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Band narrowing of avalanche phonons in paramagnetic materials

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The decay of N initially inverted spins linearly coupled to a harmonic lattice is considered. We first discuss the problem of the time dependence of the magnetization, which is approximately solved by reducing it to one with only two degrees of freedom. Successively, analytic expressions for the power spectrum of the phonons emitted are derived for two different models of the decay of the spin system. The results exhibit band narrowing in both cases. The causes of this narrowing are discussed, particularly for the narrowing which takes place after the avalanche time. In the latter range of times it is suggested that the narrowing is due to a quantum-mechanical effect operating on a macroscopic scale.

I. INTRODUCTION

In two recent papers,[1,2] we have presented a theory of the paramagnetic relaxation from negative temperature of N two-level spins linearly coupled to a harmonic lattice, which is free of statistical assumptions and which is essentially based on the well-known spin-phonon Hamiltonian

$$\mathcal{H}=\sum_k \omega_k \alpha_k^\dagger \alpha_k + \tfrac{1}{2}\omega_0 \sum_r \sigma_z^r + \sum_{k,r} \epsilon(\alpha_k \sigma_+^r e^{ikr} + \text{H. c.}) , \qquad (1.1)$$

where the α operators refer to the phonon spectrum, the σ^r to the spin at site r in the lattice, and ϵ is a simplified k-independent coupling constant. The first of these papers[1] was focused on the time dependence of the magnetization σ_z, which is defined as the difference of the spin-level populations normalized to one, while the interest of the second[2] was centered mainly on the shape of the power spectrum of the phonons emitted in the relaxation process as a function of time.

The main points which emerge from this analysis are the following.

(i) The decay of the magnetization from an initial situation of inverted population and no lattice excitation exhibits a well-pronounced knee which takes the system to a semistable situation of equalized spin-level populations in times orders-of-magnitude shorter than the normal spin-lattice relaxation time.

(ii) The quick decrease in the magnetization corresponds to an avalanche process in which phonons are generated which stimulate the decay of more spins with the generation of further phonons. This self-regenerative process stops when the populations of the spins are equalized, that is when $\sigma_z \sim 0$.

(iii) The width of the power spectrum of the emitted phonons decreases with time during the avalanche, and it is possible to distinguish two main mechanisms of narrowing. The first is due to the fact that modes near to the center of the band of phonons in speaking terms with the spins interact more efficiently than those in the wings; consequently, their populations increase faster. The second is due to the decrease of the width of the band of modes in speaking terms itself, which progressively cuts more and more modes off the region of quasiexponential increase of the phonon population.

(iv) The usual rate equations to describe normal paramagnetic relaxation are not applicable to the model, since they can be derived from Hamiltonian (1.1) only under the assumption of a time-independent phonon bandwidth.

Since the problem is self-consistent in the sense that the time variation of the population of each lattice mode influences the decay of the magnetization and vice versa, it had been necessary in the development of the previously published calculations to make several assumptions based on physical considerations in order to keep the mathematics as simple as possible and not to obscure the physical features of the model. The aim of this paper is to present more rigorously and completely the mathematical aspects of the theory. In this way a twofold objective is attained, since it shall be possible to test the limits of the model and the validity of the approximations previously made, while a neater picture of the narrowing of the power spectrum of the emitted phonons shall result.

This paper is divided into two parts, the first of which coincides with Sec. II and is dedicated to the behavior of the magnetization σ_z as a function of time; in the second part (Secs. III and IV) we shall be concerned with the shape of the power spectrum of the emitted phonons. The starting point in both cases is the equation which is obeyed by the phonon population in the kth mode n_k

$$\frac{d^2 n_k}{dt^2}=(\eta^2\sigma_z - \Delta\omega_k^2)n_k + \eta^2 \frac{1+\sigma_z}{2} , \qquad (1.2)$$

where $\eta^2 = 2N\epsilon^2$, N being the number of spins in the crystal, and where

$$\Delta\omega_k^2 = (\omega_k - \omega_0)^2 .$$

The derivation of Eq. (1.2) from Hamiltonian (1.1) has been discussed in Ref. 1. We only remark that this equation was derived by neglecting terms which in Ref. 1 were called "higher-order correlations." In fact, the interplay of the geometry of the sample and effects coming from the magnitude of relaxation time T_2 may cause these terms to become important in determining the behavior of the system. These effects shall be discussed in a forthcoming paper. Here we shall simply assume that we are in the condition to discard them.

We shall always be concerned in this paper with modes with $|\Delta\omega_k| \leqslant \eta$, and we shall assume that the modes out of this band are too far away from resonance to be affected in any relevant way by the vicissitudes of the magnetization.

II. TIME DEPENDENCE OF σ_z

In Ref. 2 we have shown that from (1.2) it is possible, under rather mild conditions involving the smoothness of $n_k(t)$ as a function of k in the neighborhood of the resonance, to derive the following integral equation for the magnetization of the system at time t, if at time 0 all the spins are in the upper state and no phonon is present in the lattice,

$$\sigma_z(t) = 1 - \frac{g(\omega_0)\pi}{N}\int_0^t \eta^2\{[1+\sigma_z(t')] + 2n_0(t')\sigma_z(t')\}\,dt' , \qquad (2.1)$$

where $g(\omega_0)$ and n_0 are, respectively, the density of states of the lattice modes and the resonant phonon population. Since the integrand in (2.1) can be seen from (1.2) to be equal to

$$\frac{2g(\omega_0)\pi}{N}\frac{d^2n_0}{dt^2} ,$$

one obtains the relation

$$\frac{dn_0}{dt} = B(1-\sigma_z) , \qquad (2.2)$$

where $B = N/2\pi g(\omega_0)$. Moreover, differentiating both sides of (2.1) one gets

$$\frac{d\sigma_z}{dt} = -A(1+\sigma_z+2n_0\sigma_z) , \qquad (2.3)$$

where $A = 2\pi g(\omega_0)\epsilon^2$. Equations (2.2) and (2.3) form a system of two coupled first-order differential equations for the unknown $n_0(t)$ and $\sigma_z(t)$. In this way we have reduced the problem quite drastically from one with N to one with only two degrees of freedom. We proceed by substituting $2\sigma_z$ to the spontaneous emission term $1+\sigma_z$ in (2.3). This should not introduce any serious error since the substituted term is important only at the very beginning of the decay when $\sigma_z \sim 1$. Then eliminating σ_z by (2.2), Eq. (2.3) becomes

$$\frac{1}{B}\frac{d^2n_0}{dt^2} = 2A(n_0+1)\left(1-\frac{1}{B}\frac{dn_0}{dt}\right) . \qquad (2.4)$$

Moreover, if we set

$$p = \frac{dn_0}{dt}, \quad \frac{d^2n_0}{dt^2} = p\frac{dp}{dn_0} ,$$

Eq. (2.4) can be put in the form

$$\frac{p}{B-p}\,dp = 2A(n_0+1)\,dn_0 ,$$

which can be integrated to give

$$B - p - B\ln(B-p) = An_0(n_0+2) + C_1 , \qquad (2.5)$$

where C_1 is a constant which we shall determine from the initial conditions. The problem is now that of integrating first-order nonlinear Eq. (2.5).

From (2.2) we find that in the range of interest ($\sigma_z > 0$) it is always $p \leqslant B$. We then divide this range into two regions where we shall use different approximations.

(a) $p \ll B$. In this range we develop the left-hand side of (2.5) in a power series of p/B, keeping terms up to $O(p^2/B^2)$ so that

$$B - p - B\ln(1-p/B) - B\ln B \simeq B(1-\ln B) + p^2/2B ,$$

and (2.5) becomes

$$2B^2(1-\ln B) + p^2 = 2ABn_0(n_0+2) + C_1 . \qquad (2.6)$$

C_1 is determined by imposing that $\sigma_z = 1$ at $t = 0$, so that $p = 0$ from (2.2). Then

$$C_1 = 2B^2(1-\ln B) ,$$

and substituting in (2.6) yields

$$dn_0/[n_0(n_0+2)]^{1/2} = (2AB)^{1/2}\,dt . \qquad (2.7)$$

Equation (2.7) can be immediately integrated to give

$$n_0 = -1 + \cosh[(2AB)^{1/2}t + C_2] .$$

In this expression C_2 must be zero, since $n_0 = 0$ at $t = 0$. Therefore, the population of the resonant mode varies like

$$n_0(t) = -1 + \cosh(2AB)^{1/2}t , \qquad (2.8)$$

up to times such that $p \ll B$. We can make the definition of this range more precise by introducing a value $p^* = KB$ with $K \leqslant \frac{1}{2}$ and constant. Thus, (2.8) should be valid for $p \leqslant p^*$; we also introduce a time τ corresponding to p^*. From (2.7) we see (neglecting 2 with respect to n_0) that the population of phonons in the resonant mode at $t = \tau$ is given by

$$n_0^* \simeq K(B/2A)^{1/2} , \qquad (2.9)$$

and from (2.8) we find

$$\tau = \frac{1}{(2AB)^{1/2}} \operatorname{arc\,cosh}(n_0^* + 1)$$
$$\simeq \frac{1}{(2AB)^{1/2}} \operatorname{arc\,cosh}\left[K\left(\frac{B}{2A}\right)^{1/2} + 1\right] . \quad (2.10)$$

(b) $p \lesssim B$. In this range the logarithmic term dominates the linear one in the left-hand side of (2.5), which we approximate to

$$B \ln(B - p) = - A n_0 (n_0 + 2) - C_1 . \quad (2.11)$$

The new value of C_1 is found by imposing that $p = p^* \equiv KB$ when $n_0 = n_0^*$. Using (2.9) we obtain from (2.11) (again neglecting 2 with respect to n_0)

$$C_1 = - B[\ln(1 - K)B + \tfrac{1}{2}K^2] . \quad (2.12)$$

Substituting in (2.11) we get within the same approximations

$$p = B[1 - (1 - K)e^{-A n_0^2/B + K^2/2}] . \quad (2.13)$$

We remark that in this range $n_0 > n_0^*$; therefore because of (2.9) the exponential in (2.13) is smaller than 1, and we approximate

$$\frac{dn_0}{dt} = B , \quad (2.14)$$

whose solution is

$$n_0 = Bt + C_2 . \quad (2.15)$$

We find C_2 in (2.15) by imposing that $n_0 = n_0^*$ when $t = \tau$. Using (2.9) and (2.10) in (2.15) we obtain

$$C_2 = (B/2A)^{1/2}\{K - \operatorname{arc\,cosh}[K(B/2A)^{1/2} + 1]\} ,$$

from which we deduce that the growth of the population of the resonant mode in this range of times is approximately of the form

$$n_0(t) = Bt + (B/2A)^{1/2}\{K - \operatorname{arc\,cosh}[K(B/2A)^{1/2} + 1]\}$$
$$= B(t - \tau) + n_0^* . \quad (2.16)$$

The linear increase of n_0 at relatively large times shown by (2.16) may seem surprising from a physical point of view. In fact, it is a consequence of the form of our starting equation (2.2). We shall discuss at length this point in the next sections where we shall consider also the populations of all the other lattice modes.

From (2.8) and (2.13) we can find the time behavior of σ_z using (2.2). We first define σ_z^* as the value assumed by σ_z when $p = p^* \equiv KB$ by

$$p^* \equiv KB = B(1 - \sigma_z^*) ,$$

so that $\sigma_z^* = 1 - K$. Then we obtain directly

$$\sigma_z(t) = 1 - (2A/B)^{1/2} \sinh(2AB)^{1/2} t \quad (\sigma_z > \sigma_z^*) ,$$
$$\sigma_z(t) = (1 - K)e^{-AB(t-\tau)^2 + K^2/2} \quad (\sigma_z < \sigma_z^*) . \quad (2.17)$$

Function (2.17) for $\sigma_z(t)$ is continuous at $t = \tau$ within the limits of the approximations we have made, that is neglecting 1 with respect to n_0^*. This is certainly legitimate since the number of phonons in the resonant mode at $t = \tau$ will be shown later to be of the order of 10^3–10^4. Expression (2.17) should constitute quite a good approximation to the solution of the problem of the time evolution of $\sigma_z(t)$ in the first part of the decay of the magnetization, which forms part of the object of the present paper. The canonical procedure now should be to substitute (2.17) into (1.2) and to solve the resulting differential equation for each mode in the lattice. Unfortunately, it is impossible to do this analytically, and we shall be content with using instead of (2.17) an expression which resembles it, and which has the advantage of being capable of an analytic treatment. Of (2.17) other than the general shape, we shall retain the parameter τ and a value of K which we shall fix on the basis of the approximation we use for $\sigma_z(t)$. In any case we shall define τ as the time at which the magnetization is reduced to one-half.

III. PHONONS FROM A STEP DECAY

We shall find it useful to consider first a relatively simple case when the decay of the magnetization σ_z is approximated by a step such that

$$\sigma_z(t) = 1 \quad (0 < t < \tau) ,$$
$$\sigma_z(t) = 0 \quad (t > \tau) .$$

In this case Eq. (1.2) reduces

$$\frac{d^2 n_k}{dt^2} = (\eta^2 - \Delta\omega_k^2) n_k + \eta^2 \quad (0 < t < \tau) ,$$
$$\frac{d^2 n_k}{dt^2} = - \Delta\omega_k^2 n_k + \tfrac{1}{2}\eta^2 \quad (t > \tau) , \quad (3.1)$$

with the usual boundary conditions

$$n_k(0) = \left.\frac{dn_k}{dt}\right|_{t=0} = 0 . \quad (3.2)$$

We shall solve Eqs. (3.1) separately and then join the solutions at $t = \tau$.

In the first part of (3.1) we put

$$n_k = m_k + c_k , \quad c_k = - \eta^2/(\eta^2 - \Delta\omega_k^2) , \quad (3.3)$$

and the equation becomes

$$\frac{d^2 m_k}{dt^2} = (\eta^2 - \Delta\omega_k^2) m_k , \quad (3.4)$$

with the boundary conditions

$$m_k(0) = \frac{\eta^2}{\eta^2 - \Delta\omega_k^2} , \quad \left.\frac{dm_k}{dt}\right|_{t=0} = 0 . \quad (3.5)$$

The solution of (3.4) with (3.5) is obviously

$$m_k = \frac{\eta^2}{\eta^2 - \Delta\omega_k^2} \cosh(\eta^2 - \Delta\omega_k^2)^{1/2} t , \quad (3.6)$$

so that the solution of (3.1) with (3.2) is

$$n_k(t) = \frac{\eta^2}{\eta^2 - \Delta\omega_k^2}[\cosh(\eta^2 - \Delta\omega_k^2)^{1/2} t - 1] \quad (0 < t < \tau) . \tag{3.7}$$

In the second part of (3.2) we put

$$n_k = m_k + c_k, \quad c_k = \eta^2 / 2\Delta\omega_k^2 , \tag{3.8}$$

and the equation becomes simply

$$\frac{d^2 m_k}{dt^2} = -\Delta\omega_k^2 m_k , \tag{3.9}$$

whose general solution is

$$m_k = a_k \cos\Delta\omega_k t + b_k \sin\Delta\omega_k t , \tag{3.10}$$

where $\Delta\omega_k = |\omega_k - \omega_0|$. Correspondingly, the general solution of the second part of (3.1) is

$$n_k = a_k \cos\Delta\omega_k t + b_k \sin\Delta\omega_k t + \eta^2/2\Delta\omega_k^2 . \tag{3.11}$$

Joining (3.7) and (3.11) at $t = \tau$ leads to

$$n_k(t) = \frac{\eta^2}{\eta^2 - \Delta\omega_k^2}[\cosh(\eta^2 - \Delta\omega_k^2)^{1/2}\tau - 1]\cos\Delta\omega_k(t-\tau) + \frac{\eta^2}{\eta^2 - \Delta\omega_k^2}\left(\frac{(\eta^2 - \Delta\omega_k^2)^{1/2}}{\Delta\omega_k}\sinh(\eta^2 - \Delta\omega_k^2)^{1/2}\tau\right)\sin\Delta\omega_k(t-\tau) + \frac{\eta^2}{\Delta\omega_k^2}\sin^2\frac{\Delta\omega_k}{2}(t-\tau) \quad (t > \tau) . \tag{3.12}$$

Expressions (3.7) and (3.12) give the sought-for solutions for the phonon population as a function of time, in the sense that they are solutions of (3.1) and (3.2). One should be cautious about the meaning of these results, however, since the model we have chosen for the decay of σ_z is very extreme, and for $t < \tau$ entirely nonrealistic from the point of view of the conservation of the total number of excitations. In fact we see that according to (3.7) phonons are created in the lattice before the spins have moved from the upper state. This is clearly unphysical, and it is evident that this model is not capable of describing the conservation of the excitations in the system, which we know may be derived directly from (1.1). In spite of this fact, however, the true equations for each of the modes should not be very different from (3.1) for $t < \tau$, and therefore we expect that the solutions (3.7) should be fairly realistic. In fact if we consider in (3.7) the resonant case, we obtain exactly expression (2.8). The real meaning of this is that if we make a reasonably small-percentage error in choosing our $\sigma_z(t)$, this results in reasonably small errors in each of the n_k; but if we sum all these small errors in the phonon populations when we wish to recover σ_z and eventually conservation of the excitation number, we may end up with a large unbalance due to the very large number of lattice modes. We may therefore expect that even the very extreme behavior of σ_z we have considered yields a reasonable phonon spectrum for $t < \tau$ in the sense that the order of magnitude of the phonon populations should be the right one in this range of times. Moreover, for $t > \tau$ solutions (3.12) have a queer appearance since they show an oscillating behavior and become negative at times which depend on the magnitude of $\Delta\omega_k$. There may be two approximations responsible for this unphysical behavior. The first is higher-order correlations which tend to redistribute the energy between different lattice modes and which we had discarded[1] in the derivation of (1.2) from (1.1). These are certainly small for $t < \tau$ since most of the spins are parallel; later on, however, they might play a role—particularly when the other terms governing the second time derivative of the population of the kth mode become small because of the oscillations, that is when n_k tends to zero. The second is the approximation of $\sigma_z(t)$ which has already been discussed for $t < \tau$; when $t > \tau$ it may give errors in the population of the modes which accumulate in time. In fact, one might think that if we could solve (1.3) with the right self-consistent $\sigma_z(t)$ then the negative phonon populations would disappear. If this is the main source of our difficulties, we may get over it in the following way. We have previously[1] shown that in the Heisenberg representation

$$\epsilon_k \sum_r (\alpha_k \sigma_+^r e^{ikr} + \alpha_k^\dagger \sigma_-^r e^{-ikr}) = -(\omega_k - \omega_0)\alpha_k^\dagger \alpha_k \tag{3.13}$$

is rigorously valid along the spontaneous evolutionary path of our system. If the number of phonons in the kth mode ever happens to become zero at time t', then from (3.13) we deduce that at this time it is also

$$\langle i | \sum_r (\alpha_k \sigma_+^r e^{ikr} + \alpha_k^\dagger \sigma_-^r e^{-ikr}) | i \rangle = \langle i | \sum_r \alpha_k^\dagger \sigma_-^r e^{-ikr} | i \rangle = 0 , \tag{3.14}$$

where $|i\rangle$ is the state with all the spins up and no phonon in the lattice. On the other hand, the first derivative of $\alpha_k^\dagger \alpha_k$ with respect to time is given by

$$\frac{d}{dt}\alpha_k^\dagger \alpha_k = i[\alpha_k^\dagger \alpha_k, \mathcal{H}] = i\epsilon_k \sum_r (-\alpha_k \sigma_+^r e^{ikr} + \alpha_k^\dagger \sigma_-^r e^{-ikr}) , \tag{3.15}$$

and taking the average value of (3.15) at time t' we find from (3.14)

$$\frac{d}{dt}\langle i|\,\alpha_k^\dagger\alpha_k\,|i\rangle|_{t=t'}=0\ .$$

Therefore, we know quite generally that if $n_k(t')$ is zero, also $(dn_k/dt)|_{t=t'}$ must be zero. We may use this information as initial conditions and again solve (1.2) for $t>t'$. Since for most of the modes n_k becomes zero only for $t'>\tau$ when σ_z is practically zero, solutions of (1.2) consist of harmonic oscillations in the population of the kth mode of frequency $\Delta\omega_k$ and of amplitude $\eta^2/2\Delta\omega_k^2$ about the central value $\eta^2/2\Delta\omega_k^2$. These are small amplitude variations of n_k compared to the growth of the populations of the modes near resonance. Therefore, we shall keep to the rule of following the modes only up to the point when n_k becomes zero for the first time, and of neglecting the population of these modes at larger times.

It is interesting to take the limit $\Delta\omega_k\to 0$ in (3.12) since this gives the population of the resonant mode $n_0(t)$ for $t>\tau$. We obtain

$$n_0(t)=n_0^*+\eta(t-\tau)\sinh\eta\tau+\tfrac{1}{4}\eta^2(t-\tau)^2\quad(t>\tau)\ ,\tag{3.16}$$

and this confirms the linear growth of the number of resonant phonons, which we had found in Sec. II, up to times such that

$$\eta\sinh\eta\tau\gtrsim\tfrac{1}{4}\eta^2(t-\tau)\ .\tag{3.17}$$

On the other hand, since up to time τ it is always $\sigma_z=1$, we are entitled from (2.2) to use $K=1$. Then $p^*=B$ and we get $\eta\sinh\eta\tau=B=N/2\pi g(\omega_0)$. Recalling $\eta^2=2N\epsilon^2$, condition (3.17) assumes the form

$$t-\tau\lesssim 2A\ .$$

Since $A^{-1}=T_1$, the ordinary spin-lattice relaxation time, we find that the last term in (3.12) is never likely to play any role because at times when it might be numerically important, the magnetization of the system is decaying in entirely different conditions below $\sigma_z=0$, and the whole treatment given in this paper breaks down. In any case, it is evident from (3.12) that there is an evolution of the whole spectrum of emitted phonons after time τ and that the distribution of phonons tends to become more and more peaked about the resonant frequency. In fact we see that when t is large enough but smaller than T_1 the dominant term on the right-hand side of (3.12) is the second part which becomes of the form

$$\frac{\eta^2}{\eta^2-\Delta\omega_k^2}\left[(\eta^2-\Delta\omega_k^2)^{1/2}\sinh(\eta^2-\Delta\omega_k^2)^{1/2}\tau\right]\pi\delta(\Delta\omega_k)\ .\tag{3.18}$$

Integration of (3.18) over the whole range of $\Delta\omega_k$ is immediate, and we get the total number of phonons in the lattice as

$$\pi g(\omega_0)\eta\sinh\eta\tau=\pi g(\omega_0)B=\tfrac{1}{2}N\ ,\tag{3.19}$$

which is independent of time and coincides with the number of spins which have decayed to the ground state. We see that at relatively large times conservation of the total number of excitations is entirely recovered.

We wish to conclude this section by pointing out one more merit of the treatment we have given. In the Introduction we have recalled that there are two mechanisms which contribute to the narrowing of the power spectrum of the phonons emitted during the avalanche.[2] It is easy to convince oneself that the second of the two, due to the gradual decrease of the width of the band of modes in speaking terms caused by the gradual decrease in $\sigma_z(t)$, is not operative when the spins decay in a step-like fashion. This permits us to isolate the effects of the first mechanism, which is the only one operative here, and this fact shall be found useful in Sec. IV. Moreover, a third source of narrowing is suggested by the present treatment, which operates at larger times after the avalanche is concluded, but when the system has not yet resumed the normal decay. As it is evident from (3.12) and (3.18), this is more remindful of an interference narrowing, which causes power to be transferred from the wings to the central part of the phonon spectrum and which can account for the linear increase of the population of the resonant mode. In other words, the present treatment seems to suggest that the mentioned increase takes place at the expenses of the other modes, whose energy content is progressively exhausted.

IV. PHONONS FROM A FERMI DECAY

A more realistic decay law for the magnetization[3] which is suitable for an analytic treatment, is

$$\sigma_z(t)=1/(e^{\eta(t-\tau)}+1)\ ,\tag{4.1}$$

which we call Fermi like because (4.1) resembles the familiar Fermi distribution low if η^{-1} is KT, t is the energy, and τ the chemical potential. When we operate as in Sec. III

$$n_k=m_k+c_k,\quad c_k=-\eta^2/(\eta^2-\Delta\omega_k^2)\ ,\tag{4.2}$$

Eq. (1.2) becomes

$$\frac{d^2m_k}{dt^2}=(\eta^2\sigma_z-\Delta\omega_k^2)m_k+\frac{1+R_k^2}{1-R_k^2}\,\frac{1-\sigma_z}{2}\,\eta^2\ ,\tag{4.3}$$

where $R_k^2=(\Delta\omega_k/\eta)^2$ varies between 0 and 1. The last term is zero at $t=0$, and at later times is always smaller than the first term on the right-hand side of (4.3), except perhaps in a small region of modes (amounting to a few percent of the total) at the extreme wings of the band of modes in speaking terms with the spin system, where $R_k\simeq 1$. Since the population of these modes is never likely to reach noticeable values, we shall discard the last term in (4. 3) and study the well-known equation[4]

$$\frac{d^2 m_k}{dt^2}=\left(\frac{\eta^2}{e^{\eta(t-\tau)}+1}-\Delta\omega_k^2\right)m_k\ , \tag{4.4}$$

with the same boundary conditions as in Sec. III

$$m_k(0)=\frac{\eta^2}{\eta^2-\Delta\omega_k^2}\ ,\quad \left.\frac{dm_k}{dt}\right|_{t=0}=0; \tag{4.5}$$

the change of variable

$$\xi=-e^{-\eta(t-\tau)}\ , \tag{4.6}$$

takes (4. 4) into the form

$$\xi^2\frac{d^2 m_k}{d\xi^2}+\xi\frac{dm_k}{d\xi}=\left(\frac{1}{1-1/\xi}-R_k^2\right)m_k\ .$$

This equation, after substituting

$$m_k=u_k\xi^{-iR_k}\ , \tag{4.7}$$

becomes

$$\xi(1-\xi)\frac{d^2u_k}{d\xi^2}+(1-2iR_k)(1-\xi)\frac{du_k}{d\xi}+u_k=0\ . \tag{4.8}$$

Equation (4. 8) is a hypergeometric equation whose solution analytic at $\xi=0$ is

$$u_k^{(1)}=F[-iR_k+(1-R_k^2)^{1/2},\ -iR_k-(1-R_k^2)^{1/2};\ 1-2iR_k\ ;\ -e^{-\eta(t-\tau)}]\ . \tag{4.9}$$

A second solution of (4. 8) linearly independent from (4. 9) is, for $R_k\neq 0$,

$$u_k^{(2)}=e^{-2i\Delta\omega_k(t-\tau)}F[iR_k-(1-R_k^2)^{1/2},\ iR_k+(1-R_k^2)^{1/2};\ 1+2iR_k\ ;-e^{-\eta(t-\tau)}]\ . \tag{4.10}$$

The case $R_k=0$ (resonant model) shall be treated separately at the end of this section. Using (4. 9) and (4. 10) we find two linearly independent solutions of (4. 4) from (4. 7) as

$$m_k^{(1)}=e^{i\Delta\omega_k(t-\tau)}F[-iR_k+(1-R_k^2)^{1/2},\ -iR_k-(1-R_k^2)^{1/2};\ 1-2iR_k\ ;-e^{-\eta(t-\tau)}]\ , \tag{4.11}$$

$$m_k^{(2)}=e^{-i\Delta\omega_k(t-\tau)}F[iR_k-(1-R_k^2)^{1/2},\ iR_k+(1-R_k^2)^{1/2};\ 1+2iR_k\ ;-e^{-\eta(t-\tau)}]\ . \tag{4.12}$$

We note that $m_k^{(2)}=[m_k^{(1)}]^*$ because $F(a,b;c;z)=F(b,a;c;z)$. Imposing boundary conditions (4. 5) we shall now find the linear combination of (4. 11) and (4. 12) suitable for our purposes. The equations in (4. 5) are initial boundary conditions, however, and at $t=0$ the argument of both the hypergeometric functions in (4. 11) and (4. 12) are $-e^{\eta\tau}$, which is much larger than one in most of the realistic conditions, as will be shown in Sec. V that $\eta\tau=7$–10. Therefore, hypergeometric series expansion of (4. 11) and (4. 12) would not be legitimate at $t=0$, and it is convenient to continue analytically these functions out of the unit circle by the well-known relationship[5]

$$\begin{aligned}F[-iR_k+(1-R_k^2)^{1/2},-iR_k-(1-R_k^2)^{1/2};1-2iR_k;-e^{-\eta(t-\tau)}]\\ =A_k e^{[-iR_k+(1-R_k^2)^{1/2}]\eta(t-\tau)}F[-iR_k+(1-R_k^2)^{1/2},iR_k+(1-R_k^2)^{1/2};1+2(1-R_k^2)^{1/2};-e^{\eta(t-\tau)}]\\ +B_k e^{[-iR_k-(1-R_k^2)^{1/2}]\eta(t-\tau)}F[-iR_k-(1-R_k^2)^{1/2},iR_k-(1-R_k^2)^{1/2};1-2(1-R_k^2)^{1/2};-e^{\eta(t-\tau)}]\ ,\end{aligned} \tag{4.13}$$

and by its complex conjugate, where we have put for simplicity

$$A_k=\frac{\Gamma(1-2iR_k)\Gamma[-2(1-R_k^2)^{1/2}]}{\Gamma[-iR_k-(1-R_k^2)^{1/2}]\Gamma[1-iR_k-(1-R_k^2)^{1/2}]}\ ,$$

$$B_k=\frac{\Gamma(1-2iR_k)\Gamma[2(1-R_k^2)^{1/2}]}{\Gamma[-iR_k+(1-R_k^2)^{1/2}]\Gamma[1-iR_k+(1-R_k^2)^{1/2}]}\ . \tag{4.14}$$

The following exact relationship between these coefficients shall be found useful in the future:

$$A_kB_k^*-A_k^*B_k=iR_k/(1-R_k^2)^{1/2}\ . \tag{4.15}$$

Expression (4. 13) and its complex conjugate are now suitable for expansion by hypergeometric series in the neighborhood of $t=0$; in view of the smallness of the exponential at this time we retain only the first term in the series and obtain

$$F[-iR_k+(1-R_k^2)^{1/2},-iR_k-(1-R_k^2)^{1/2};1-2iR_k;-e^{-\eta(t-\tau)}]\simeq e^{-i\Delta\omega_k(t-\tau)}(A_k e^{(1-R_k^2)^{1/2}\eta(t-\tau)}+B_k e^{-(1-R_k^2)^{1/2}\eta(t-\tau)})\ . \tag{4.16}$$

Substitution of (4. 16) and its complex conjugate in (4. 11) and (4. 12) yields the following expressions valid in the neighborhood of $t\sim 0$:

$$\begin{aligned}m_k^{(1)}&\simeq A_k e^{(1-R_k^2)^{1/2}\eta(t-\tau)}+B_k e^{-(1-R_k^2)^{1/2}\eta(t-\tau)}\ ,\\ m_k^{(2)}&\simeq A_k^* e^{(1-R_k^2)^{1/2}\eta(t-\tau)}+B_k^* e^{-(1-R_k^2)^{1/2}\eta(t-\tau)}\ .\end{aligned} \tag{4.17}$$

We now apply the boundary conditions (4. 5) to (4. 17) and obtain the full solution of (4. 4) as

$$m_k(t)=a_km_k^{(1)}+b_km_k^{(2)}\ , \tag{4.18}$$

where

$$a_k=b_k^*=\frac{i}{2R_k(1-R_k^2)^{1/2}}(A_k^*e^{-(1-R_k^2)^{1/2}\eta\tau}-B_k^*e^{(1-R_k^2)^{1/2}\eta\tau})\ , \tag{4.19}$$

and where $m_k^{(1)}$ and $m_k^{(2)}$ are given by (4.11) and (4.12) or by the equivalent expressions obtained by the analytic continuation (4.13). In the latter case, using (4.15) we have explicitly

$$m_k(t)=\frac{1}{2(1-R_k^2)}\left[e^{-(1-R_k^2)^{1/2}\eta t}F[-iR_k-(1-R_k^2)^{1/2},\, iR_k-(1-R_k^2)^{1/2};\, 1-2(1-R_k^2)^{1/2};\, -e^{\eta(t-\tau)}]\right.$$
$$\left.+e^{(1-R_k^2)^{1/2}\eta t}F[-iR_k+(1-R_k^2)^{1/2},\, iR_k+(1-R_k^2)^{1/2};\, 1+2(1-R_k^2)^{1/2};\, -e^{\eta(t-\tau)}]\right]. \tag{4.20}$$

Expression (4.20) is an exact solution of Eq. (4.4), and it is related to the occupation number of phonons in the kth mode by the very simple relation (4.2). Its form however, is not simple, and in order to understand its behavior as a function of time we are compelled to use approximations.

We consider first the case $\eta(t-\tau)\ll 0$, and remark that this does not necessarily imply that ηt is small, but only that we are far enough from the avalanche time. In this case we can develop the hypergeometric functions in (4.20) in a power series of $-e^{\eta(t-\tau)}$ keeping only the first two terms in the hypergeometric series. Then we have

$$m_k(t)\simeq\frac{1}{2(1-R_k^2)}\left[e^{-(1-R_k^2)^{1/2}\eta t}\left(1+\frac{e^{\eta(t-\tau)}}{-1+2(1-R_k^2)^{1/2}}\right)\right.$$
$$\left.+e^{(1-R_k^2)^{1/2}\eta t}\left(1-\frac{e^{\eta(t-\tau)}}{1+2(1-R_k^2)^{1/2}}\right)\right]. \tag{4.21}$$

We remark that the divergence in (4.21) for the modes with $R_k=\sqrt{\tfrac{3}{2}}$ is only apparent and it is due to the divergence of the first hypergeometric series in (4.20) for $1-2(1-R_k^2)^{1/2}=0$. We might eliminate it by a suitable linear transformation formula for the hypergeometric function, but this would involve a considerable algebraic labor which would obscure the implicity of the result of our approximation (4.21). We rather prefer to avoid this pseudodivergence by sampling the band of the modes in speaking terms. Consequently, we shall consider two regions of modes. The first is that of the wings where we shall approximate $(1-R_k^2)^{1/2}\simeq 0$ in the denominators of (4.21); this, however, excludes the very small number of modes which have been discussed at the beginning of this section and for which Eq. (4.4) is not valid for small times. In the second region we simplify (4.21) by substituting both denominators with $(2-R_k^2)$; this is valid in a central zone of frequencies extending from $R_k=0$ to about $R_k\sim\frac{1}{2}$. Then we immediately get from (4.21) in the wings of the band

$$m_k(t)\simeq\frac{1}{1-R_k^2}\cosh[(1-R_k^2)^{1/2}\eta t](1-e^{\eta(t-\tau)}) . \tag{4.22}$$

This has an interesting appearance, since it shows that the phonon population in the wings, far from the avalanche, is smaller than in the case of the step-like decay. This is to be ascribed to the slow decrease of σ_z from the initial value which is not present in the case of Sec. III. This difference tends to increase with time. In the central part of the band we obtain

$$m_k(t)\simeq\frac{1}{1-R_k^2}\left(\cosh(1-R_k^2)^{1/2}\eta t-\frac{e^{\eta(t-\tau)}}{2-R_k^2}\sinh(1-R_k^2)^{1/2}\eta t\right), \tag{4.23}$$

which shows the same effect of reduced growth. However, since

$$\frac{\sinh(1-R_k^2)^{1/2}\eta t}{2-R_k^2}<\cosh(1-R_k^2)^{1/2}\eta t ,$$

the reduction in the growth of the modes near the center is smaller than that of the modes in the wings of the band. Consequently, this indicates the beginning of a narrowing due to the decrease of σ_z. If we assume that $\eta t>1$, then we may approximate

$$\cosh(1-R_k^2)^{1/2}\eta t\sim\sinh(1-R_k^2)^{1/2}\eta t$$

in (4.23) and we may give a qualitative idea of the growth of the phonons in both ranges of frequencies by adopting

$$m_k(t)\simeq\frac{1}{1-R_k^2}\cosh[(1-R_k^2)^{1/2}\eta t]\left(1-\frac{e^{\eta(t-\tau)}}{2-R_k^2}\right) \tag{4.24}$$

instead of (4.22) and (4.23). Comparison of (4.24) with (3.6) suggests us to define a scaling function $F(t, R_k)$ which may be taken as a measure of the narrowing of the power spectrum of the phonons due to the gradual decrease of σ_z prior to the avalanche as

$$F(t, R_k)=1-e^{\eta(t-\tau)}/(2-R_k^2) . \tag{4.25}$$

We see that F as a function of R_k tends to become more and more narrow as time increases, and also that even the resonant mode is affected.

We now wish to consider the region of times up to $\eta(t-\tau)\sim 1$, that is when the avalanche is already well developed. Consequently, we cannot use the expansion which led to (4.21), and we have to resort to other approximations. In the wings of the band we approximate

$$F[-iR_k-(1-R_k^2)^{1/2},\, iR_k-(1-R_k^2)^{1/2};\, 1-2(1-R_k^2)^{1/2};\, -e^{\eta(t-\tau)}]\simeq F(-iR_k,\, iR_k;\, 1;\, -e^{\eta(t-\tau)}) ,$$

and use

$$F(-iR_k, iR_k; 1; -e^{\eta(t-\tau)}) \simeq \tfrac{1}{2}F(-iR_k, iR_k; \tfrac{1}{2}; -e^{\eta(t-\tau)}) + \tfrac{1}{2} ,$$

which is an approximate relation and can be obtained by expanding both sides by the hypergeometric series for $t<\tau$. This allows us to use[5]

$$F(-a, a; \tfrac{1}{2}; \sin^2 z) = \cos 2az ,$$

where we put

$$\sin^2 z = -e^{\eta(t-\tau)} ,$$

$$\cos 2az = \cos\{2R_k \ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\} .$$

Consequently, in this range of frequencies we approximate both hypergeometric functions in (4.20) by

$$\tfrac{1}{2}\cos\{2R_k \ln[e^{(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\} + \tfrac{1}{2} . \quad (4.26)$$

Expression (4.20) then becomes

$$m_k(t) \simeq \frac{1}{1-R_k^2}\cosh[(1-R_k^2)^{1/2}\eta t] \times (\tfrac{1}{2}\cos\{2R_k \ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\} + \tfrac{1}{2}) . \quad (4.27)$$

The scaling function $F(t, R_k)$ that can be deduced from (4.27) for times up to $\eta(t-\tau)\sim 1$ and for the modes in the wings is therefore given by (4.26). For the central region, we approximate in (4.20)

$$F[-iR_k-(1-R_k^2)^{1/2}, iR_k-(1-R_k^2)^{1/2}; 1-2(1-R_k^2)^{1/2}; -e^{\eta(t-\tau)}] \sim \Gamma[1-2(1-R_k^2)^{1/2}]\tfrac{1}{2}R_k^2 e^{2\eta(t-\tau)} \times F(1-iR_k, 1+iR_k; 3; -e^{\eta(t-\tau)}) , \quad (4.28)$$

where use has been made of[5]

$$\lim_{c\to -1}\frac{1}{\Gamma(c)}F(a, b; c; z) = \tfrac{1}{2}a(a+1)b(b+1)z^2F(a+2, b+2; 3; z) .$$

Laurent expansion[6] of $\Gamma[1-2(1-R_k^2)^{1/2}]$ about the singularity at $R_k=0$ shows that the product $R_k^2\Gamma[1-2(1-R_k^2)^{1/2}]$ remains finite and near to -1 for all the modes in the range we are considering. We approximate also

$$F[-iR_k+(1-R_k^2)^{1/2}, iR_k+(1-R_k^2)^{1/2}; 1+2(1-R_k^2)^{1/2}; -e^{\eta(t-\tau)}] \simeq F(1-iR_k, 1+iR_k; 3; -e^{\eta(t-\tau)}) . \quad (4.29a)$$

When (4.28) and (4.29a) are substituted in (4.20), one can easily see that the contribution coming from the former is a factor $e^{-2\eta\tau}$ smaller than the contribution of (4.29a) and can be discarded. Therefore, since $\eta t \gg 1$ and

$$e^{(1-R_k^2)^{1/2}\eta t} \simeq 2\cosh(1-R_k^2)^{1/2}\eta t ,$$

we may write

$$m_k(t) \simeq \frac{1}{1-R_k^2}\cosh[(1-R_k^2)^{1/2}\eta t] \times F(1-iR_k, 1+iR_k; 3; -e^{\eta(t-\tau)}). \quad (4.29b)$$

From (4.29b) we deduce a scaling factor

$$F(t, R_k) = F(1-iR_k, 1+iR_k; 3; -e^{\eta(t-\tau)}) .$$

Moreover,

$$F(1-iR_k, 1+iR_k; 3; -e^{\eta(t-\tau)}) \simeq \tfrac{1}{2}F(1-iR_k, 1+iR_k; \tfrac{3}{2}; -e^{\eta(t-\tau)}) + \tfrac{1}{2} , \quad (4.30)$$

as can be checked by expanding both sides by the hypergeometric series for $t<\tau$. Using the well-known relation[6]

$$F(1+\tfrac{1}{2}a, 1-\tfrac{1}{2}a; \tfrac{3}{2}; \sin^2 z) = 2\sin az/a\sin 2z ,$$

where

$$a = -2iR_k, \quad z = i\ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}] ,$$

expression (4.30) becomes

$$F(1-iR_k, 1+iR_k; \tfrac{3}{2}; -e^{\eta(t-\tau)}) \simeq \frac{\sin\{2R_k\ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\}}{R_k\sinh\{2\ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\}} ,$$

and the scaling factor for the central modes up to times such that $\eta(t-\tau)\sim 1$ is

$$F(t, R_k) \simeq \frac{\sin\{2R_k\ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\}}{2R_k\sinh\{2\ln[e^{\eta(t-\tau)/2} + (1+e^{\eta(t-\tau)})^{1/2}]\}} + \tfrac{1}{2} . \quad (4.31)$$

When (4.26) and (4.31) are plotted as functions of R_k for different times, they reveal a progressive narrowing of the power spectrum of the emitted phonons up to times just after the main part of the avalanche; the rate of narrowing is more marked in this region of times before as expected, since most of the modes we are considering are cut off the region of quick growth by the decrease of σ_z near $t\sim\tau$.

We now consider the situation for large times when $\eta(t-\tau)\gg 1$, and we concentrate on the central part of the band where R_k is small. At these times expressions (4.20) is not a manageable starting point however, since the argument of the hypergeometric functions appearing in it is large, and we turn to (4.18) with $m_k^{(1)}$ and $m_k^{(2)}$ given by (4.11) and (4.12). We need first explicit expressions for a_k and b_k. When the various $\Gamma(z)$ in (4.14) are expanded in Laurent series about the appropriate singularities,[5] the following approximate expressions are obtained:

$$A_k \simeq \tfrac{1}{2}\Gamma(1-2iR_k), \quad B_k \simeq \frac{\Gamma(1-2iR_k)}{\Gamma(1-iR_k)\Gamma(2-iR_k)} ,$$

which are regular in the neighborhood of $R_k=0$. Consequently, in (4.19) we may neglect the term with the negative exponential as a factor, due to the magnitude of $\eta\tau$. We thus obtain

$$a_k \simeq -\frac{i\Gamma(1+2iR_k)}{2R_k\Gamma(1+iR_k)\Gamma(2+iR_k)} e^{(1-R_k^2)^{1/2}\eta\tau}; \quad b_k = a_k^* . \tag{4.32}$$

Using expressions (4.32) to weigh (4.11) and (4.12) in (4.13) we find

$$m_k(t) \simeq -e^{(1-R_k^2)^{1/2}\eta\tau} \mathrm{Re}\left(\frac{i\Gamma(1+2iR_k)}{R_k\Gamma(1+iR_k)\Gamma(2+iR_k)}\right) \times e^{i\Delta\omega_k(t-\tau)} F(1-iR_k, 1+iR_k; \times 1-2iR_k; -e^{-\eta(t-\tau)}) . \tag{4.33}$$

This is still a rather opaque expression, but it reduces to a familiar one if we neglect the imaginary parts in the arguments of the Γ functions because of the smallness of R_k and if we retain only the first term in the hypergeometric series, which is one, because of the smallness of the exponential. Then (4.33) reduces to

$$m_k(t) \simeq \eta e^{(1-R_k^2)^{1/2}\eta\tau} \frac{\sin\Delta\omega_k(t-\tau)}{\Delta\omega_k} \underset{t\to\infty}{\longrightarrow} \eta e^{\eta\tau}\pi\delta(\omega_k-\omega_0) . \tag{4.34}$$

Since τ is the time at which $\sigma_z = \frac{1}{2}$, we use $K = \frac{1}{2}$ and from Sec. II we get approximately

$$e^{\eta\tau} \sim (B/2A)^{1/2}, \quad \eta = (2AB)^{1/2} ,$$

so that (4.34) can be written as

$$m_k(t) \sim n_k(t) \sim \pi B\delta(\omega_k - \omega_0) .$$

From this we easily find

$$\int_{-\eta}^{\eta} n_k(t) g(\omega_k) d(\Delta\omega_k) \simeq \pi B g(\omega_0) = \tfrac{1}{2}N , \tag{4.35}$$

which is analogous to (3.19) and reassures us about conservation of the number of excitations at large times.

We now go back to investigate the resonant mode, which we call the degenerate case because solutions (4.9) and (4.10) coincide for $R_k = 0$. In these circumstances $m_0 = u_0$ and the differential equation becomes

$$\frac{d^2 m_0}{dt^2} = \eta^2 \sigma_z m_0 . \tag{4.36}$$

Two linearly independent solutions of (4.36) are[6]

$$m_0^{(1)} = F(1, -1; 1; -e^{-\eta(t-\tau)}) = F(-1, 1; 1; -e^{-\eta(t-\tau)}),$$
$$m_0^{(2)} = e^{2\eta(t-\tau)}(1+e^{-\eta(t-\tau)}) F(2, 2; 3; -e^{\eta(t-\tau)}) , \tag{4.37}$$

the first of which reduces simply to

$$m_0^{(1)} = 1 + e^{-\eta(t-\tau)} . \tag{4.38}$$

In the second we use[5]

$$F(2, 2; 3; z) = 2\frac{d}{dz} F(1, 1; 2; z) = -2\frac{d}{dz}\left(\frac{\ln(1-z)}{z}\right) ,$$

and we find

$$m_0^{(2)} = 2[-1 + (1+e^{-\eta(t-\tau)})\ln(1+e^{\eta(t-\tau)})] . \tag{4.39}$$

For small t, $(t-\tau)$ is large and negative, and (4.39) can be approximated by

$$m_0^{(2)} \simeq e^{\eta(t-\tau)} . \tag{4.40}$$

Boundary conditions (4.5) can be imposed on the general solution

$$m_0(t) \simeq a m_0^{(1)}(t) + b m_0^{(2)}(t) ,$$

where $m_0^{(1)}$ and $m_0^{(2)}$ are given by (4.38) and (4.40). We find

$$a = (1 + 2e^{\eta\tau})^{-1}, \quad b = (e^{-2\eta\tau} + 2e^{-\eta\tau})^{-1} ,$$

and consequently

$$m_0(t) \simeq \frac{1+e^{-\eta(t-\tau)}}{1+2e^{\eta\tau}} + \frac{2}{e^{-2\eta\tau}+2e^{-\eta\tau}}[-1 + (1+e^{-\eta(t-\tau)})\ln(1+e^{\eta(t-\tau)})] . \tag{4.41}$$

When t is small, neglecting $e^{-2\eta\tau}$ with respect to $e^{-\eta\tau}$, (4.41) reduces to

$$m_0(t) \simeq \cosh\eta t ,$$

which gives the same time dependence for $n_0(t)$ as in the case of step-like decay before the avalanche. At $t = \tau$ (4.41) gives

$$m_0(\tau) \simeq e^{\eta\tau}(2\ln 2 - 1) \simeq 0.4 e^{\eta\tau} ,$$

which should be compared with the value of $0.5e^{\eta\tau}$ deducible from (3.12) for the step-like decay. At times t such that $\eta(t-\tau) \gg 1$ expression (4.41) can be approximated by

$$m_0(t) \simeq e^{\eta\tau}[\eta(t-\tau) - 1] ,$$

which again confirms the linear growth of the phonons in the resonant mode at large time and is in good agreement with the limiting case that can be found from (4.34) when $\Delta\omega_k$ tends to zero.

V. DISCUSSION AND CONCLUSIONS

We wish first to calculate relative orders of magnitude of the most relevant parameters we have used in this paper in order to show the consistency of the approximations we have done at various stages during the development of the theory. We observe that our model does not include direct dipolar interactions between the spins. Consequently, our conclusions should be valid for systems where the phonon interruption time is shorter than the spin-spin relaxation time T_2. An interesting example is that of the Cu salt studied by Giordmaine *et al.*[7] which we have discussed in Ref. 2. Here we shall consider two cases; in both of them we take $K=\frac{1}{2}$ and the number of paramagnetic centers per unit volume $N=10^{18}$. In the first case we assume $2\pi g(\omega_0)=5\times10^5$ sec and $T_1=A^{-1}=10^{-5}$ sec, and we find

$$B=N/2\pi g(\omega_0)=2\times10^{12}\ \text{sec}^{-1}\ ,$$

$$\eta=(2AB)^{1/2}\simeq 6.5\times10^{8}\ \text{sec}^{-1}\ ,$$

$$n_0^*=K(B/2A)^{1/2}\simeq 1.5\times10^{3}\ ,$$

$$\tau\simeq 1.2\times10^{-8}\ \text{sec}\ .$$

In the second case we assume $2\pi g(\omega_0)=5\times10^6$ sec and $T_1=10^{-2}$ sec and we get

$$B\simeq 2\times10^{11}\ \text{sec}^{-1},\quad \eta\simeq 6.3\times10^{6}\ \text{sec}^{-1}\ ,$$

$$n_0^*\simeq 1.6\times10^{4},\quad \tau\simeq 1.6\times10^{-6}\ \text{sec}\ .$$

The numbers given here show that the avalanche may develop up to times of the order of microseconds, while the equivalent temperature of phonons in the resonant mode at the avalanche time may become of the order of $(10^3-10^4)\,^\circ$K. We remark that the rate of growth of the phonons in the resonant mode after the avalanche is quite large, being given by B, and this results in a significant increase in the number of resonant phonons already at time 2τ after the avalanche. These conclusions are also valid for the Fermi-like decay, where we may also see that it is legitimate to neglect $e^{-\eta\tau}$ with respect to one, since $\eta\tau$ is of the order of 10.

We now wish to propose an interpretation of our results from the point of view of the time-energy uncertainty principle. It is well known[8] that if we have a single spin in the lattice, which decays from the upper to the ground state, and if we assume that the magnetic energy of these two states of the spin is well defined, then the quantum of radiation in the elastic field will be emitted preferably in a band of frequencies of width $A=T_1^{-1}$ where T_1 is the characteristic time for the decay process. This is a consequence of the time-energy uncertainty principle, since in this case the duration of the interaction between the spin and the emitted phonon is on the average of T_1 seconds, the phonons being free to escape far from the spin after this time. In the case of N spins which form the argument of this paper, the situation is not the same even for the decay of the first few spins, since the phonons emitted cannot escape into a region devoid of interaction and the time a phonon lives before turning another spin, becomes very short, giving rise to a spread in the spectrum of the phonon occupation numbers which initially is of the order of η.

After this initial stage, but before the avalanche time τ, it is convenient to distinguish two effects. First, we observe that each bunch of spins which may be assumed to emit at the same time finds a situation which is different from that of the other bunches because of the presence of a different number of stimulating phonons; more precisely, the characteristic time associated with the decay of a bunch of spins is proportional to the inverse of the number of resonant phonons present in the lattice, and consequently it is shorter than that of the preceding bunches. Therefore, we expect that because of this effect, the width of the power spectrum of the emitted phonons be larger than $1/t$ as one would predict from a naive use of the uncertainly principle with the equality sign, since the whole spectrum should result from the superposition of phonons emitted at different times and with decreasing characteristic times up to time t. The second effect is the spread due to the interruption time, which is still small in this range of times in spite of the decrease of σ_z; this would tend to decrease the width of the spectrum of the emitted phonons, but we have shown that it does not play a noticeable role up to about time τ. The net result at times before the avalanche should then be a band larger than $1/t$, and in fact we have shown in a preceding paper [Ref. 2, Eq. (15)] that the bandwidth satisfies

$$\langle\Delta\omega\rangle(\eta t)^{1/2}\simeq\sqrt{2}\,\eta.$$

This tells us that after $\eta t=1$ the band becomes larger than the miminum required by a formulation of the uncertainty principle in which the interaction which causes the decay is assumed to begin at $t=0$ for all spins which have been reversed.

The situation is different late after the avalanche, since then the average value of σ_z is near zero, and the phonon interruption time has become much larger than before τ. Consequently, this contribution to the linewidth is silenced and the bandwidth is essentially influenced by the fact that the great majority of phonons are emitted at time τ. From expressions (3.12) and (4.34) for the number of phonons in the two models we have studied, we find that after the avalanche the width of the phonon distribution is approximately given by

$$\langle\Delta\omega\rangle\simeq(t-\tau)^{-1}\ .$$

This indicates that $\langle\Delta\omega\rangle$ is larger than would be expected if the beginning of the interaction causing the decay were set at $t=0$. It is as if the uncertainty principle were valid with the equality sign, but the beginning of the interaction were shifted at the avalanche time. In other words, since we are dealing with a regenerative process, the effective strength of the interaction increases suddenly at τ with the number of emitted phonons, and the system behaves as if the interaction had actually started at this time. Since the interaction between spins and phonons goes on after τ, the narrowing of the power spectrum of the emitted phonons at large times should be interpreted as a quantum-mechanical effect on a macroscopic scale.

In conclusion, in this paper we have shown that the problem of the decay of the magnetization from negative temperatures may, by suitable approximations, be reduced from a problem in which $O(N)$ lattice modes are coupled to the spins, to a problem with only the resonant modes coupled to the spin, and we have approximately solved the latter, giving explicit time dependences for both the magnetization and the number of phonons in the resonant modes. Furthermore, we have presented detailed analytic calculations of the power spectrum of phonons emitted by N paramagnetic spins $s=\frac{1}{2}$ during an avalanche. We have performed two sets of calculations, in each of them assuming a different decay law for the magnetization of the spin system: we have called these two models the step-like and the Fermi-like decay, respectively. In this way it has been possible to confirm quantitatively that the parameters of the spectrum of the emitted phonons vary during and after the avalanche, showing a definite tendency towards narrowing. By comparing the two types of decay considered, we have been able to separate two mechanisms for this narrowing, which operate before and during the first part of the avalanche, and which we had discussed in a semiquantitative fashion in another paper.[2] Moreover, we have pointed out the possibility of further narrowing after the avalanche is concluded and we have suggested that this third source of narrowing may be the manifestation on a macroscopic scale of a quantum-mechanical effect. Since our model does not include any spin-spin interaction other than that mediated by the phonons in the lattice, the present theory should be applicable only to crystals where the phonon interruption time is smaller than T_2. We cannot exclude, however, that the validity of the present theory goes beyond these limits,[9] since the stirring in the spin system caused by the avalanche phonons might considerably reduce the effects of some of the relevant parameters which we have not taken into account.

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