Letter

No eigenstate of the critical transverse-field Ising chain satisfies the area law

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We argue that, in a basis common to all one-site shift-invariant conserved charges, there is no eigenstate of a noninteracting local spin-$\frac{1}{2}$ chain Hamiltonian that satisfies the area law if the ground state has half-integer central charge. That is to say, in those models all (quasi)local one-site shift invariant conserved operators are gapless. Using both analytical and numerical techniques, indeed, we calculate the bipartite Rényi entropies and show that there are three distinct one-site shift-invariant noninteracting models, two of which are equivalent to the XX model (for one of them the transformation breaks one-site shift invariance) and the other to the critical Ising model. The former class has two locally distinct one-site shift-invariant excited states satisfying the area law; the latter two classes have none.

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The ground states of shift-invariant spin-chain systems with gapped local Hamiltonians are low entangled: the entropy of a block of spins has only a subleading dependence on the block's length [1]. This is known as "area law" and applies to systems in higher dimensions as well [2]. In 1D the area law generally breaks down at quantum phase transitions, where the entropy of a spin block can develop a logarithmic dependence on its length [3]. In contrast, excited states are expected to follow a volume law [4]: the entropy of a block of spins is proportional to the block's length. Exceptions to this rule (nearly) define the so-called quantum many-body scars [5], which are excited states with (generally) anomalously low bipartite entanglement. Integrable systems are quite exceptional in this respect, as they exhibit infinitely many excited states with subextensive entropies and energies that are extensively larger than the ground state one [6,7]. This is a direct consequence of the existence of infinitely many local conservation laws, indeed the ground state of any conserved charge with fast enough decaying interactions is an eigenstate of the Hamiltonian with subextensive bipartite entanglement. Both the typical properties of excited states [6–36] and the critical properties of low-entangled ones [37–43] have attracted some attention. How many locally different excited states satisfy, however, the area law? This is a tricky, somewhat ill-defined question that has been either overlooked or addressed just incidentally and/or carelessly. On one hand, the answer could depend on the basis chosen to diagonalize the Hamiltonian and degeneracies could also be sensitive to the system's size. On the other hand, the choice of the basis can be physical, for example, in investigations into the stationary behavior of observables when the system is prepared in some nonequilibrium initial state. That was our very motivation for starting this investigation. In contrast to the ordinary behavior reported in Refs. [44,45], we have indeed observed an unusually slow decay of connected correlations at late times after a quench between one-site shift-invariant ("1-ß" in the following) critical Hamiltonians in the Ising universality class. We then wondered whether something special characterizes the excited states of the critical Ising model. In 1-ß noninteracting spin chains that are mapped to free fermions by a Jordan-Wigner transformation, such as the quantum Ising model, a relevant basis of excited states consists of 1-ß Slater determinants for the Jordan-Wigner fermions. In any such excited state, the bipartite entanglement entropies can be easily computed numerically, and the entropies of large subsystems can be predicted on the basis of the asymptotic behaviors of determinants of Toeplitz and block-Toeplitz matrices [46], which have been thoroughly investigated (see, e.g., Refs. [47,48]). This opportunity is definitely rare and has already been exploited to quantify the picture summarized above [6,41,49]. In particular, it was shown that there are infinitely many excited states that can be described by zero-temperature conformal field theories (CFTs) with half-integer or integer central charges. More vague are the statements about excited states satisfying the area law. It looks like there is a common belief that one should expect infinitely many states of that kind. We point out here that the picture is quite different. Specifically, only a finite number of locally different excited states seem to satisfy the area law, and there are notable cases, such as the critical Ising model, where there are none. We argue that this no-go theorem applies to every noninteracting 1-ß local Hamiltonian with a critical ground state described by a CFT with half-integer central charge. We show it in two steps. First, we explicitly study the XX model (XX)

$$\mathbf{H}_{\text{XX}} = J\sum_{\ell} \boldsymbol{\sigma}^x_\ell \boldsymbol{\sigma}^x_{\ell+1} + \boldsymbol{\sigma}^y_\ell \boldsymbol{\sigma}^y_{\ell+1}, \tag{1}$$

the critical Ising model (CI)

$$\mathbf{H}_{\text{CI}} = -J\sum_{\ell} \boldsymbol{\sigma}^x_\ell \boldsymbol{\sigma}^x_{\ell+1} + \boldsymbol{\sigma}^z_\ell, \tag{2}$$

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and the strongly anisotropic XY model (X-X)

$$\mathbf{H}_{\text{X-X}} = J\sum_{\ell} \boldsymbol{\sigma}^x_\ell \boldsymbol{\sigma}^x_{\ell+1} - \boldsymbol{\sigma}^y_\ell \boldsymbol{\sigma}^y_{\ell+1}. \tag{3}$$

We aver that, among them, only the XX model has eigenstates satisfying the area law (in the 1-ß basis); and no excited state is described by a CFT with half-integer central charge. We then prove that every 1-ß noninteracting Hamiltonian can be mapped into a 1-ß conservation law of XX, CI, or X-X by a discrete and/or a continuous local unitary transformation. Such transformations do not affect the asymptotic dependency of the bipartite entropies on the subsystem's length, hence the generality of our findings.

Noninteracting spin chains. We consider a spin-$\frac{1}{2}$ chain with L sites described by a generic 1-ß Hamiltonian (with periodic boundary conditions) that can be mapped to free fermions by a Jordan-Wigner transformation $\mathbf{a}_{2\ell-1} = \prod_{j<\ell} \boldsymbol{\sigma}^z_j \boldsymbol{\sigma}^x_\ell$, $\mathbf{a}_{2\ell} = \prod_{j<\ell} \boldsymbol{\sigma}^z_j \boldsymbol{\sigma}^y_\ell$, where $\mathbf{a}_j$ are Majorana fermions ($\{\mathbf{a}_i, \mathbf{a}_j\} = 2\delta_{ij}$), which in the following will be imagined as sitting on pseudosites (j in $\mathbf{a}_j$). The excited states $|\{p\}\rangle$ split in two sectors, usually called Ramond (+) [50] and Neveu-Schwarz (–) [51], differing only in the quantization conditions satisfied by the momenta p of the excitations: $e^{iLp} = 1$ in Ramond and $e^{iLp} = -1$ in Neveu-Schwarz. An excited state is in Ramond if $\prod_{\ell=1}^L \boldsymbol{\sigma}^z_\ell |\{p\}\rangle = -|\{p\}\rangle$ and in Neveu-Schwarz if $\prod_{\ell=1}^L \boldsymbol{\sigma}^z_\ell |\{p\}\rangle = |\{p\}\rangle$; in the fermionic picture it is an eigenstate of a quadratic form of fermions $\mathbf{H}^\pm = \frac{1}{4}\sum_{\ell,n} \mathbf{a}_\ell \mathcal{H}^\pm_{\ell n} \mathbf{a}_n$ with periodic (+) or anti-periodic (−) boundary conditions (i.e., $\mathcal{H}^+$ is block circulant and $\mathcal{H}^-$ block anticirculant), respectively. As such, each excited state is fully characterized by the fermionic two-point correlations, which are customarily organized in a matrix $\Gamma_{\{p\}} = \mathrm{I} - \langle\{p\}|\mathbf{a}\otimes\mathbf{a}|\{p\}\rangle$, known as correlation matrix. Translational invariance makes it convenient to work in the Fourier space. We refer the reader to Ref. [52] for a review of some free-fermion techniques; in the following we only list some useful known results. The 2-by-2 block-Fourier transform (also known as symbol) $\hat{\Gamma}^{(2)}(k)$ of a correlation matrix Γ is 2π periodic and satisfies $\hat{\Gamma}^{(2)}(k) = [\hat{\Gamma}^{(2)}(k)]^\dagger = -[\hat{\Gamma}^{(2)}(-k)]^t$ (by the fermionic algebra) [53]. Restricting to excited states, $\hat{\Gamma}^{(2)}_{\{p\}}(k)$ can be written in terms of the block-Fourier transform $\hat{\mathcal{H}}^{(2)}(k)$ of $\mathcal{H}^\pm$ as follows. First, $\hat{\Gamma}^{(2)}_{\{p\}}(k)$ commutes with $\hat{\mathcal{H}}^{(2)}(k)$ and, for given k, has eigenvalues ± 1; each choice of eigenvalues associated with the eigenvectors corresponds to a different excited state, and, in particular, for the ground state we have $\hat{\Gamma}^{(2)}_{\text{GS}}(k) = -\text{sgn}(\hat{\mathcal{H}}^{(2)}(k))$. Second, in the basis diagonalizing $\hat{\mathcal{H}}^{(2)}(k)$ in such a way that the first eigenvalue is identified with the excitation energy $\varepsilon(k)$ [and the second is, in turn, $-\varepsilon(-k)$], adding an excitation with momentum $\bar{p}$ to $|\{p\}\rangle$ ($\bar{p} \notin \{p\}$) has a double effect [54], $\Sigma_{\{p\}\cup\{\bar{p}\}}(\pm\bar{p}) = \mp\sigma^z\Sigma_{\{p\}}(\pm\bar{p})$, where $\Sigma_{\{p\}\cup\{\bar{p}\}}(\pm\bar{p})$ is the diagonal matrix equivalent to the symbol of the correlation matrix (see also Ref. [6]).

Entanglement entropies. The entanglement properties of a bipartition $A\cup\bar{A}$ in a pure state can be measured by the von Neumann entropy $S_1(A) = -\text{tr}[\rho_A \log\rho_A]$ [55], or, more generally, by the Rényi entropies $S_\alpha(A) = \log\text{tr}[\rho_A^\alpha]/(1-\alpha)$ [56]. Let $\mathbf{O}_A$ be an observable acting nontrivially only in a connected subsystem A. By the Wick's theorem, its expectation value in an excited state $|\{p\}\rangle$ can be expressed in terms of the submatrix $\Gamma_{\{p\},A}$ of $\Gamma_{\{p\}}$ corresponding to restricting the indices to the sites in A. A's correlation matrix $\Gamma_{\{p\},A}$ is in turn block-Toeplitz with symbol $\hat{\Gamma}^{(2)}_{\{p\}}(k)$. From now on we use the notation $\mathrm{T}^{(n)}_N$ to indicate (nN)-by-(nN) block-Toeplitz matrices with n-by-n blocks. Several representations of the Rényi entropies of a spin block A in terms of $\Gamma_{\{p\},A}$ are known. Here we use a formula that naturally arises when computing the Rényi entropies in free quantum field theories [45,57]

$$S_\alpha(A) = \sum_{j=0}^{\alpha-1} \frac{\log\det\left|\varrho^{(2)}_A\left(\frac{\pi(j+\frac{1}{2})}{\alpha}\right)\right|}{2(1-\alpha)}, \tag{4}$$

where $\varrho^{(2)}_A(\phi)$ has the 2-by-2 symbol

$$\hat{\varrho}^{(2)}(k;\phi) = \sin\phi\,\mathrm{I}_2 - i\cos\phi\,\hat{\Gamma}^{(2)}_{\{p\}}(k). \tag{5}$$

This allows us to carry out a qualitative analysis of bipartite entanglement without specifying the Rényi index α. Before starting on it, we warn the reader that the entanglement properties of excited states could become basis dependent especially in superintegrable (also known as non-Abelian integrable) systems, such as the quantum XY model in zero field [58], or in integrable systems with Hilbert-space fragmentation [59], such as the dual folded XXZ model [60], in which there are exponentially large degenerate sectors spanned by low-entangled states. In our case degeneracies can be originated by the symmetries of the dispersion relation or by accidental equivalences of multiparticle energies. Generally, both kinds of degeneracies can be lifted by adding a 1-ß conservation law to the Hamiltonian and/or by changing the chain's length. This selects a natural 1-ß basis of excited states.

Our first step will be an analytical study, which is mainly a reorganization of results that are already known or that can be readily derived from what is known.

XX model. The XX model has the $U(1)$ symmetry of rotations about z, i.e., the Hamiltonian commutes with $\mathbf{S}^z = \frac{1}{2}\sum_\ell \boldsymbol{\sigma}^z_\ell$. This is enough to conclude that there are at least two locally different excited states satisfying the area law, $|\cdots\uparrow\uparrow\cdots\rangle$ and $|\cdots\downarrow\downarrow\cdots\rangle$. Our goal in this case is to establish that any other state of that kind is locally equivalent to one or to the other. Let us start by writing the symbol of the correlation matrix in $|\{p\}\rangle$,

$$\hat{\Gamma}^{(2)}_{\{p\}}(k) = \frac{\hat{\Sigma}^{(1)}_{\{p\}}(k) - \hat{\Sigma}^{(1)}_{\{p\}}(-k)}{2}\mathrm{I}_2 + \frac{\hat{\Sigma}^{(1)}_{\{p\}}(k) + \hat{\Sigma}^{(1)}_{\{p\}}(-k)}{2}\sigma^y, \tag{6}$$

where $\hat{\Sigma}^{(1)}_{\{p\}}(k) \in \{-1, 1\}$ and, in particular, $\hat{\Sigma}^{(1)}_{\emptyset}(k) = -\text{sgn}(\cos k)$. The simple structure of the symbol is translated into a simple structure of the correlation matrix, which allows one to express the entropies as determinants of $|A|$-by-$|A|$ Toeplitz matrices,

$$\det\varrho^{(2)}_A(\phi) = \left|\det\left|\sin\phi\,\mathrm{I}_{|A|} + i\cos\phi\,\Sigma^{(1)}_A\right|\right|^2, \tag{7}$$

where $\Sigma^{(1)}_A$ has symbol $\hat{\Sigma}^{(1)}_{\{p\}}(k)$ (see B in the Supplemental Material, SM [61]). In the thermodynamic limit we can apply the result proved by Szegö in Ref. [62], and obtain the asymptotic behavior as $|A| \to \infty$,

$$\frac{\log\det \varrho_A^{(2)}(\phi)}{|A|} \to \int_{-\pi}^{\pi}\frac{dk}{2\pi}\log|\sin^2\phi+\cos^2\phi[\hat\Sigma^{(1)}(k)]^2|. \tag{8}$$

Importantly, $\hat\Sigma^{(1)}(k)$ captures the thermodynamic limit of the symbol of the excited state, which, to be precise, should be referred to as a macrostate, in that it represents infinitely many locally equivalent excited states. The symbol, in turn, can assume any value in $[-1, 1]$.

The excited states with minimal entropy are subextensive and, in turn, their $\hat\Sigma^{(1)}(k)$ should make the right-hand side of (8) vanish, that is to say, $\hat\Sigma^{(1)}(k) \in \{-1, 1\}$. All sequences of excited states with subextensive entropy are therefore characterized by a piecewise constant symbol $\hat\Sigma^{(1)}(k)$ with a given number n of discontinuities. If the latter is finite, the discontinuities are simple examples of Fisher-Hartwig singularities [47,63]. Specifically, each discontinuity gives the additive asymptotic contribution $-(\frac{\phi}{\pi}-\frac{1}{2})^2\log|A|$ to $\log\det|\sin\phi \mathrm{I}_{|A|}+i\cos\phi\Sigma_A^{(1)}|$ in (7). The corresponding states behave as ground states of CFTs with integer central charges $\frac{n}{2}$. Their symbol can indeed be reinterpreted as the sign of the symbol of a quasilocal quadratic form of fermions and vice versa. The area law can be satisfied only if there are no discontinuities at all, i.e., $\hat\Sigma^{(1)}(k) = \pm 1$. These are the symbols of the states that are locally equivalent to $|\cdots\uparrow\uparrow\cdots\rangle$ and $|\cdots\downarrow\downarrow\cdots\rangle$. Consequently, the ground state of any gapped conservation law with exponentially decaying interactions is either $|\cdots\uparrow\uparrow\cdots\rangle$ or $|\cdots\downarrow\downarrow\cdots\rangle$. It can also be argued (see E in the SM [61]) that there should not be exceptions in the presence of infinitely many discontinuities in the symbol, as the bipartite entanglement tends to increase with their number (such states can also be interpreted as ground states of long-range Hamiltonians [64]); a rigorous proof of it is, however, still lacking.

Critical Ising model. The critical Ising model is special as $\mathcal{H}^{\pm}$ is not only antisymmetric block-(anti)circulant, but it is antisymmetric (anti)circulant (antisymmetry manifests the fermionic algebra). This property can be almost directly translated to the correlation matrices of the excited states. There is only a subtlety due to the twofold degeneracy of the eigenstates in the Ramond sector, which break the invariance under one pseudo-shift (see A in the SM [61]). Since our search for states satisfying the area law is not affected by this issue, when needed, we will consider the incoherent superposition of the two degenerate states, which is described by a circulant matrix $\Gamma^{(1)}$ with symbol $\hat\Gamma^{(1)}_{\{p\}}(k) \in \{-1, 1\}$ for $k \neq 0, \pi$ and $\hat\Gamma^{(1)}(0) = \hat\Gamma^{(1)}(\pi) = 0$. We then have (see B in the SM [61]) that the entropy is computed from

$$\det\varrho_A^{(2)}(\phi) = \det\left|\sin\phi \mathrm{I}_{2|A|}+i\cos\phi\Gamma_A^{(1)}\right|. \tag{9}$$

In the thermodynamic limit the Szegö lemma gives

$$\frac{\log\det \varrho_A^{(2)}(\phi)}{|A|} \to \int_{-\pi}^{\pi}\frac{dk}{2\pi}\log|\sin^2\phi+\cos^2\phi[\hat\Gamma^{(1)}(k)]^2|, \tag{10}$$

which matches the expression for the XX model provided that $\Sigma^{(1)}(k)$ is replaced by $\Gamma^{(1)}(k)$. The qualitative analysis is therefore almost identical. The unique but important deviation comes from the fact that, while $\Sigma^{(1)}(k)$ could be any function with image in $[-1, 1]$, $\Gamma^{(1)}(k)$ is forced to be odd by the fermionic algebra. Consequently, the smallest number of discontinuities in the symbol is 2 (one at $k = 0$ and one at $k = \pi$) and corresponds to the ground state or to the state with maximal energy. These states break the area law with a logarithmic term corresponding to a CFT with central charge $\frac{1}{2}$. Contrary to the XX model, there is no excited state satisfying the area law that can be interpreted as the ground state of a quasilocal Hamiltonian, that is to say, every quasilocal conservation law of the critical Ising model is gapless. As also observed in Ref. [41], we find that the central charges of the CFTs describing the low-entangled excited states of CI can be both half-integer and (nonzero) integer.

X-X model. The X-X model, described by Hamiltonian (3), is equivalent to the XX model, indeed $\mathbf{H}_{\text{X-X}} = \mathbf{\Pi}^{zy}\mathbf{H}_{\text{XX}}\mathbf{\Pi}^{zy}$, where $\mathbf{\Pi}^{zy} = \prod_j \boldsymbol{\sigma}^z_{2j-1}\boldsymbol{\sigma}^y_{2j}$. The similarity transformation, however, breaks one-site shift invariance, hence only the 1-ß charges of the XX model that are even under $\mathbf{\Pi}^{zy}$ are mapped to 1-ß charges of the X-X model. The remaining 1-ß charges of X-X correspond to conservation laws of XX that break one-site shift invariance into a two-site one, whose existence was established in Ref. [58]. Thus, we are forced to treat the X-X model independently of the XX one.

The symbol of the correlation matrix in $|\{p\}\rangle$ reads

$$\Gamma^{(2)}_{\{p\}}(k) = \frac{\hat\omega^{(1)}_{+,\{p\}}(k)+\hat\omega^{(1)}_{-,\{p\}}(k)}{2}\mathrm{I} + \frac{\hat\omega^{(1)}_{+,\{p\}}(k)-\hat\omega^{(1)}_{-,\{p\}}(k)}{2}\sigma^x, \tag{11}$$

where $\hat\omega^{(1)}_{s,\{p\}}(k) \in \{-1, 1\}$ are odd in k. Just as we did in CI, we are overlooking the zero-energy modes $k = 0, \pi$, which would give a contribution $(s_0\delta_{k0} + s_\pi\delta_{k\pi})\sigma^y$ (see A in the SM [61]). We can again (see B in the SM [61]) decompose the block-Toeplitz matrix so as to express the entropies in terms of determinants of Toeplitz matrices,

$$\det\varrho_A^{(2)}(\phi) = \prod_{s=\pm1}\det\left|\sin\phi \mathrm{I}_{|A|}+i\cos\phi\omega^{(1)}_{s,A}\right|. \tag{12}$$

The right-hand side is the product of two determinants of the Ising kind (9), therefore the entanglement entropies are the sum of the entropies of two excited states of the Ising model. The absence of a state satisfying the area law in the critical Ising model implies in turn the same in the X-X model. Remarkably, also here the central charges of the CFTs describing the low-entangled excited states can be both half-integer and integer.

Numerical analysis. Although persuasive, our analytical hints are based on the assumption that the local properties of any excited state can be described with a symbol that does not depend on the subsystem's length; but this is too restrictive. We can indeed easily imagine exotic sequences of excited states that do not fit in the classification above: e.g., for any given $|A|$, we can find excited states in which the density of excitations is $\sim\frac{\log|A|}{|A|}$, contaminating the family of states that break the area law logarithmically by states that satisfy the volume law. In order to rule out unconventional states satisfying the area law, we resort to a numerical analysis.

We start considering finite chains and evaluate the half-chain von Neumann entropy of all excited states. Those

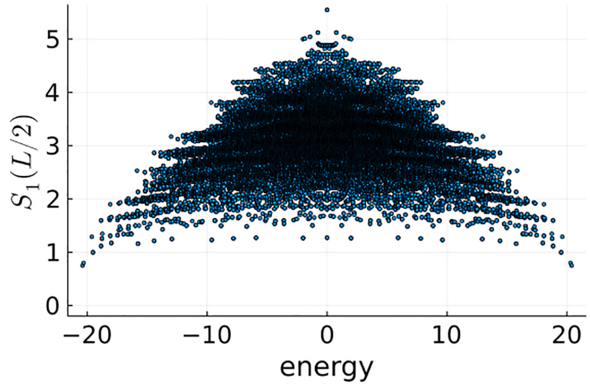

FIG. 1. Entropy of all the 1-ß eigenstates of CI as a function of their energy for $L = 16$.

satisfying the area law correspond to minima of such an entropy landscape that remain bounded even increasing the chain's length L. For small L, we can recognize the global minima by a brute-force search. This is shown in Fig. 1 for CI, which suggests that the ground state and the maximal energy state have the lowest bipartite entanglement and points, in turn, at the absence of excited states satisfying the area law. A real quantitative difference between states satisfying or breaking the area law appears, however, only in much larger chains. The number of eigenstates, on the other hand, diverges exponentially with L, thus computing the entropy of all eigenstates becomes soon unfeasible. We circumvent this problem by means of simulated annealing: We initialize the system in a random eigenstate $|\{p\}\rangle$ generated from a uniform distribution. At each step we make a particle-hole transformation at a random momentum and accept the new excited state $|\{p'\}\rangle$ with probability $\min\{\exp(\frac{S_1(\frac{L}{2};\{p\})-S_1(\frac{L}{2};\{p'\})}{\mathfrak{s}}), 1\}$ in analogy with the Metropolis-Hastings algorithm. Here $\mathfrak{s}$ is a parameter that depends on the iteration and is chosen so as to approach 0 at the end of the simulation (more details about the algorithm are reported in C in the SM [61]). Figure 2 shows a run of it in CI. In a reasonable number of iterations the excited state becomes of the form that we investigated analytically (i.e., the number of discontinuities in its symbol becomes independent of L) and finally the algorithm selects the ground state or the maximal energy state. Similar analyses for XX and X-X have confirmed the pictures drawn before and are reported in D in the SM [61].

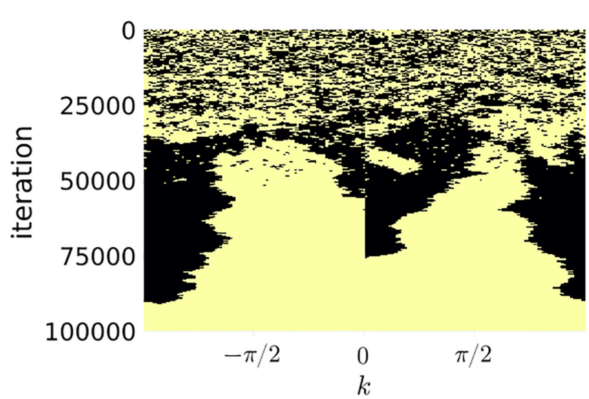

FIG. 2. A single run of simulated annealing converging to the ground state of CI in the Neveu-Schwarz sector with $L = 200$. In yellow (black) the particle (hole) momenta.

A lower bound. The last proof we give is based on a conjecture. In E in the SM [61] we provide evidence that the half-chain von Neumann entropy of an XX excited state for which $\hat{\Sigma}^{(1)}_{\{p\}}(k)$ has n changes of signs satisfies

$$S_1(A) \geqslant \left(\frac{\log 3}{2} - \frac{\log 2}{3}\right)n, \tag{13}$$

where $|A| = \frac{L}{2}$. For CI, if we consider the discontinuities of the scalar symbol, an analogous expression applies with an overall factor 1/2. The area law can only hold if n is finite. Since two locally different states differ in an extensive number of excitations, such a state would be always locally equivalent to one of the states that we addressed in the thermodynamic limit with a finite number of discontinuities smaller than or equal to n. Thus, (13) implies our conclusions.

Equivalence of 1-ß quasilocal models. Reference [65] showed that a local operator that time evolves for a finite time under a local Hamiltonian can still be approximated by another local operator with an error (in operator norm) that decreases exponentially with its range. Such transformations are used to define topological phases of matter [66] and will be denoted by $\mathfrak{L}$. They are relevant to us because they preserve the asymptotic dependency of the bipartite entropies on the subsystem's length. To that aim, however, one can also consider discrete transformations, denoted by $\mathfrak{D}$, that do not have a local generator but act as $\mathfrak{L}$ on local operators. An example is a shift by a finite number of sites. Since we are studying the entropies of connected blocks, we can overlook the difference between spins and fermions [67,68] and preserve locality in whichever of the two spaces. This allows us to include also transformations such as a shift by one pseudosite. In F in the SM [61] we show

$$\hat{\mathcal{H}}^{(2)}(k) \overset{\mathfrak{L}}{\mapsto} \begin{cases} e^{ink\sigma^z}\sigma^y & \overset{\mathfrak{D}}{\mapsto} \quad \sigma^y \\ \sin k e^{ink\sigma^z}\sigma^x & \overset{\mathfrak{D}}{\mapsto} \quad \sin k\sigma^x \\ \cos\frac{k}{2}e^{i(n-\frac{1}{2})k\sigma^z}\sigma^y & \overset{\mathfrak{D}}{\mapsto} \quad \sin\frac{k}{2}e^{i\frac{k}{2}\sigma^z}\sigma^x \\ \sin\frac{k}{2}e^{i(n-\frac{1}{2})k\sigma^z}\sigma^x & \overset{\mathfrak{D}}{\mapsto} \quad \sin\frac{k}{2}e^{i\frac{k}{2}\sigma^z}\sigma^x \end{cases} \tag{14}$$

where all symbols are defined up to multiplication by a smooth even function. In other words, we recognize four families of topologically different models. After additional discrete transformations, the first family is mapped to XX, the second one to X-X, and the last two of them to CI. Since only in XX there are excited states satisfying the area law but there are no quasilocal conserved operators whose ground state is described by a half-integer central charge, we can conclude that, quite generally, there are no excited states satisfying the area law when the ground state of a 1-ß noninteracting Hamiltonian is described by a CFT with half-integer central charge.

Discussion. We have reported a no-go theorem connecting the central charge of the conformal field theory describing the low-energy properties of a noninteracting spin chain with the existence of excited states satisfying the area law. Concerning the generality of our findings, there are several open questions. Can the assumption of one-site shift invariance be partially relaxed? Does something similar apply also in the presence of interactions? In the thermodynamic limit any linear combination of excited states in a collapsing energy shell is conserved to all intents and purposes; does the result hold true also in such quasistationary states?

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[1] M. B. Hastings, An area law for one-dimensional quantum systems, J. Stat. Mech.: Theory Exp. (2007) P08024.
[2] J. Eisert, M. Cramer, and M. B. Plenio, *Colloquium*: Area laws for the entanglement entropy, Rev. Mod. Phys. **82**, 277 (2010).
[3] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A: Math. Theor. **42**, 504005 (2009).
[4] P. Hayden, D. W. Leung, and A. Winter, Aspects of generic entanglement, Commun. Math. Phys. **265**, 95 (2006).
[5] M. Serbyn, D. A. Abanin, and Z. Papić, Quantum many-body scars and weak breaking of ergodicity, Nat. Phys. **17**, 675 (2021).
[6] V. Alba, M. Fagotti, and P. Calabrese, Entanglement entropy of excited states, J. Stat. Mech.: Theory Exp. (2009) P10020.
[7] W. Beugeling, A. Andreanov, and M. Haque, Global characteristics of all eigenstates of local many-body Hamiltonians: participation ratio and entanglement entropy, J. Stat. Mech.: Theory Exp. (2015) P02002.
[8] J. Bhattacharya, M. Nozaki, T. Takayanagi, and T. Ugajin, Thermodynamical property of entanglement entropy for excited states, Phys. Rev. Lett. **110**, 091602 (2013).
[9] G. Wong, I. Klich, L. A. P. Zayas, and D. Vaman, Entanglement temperature and entanglement entropy of excited states, J. High Energy Phys. 12 (2013) 020.
[10] T. Pálmai, Excited state entanglement in one-dimensional quantum critical systems: Extensivity and the role of microscopic details, Phys. Rev. B **90**, 161404(R) (2014).
[11] J. Mölter, T. Barthel, U. Schollwöck, and V. Alba, Bound states and entanglement in the excited states of quantum spin chains, J. Stat. Mech.: Theory Exp. (2014) P10029.
[12] F. Ares, J. G. Esteve, and F. Falceto, Rényi entanglement entropy in fermionic chains, Int. J. Geom. Methods Mod. Phys. **12**, 1560002 (2015).
[13] H.-H. Lai and K. Yang, Entanglement entropy scaling laws and eigenstate typicality in free fermion systems, Phys. Rev. B **91**, 081110(R) (2015).
[14] V. Alba, Eigenstate thermalization hypothesis and integrability in quantum spin chains, Phys. Rev. B **91**, 155123 (2015).
[15] J. P. Keating, N. Linden, and H. J. Wells, Spectra and eigenstates of spin chain Hamiltonians, Commun. Math. Phys. **338**, 81 (2015).
[16] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, Entanglement entropy of eigenstates of quadratic fermionic Hamiltonians, Phys. Rev. Lett. **119**, 020601 (2017).
[17] L. Vidmar and M. Rigol, Entanglement entropy of eigenstates of quantum chaotic Hamiltonians, Phys. Rev. Lett. **119**, 220603 (2017).
[18] S. Moudgalya, N. Regnault, and B. A. Bernevig, Entanglement of exact excited states of Affleck-Kennedy-Lieb-Tasaki models: Exact results, many-body scars, and violation of the strong eigenstate thermalization hypothesis, Phys. Rev. B **98**, 235156 (2018).
[19] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, Volume law and quantum criticality in the entanglement entropy of excited eigenstates of the quantum Ising model, Phys. Rev. Lett. **121**, 220602 (2018).
[20] M. Mestyán, V. Alba, and P. Calabrese, Rényi entropies of generic thermodynamic macrostates in integrable systems, J. Stat. Mech.: Theory Exp. (2018) 083104.
[21] W. Beugeling, A. Bäcker, R. Moessner, and M. Haque, Statistical properties of eigenstate amplitudes in complex quantum systems, Phys. Rev. E **98**, 022204 (2018).
[22] L. Hackl, L. Vidmar, M. Rigol, and E. Bianchi, Average eigenstate entanglement entropy of the XY chain in a transverse field and its universality for translationally invariant quadratic fermionic models, Phys. Rev. B **99**, 075123 (2019).
[23] C. Murthy and M. Srednicki, Structure of chaotic eigenstates and their entanglement entropy, Phys. Rev. E **100**, 022131 (2019).
[24] T. LeBlond, K. Mallayya, L. Vidmar, and M. Rigol, Entanglement and matrix elements of observables in interacting integrable systems, Phys. Rev. E **100**, 062134 (2019).
[25] S. Murciano, P. Ruggiero, and P. Calabrese, Entanglement and relative entropies for low-lying excited states in inhomogeneous one-dimensional quantum systems, J. Stat. Mech.: Theory Exp. (2019) 034001.
[26] G. C. Levine, Nonlocal fermions and the entropy volume law, Phys. Rev. D **100**, 025017 (2019).
[27] T.-C. Lu and T. Grover, Rényi entropy of chaotic eigenstates, Phys. Rev. E **99**, 032111 (2019).
[28] S. Moitra and R. Sensarma, Entanglement entropy of fermions from Wigner functions: Excited states and open quantum systems, Phys. Rev. B **102**, 184306 (2020).
[29] P. Łydżba, M. Rigol, and L. Vidmar, Eigenstate entanglement entropy in random quadratic Hamiltonians, Phys. Rev. Lett. **125**, 180604 (2020).
[30] J. Angel-Ramelli, Entanglement entropy of excited states in the quantum Lifshitz model, J. Stat. Mech.: Theory Exp. (2021) 013102.
[31] Q. Miao and T. Barthel, Eigenstate entanglement: Crossover from the ground state to volume laws, Phys. Rev. Lett. **127**, 040603 (2021).
[32] Q. Miao and T. Barthel, Eigenstate entanglement scaling for critical interacting spin chains, Quantum **6**, 642 (2022).
[33] M. Haque, P. A. McClarty, and I. M. Khaymovich, Entanglement of midspectrum eigenstates of chaotic many-body systems: Reasons for deviation from random ensembles, Phys. Rev. E **105**, 014109 (2022).
[34] E. Bianchi, L. Hackl, M. Kieburg, M. Rigol, and L. Vidmar, Volume-law entanglement entropy of typical pure quantum states, PRX Quantum **3**, 030201 (2022).
[35] P. Buividovich, Quantum chaos in supersymmetric Yang-Mills-like model: equation of state, entanglement, and spectral form-factors, PoS **(LATTICE2022)**, 246 (2022).

[36] X.-H. Yu, Z. Gong, and J. I. Cirac, Free-fermion page curve: Canonical typicality and dynamical emergence, Phys. Rev. Res. **5**, 013044 (2023).
[37] F. C. Alcaraz, M. I. Berganza, and G. Sierra, Entanglement of low-energy excitations in conformal field theory, Phys. Rev. Lett. **106**, 201601 (2011).
[38] M. I. Berganza, F. C. Alcaraz, and G. Sierra, Entanglement of excited states in critical spin chains, J. Stat. Mech.: Theory Exp. (2012) P01016.
[39] M. Storms and R. R. P. Singh, Entanglement in ground and excited states of gapped free-fermion systems and their relationship with Fermi surface and thermodynamic equilibrium properties, Phys. Rev. E **89**, 012125 (2014).
[40] B. Herwerth, G. Sierra, H.-H. Tu, and A. E. B. Nielsen, Excited states in spin chains from conformal blocks, Phys. Rev. B **91**, 235121 (2015).
[41] A. Jafarizadeh and M. A. Rajabpour, Bipartite entanglement entropy of the excited states of free fermions and harmonic oscillators, Phys. Rev. B **100**, 165135 (2019).
[42] J. Zhang and M. A. Rajabpour, Corrections to universal Rényi entropy in quasiparticle excited states of quantum chains, J. Stat. Mech.: Theory Exp. (2021) 093101.
[43] J. Zhang and M. A. Rajabpour, Entanglement of magnon excitations in spin chains, J. High Energy Phys. 02 (2022) 072.
[44] V. Marić and M. Fagotti, Universality in the tripartite information after global quenches, Phys. Rev. B **108**, L161116 (2023).
[45] V. Marić and M. Fagotti, Universality in the tripartite information after global quenches: (generalised) Quantum XY models, J. High Energ. Phys. 06 (2023) 140.
[46] A. R. Its and V. E. Korepin, The Fisher-Hartwig formula and entanglement entropy, J. Stat. Phys. **137**, 1014 (2009).
[47] P. Deift, A. Its, and I. Krasovsky, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities, Ann. Math. **174**, 1243 (2011).
[48] E. Basor, Asymptotics of determinants of block Toeplitz matrices, Random Matrices: Theory Appl. **06**, 1740003 (2017).
[49] F. Ares, J. G. Esteve, F. Falceto, and E. Sánchez-Burillo, Excited state entanglement in homogeneous fermionic chains, J. Phys. A: Math. Theor. **47**, 245301 (2014).
[50] P. Ramond, Dual theory for free fermions, Phys. Rev. D **3**, 2415 (1971).
[51] A. Neveu and J. Schwarz, Factorizable dual model of pions, Nucl. Phys. B **31**, 86 (1971).
[52] M. Fagotti, Charges and currents in quantum spin chains: Late-time dynamics and spontaneous currents, J. Phys. A: Math. Theor. **50**, 034005 (2017).
[53] We are defining a block-Fourier transform rather than a simple Fourier transform because there are two Majorana fermions per site.
[54] Note that only an even number of excitations preserves the sector.
[55] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).
[56] A. Rényi, *Probability Theory* (North-Holland Publishing, Amsterdam, 1970), p. 670.
[57] H. Casini and M. Huerta, Entanglement entropy in free quantum field theory, J. Phys. A: Math. Theor. **42**, 504007 (2009).
[58] M. Fagotti, On conservation laws, relaxation and pre-relaxation after a quantum quench, J. Stat. Mech. (2014) P03016.
[59] S. Moudgalya and O. I. Motrunich, Hilbert space fragmentation and commutant algebras, Phys. Rev. X **12**, 011050 (2022).
[60] L. Zadnik and M. Fagotti, The folded spin-1/2 XXZ Model: I. Diagonalisation, jamming, and ground state properties, SciPost Phys. Core **4**, 010 (2021).
[61] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.109.L201116 for (A) Zero energy modes; (B) Determinant formulas for the entropies; (C) Numerical minimization procedure; (D) Additional numerical findings; (E) Lower bounds for the von Neumann entropy; (F) Classification of noninteracting models.
[62] G. Szegő, Beiträge zur theorie der toeplitzschen formen, Math. Z. **6**, 167 (1920).
[63] E. Basor and H. Widom, Toeplitz and Wiener-Hopf determinants with piecewise continuous symbols, J. Funct. Anal. **50**, 387 (1983).
[64] G. Gori, S. Paganelli, A. Sharma, P. Sodano, and A. Trombettoni, Explicit Hamiltonians inducing volume law for entanglement entropy in fermionic lattices, Phys. Rev. B **91**, 245138 (2015).
[65] S. Bravyi, M. B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, Phys. Rev. Lett. **97**, 050401 (2006).
[66] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, *Quantum Information Meets Quantum Matter* (Springer, New York, 2019).
[67] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in quantum critical phenomena, Phys. Rev. Lett. **90**, 227902 (2003).
[68] M. Fagotti and P. Calabrese, Entanglement entropy of two disjoint blocks in XY chains, J. Stat. Mech.: Theory Exp. (2010) P04016.