Letter

Floquet-Anderson localization in the Thouless pump and how to avoid it

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We investigate numerically how on-site disorder affects conduction in the periodically driven Rice-Mele model, a prototypical realization of the Thouless pump. Although the pump is robust against disorder in the fully adiabatic limit, much less is known about the case of finite period time T, which is relevant also in light of recent experimental realizations. We find that, at any fixed period time and nonzero disorder, increasing the system size $L \to \infty$ always leads to a breakdown of the pump, indicating Anderson localization of the Floquet states. Our numerics indicate, however, that, in a properly defined thermodynamic limit, where L/T^{θ} is kept constant, Anderson localization can be avoided, and the charge pumped per cycle has a well-defined value — as long as the disorder is not too strong. The critical exponent θ is not universal, rather, its value depends on the disorder strength. Our findings are relevant for practical, experimental realizations of the Thouless pump, for studies investigating the nature of its current-carrying Floquet eigenstates, as well as the mechanism of the full breakdown of the pump, expected if the disorder exceeds a critical value.

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Introduction. The Thouless pump [1] was instrumental in understanding the role topology plays in the theory of the quantum Hall effect. Its simplest form is that of a two-band, gapped fermionic chain whose parameters are slowly and periodically varied in time. In the half-filled state, where the lower/upper band is completely filled/empty, and in the adiabatic limit, an integer number Q of fermions are pumped in the lower band, where Q is a topological invariant, equal to the Chern number C of the pump sequence.

While it started out as a thought experiment, the Thouless pump can now be found in the laboratory [2]. After its demonstration in photonic systems [3–6], topological pumping has been realized in a variety of platforms, such as mechanical metamaterials [7,8], ultracold atoms [9–11], and other quantum systems [12]. More recently, an electrical circuit demonstration has been put forward [13].

The real-life Thouless pump has a finite size, L, a finite period time, T, and it is disordered. This raises the question whether these, either separately or in combination, can prove detrimental to its robustness, namely, to the quantization of the pumped charge. For instance, even without disorder, a finite period time gives corrections to the quantized value of the pumped charge. For a quench-like switching on of the pump, the authors of Ref. [14] found these to scale as $1/T^2$, but they should be greatly reduced for a smoother switching-on of the periodic driving cycle. The breaking of adiabaticity, however, is not always sufficient to destroy the Thouless pump. As shown in Ref. [15], if the disorder-free pump is made longer and longer, then adiabaticity will be strongly broken for any finite T, no matter how large. In spite of this, they found that the quantization of the pumped charge survives in the steady-state regime, when the pump performs cycle after cycle. Similarly, in the adiabatic $T \to \infty$ limit, the quantization of the pumped charge should be robust against small disorder, as discussed by Thouless and Niu [16]. Here, eigenstates of the time-dependent Hamiltonian are all Anderson localized, and in the adiabatic limit an intuitive (although possibly misleading) picture is that periodic modulation pumps charge between them.

In this Letter we focus on the effect of onsite disorder on an actual Thouless pump (finite T rather than the adiabatic limit). On the one hand, disorder can even result in a suppression of finite-T corrections, and a higher pumped charge [17]. On the other, adding too much disorder has to result in a breakdown of the pump, via an Anderson localization transition — a few works have already studied this numerically [17,18].

On-site disorder on the Thouless pump is particularly interesting because of the connection to the "levitation and annihilation" in Chern insulators [19]. Disorder in the Chern insulator localizes its eigenstates, but each topological band has (at least) one state that remains extended in the thermodynamic limit, which "carries the Chern number" [20–22]. As disorder is increased, full Anderson localization happens by these robustly extended states "levitating" towards each other in energy and "annihilating." Can such phenomena be observed in the Floquet states of the disordered Thouless pump?

Numerical results [17] are consistent with this, and have even identified a critical exponent for this Anderson localization transition, obtained by scaling up the size of the pump at a constant (and large) period time.

Here, we uncover a previously unexplored behavior of the disordered Thouless pump, specifically with regard to its thermodynamic limit. One might think that for a thermodynamic limit, $L \to \infty$ and $T \to \infty$ should be taken one after the other. However, we argue this is inadequate. Taking $T \to \infty$ first is problematic, since the charge pump becomes infinitely slow. Moreover, in this "ultra-adiabatic" limit transitions occur between distant Anderson localized eigenstates, thus computation of Q needs open boundary conditions [23]. Taking $L \to \infty$ first is often (sometimes tacitly) assumed. However, as we show in the following, this leads to a breakdown of the pump due to the Anderson localization of the Floquet eigenstates. In this Letter we suggest a properly defined way to take $L \to \infty$ and $T \to \infty$.

The model. We consider the periodically driven Rice-Mele model [24] with an on-site potential disorder that is independent of time. Spinless fermions hop on a closed chain of $L = 2N$ sites, with the unitary time evolution governed by the Hamiltonian

$$\hat{H}(t) = -\sum_{m=1}^{L}\left[J + (-1)^m \tilde{J}\cos\frac{2\pi t}{T}\right]\hat{c}_m^\dagger \hat{c}_{m+1} + \text{H.c.} \\ -\sum_{m=1}^{L}\left[(-1)^m \Delta \sin\frac{2\pi t}{T} + W\zeta_m\right]\hat{c}_m^\dagger \hat{c}_m, \qquad (1)$$

where $\hat{c}_m$ annihilates a fermion on site m, with $\hat{c}_{L+1} = \hat{c}_1$, i.e., periodic boundary conditions, $J/\tilde{J}$ are uniform/staggered components of the nearest-neighbor hopping, Δ is a staggered on-site potential, and t and T are time and period time. The on-site disorder has amplitude W and the ζ_m's are real random numbers uniformly distributed on $[-1/2, 1/2]$. We set $\hbar = 1$ for convenience. In this noninteracting model, all quantitites of interest can be computed from the single-particle $L \times L$ Hamiltonian matrix $H(t)$, with $\hat{H}(t) = \sum_{l,m=1}^{L} \hat{c}_l^\dagger H_{lm}(t)\hat{c}_m$.

We use the basis of Floquet states: eigenstates $|\psi_n\rangle$ of the single-particle (Floquet) unitary operator $\hat{U}$ for one period of time evolution, $\hat{U}\,|\psi_n\rangle = e^{-i\varepsilon_n T}\,|\psi_n\rangle$. Here $\hat{U} = \mathcal{T} e^{-i\int_0^T dt \hat{H}(t)}$, where $\mathcal{T}$ is time ordering, $n = 1, 2, \ldots, L$ is the eigenstate index and ε_n is the quasienergy. Floquet states evolve periodically in time, up to a phase factor

$$|\psi_n(t)\rangle = \mathcal{T} e^{-i\int_0^t dt' \hat{H}(t')}\,|\psi_n\rangle = e^{i\varepsilon_n T}\,|\psi_n(t+T)\rangle\,. \qquad (2)$$

If the disorder is weak and the pump is run slowly enough, Floquet states can be assigned to bands according to their average energy

$$\overline{E_n} = \frac{1}{T}\int_0^T dt\,\langle\psi_n(t)|\,\hat{H}(t)\,|\psi_n(t)\rangle\,. \qquad (3)$$

Floquet states carry current whose integral over the time period gives the pumped charge in that state

$$Q_n = 2\int_0^T dt\left(J + \tilde{J}\cos\frac{2\pi t}{T}\right)\text{Im}[\psi_{n,2}^*(t)\psi_{n,1}(t)]. \qquad (4)$$

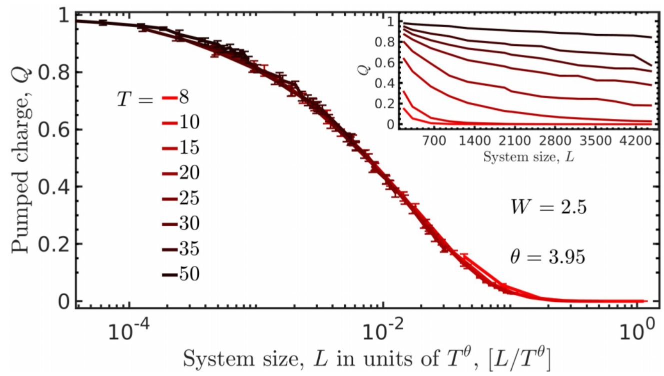

FIG. 1. Pumped charge Q decreases as chain length L is increased from 80 to 4480, with disorder $W = 2.5$ (average of 20 disorder realizations). Results for various period times T fall onto each other if rescaling the system size as L/T^θ, with $\theta = 3.95$. Inset: unscaled data.

Here we take the current between sites 1 and 2, with $\psi_{n,j}$ denoting the amplitude of the nth Floquet state on site j, but the position does not matter, due to the periodicity of the time evolution of Floquet states.

We calculate the charge pumped in the so-called sustained pumping limit of a filled lower band [17,25]: The system is initialized at $t = 0$, with the $L/2$ lowest-energy eigenstates $|\phi_l\rangle$ of the instantaneous Hamiltonian fully occupied, and then is time evolved. After many cycles, this results effectively in a Floquet diagonal ensemble [25], i.e., an incoherent mixture where Floquet states are populated with the same weights as at $t = 0$. Thus, the charge pumped per cycle in this limit is

$$Q = \sum_{n=1}^{L} Q_n \sum_{l=1}^{L/2} |\langle\phi_l|\,\psi_n\rangle|^2. \qquad (5)$$

The numerical method. We compute the time evolution of the Floquet states, needed for Eq. (4), as a matrix product of time-slices of the timestep operator. For the time-slices, we used a recently developed method based on the Chebyshev polynomial representation of skew Hermitian matrices, $e^{-iHdt} \approx \alpha_0 - iz_0 H dt - \alpha_1[Hdt]^2 + iz_1[Hdt]^3 + \alpha_2[Hdt]^4 - iz_2[Hdt]^5$, with constants specified to 20 decimals [26]. This gives the matrix exponential to numerical accuracy, as long as $||H(t)dt||_1 < 1.17 \times 10^{-2}$. We could reach chains lengths up to $L = 10\,000$, more than a factor of 10 larger than previous works [17]. We use the hopping J as our energy scale, and set parameters as

$$J = 1, \quad \tilde{J} = 1/2, \quad \Delta = 1.5, \qquad (6)$$

for a well-defined gap with a Chern number $C = 1$. Thus, $Q = 1$ in the adiabatic limit, as long as the instantaneous Hamiltonian is gapped, i.e., $W \lesssim 3.5$ [17].

Results. We find that the disordered Thouless pump breaks down when increasing the length L of the chain, keeping the period time T constant. Examples are shown in Fig. 1, for disorder $W = 2.5$, for $T = 8$ to 50, and $L = 80$ up to 4480. In these and all cases we studied, the pumped charge decreased as the length was increased. This suggests that Floquet-Anderson localization does not only set in when the disorder is large ($W > 3.5$), but occurs for any on-site

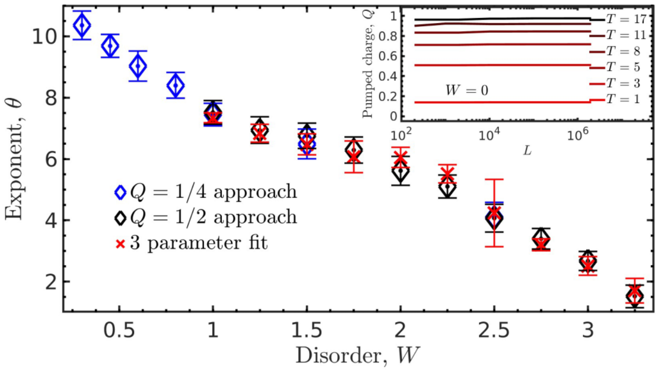

FIG. 2. The scaling exponent θ, obtained by three numerical approaches for each value of disorder W (see main text for details). The exponent decreases as W increases, while it appears to diverge at $W = 0$, consistent with the pumped charge Q being independent of the system size in the clean case — this later item is shown in the inset. For a detailed description of how the exponent was extracted see the Supplemental Material [27].

disorder, $W > 0$. The length L where the pumped charge decreases significantly (e.g., $Q = 1/2$, or $Q = 1/4$) provides an estimate for the Floquet localization length ζ_F.

We find that slower driving (longer period times T) leads to more resilient Thouless pumps, with an apparent power-law relation

$$Q(L, T, W) = Q(L', (L'/L)^{1/\theta(W)}T, W). \tag{7}$$

This is suggested by the good collapse of the numerically measured Q values when using the above scaling relation, as shown in Fig. 1. Thus, the Floquet localization length appears to scale with the period time as $\zeta_F \propto T^\theta$.

We find that the exponent θ of Eq. (7) does not take on a universal value, but depends continuously on the disorder W, as shown in Fig. 2. We extracted the exponent by three different methods. First and second, by identifying ζ_F with the system size where the pumped charge is $Q = 1/2$, and $Q = 1/4$, respectively. Third, we took all the data for a fixed W, and various T and L values, and fitted it with a three-parameter Ansatz — detailed in the Supplemental Material (SM, [27]). These methods agree, and give a disorder-dependent exponent θ, which approaches $\theta \approx 2$ near the critical disorder $W \approx 3.5$. Note, however, that the numerical evidence for the power-law scaling is strong only for the case of moderate disorder. For smaller disorders, $W \lesssim 1.5$, we have $1/\theta \ll 1$, thus in the numerically available range the evidence for the power-law scaling here is not conclusive, as discussed in the SM [27].

For a more complete picture of the breakdown of the Thouless pump as a function of disorder W, chain length L, and period time T, we show the numerically obtained map of pumped charge Q in Fig. 3. The colors show Q values for $L = 320$, and results for other lengths are shown as $Q = 1/4$ isolines. These reveal four qualitatively different regimes of the charge pump. For small disorder, $W \lesssim 0.5$, the Floquet localization length ζ_F decreases sharply as the disorder is increased. For $0.5 \lesssim W \lesssim 2$, and period times $10 \lesssim T \lesssim 20$, ζ_F does not depend much on the disorder strength. For larger disorder, $2 \lesssim W \lesssim W_c = 3.5$, we have a sharp decrease of

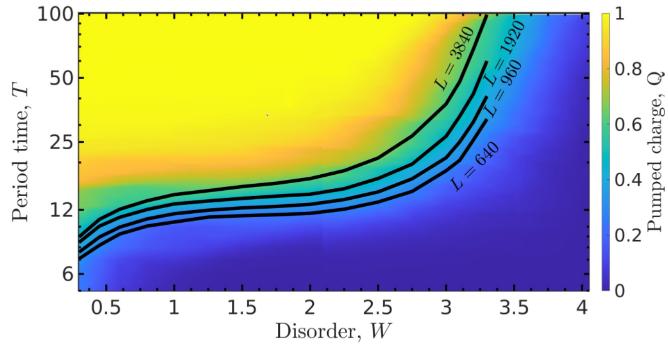

FIG. 3. Colormap: Pumped charge Q, for $L = 320$, for various period times T and disorders W (average over 20 disorder realizations). The Thouless pump works well (light area) for small W and large T, and breaks down both if T decreases or W increases. For $W \gg W_c \approx 3.5$, where there is no gap in the instantaneous energy spectrum, the pumped charge is very close to 0 for any T. Black lines: For system sizes L fixed at larger values than $L = 320$, we show the $T(W)$ curves along which $Q = 1/4$. These show that the pump breaks down easier if the chain is longer.

the ζ_F as W is increased. Finally, above the critical disorder value, $3.5 \lesssim W$, we observe the charge pump breaking down completely.

Thermodynamic limit. For a deeper understanding of how disorder impacts the Thouless pump, we define an alternate thermodynamic limit: $L \to \infty$ and $T \to \infty$ together, with L/T^θ kept constant. This is needed, e.g., to explore the extended/localized nature of the Floquet states, using the inverse participation ratio [17] (IPR) $P_{2,n} = \sum_{m=1}^{L} |\psi_{n,m}|^4$. To show how this limit avoids the problem of the Anderson localization of Floquet states, see Fig. 4. We choose parameters

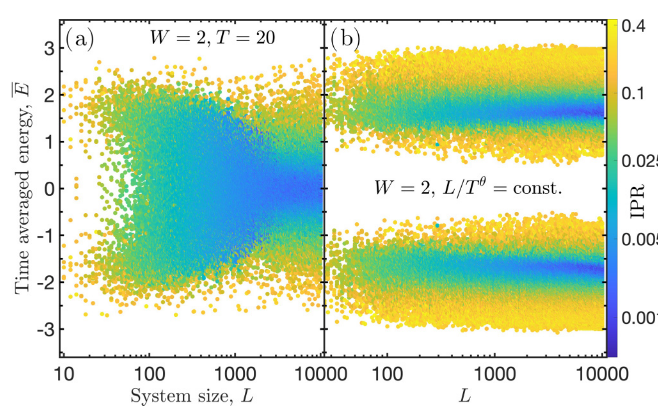

FIG. 4. Spectrum of the average energy of Floquet states in a pump with disorder $W = 2$, for various system lengths, and with color representing the IPR value of the states (single disorder realizations). In (a), the period time fixed, $T = 20$, increasing the length leads to a qualitative change in the spectrum: For $L \lesssim 100$, two bands of extended states can be seen, which carry current right/left in the lower/upper band; for larger lengths the gap between the bands is closed. In (b), the period time is increased together with the system size, keeping L/T^θ constant, where $\theta = 6.01$ was used. Here there is no qualitative change in the spectrum as L is increased, however, the range of extended states becomes narrower.

so that for short lengths, $L \lesssim 100$, the Thouless pump works well, and Floquet states form two well-separated bands. If the length is increased at fixed T (panel a), the two bands merge and the pump begins to break down. In contrast, if L/T^θ is kept constant [Fig. 4(b)], the spectrum of Floquet states shows no qualitative change up to the largest system sizes that we were able to access numerically. For each band, states in the band center are more extended (lower IPR, decreasing with L) and at the band edges more localized (higher IPR, independent of L). More detailed analysis of the IPR values, and examples for shorter period times T, where the breakdown of the pump is even more visible, are shown in the SM [27]. Also there we present additional numerical simulations highlighting that this phenomenology does not occur if the pump is topologically trivial.

Discussion and conclusions. We find that on-site potential disorder in the periodically driven Rice-Mele model results in a breakdown of the Thouless pump in the $L \to \infty$, constant-T limit due to the Anderson localization of the Floquet states. This can be avoided by taking $L \to \infty$ and $T \to \infty$ together, keeping L/T^θ constant, where θ is a disorder-dependent critical exponent. Although we expected $\theta = 2$ based on the corrections to adiabaticity, we find that this is not the case, rather, θ depends on disorder strength continuously. It is an interesting open problem to find an analytical explanation of this phenomenon.

Our work is a starting point for a more systematic investigation of the relation between Anderson localization in the Thouless pump and the "levitation and annihilation" of extended states in Chern insulators. By finding a suitable thermodynamic limit, we open the way to studying numerically the conduction in the disordered Thouless pump, as well as the mechanism of its breakdown as disorder is increased. Our preliminary results, shown in the SM [27], suggest that the current-carrying states here have a fractal nature since the corresponding IPR values seem to scale with a power of system size with a noninteger exponent, but there are many open questions here including whether in the thermodynamic it is only a single Floquet state per band that carries the current.

From a broader perspective, the Thouless pump is the oldest among a large family of topological pumps, which by now go well beyond Chern insulators. Topological pumping has been associated with a wide range of topological insulators and superconductors [28–30], and has been proposed to occur also between topological defects [31,32]. It has been further extended to higher-order topological phases [33–35], which can lead to dipole or to quadrupole pumps [36]. Very recently, topological pumping between the corners of a two-dimensional sample has been shown to occur both theoretically and experimentally, with the pump working either via bulk states [37,38] or via edge states [39]. Our work opens a new direction of research in this field, consisting in the study of the thermodynamic limits associated to this large family of pumps and the critical exponents characterizing them.

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