Letter

d-wave Hall effect and linear magnetoconductivity in metallic collinear antiferromagnets

Dmitrii L. Vorobev[1,2] and Vladimir A. Zyuzin[2]
[1]*Higher School of Economics University, 101000 Moscow, Russia*
[2]*L.D. Landau Institute for Theoretical Physics, 142432, Chernogolovka, Russia*

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In this Letter we theoretically predict a distinct class of anomalous Hall effects occurring in metallic collinear antiferromagnets. The effect is quadratic and d-wave symmetric in an external magnetic field. In addition, the electric current, transverse to the current voltage drop and the magnetic field in the predicted effect are all in the same plane. The studied theoretical model consists of two-dimensional fermions interacting with the Néel order through momentum-dependent exchange interaction having a d-wave symmetry. We also find unusual linear magnetoconductivity in this model.

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The anomalous Hall effect (AHE) is one of the experimental tools which sheds light on the symmetries of the crystal structure of the material. The effect primarily requires time-reversal symmetry breaking. The details of the crystal structure and magnetic order can vary the AHE. For example, in ferromagnets with magnetization $\mathbf{M}$ (or the Zeeman part of external magnetic field $\mathbf{B}$) it is expected [1] there will be

$$\mathbf{j}_{\text{AHE}} \propto [\mathbf{M} \times \mathbf{E}], \tag{1}$$

electric current response, where $\mathbf{E}$ is the electric field corresponding to the voltage drop. This is the ferromagnetic analog of the regular Hall effect [2,3]. The main ingredients of AHE in ferromagnets is the combination of the exchange (momentum-independent) spin splitting and the spin-orbit coupling [4–9] of the conducting fermions. For example, a two-dimensional fermion system with Rashba spin-orbit coupling and Zeeman-like ferromagnetic exchange is one of the most studied models in relation to the anomalous Hall effect [10–13]. One of the main mechanisms of the AHE is the anomalous part of the fermion velocity [1], which, however, is due to the Berry curvature [14,15] of the conducting fermions.

It is then possible that in systems with C_{1v} and C_{3v} symmetry the magnetic-field-driven AHE can have in-plane configuration [16–21], where all three vectors, namely, the electric current, transverse voltage drop, and external magnetic field, are in the same, as in the example below, x-y plane

$$\mathbf{j}_{\text{IPHE}} = \sigma_{\text{IPHE}}[[\mathbf{B} \times \mathbf{e}_z] \times \mathbf{e}_y] \times \mathbf{E}, \tag{2}$$

here the unit vectors $\mathbf{e}_z$ and $\mathbf{e}_y$ (in the z and y directions, respectively) are defined by the spin-orbit coupling (see Ref. [19] for details), and σ_{IPHE} is the in-plane Hall conductivity. The first two vector products filter out only the B_y component multiplied by the $\mathbf{e}_z$ unit vector. This effect was experimentally observed in Refs. [22–24]. In addition, the authors of Refs. [17–19] suggested that, in systems with C_{3v} symmetry $\sigma_{\text{IPHE}} \propto B_y^2 - 3B_x^2$ vanishing at $B_y = \pm\sqrt{3}B_x$ overall making the $\mathbf{j}_{\text{IPHE}}$ current $\frac{2\pi}{3}$ periodic in the in-plane magnetic field, which was experimentally observed in Ref. [24].

The situation with AHE in collinear antiferromagnets is currently under study [25,26]. In simple collinear antiferromagnets with two sublattices, there is a symmetry under a combination of time-reversal and translation operations which does not allow for spin-splitting of conducting fermions, therefore, making the anomalous Hall effect vanish. However, microscopic surroundings of each sublattice [26] may make a difference. For example, the aforementioned symmetry will be broken if the surroundings of the spin-up sublattice are different from the surroundings of the spin-down one. See the left plot in Fig. 1 for schematics. The remaining symmetry is a combination of time-reversal and $\frac{\pi}{2}$ rotation operations, which will allow for the spin-splitting of the conducting fermions shown in the right plot in Fig. 1. In the level of the Hamiltonian, such spin-splitting can be understood as the momentum-dependent g-factor which obeys the symmetry of the lattice. Research of materials with momentum-dependent g-factor have been studied for decades [27–33]. However, the existence of such a g-factor in antiferromagnetic materials was pointed out only recently [26,34–41]. It is understood that conducting fermions in, for example, RuO_2, MnF_2, $FeSb_2$, MnTe Néel-ordered antiferromagnets and many more can be described by such spin-splitting [26,34,36–38]. Experimentally measured [39] spin-filtered electric transport is one of the manifestations of such spin-splitting [40]. We note that there are other proposals [42] on how the asymmetry of the sublattices in Néel-ordered antiferromagnets may result in a nonzero AHE.

In this paper we show that, in addition to the known cases of the anomalous Hall effect, given by Eqs. (1) and (2), metallic antiferromagnets with the spin-split conducting fermions described above may show very unusual magnetic-field-driven anomalous Hall effect proportional to the second power of the magnetic field, given by

$$\mathbf{j}_{\text{DWHE}} \propto B_x B_y [\mathbf{e}_z \times \mathbf{E}], \tag{3}$$

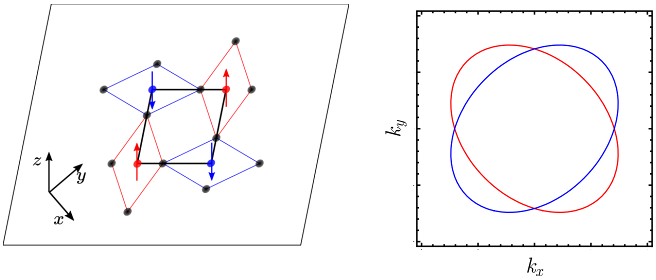

FIG. 1. Left: Schematics of two-dimensional square lattice in the x-y plane with Néel order in the z-direction. Red/blue circles correspond to two atoms in the unit cell with spin-up/down order. Local environment composed of black circles is different around red/blue atoms. There is a symmetry of $\frac{\pi}{2}$ rotation together with time reversal: the $\frac{\pi}{2}$ rotation connects the local environment of the two elements while the time-reversal flips the spin. Right: Fermi surface of fermions corresponding to Hamiltonian Eq. (5) with $\lambda = 0$. Red corresponds to spin up, while blue to spin down.

where $\mathbf{e}_z$ is defined by the direction of the Néel vector. We will refer to it as the d-wave Hall effect, hence the DWHE abbreviation in Eq. (3). Indeed the B_xB_y product has the aforementioned symmetry. It is notable that just like in the in-plane Hall effect in Eq. (2) all three vectors, namely, the electric current, transverse voltage drop, and the magnetic field, are in the same plane. Such a response is not prohibited by the Onsager relation, as it is overall cubic in time-reversal symmetry breaking fields since $\mathbf{e}_z \to -\mathbf{e}_z$ in Eq. (3) under the time-reversal operation.

In addition to Eq. (3) we find another experimentally relevant response, namely, the linear magnetoconductivity (LMC)

$$\mathbf{j}_{\rm LMC} \propto B_z(E_x\mathbf{e}_y + E_y\mathbf{e}_x), \tag{4}$$

which, together with the regular Hall effect, will result in anisotropic Hall conductivity, i.e., $\sigma_{xy} \neq \sigma_{yx}$. Again, this effect is allowed by the Onsager relation because the response in Eq. (4) is actually quadratic in time-reversal symmetry breaking fields since B_z is selected by Néel order $\mathbf{e}_z$ as $B_z \to (\mathbf{e}_z \cdot \mathbf{B})$ and both change sign under time reversal.

To show the effect, we study a two-dimensional metallic antiferromagnet system shown in the left plot in Fig. 1. The Hamiltonian of the conducting fermions interacting with the Néel vector and consistent with the lattice symmetry are written as

$$\hat{H}_0 = \frac{\mathbf{k}^2}{2m} + \lambda(k_x\sigma_y - k_y\sigma_x) + \beta\sigma_z k_x k_y, \tag{5}$$

where σ are Pauli matrices describing the spin of the conducting fermions. The term with β is the interaction of the conducting fermions with the antiferromagnetic Néel vector (coined as the altermagnetism by the authors of Refs. [26,37,38]). This term breaks the time-reversal symmetry. As noted, it is a combination of difference in atomic configurations around ordered spins and the antiferromagnetic order which generates this term, not just the antiferromagnetic order alone. For an example see the left plot in Fig. 1, where a combination of translation and time-reversal symmetries is broken due to the local configuration, while a $\frac{\pi}{2}$ rotation together with the time reversal is the symmetry of the lattice which supports the term β. We include Rashba spin-orbit coupling denoted by λ. In addition, we apply an external magnetic field in the x-y plane which acts only on spins of fermions (Zeeman magnetic field)

$$\hat{H}_{\rm Z} = h_x\sigma_x + h_y\sigma_y, \tag{6}$$

where $\mathbf{h} = \frac{1}{2}g\mu_{\rm B}\mathbf{B}$, g is the g-factor and $\mu_{\rm B}$ is the Bohr magneton. Both terms will be needed in our analysis of the electric current responses. In addition, the orbital part of the magnetic field in the z-direction will be considered. We assume that finite electron density with chemical potential $\mu > 0$ does not suppress the antiferromagnetic order, as well as the fact that the external Zeeman magnetic field does not cant the antiferromagnetically ordered spins in any direction. This canting is plausible if the field is smaller than the magnetic anisotropy which favors the z-direction for the Néel vector.

The Hamiltonian Eq. (5) has not been derived by us from microscopics; it is an effective model which corresponds to the lattice symmetry [26,37,38]. However, one may think of it as a model of correlated fermions on a square lattice. Then an antiferromagnetic order develops in the vicinity of the half-filling, gapping out the fermions. The slight doping with fermions populates the conduction band with quadratic spectrum [first term in Eq. (5)]. If, in addition, there is a different local environment of spins shown in the left figure in Fig. 1, the spectrum of fermions will acquire a term with β in Eq. (5). One may also think of a term with β as a d-wave Pomeranchuk or Stoner-like magnetic instability [30,31].

We are interested in the electric current responses of this system to external electric field. Let us first understand what kind of responses can be deduced from the symmetry argument. The symmetry group corresponding to the unperturbed system, which is the fermions on a simple square lattice without anything else, is the $D_{4\rm h}$ group. The other terms are treated as perturbations, and in this particular symmetry group [43], field β transforms as the $\Gamma_3^{(+)}$ element of the group, magnetic field B_z as $\Gamma_2^{(+)}$, and $B_{x/y}$ as $\Gamma_5^{(+)}$, the electric field as $\Gamma_5^{(-)}$, λ as $\Gamma_2^{(-)}$, and the electric current transforms as $\Gamma_5^{(-)}$. The elements of the group obey the multiplication rules, for example, listed in Refs. [43–45]. We must find all products that are linear in electric field, linear in β, to second order in the magnetic field, and to whatever order in λ that transform as $\Gamma_5^{(-)}$ and hence can be a part of the electric current. In addition, we require the Onsager relation for the conductivity $\sigma_{ij}(\mathbf{B}, \beta) = \sigma_{ji}(-\mathbf{B}, -\beta)$ to satisfy. By performing the exercise of multiplying the group elements we get for the electric current

$$\begin{aligned}\mathbf{j} = {} & \sigma_{\rm D}\mathbf{E} + \sigma_{\rm H}[\mathbf{E}\times\mathbf{B}] + \sigma_2[(\mathbf{E}\cdot\mathbf{B})\mathbf{B} - \mathbf{B}^2\mathbf{E}] \\ & + \sigma_{\rm LMC}\beta B_z(E_x\mathbf{e}_y + E_y\mathbf{e}_x) + \sigma_{\rm DWHE}\beta B_xB_y[\mathbf{e}_z\times\mathbf{E}].\end{aligned} \tag{7}$$

The first line here is consistent with Refs. [3,46,47]. A term with $\sigma_{\rm D}$ is the regular Drude conductivity, a term with $\sigma_{\rm H} = \frac{1}{B}\omega_{\rm c}\tau\sigma_{\rm D}$, where $\omega_{\rm c} = \frac{eB}{mc}$ is the cyclotron frequency, is the regular Hall effect due to the Lorentz force [2], σ_2 is due to the Lorentz force as well (in Weyl semimetals this term can be due to the chiral anomaly, for an example see Ref. [48], and in ferromagnets it is called the planar Hall effect if $\mathbf{B}$ is replaced by the magnetization $\mathbf{M}$ [49]) and exists in any three-dimensional electron system [3,46,47], $\sigma_{\rm LMC}$ is the

LMC expected in the time-reversal symmetry-broken systems [48,50–55], and finally a term with $\sigma_{\rm DWHE}$ is the DWHE. The last two terms in Eq. (7) are unique to the system described by $\hat{H}_0 + \hat{H}_{\rm Z}$. Let us demonstrate how they appear (please see the Supplemental Material [56] for more details). According to the multiplication table given in Refs. [43–45], $E_{x/y}B_z\beta$ transform as $\Gamma_5^{(-)} \times \Gamma_2^{(+)} \times \Gamma_3^{(+)} = \Gamma_5^{(-)}$ while $E_{x/y}B_xB_y\beta$ as $\Gamma_5^{(-)} \times \Gamma_5^{(+)} \times \Gamma_5^{(+)} \times \Gamma_3^{(+)} = \Gamma_5^{(-)}$. Indeed, the two combinations transform as electric current. In addition to the Lorentz force contribution to $\sigma_{\rm H}$ there might be a contribution from the regular anomalous Hall effect given by Eq. (1) if $\mathbf{M}$ there is replaced by $B_z\mathbf{e}_z$.

If tge mechanism behind each term in the first line of Eq. (7) is understood [3], the terms in the second line have not been discussed anywhere before and are the subjects of the analysis below. We first introduce the notations. The spectrum for $s = \pm$ branches out corresponding to the $\hat{H}_0 + \hat{H}_{\rm Z}$ Hamiltonian reads

$$\varepsilon_{\mathbf{k}}^{(\pm)} = \frac{k^2}{2m} \pm \sqrt{\Delta_{\mathbf{k}}^2 + \lambda^2\tilde{k}^2}, \tag{8}$$

where $\lambda\tilde{k}_x = \lambda k_x + h_y$ and $\lambda\tilde{k}_y = \lambda k_y - h_x$ and $\Delta_{\mathbf{k}} = \beta k_x k_y$ were introduced for brevity. The spinors are $\Psi_{\mathbf{k},+} = [\cos(\frac{\xi_{\mathbf{k}}}{2})e^{i\chi_{\mathbf{k}}}, -\sin(\frac{\xi_{\mathbf{k}}}{2})]^{\rm T}$ and $\Psi_{\mathbf{k},-} = [\sin(\frac{\xi_{\mathbf{k}}}{2})e^{i\chi_{\mathbf{k}}}, \cos(\frac{\xi_{\mathbf{k}}}{2})]^{\rm T}$, where $[\dots]^{\rm T}$ is the transposition, $\cos(\xi_{\mathbf{k}}) = \frac{\Delta_{\mathbf{k}}}{\sqrt{\Delta_{\mathbf{k}}^2+\lambda^2\tilde{k}^2}}$ and $\chi_{\mathbf{k}} = \arctan(\frac{\tilde{k}_y}{\tilde{k}_x})$ is the phase.

The anomalous Hall effect [10] as well as LMC [48,50,51] are defined by the nontrivial Berry phase of conducting fermions. The intrinsic mechanism [10] of the anomalous Hall effect is given by

$$\mathbf{j}_{\rm DWHE} = e^2\left[\int \frac{d\mathbf{k}}{(2\pi)^2} \sum_{n=\pm} \mathbf{\Omega}_{\mathbf{k}}^{(n)} \mathcal{F}(\epsilon_{\mathbf{k},n})\right] \times \mathbf{E}, \tag{9}$$

where $\mathcal{F}(\epsilon)$ is the Fermi-Dirac distribution function. Following the lines of Ref. [19] the Berry curvature

$$\begin{aligned}\Omega_{z;\mathbf{k}}^{(\pm)} &= 2{\rm Im}(\partial_{k_x}\Psi_{\mathbf{k},\pm}^{\dagger})(\partial_{k_y}\Psi_{\mathbf{k},\pm}) \\ &= \mp\frac{\lambda^2}{2\left(\Delta_{\mathbf{k}}^2 + \lambda^2\tilde{k}^2\right)^{3/2}}(\Delta_{\mathbf{k}} - \tilde{k}_x\partial_x\Delta_{\mathbf{k}} - \tilde{k}_y\partial_y\Delta_{\mathbf{k}})\end{aligned} \tag{10}$$

in our model is derived to be

$$\Omega_{z;\mathbf{k}}^{(\pm)} = \mp\frac{\lambda^2\Delta_{\mathbf{k}} - \lambda\beta(k_xh_x - k_yh_y)}{2\left(\Delta_{\mathbf{k}}^2 + \lambda^2\tilde{k}^2\right)^{3/2}}. \tag{11}$$

It is clear that if $h_x = h_y = 0$ the integral of the Berry curvature over the angles vanishes because of the d-wave symmetry. Thus, the anomalous Hall effect is absent in this case. We define $\sigma_{\rm DWHE}$ as in Eq. (7), i.e., as $\mathbf{j}_{\rm DWHE} = \sigma_{\rm DWHE}\beta B_xB_y[\mathbf{e}_z \times \mathbf{E}]$. When $h_x \neq 0$ and $h_y \neq 0$ the DWHE is nonzero and we plot it in Fig. 2. In addition, we give approximate analytical expressions for various limits of the physical parameters. In the limit of $h_{x/y} \ll \lambda k_{\rm F}$ we have

$$\sigma_{\rm DWHE} \approx \sigma_0|\lambda|k_{\rm F}\frac{\left(\beta k_{\rm F}^2\right)^4 + 11\left(\beta k_{\rm F}^2\right)^2(\lambda k_{\rm F})^2 + 16(\lambda k_{\rm F})^4}{\left[\left(\beta k_{\rm F}^2\right)^2 + (2\lambda k_{\rm F})^2\right]^{5/2}}, \tag{12}$$

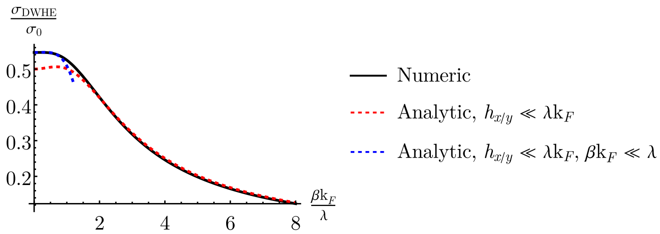

FIG. 2. Plot of the d-wave Hall conductivity $\sigma_{\rm DWHE}$ given by Eqs. (12) and (13). Values of the parameters used in the numerical calculation are $\frac{h_x}{\lambda k_{\rm F}} = 0.25$, $\frac{h_y}{\lambda k_{\rm F}} = 0.15$, and $T = 0$. We define $\sigma_0 = (\frac{1}{2}g\mu_{\rm B})^2\frac{e^2\nu_F}{(\lambda k_{\rm F})^2}$ for brevity.

where $\nu_{\rm F} = \frac{m}{\pi}$ is the density of states, $k_{\rm F} = \sqrt{2m\mu}$ is the Fermi momentum, and where we define $\sigma_0 = (\frac{1}{2}g\mu_{\rm B})^2\frac{e^2\nu_F}{(\lambda k_{\rm F})^2}$ for brevity. This dependence is shown in red in Fig. 2. In the same limit $h_{x/y} \ll \lambda k_{\rm F}$, but for $\beta k_{\rm F}^2 < \lambda k_{\rm F}$ we approximate

$$\begin{aligned}\sigma_{\rm DWHE} \approx \sigma_0\bigg[&\frac{(\lambda k_F)^2}{(\lambda k_F)^2 - h^2} + \frac{1}{16}\frac{(\lambda k_F)^2 - 8h^2}{(\lambda k_F)^4}\left(\beta k_F^2\right)^2 \\ &- \frac{1}{64}\frac{6(\lambda k_F)^2 + h^2}{(\lambda k_F)^6}\left(\beta k_F^2\right)^4\bigg].\end{aligned} \tag{13}$$

This dependence is shown in blue in Fig. 2. When both $h_x \gg \beta k_{\rm F}^2$ and $h_y \gg \beta k_{\rm F}^2$ we approximate

$$\sigma_{\rm DWHE} \approx \sigma_0\frac{(\lambda k_{\rm F})^4}{h^4}, \tag{14}$$

where $h^2 = h_x^2 + h_y^2$. Thus the magnitude of the corresponding part of the electric current decays with the magnetic field as an inverse square of the field. Let us now briefly mention the insulating case. We set $\frac{k^2}{2m} \to 0$ by assuming $m \to \infty$ and $\mu = 0$, then the spectrum (8) becomes $\varepsilon_{\mathbf{k}}^{(\pm)} = \pm\sqrt{\Delta_{\mathbf{k}}^2 + \lambda^2\tilde{k}^2}$. By setting $\mu = 0$ the system becomes insulating with a gap equal to $2\sqrt{h_x^2 + h_y^2}$ at $k_x = k_y = 0$ and to $2\beta\frac{h_xh_y}{\lambda^2}$ at $\tilde{k}_x = \tilde{k}_y = 0$. In Fig. 3 we plot $\sigma_{\rm DWHE}\beta B_xB_y$, the proportionality coefficient in the right-hand side of Eq. (9) between the current and the electric field. Only the valence band contributes at $T = 0$ to the current. The conductivity is quantized as $\frac{e^2}{2}$ as expected [10,15], vanishes when either h_x or h_y is zero, and changes sign in accordance with d-wave symmetry.

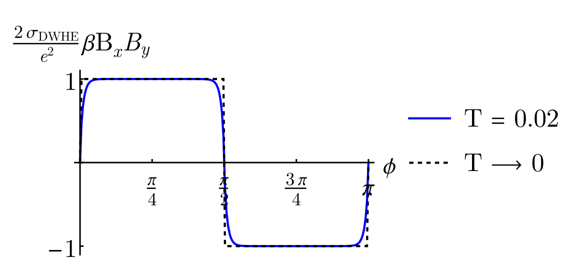

FIG. 3. Quantized DWHE in insulating system. Insulator is achieved by setting $\frac{k^2}{2m} \to 0$ and $\mu = 0$. Plot of the anomalous Hall conductivity $\sigma_{\rm DWHE}$ as a function of the angle ϕ of the in-plane Zeeman magnetic field. Here $h_x = h\cos(\phi)$ and $h_y = h\sin(\phi)$. Values of the parameters used in the numerical calculation are $h = 0.5$, $\mu = 0$, $\lambda = 0.5$, and $\beta = 1.5$.

We note that the predicted DWHE in the three-dimensional system will be experimentally measured together with the $\propto (\mathbf{E}\cdot\mathbf{B})\mathbf{B} = (E_xB_x + E_yB_y)(B_x\mathbf{e}_x + B_y\mathbf{e}_y)$ term in the electric current. In this term it will appear that the $B_xB_y(E_x\mathbf{e}_y + E_y\mathbf{e}_x)$ part is looking like the predicted DWHE, however, the latter is $\propto \beta B_xB_y(E_x\mathbf{e}_y - E_y\mathbf{e}_x)$, and one will have to filter it out from the former.

Finally, we note that there are other mechanisms contributing to the anomalous Hall effect [10,12,13]. They are the skew-scattering and side-jump scattering processes due to the impurities, which are expected to alter the amplitude of the predicted-here DWHE but not its symmetry. Their consideration is left for future research.

Let us now discuss linear magnetoconductivity. Although, as we show above, the integral of the Berry curvature over the angles vanishes when $h_x = h_y = 0$, the Berry curvature can still contribute to the electric current through, for example, modification of the density of states [15]. To study the electric current, we employ the method of the kinetic equation

$$\frac{\partial n_{\mathbf{k}}^{(s)}}{\partial t} + \dot{\mathbf{k}}^{(s)}\frac{\partial n_{\mathbf{k}}^{(s)}}{\partial \mathbf{k}} + \dot{\mathbf{r}}^{(s)}\frac{\partial n_{\mathbf{k}}^{(s)}}{\partial \mathbf{r}} = I_{\mathrm{coll}}\big[n_{\mathbf{k}}^{(s)}\big], \tag{15}$$

with equations of motion updated in the presence of the Berry curvature [15], $\dot{\mathbf{r}}^{(s)} = \frac{\partial \epsilon_{\mathbf{k}}^{(s)}}{\partial \mathbf{k}} + \dot{\mathbf{k}}^{(s)} \times \boldsymbol{\Omega}_{\mathbf{k}}^{(s)}$, and $\dot{\mathbf{k}}^{(s)} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{r}}^{(s)} \times \mathbf{B}$. The current is given by $\mathbf{j} = e\sum_{s=\pm}\int_{\mathbf{k}}[1 + \frac{e}{c}(\mathbf{B}\cdot\boldsymbol{\Omega}_{\mathbf{k}}^{(s)})]\dot{\mathbf{r}}_{\mathbf{k}}^{(s)}n_{\mathbf{k}}^{(s)}$. We approximate the kinetic equation only by intraband scattering, $I_{\mathrm{coll}}[n_{\mathbf{k}}^{(\pm)}] = (\bar{n}^{(\pm)} - n_{\mathbf{k}}^{(\pm)})\tau^{-1}$, where $\bar{n}^{(s)} = (4\pi)^{-1}\int \sin(\theta)d\theta d\phi[1 + \frac{e}{c}(\mathbf{B}\cdot\boldsymbol{\Omega}_{\mathbf{k}}^{(s)})]n_{\mathbf{k}}^{(s)}$ is the distribution function averaged over the angles and τ is the fermion's lifetime due to the elastic scattering on impurities. Interband scatterings are also allowed, but only by virtue of the spin-orbit coupling λ since, without it, the bands are spin polarized and there is no scattering between them. Then these processes will contribute in higher order in spin-orbit coupling than what we will derive. Besides, there is no chiral anomaly in the system and, therefore, interband scattering processes are not important.

The kinetic equation is approximated as usual; we follow the lines of Refs. [48,51] to obtain for the LMC defined as $\mathbf{j}_{\mathrm{LMC}} = \sigma_{\mathrm{LMC}}\beta B_z(E_x\mathbf{e}_y + E_y\mathbf{e}_x)$, the following expression:

$$\sigma_{\mathrm{LMC}} = -2e^2 v_F \frac{e\tau}{mc}\frac{|\lambda|k_{\mathrm{F}}}{\sqrt{(2\lambda k_{\mathrm{F}})^2 + \left(\beta k_{\mathrm{F}}^2\right)^2}}. \tag{16}$$

We note that it is the correction to the density of states [15] due to the $\frac{e}{c}(\mathbf{B}\cdot\boldsymbol{\Omega}_{\mathbf{k}}^{(s)})$ nonzero product that contributes to this current. In Ref. [51] it was shown that linear magnetoconductivity in ferromagnets can be $\delta\mathbf{j} = \alpha_1(\mathbf{E}\cdot\mathbf{B})\mathbf{M} + \alpha_2(\mathbf{E}\cdot\mathbf{M})\mathbf{B} + \alpha_3(\mathbf{M}\cdot\mathbf{B})\mathbf{E}$, where the Onsager relation requires $\alpha_1 = \alpha_2$ (see also the comment in Ref. [52]) whose parts were recently experimentally observed in Refs. [53–55]. Here we find a distinct structure of the LMC. There is strong evidence that the predicted-here $\mathbf{j}_{\mathrm{LMC}}$ was already observed in Ref. [41] (the red arrows in the central figure in Fig. 2 in Ref. [41]). There, a sign of the voltage drop transverse to the passed current has a d-wave symmetry with respect to the direction of the current. In the AHE given in Eqs. (1), (2), and (3), the sign does not depend on the direction of the passed current, while Eq. (4) does and it has the observed d-wave symmetry. Indeed, according to Eq. (4), the transverse to the current voltage drop vanishes when the current is passed at $\frac{\pi}{4} + \frac{\pi}{2}n$ angles.

We see that the results decay as a power law in the high density limit. On the other hand, antiferromagnetism does not survive extensive doping of the system with conducting electrons. Therefore, our results are expected to be experimentally observed in the low-doping regime of antiferromagnets. We speculate that our predicted DWHE might be relevant to the polar Kerr effect observed in the pseudogap phase of cuprates [57]. Indeed, either the polar Kerr effect or Faraday rotation is due to the off-diagonal elements of the dielectric tensor, which are defined by the Hall effect in the medium. Then the question becomes which of the anomalous Hall effects, Eqs. (1), (2), or (3), contributes.

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