Letter

Boundary-deconfined quantum criticality at transitions between symmetry-protected topological chains

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Decades of research have revealed a deep understanding of topological quantum matter with protected edge modes. We report that even richer physics emerges when tuning between two topological phases of matter whose respective edge modes are incompatible. The frustration at the edge leads to novel boundary physics, such as symmetry-breaking phases with exotic non-Landau transitions—even when the edge is zero-dimensional. As a minimal case study, we consider spin chains with $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry, exhibiting two nontrivial symmetry-protected topological (SPT) phases. At the bulk 1+1D critical transition between these SPT phases, we find two stable 0+1D boundary phases, each spontaneously breaking one of the $\mathbb{Z}_3$ symmetries. Furthermore, we find that a single boundary parameter tunes a non-Landau boundary critical transition between these two phases. This constitutes a 0+1D version of an exotic phenomenon driven by charged vortex condensation known as deconfined quantum criticality. This work highlights the rich unexplored physics of criticality between nontrivial topological phases and provides insights into the burgeoning field of gapless topological phases.

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Symmetry-protected edge modes in phases of matter are well-understood when there is a finite energy gap to creating excitations in the bulk [1–3]. For instance, in 1D systems [4–8] this leads to topologically protected ground state degeneracies which are exponentially localized near the endpoints [9,10]. However, edge modes at phase transitions and criticality [11–42] remain a fertile area of study. Although such edge modes delocalize and disappear at phase transitions to the *trivial* phase [Fig. 1(a)] [43–49], it has been realized that transitions to *other* phases—such as spontaneous symmetry-breaking phases [Fig. 1(b)] [28,37,50–53]—can leave part of the edge mode intact.[1]

This raises a question which is fundamental for understanding the interplay between topology and criticality: What is the fate of edge modes when the system transitions from one topological phase to *another* nontrivial topological phase? To what extent do edge modes survive at the critical point? While previous work has studied this question in the noninteracting fermion case [33,37,54], here we explore a more generic and richer framework. Importantly, in our work, the edge modes of one phase are incompatible with those of the other phase due to differences in how they realize the symmetry action. The resulting frustration gives rise to fascinating boundary effects when both types of edge modes are forced to coexist and compete at criticality.

As a minimal example of this scenario, we study a transition between two $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric spin chain Hamiltonians [55], each phase hosting protected edge modes transforming under *distinct* projective representations [4]. These two gapped phases are simple examples of the more general phenomenon of symmetry-protected topological (SPT) phases. We find that edge degeneracy typically persists at the critical point in two possible ways; more precisely, there are two conformal boundary conditions each spontaneously breaking one of the two $\mathbb{Z}_3$ symmetries [Fig. 1(c)]. These can be thought of as spontaneous symmetry-breaking phases in *zero* spatial dimensions. Moreover, we find a direct continuous boundary transition between these two [Fig. 1(d)], where one symmetry breaks exactly when another is restored. This is a stark violation of the conventional Landau paradigm of phase transitions which posits that symmetry subgroups only break one at a time. In fact, it is a 0+1D manifestation of a deconfined quantum critical point (DQCP), an exotic phenomenon originally proposed for 2+1D [56–76] and recently explored in 1+1D [77–84]. Indeed, we discuss how even the mechanism is quite similar to that in higher dimensions, namely, condensing defects for one symmetry-breaking order gives rise to long-range order for the other [58].

Moreover, we show that the bulk critical point itself has a nontrivial topological invariant—making it an instance of gapless SPT or symmetry-enriched criticality [37]. The conventional lore for topologically nontrivial SPT phases, shown rigorously in the gapped case, is that edge modes are guaranteed by a bulk-boundary correspondence. However, at the boundary critical point reported in this work, edge modes disappear. This shows that the notion of bulk-boundary correspondence is more subtle for gapless SPT phases, opening up exciting future research directions.

$\mathbb{Z}_3 \times \mathbb{Z}_3$ cluster SPT chains. Define shift and clock matrices

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{1}$$

[1]For gapped SPT phases its value depends on the choice of ground state. In contrast, gapless phases can have robust 0+1D symmetry breaking on the edge [28,37,53].

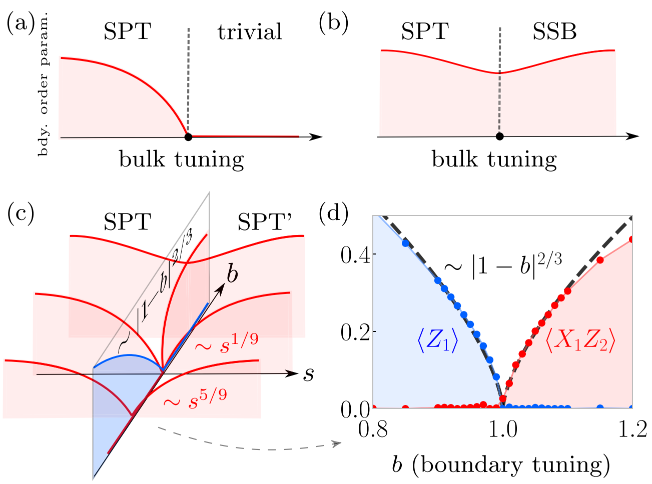

FIG. 1. SPT edge modes, criticality, and boundary DQCP. Panels highlight what happens to SPT edge modes when tuning to quantum criticality. Based on end-to-end long-range order of boundary order parameters: (a) tuning to trivial phase destroys edge modes and (b) edge modes can persist upon tuning toward a nontrivial phase, like a symmetry-breaking phase [28,37,53]. (c) In this work, we show a richer phenomenology at a transition between two distinct SPT phases protected by $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry [Eq. (5)]; there are distinct symmetry-breaking boundary conditions at criticality. (d) Moreover, there is a direct continuous transition ("DQCP") between these two by tuning a boundary parameter.

where $\omega = e^{2\pi i/3}$. We consider quantum chains respecting the $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry generated on even and odd sublattices:

$$U^e = \prod_j X_{2j} \quad \text{and} \quad U^o = \prod_j X_{2j+1}. \tag{2}$$

Following Ref. [55], we define "cluster Hamiltonians" [51] for two distinct nontrivial $\mathbb{Z}_3 \times \mathbb{Z}_3$ SPT phases

$$H_\omega = -\sum_j (Z_{2j-1} X_{2j} Z_{2j+1}^\dagger + Z_{2j} X_{2j+1}^\dagger Z_{2j+2}^\dagger + \text{H.c.}),$$
$$H_{\bar{\omega}} = -\sum_j (Z_{2j-1} X_{2j}^\dagger Z_{2j+1}^\dagger + Z_{2j} X_{2j+1} Z_{2j+2}^\dagger + \text{H.c.}). \tag{3}$$

The effective low-energy action of U^e and U^0 on each boundary is such that they commute only up to a projective phase ω or $\bar{\omega}$, leading to a threefold degenerate ground state space (per edge) [55].

The edge projective symmetry action is detectable via bulk string order parameters. That is, among operators of the form $\cdots X_{2j-6} X_{2j-4} X_{2j-2} \mathcal{O}_{2j}$ (a $\mathbb{Z}_3^e$-string operator), only those with $\mathbb{Z}_3^o$-charged $\mathcal{O}_{2j}$ have have long-range order (LRO) [85], and vice versa for the other symmetry. For instance, H_ω has LRO in an ω-charged $\mathbb{Z}_3^e$-string operator

$$\lim_{|k-j|\to\infty} \langle Z_{2j-1} X_{2j} X_{2j+2} \cdots X_{2k} Z_{2k+1}^\dagger \rangle = 1, \tag{4}$$

while $H_{\bar{\omega}}$ has an $\bar{\omega}$-charged $\mathbb{Z}_3^e$-string operator. While the left hand side of Eq. (4) is unity only for the fixed-point Hamiltonian H_ω, it remains nonzero throughout the SPT phase [85].

Numerical method. We confirm our CFT analysis using density matrix renormalization group (DMRG) simulations [86,87] on finite chains of lengths $25 \lesssim L \lesssim 125$. At each length, we considered the limit of bond dimesnion $\chi \to \infty$, with simulations run up to $\chi = 170$ found to sufficiently guarantee convergence for ground state end-to-end correlators and excited state energy levels. For technical efficiency, instead of implementing the cluster Hamiltonian directly, we simulated a unitarily equivalent three-state Potts chain as described in Ref. [88].

Criticality and boundary symmetry breaking. We study a linear interpolation between the two nontrivial cluster Hamiltonians (3):

$$H(s, b) = (1+s)H_\omega + (1-s)H_{\bar{\omega}} - b(X_1 + X_{2N+1} + \text{H.c.}). \tag{5}$$

Since we are interested in edge behavior, we have open boundary conditions with $j \in [1, 2N+1]$ and boundary tuning parameter b to explore generic boundary behavior.

This model exhibits a direct transition at the midpoint $s = 0$. In fact, a local unitary (the SPT entangler [89]) maps $H_1 \mapsto H_\omega \mapsto H_{\bar{\omega}} \mapsto H_1$, where $H_1 = -\sum_j X_j + \text{H.c.}$ is a trivial phase. So the bulk critical point can be mapped to one between the trivial and SPT phase, $H_1 + H_\omega$, which in Ref. [44] was found to be be described by a certain orbifold of two copies of the three-state Potts conformal field theory (Potts^2 CFT). However, these entangler transformations do not apply for open boundary conditions, and we will find $H_\omega + H_{\bar{\omega}}$ has much richer boundary criticality than $H_1 + H_\omega$; we will also discuss how a bulk symmetry-protected topological invariant detects this difference. We note that this $s = 0$ critical point belongs to a one-parameter family of theories stabilized by $\mathbb{Z}_3 \times \mathbb{Z}_3$, translation, and charge conjugation (see Refs. [83,88,90,91] therein for details on adjacent bulk phases).

Unlike in gapped SPT phases, string operators [Eq. (4)] no longer have LRO at criticality. Instead they decay algebraically with universal exponents distinguishing $H_\omega + H_{\bar{\omega}}$ and $H_1 + H_\omega$. For example, considering charges of the 'lightest' string operators, i.e., those with the smallest such exponents, $H_\omega + H_{\bar{\omega}}$ has two degenerate U^e-string operators with U^o charges $\{\omega, \bar{\omega}\}$ [e.g., the lattice string operator Eq. (4)], while $H_1 + H_\omega$ has charges $\{1, \omega\}$; these correspond to string operators with LRO in the nearby symmetric phases. This bulk topological invariant proves that these two CFTs cannot be connected by a $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric path without passing through a multi-critical point or tuning off criticality.

In gapped SPTs, LRO of charged strings (4) directly imply edge modes. Analogously, we might expect a similar bulk-boundary correspondence can distinguish the "trivial" transition $H_1 + H_\omega$ from the "topological" one $H_\omega + H_{\bar{\omega}}$. To explore this, we turn to a more concrete analysis of Eq. (5), using analytic and numerical methods inspired by Ref. [37].

In the fine-tuned case $b = 0$, zero mode operators Z_1 and Z_{2N+1} commute with H [Eq. (5)]. Their $\mathbb{Z}_3^o$ charge implies a threefold degenerate spectrum. Morally, Z_1 and Z_{2N+1} are order parameters for a spontaneous symmetry-breaking (SSB) boundary and are indeed LRO in time. They are also phase-locked across the critical bulk: $\langle Z_1\rangle = \langle Z_{2N+1}\rangle$ in all three ground states (i.e., unlike in gapped SPTs, degeneracies are

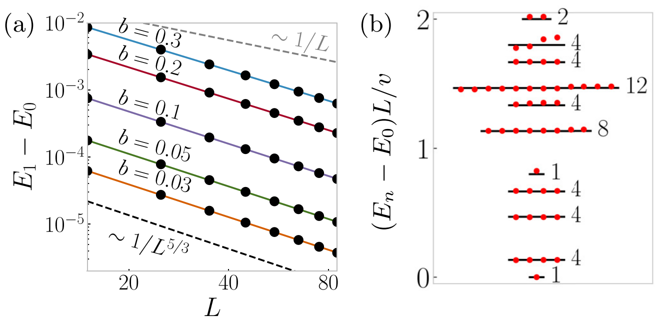

FIG. 2. Edge modes and boundary dissolution at SPT criticality. We consider Eq. (5) with open boundaries. (a) For $b < 1$, the boundary spontaneously breaks U^o. This degeneracy's finite-size splitting matches the CFT prediction $\sim L^{-5/3}$. Edge modes become exactly degenerate in the CFT limit. (b) At $b = 1$, the boundary undergoes a transition between two distinct symmetry-breaking phases. Here, we find a unique ground state. Red dots denote the numerically extracted universal finite-size spectrum (for $L = 25$; here $v = 3\sqrt{3}\pi$). Remarkably, this matches the spectrum of Potts CFT *without boundaries* (black lines). This signifies that at this point, the bulk-boundary distinction is blurred (see main text).

not independent for both edges). We call this boundary phase "*o*-SSB." Note that bulk gaplessness is what ensures a well-defined boundary SSB; in contrast, in gapped SPT phases end-to-end LRO requires a certain basis of degenerate ground states. Indeed, gapped SPT edge modes, being genuine zero-dimensional systems, have no robust notion of "phase of matter."

Adding nonzero b splits degeneracy for finite systems, similar to the exponentially small finite-size splitting of gapped SPTs [10,92]. At criticality ($s = 0$), edge modes split *algebraically* $\sim 1/N^{\alpha}$ with boundary-condition-dependent exponent α. Crucially $\alpha > 1$, such that degeneracy is relative to *bulk* finite-size splitting $\sim 1/N$ [37]. We numerically confirm this faster-than-$1/N$ splitting, and hence boundary stability, in Fig. 2(a). Later we derive a mapping to the Potts model, implying the universal exponent $\alpha = 5/3$.

The other easy limit, $b \to \infty$, projects $X_1 = X_{2N+1} = 1$, i.e., throws out sites 1 and $2N + 1$ and operators acting on them. One has the same model as when $b = 0$, but with $j \in [2, 2N]$. The story above repeats, except now U^e is spontaneously broken at the boundary, not U^o. This is a distinct boundary phase ("*e*-SSB") from $b = 0$, raising the question of what boundary transition occurs as we tune the boundary coupling.

We note this perturbatively stable boundary symmetry breaking requires the exotic topological nature of the model $H_\omega + H_{\bar{\omega}}$. For comparison, the trivial gapless theory $H_1 + H_\omega$ has a generically nondegenerate conformal spectrum and no boundary symmetry breaking except at some unstable fine-tuned boundary points. The difference lies in the so-called *boundary disorder operators*, i.e., operators toggling between superselection sectors of 0+1D SSB ground states. For $H_1 + H_\omega$, perturbing a fine-tuned degenerate edge with infinitesimal X_1 will disorder the 0+1D SSB and flow to a unique symmetry-preserving edge. In contrast, $H_\omega + H_{\bar{\omega}}$ has no RG-relevant symmetry-allowed boundary disorder operator with which we can perturb the edge. This intuitively

TABLE I. The most relevant disorder operators of the odd symmetry-breaking boundary ($b < 1$) are order parameters of the even symmetry-breaking boundary ($b > 1$) and vice versa. Here we show left boundary lattice expressions [1]. Restoring one symmetry requires condensing said disorder operator, thereby spontaneously breaking the other symmetry; this is the mechanism leading to the 0+1D "non-Landau" DQCP.

	Order operator	Disorder operator
o-SSB	Z_1	$X_1 Z_2^{(\dagger)}$
e-SSB	$X_1 Z_2^{(\dagger)}$	Z_1 (or $X_2 Z_3^{(\dagger)}$)

[1]These are lattice expressions for the boundary disorder operators at the extreme limits $b = 0$ and $b = \infty$; moreover, for generic b their expansion in continuum field is dominated by a boundary disorder operator. We also mention that Z_1 is identically zero at $b = \infty$, but for finite $b > 1$ is in the same universality as $X_2 Z_3$.

matches the bulk topological invariants and also follows from CFT (explained in Refs. [88,93]) The relevant perturbation, shown in Table I, carries nontrivial charge under the unbroken symmetry and thus cannot be generated under RG!

Finally, we remark that the boundary order parameters and disorder operators for the gapless regime match localized projective symmetry generators from the adjacent gapped SPT phases. Although the gapped SPT phase is agnostic with respect to automorphisms of $\mathbb{Z}_3 \times \mathbb{Z}_3$, the gapless theory selects a specific choice of projective symmetry generators to play the role of boundary order parameter or disorder operator. This physically corresponds to the fact that the boundary of the gapless phase has genuine 0+1d SSB, in contrast to the edge of a gapped SPT phase.

DQCP in zero dimensions. To recap, for $b \approx 0$, $H_\omega + H_{\bar{\omega}}$ with open boundaries spontaneously breaks the odd-sublattice $\mathbb{Z}_3$ symmetry, while for $b \to \infty$, it breaks the even one. It turns out these two phases persist for all b, *except at* $b = 1$, where there is a direct transition. This boundary transition is continuous, and both symmetries are unbroken there. Indeed, for $b = 1$, we find *no* ground state degeneracy (Fig. 2), contrary to a naive expectation from the bulk topological invariant.

Tuning left and right boundary couplings simultaneously [see Eq. (5)] lets us use order parameters' end-to-end correlations to detect the transition, which occurs independently on both edges. In particular $\langle Z_1 Z_{2N+1}^{\dagger} \rangle$ is nonzero in the *o*-SSB boundary phase ($0 \leqslant b < 1$) and zero in the *e*-SSB boundary phase ($b > 1$) and vice versa for $\langle X_1 Z_2 Z_{2N}^{\dagger} X_{2N+1}^{\dagger} \rangle$. The square root gives the boundary vacuum expectation value (vev). Using DMRG, we obtain Fig. 1(d) and clearly see the direct continuous transition at $b = 1$. Later, we analytically show both vevs vanish at $b = 1$ with unbroken symmetry and no ground state degeneracy.

This continuous SSB-to-SSB transition resembles deconfined quantum criticality points (DQCP) in higher dimensions. A key feature of DQCP is that the "vortex" in one ordered phase is charged under the symmetry broken in the other. Thus they cannot simultaneously condense, leading to a Landau-forbidden transition. The same mechanism prevails here, with the role of vortices played by relevant boundary disorder operators of Table I. Another salient DQCP feature is an

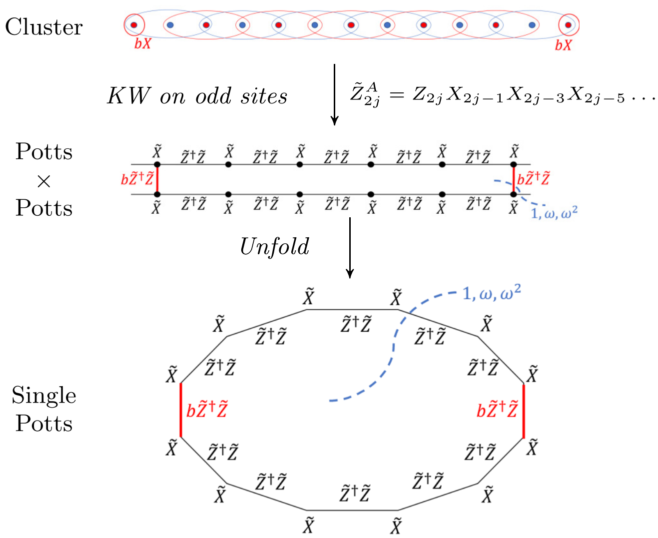

FIG. 3. Mapping critical $\mathbb{Z}_3 \times \mathbb{Z}_3$ cluster chain with boundaries to a single Potts chain. There is an exact unitary map from a finite open cluster chain to a finite *closed* Potts chain with defects. First we apply a Kramers Wannier transformation on odd sites and appropriately parametrize the resulting even sites to have the form of two $c = 4/5$ Potts chains only coupled at their boundaries by the boundary perturbation. Then we "unfold" this doubled system by simply viewing it as a single Potts system on a closed loop with defects and twisted sectors.

emergent symmetry exchanging nearby SSB phases, which we will show indeed occurs at $b = 1$. Despite these similarities, an anomalous symmetry is arguably missing. Indeed, a *bona fide* zero-dimensional anomaly is usually understood to be a projective representation; since this implies degeneracy, it cannot be present at $b = 1$. Thus, following Ref. [83], we use the term DQCP in a slightly broader context, namely, a non-Landau transition between distinct SSB phases stabilized by condensing charged defect operators.

We numerically verify symmetry restoration and nondegeneracy at $b = 1$ by computing a finite chain's spectrum, Fig. 2(b). Remarkably, this spectrum coincides with the known analytic result for a *single* Potts chain with *periodic* (twisted) boundary conditions [94]. This is no coincidence, as we now demonstrate.

Mapping to single Potts chain. Remarkably, the open chain in Eq. (5) is unitarily equivalent to a single three-state Potts chain on a ring with some defects depending on b. The mapping is summarized in Fig. 3 (further details are provided in Ref. [88]). The $\mathbb{Z}_3^e$ physical symmetry is the global $\mathbb{Z}_3$ symmetry of the Potts ring, while the eigenvalues of the $\mathbb{Z}_3^o$ generator label the $\mathbb{Z}_3$ twisted boundary conditions of the Potts ring. The result is that b tunes the strength of a single exchange term on opposite sides of the ring. The DQCP at $b = 1$ corresponds to translation symmetry, where the spectrum matches that of the Potts chain on a ring, which is nondegenerate.

With this mapping, dominant boundary operators are identified through the Potts defect conformal field theory [95–100] confirming our claims in Fig. 1. For example, at $b < 1$, the dominant symmetry-allowed boundary perturbation corresponds to an irrelevant $\psi^A \psi^{B\dagger}$ CFT operator of dimension 4/3 coupling the two chains' endpoints (Fig. 3), while the $\mathbb{Z}_3^e$-charged disorder operator corresponds to ψ^A of dimension 2/3 leading to the $|s|^{5/9}$ scaling of Fig. 1(c). Similary, at the DQCP, scalings of Figs. 1(c) and 1(d) arise from the 2/15 dimensional $\mathbb{Z}_3^e$ ($\mathbb{Z}_3^o$) charged boundary operators $\sigma\bar{\sigma}$ ($\mu\bar{\mu}$) and the 4/5 dimensional the symmetric boundary perturbation $\epsilon\bar{\epsilon}$.

Furthermore, the Potts chain's Kramers-Wannier duality interchanges U^e and U^o symmetries and all order and disorder operators. It sends $b = 1 + \delta b$ to $b = 1 - \delta b$ for $\delta b \ll 1$, acting as an emergent duality in the boundary phase diagram. Two such transformations translate the Potts ring. At $b = 1$, the emergent translation symmetry relates boundary degrees of freedom to bulk degrees of freedom. Thus the boundary critical point is also a "delocalized" QCP.

Outlook. We studied a minimal example of competition between two inequivalent types of topologically protected edge mode with $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. We found that as a result of this competition, there are effectively fewer edge modes at criticality, and they organize themselves into one of two distinct boundary-symmetry-breaking phases breaking only a three-fold subgroup. Most strikingly, there is an unconventional direct boundary transition between these two symmetry-breaking regimes. At this boundary transition, edge modes disappear and emergent features of a deconfined quantum critical point appear. These results were obtained using conformal field theory and tensor network simulations on a critical one-dimensional open-chain lattice model on an open chain with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.

SPT transitions and edge modes of gapless systems merit further study. Our results encourage exploring other direct transitions between nontrivial SPT phases, where, as we have exemplified, novel boundary physics is expected. Examples include boundaries of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^T$ SPT-SPT' transition in Eq. (28) of Ref. [37] and the $c = 2$ multicritical point where all three $\mathbb{Z}_3 \times \mathbb{Z}_3$ SPT phases meet. Another major open question regards bulk-boundary correspondence for gapless SPT phases. Remarkably we have found that even with a nontrivial bulk topological invariant, boundary edge modes can disappear in a boundary DQCP. It remains unknown how general this phenomenon is. Insights might also be gained by understanding boundary conditions as RG flows to 1+1D gapped phases [101–103]. Another open question is to explore higher-dimensional analogs, such as transitions between nontrivial 2+1D $\mathbb{Z}_n$ SPTs.

Finally, it would be exciting to explore these phenomena in experiment. Intriguingly, $\mathbb{Z}_3 \times \mathbb{Z}_3$ SPT phases have been predicted in optical lattices of cold alkaline-earth atoms [104,105]. While numerical simulations found a direct first-order transition between the two nontrivial SPT phases, our work suggests a broader phase diagram can have a direct continuous transition, where one would observe 0+1D boundary DQCP. To facilitate such experimental explorations, one can map the three-body cluster Hamiltonian to a two-body interacting system [106], similar to what has been done for the $\mathbb{Z}_2$ case [107].

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