilde{\sigma}_2\,\tilde{\sigma}_2] \\ &- w_1\int_{\vec{q}}\int_{\vec{q}'}\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_2 - w_4\int_{\vec{q}}\int_{\vec{q}'}\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1 \\ &- u_{11}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1. \end{aligned} \quad (5.2)$$

We may now again integrate out the $\tilde{\sigma}_2$ variables which appear only quadratically to obtain the reduced Hamiltonian

$$\begin{aligned}\bar{\mathcal{H}}_{\text{red}} = &-\frac{1}{2}\int_{\vec{q}}(\tilde{r}+q^2)\,\tilde{\sigma}_1\,\tilde{\sigma}_1 - \tilde{w}\int_{\vec{q}}\int_{\vec{q}'}\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1 \\ &- \tilde{u}\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1\,\tilde{\sigma}_1, \end{aligned} \quad (5.3)$$

depending only on the $\tilde{\sigma}_{1,\vec{q}}$ variables. The reduced parameters are now

$$\tilde{r} = r_{11} - r_{12}^2/r_{22}, \quad \tilde{w} = w_4 - w_1 r_{12}/r_{22}, \tag{5.4}$$

while $\tilde{u}$ is still given by (4.30). However, it must be borne in mind that there have now been a series of shifts in $\tilde{\sigma}_{1,\vec{q}}$ in deriving (5.2). In these shifts the original sixth-order term in $\sigma_{1,\vec{q}}$ contributes towards the value of u_{11} in (5.2) in such a way as to preserve stability. Accordingly, we may assume that $\tilde{u}$ in (5.3) is also positive even though close to the tricritical point (which occurs when $M_1 = 0$) $\tilde{u}$ is vanishing.

Now for the reduced Hamiltonian (5.3) to be at criticality, the coefficient $\tilde{w} = \tilde{w}(H, H^\dagger, T)$ must vanish and $\tilde{r} = \tilde{r}(H, H^\dagger, T)$ must lie on the surface of criticality,[5] which, neglecting higher-order irrelevant variables, we may write $\tilde{r} = \tilde{r}_c(\tilde{u})$. These two constraints determine a line of critical points in the $(H, H^\dagger, T)$ space. In order ϵ^0 these loci may be found from the original Hamiltonian by applying a Landau-type analysis; they are seen to form the edges of the two "wings" in the full tricritical phase diagram.[1] For $\epsilon > 0$ the location of the wings will, of course, shift but they will still be determined in part by irrelevant variables. However, it is clear from the single-component nature of $\overline{\mathcal{K}}_{\text{red}}$ that the critical exponents on the wings must again be Ising-like. It is also clear that the directions in $(H, H^\dagger, T)$ space corresponding to relevant and irrelevant eigenoperators about this Ising fixed point will be skewed[11] relative to their simple directions which are determined by the symmetry of the original Hamiltonian alone when $H = H^\dagger = 0$. We hope to return in the future to study these points, and in particular to determine the tricritical equation of state.

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APPENDIX A: LEADING RECURSION RELATIONS

We quote here the basic recursion relations for the parameters r_{ii}, w_i, and u_{ij} complete to orders ϵ, $\epsilon^{3/2}$, and ϵ^2, respectively, assuming $r_{ii} = O(\epsilon)$, $w_i = O(\epsilon^{1/2})$, $u_{ij} = O(\epsilon)$. These complete the truncated expressions (4.1) to (4.9) given in the text.

Before making the secondary shift discussed in Sec. III, the renormalized interaction parameters are given by

$$\overline{r}_{11} = \hat{c}_1^2 b^{-d}(r_{11} + 12u_{11}A_{10} + 4u_{12}A_{01} - 4w_1^2 A_{11}), \tag{A1}$$

$$\overline{r}_{22} = \hat{c}_2^2 b^{-d}(r_{22} + 12u_{22}A_{01} + 4u_{12}A_{10} - 18w_2^2 A_{02} - 2w_1^2 A_{20}), \tag{A2}$$

$$\overline{w}_1 = \hat{c}_1^2 \hat{c}_2 b^{-2d}(w_1 - 12w_2 u_{12}A_{02} - 12w_1 u_{11}A_{20} - 16w_1 u_{12}A_{11} + 4w_1^3 A_{21} + 12w_1^2 w_2 A_{12}), \tag{A3}$$

$$\overline{w}_2 = \hat{c}_2^3 b^{-2d}(w_2 - 36w_2 u_{22}A_{02} - 4w_1 u_{12}A_{20} + \tfrac{4}{3} w_1^3 A_{30} + 36w_2^3 A_{03}), \tag{A4}$$

$$\overline{u}_{11} = \hat{c}_1^4 b^{-3d}(u_{11} - 36u_{11}^2 A_{20} - 4u_{12}^2 A_{02} + 24u_{11} w_1^2 A_{21} + 8u_{12} w_1^2 A_{12} - 4w_1^4 A_{22}), \tag{A5}$$

$$\overline{u}_{12} = \hat{c}_1^2 \hat{c}_2^2 b^{-3d}(u_{12} - 12u_{11}u_{12}A_{20} - 12u_{22}u_{12}A_{02} - 16u_{12}^2 A_{11} + 8u_{12} w_1^2 A_{21} + 72u_{12} w_2^2 A_{03} + 24u_{11} w_1^2 A_{30} + 24u_{22} w_1^2 A_1 - 8w_1^4 A_{31} - 72w_1^2 w_2^2 A_{13}), \tag{A6}$$

$$\overline{u}_{22} = \hat{c}_2^4 b^{-3d}(u_{22} - 36u_{22}^2 A_{02} - 4u_{12}^2 A_{20} + 8u_{12} w_1^2 A_{30} + 216u_{22} w_2^2 A_{03} - 162w_2^4 A_{04} - 2w_1^4 A_{40}), \tag{A7}$$

There is also a spontaneously generated field $\overline{h}$ appearing as the coefficient of the spin variable $\tilde{\sigma}_{2,\vec{q}}$, namely,

$$\overline{h} = \hat{c}_2(3w_2 A_{01} + w_1 A_{10}). \tag{A8}$$

The various integrals A_{lm} were defined in Eq. (4.10).

To eliminate the linear term (A8), we make a secondary shift in the spin $\tilde{\sigma}_{2,\vec{q}}$. If we let

$$\tilde{\sigma}_{2,\vec{q}} \to \tilde{\sigma}_{2,\vec{q}} + N_a M \delta_{\vec{0},\vec{q}}, \tag{A9}$$

the final recursions relations for r_{ii}, w_i, and u_{ij} are then

$$r'_{11} = \overline{r}_{11} + 2\overline{w}_1 M + 4\overline{u}_{12} M^2, \tag{A10}$$

$$r'_{22} = \overline{r}_{22} + 6\overline{w}_2 M + 12\overline{u}_{22} M^2, \tag{A11}$$

$$w'_1 = \overline{w}_1 + 4\overline{u}_{12} M, \tag{A12}$$

$$w'_2 = \overline{w}_2 + 4\overline{u}_{22} M, \tag{A13}$$

$$u'_{ij} = \overline{u}_{ij}. \tag{A14}$$

The term linear in $\tilde{\sigma}_{2,\vec{q}}$ is eliminated provided M satisfies

$$\overline{h} + 2\overline{r}_{22} M + 3\overline{w}_2 M^2 + 4\overline{u}_{22} M^3 = 0. \tag{A15}$$

In the text, a spin rescaling is chosen such that w_2 and u_{22} go rapidly to zero and r_{22} remains constant as one iterates. Thus, after many iterations one can approximate M by

$$M \approx -\overline{h}/2\overline{r}_{22}. \tag{A16}$$

Insertion of this expression into (A10)–(A14), plus truncation of the irrelevant variables in the resulting recursion relations, leads to the equations quoted in (4.1)–(4.9). The recursion relations quoted in (4.3) and (4.4) for e_1, e_2, and κ have not been treated in detail here, but are adequate for our purposes as given in the text.

APPENDIX B: MOMENTUM DEPENDENCE OF A FEYNMAN GRAPH

We wish to show that the graph in Fig. 2(d) is independent of q^2 to zero order in ϵ. Thus, we must consider the integral

$$I(q)=\int_{\vec{k}}^{>}\frac{1}{(r_2+k^2)[r_1+(\vec{q}-\vec{k})^2]}. \quad \text{(B1)}$$

We may neglect $r_1=O(\epsilon)$ and consider the integral in four dimensions. In the limit of small $\vec{q}$ we can write

$$I(q)=\frac{1}{4\pi^3}\int_{\pi/ab}^{\pi/a}\frac{k^3\,dk}{r_2+k^2}I_1(k,q), \quad \text{(B2)}$$

with

$$I_1(k,q)=\int_0^{\pi}\frac{\sin^2\theta\,d\theta}{q^2+k^2+2qk\cos\theta}. \quad \text{(B3)}$$

But this angular integral has the value

$$I_1(k,q)=\tfrac{1}{2}\pi\min(q^{-2},\,k^{-2}), \quad \text{(B4)}$$

which is independent of q since k satisfies $\pi/ab<k\leq\pi/a$ while we are only concerned with $q\leq\pi/ab$.

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