Letter

Dynamical localization and slow dynamics in quasiperiodically driven quantum systems

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(Received 6 March 2023; accepted 13 March 2024; published 3 April 2024)

We investigate the role of a quasiperiodically driven electric field in a disordered fermionic chain. In the clean noninteracting case, we show the emergence of dynamical localization—a phenomenon previously known to exist only for a perfect periodic drive. In contrast, in the presence of disorder, where a high-frequency periodic drive preserves Anderson localization, we show that the quasiperiodic drive destroys it and leads to slow relaxation. Considering the role of interactions, we uncover the phenomenon of quasiperiodic driving-induced logarithmic relaxation, where a suitably tuned drive (corresponding to dynamical localization in the clean, noninteracting limit) slows down the dynamics even when the disorder is small enough for the system to be in the ergodic phase. This is in sharp contrast to the fast relaxation seen in the undriven model, as well as the absence of thermalization (drive-induced many-body localization) exhibited by a high-frequency periodic drive.

DOI: 10.1103/PhysRevB.109.L161104

Introduction. The nonequilibrium properties of a quantum system subjected to a time-dependent drive have been a topic of great interest [1–10]. Some notable phenomena associated with such driven systems include dynamical localization in kicked rotors [11–13], dynamical freezing [14–19], Floquet topological insulators [20–22], Floquet prethermalization [23–32] and time crystals [33–37], and Floquet many-body localization (MBL) [38–40]. An intriguing category of periodically driven systems involves an electric-field drive, which gives rise to a range of fascinating phenomena, including dynamical localization [41–45], coherent destruction of Wannier-Stark localization [45–49], and super-Bloch oscillations [50,51]. Incorporating many-body interactions further opens up fascinating possibilities, such as Stark-MBL [52–61], drive-induced MBL [49,62], and Stark time crystals [63,64]. A crucial question that arises is whether these characteristics persist in the absence of a perfectly periodic drive.

While an enormous body of work has been devoted to periodic drives, the exploration of the role of quasiperiodic driving has recently gained traction [65–81]. Features such as prethermalization [73–75], coherence restoration [82], and quasitime crystals [68,70,74,83] are associated with such drives. Furthermore, experiments have realized dynamical phases employing quasiperiodic driving [84]. In this Letter, we explore the properties of a system driven by a quasiperiodic electric field. Specifically, we address the question of whether a quasiperiodic electric-field drive can give rise to dynamical localization, and if it does, what the effect of interactions and disorder on dynamical localization would be. We address these questions by considering two discrete forms of quasiperiodic driving: Fibonacci and Thue-Morse [67,69,72]. For these sequences, we find that in the noninteracting limit, the phenomenon of dynamical localization occurs when the parameters of the drive are tuned appropriately. We derive the conditions under which it is realized and numerically demonstrate this using the dynamics of return probability and entanglement entropy. In the presence of disorder, where the undriven system exhibits Anderson localization [85,86], we find that the quasiperiodic drive destroys it and leads to a slow relaxation of the return probability together with a sublinear growth of entanglement entropy.

In the presence of many-body interactions, dynamical localization is lost, and the system approaches the infinite-temperature state. However, the approach is notably slower when the parameters are specifically tuned at the dynamical localization point, as opposed to arbitrary parameter choices. This follows from the fact that the drive suppresses the hopping significantly at the dynamical localization point. Next, we demonstrate how the hopping suppression can be exploited to manipulate the dynamical behavior of a disordered many-body interacting system. Starting from the ergodic phase in the weak disorder limit of the undriven model, where the autocorrelation function is known to exhibit power-law decay [87], we show that a quasiperiodic electric-field drive can in fact lead to a slow logarithmic relaxation. This is in sharp contrast to the drive-induced MBL [62] that is seen for a high-frequency periodic drive. In Table I, the chief findings from our work are summarized alongside already established results in the literature.

Model Hamiltonian and driving protocol. We consider a disordered interacting one-dimensional tight-binding chain subjected to a time-dependent electric field $\mathcal{F}(t)$. The model Hamiltonian can be written as

$$H = -\frac{\Delta}{4}\sum_{j=0}^{L-2}(c_j^\dagger c_{j+1} + \text{H.c.}) + \mathcal{F}(t)\sum_{j=0}^{L-1} j\left(n_j - \frac{1}{2}\right) + \sum_{j=0}^{L-1} h_j\left(n_j - \frac{1}{2}\right) + U\sum_{j=0}^{L-2}\left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right), \tag{1}$$

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TABLE I. Schematic of the main results for the quasiperiodic drive and a comparison with existing results for the undriven and high-frequency periodic driving cases based on an analysis of the autocorrelation function and entanglement entropy. The third row contains the central findings of our work. Here, W, U refers to disorder and interaction strength, respectively.

Driving Protocol	$W = 0$ $U = 0$	$W \neq 0$ $U = 0$	$W \neq 0,\ U \neq 0$ Ergodic regime
Undriven	Fast relaxation	Anderson localization [85]	Fast relaxation [87]
Periodic	Dynamical localization (Periodic oscillation) [41]	Anderson localization [88]	Drive-induced MBL [62]
Quasiperiodic	Dynamical localization (Periodic oscillation)	Slow relaxation $S(t) \propto t^{\gamma}, \gamma < 1$	Drive-induced logarithmic relaxation $S(t) \propto \log t$

where c_j and $c_j^\dagger$ are fermionic annihilation and creation operators, respectively, n_j are number operators, Δ is the hopping strength, U is the strength of the nearest-neighbor interaction, and h_j is the on-site potential taken from a uniform distribution, $h_j \in [-W, W]$. For the undriven case ($\mathcal{F}(t) = 0$), Eq. (1) is the standard model of MBL where a transition from an ergodic phase to an MBL phase occurs on varying the disorder strength [89–92], although the precise value of the critical disorder strength is still controversial [93,94]. In this work, we focus only on the ergodic side of the MBL transition. For a periodic drive with time period T, $\mathcal{F}(t+T) = \mathcal{F}(t)$, the clean noninteracting limit ($W = 0, U = 0$) exhibits dynamical localization for appropriately tuned driving amplitude and frequency [41,42,49], while for a randomly fluctuating field there is no dynamical localization [48]. In this work, we focus on the case where the time-dependent field is neither periodic nor random but is rather quasiperiodic in nature and contains many frequencies in the Fourier spectrum. We allow the field $\mathcal{F}$ to oscillate between $\pm F$ after each period T mimicking the Fibonacci and Thue-Morse sequences [67,68,74].

In order to describe our driving protocol, it is convenient to introduce the unitary operators $U_{B/A} = e^{-iTH_{B/A}}$ where $H_{A/B}$ are the Hamiltonians corresponding to the field $\mathcal{F}(t) = \pm F$, respectively [Eq. (1)]. For the Fibonacci sequence, we start from the unitary operators $U_0 = U_A$ and $U_1 = U_B$, and generate the subsequent evolution according to the Fibonacci sequence [72],

$$U_n = U_{n-2}U_{n-1}, \quad n \geqslant 2. \tag{2}$$

The Thue-Morse sequence (TMS), on the other hand, can be generated using the recurrence relation [74]

$$U_{n+1} = \tilde{U}_n U_n, \quad \tilde{U}_{n+1} = U_n \tilde{U}_n, \tag{3}$$

where we start with the unitary operators $U_1 = U_B U_A$, $\tilde{U}_1 = U_A U_B$. The time evolution of an initial state can be expressed as $|\psi_n\rangle = U_n|\psi(0)\rangle$.

Dynamical localization. We first consider the clean noninteracting limit ($W = 0, U = 0$). In this case, the Hamiltonian (1) can be written in terms of the unitary operators [95] $\hat{K} = \sum_{n=-\infty}^{n=\infty} |n\rangle\langle n+1|$, $\hat{K}^\dagger = \sum_{n=-\infty}^{n=\infty} |n+1\rangle\langle n|$, and $\hat{N} = \sum_{n=-\infty}^{\infty} n|n\rangle\langle n|$, as

$$\hat{H} = -\frac{\Delta}{4}(\hat{K} + \hat{K}^\dagger) + \mathcal{F}(t)\hat{N}. \tag{4}$$

These operators follow the commutation relations $[\hat{K}, \hat{N}] = \hat{K}$, $[\hat{K}^\dagger, \hat{N}] = -\hat{K}^\dagger$, $[\hat{K}, \hat{K}^\dagger] = 0$. With this form of the Hamiltonian and the commutation relations, we can write down the effective Hamiltonian H_{eff}. For Thue-Morse driving, we work out an expression for the stroboscopic unitary operator defined at the mth Thue-Morse level, $U(N = 2^m)$. With the aid of Eq. (3), we can express U as a product of a string of the unitary operators U_A and U_B, which, using the Baker-Campbell-Hausdorff formula [96], can be written in terms of H_{eff} as [97]

$$U(N = 2^m) \equiv \exp(-i2^m T H_{\text{eff}}). \tag{5}$$

Here, H_{eff} is given by

$$H_{\text{eff}} \equiv \Delta_{\text{eff}}(\hat{K} + \hat{K}^\dagger), \quad \Delta_{\text{eff}} = -\frac{\Delta}{4}\left[\frac{\sin(2F\pi/\omega)}{(2F\pi/\omega)}\right], \tag{6}$$

where $\omega = \frac{2\pi}{T}$. Thus we see from Eq. (6) that the coefficients of $\hat{K}$ and $\hat{K}^\dagger$ get renormalized. When the amplitude of the drive is tuned at $\frac{F}{\omega} = \frac{n}{2}$ ($n \in \mathbb{Z}$), Δ_{eff} vanishes and any state will return to itself after a time period T, and thus the system remains dynamically localized. Similar results are obtained for Fibonacci driving [97].

The return probability, which is a measure of the probability of finding a particle at an initially localized site n after a time t is defined as $P(t) = |\langle\psi(0)|\psi(t)\rangle|^2$; in our study, we take $n = L/2$. The dynamics of the return probability for the Thue-Morse driven system is plotted in Fig. 1(a). For the ratio $F/\omega = 2$, we find that the return probability has oscillatory behavior. We have checked using stroboscopic dynamics that it periodically returns to its original value of unity even for very long times. On the other hand, tuning the ratio at $F/\omega = 1.3$, we see that the return probability is vanishingly small. A similar observation can be made when looking at the growth of entanglement entropy defined as [98–100], $S(t) = -\text{Tr}(\rho_{L/2} \ln \rho_{L/2})$, where $\rho_{L/2} = \text{Tr}_{1 \leqslant i \leqslant L/2}\{|\psi(t)\rangle\langle\psi(t)|\}$ is the reduced density matrix of half the chain obtained by tracing out the other half of the chain. Starting with an initial Néel state $|\psi(0)\rangle = \prod_{i=1}^{L/2} \hat{c}_{2i}^\dagger|0\rangle$, we study the dynamics of the entropy in Fig. 1(c). Again, when $F/\omega = 2$, we find oscillatory behavior, while for $F/\omega = 1.3$, $S(t)$ starts to grow in time. These observations confirm the analytical prediction that when the ratio F/ω is tuned properly, the system exhibits dynamical localization; else, it behaves as a nearest-neighbor chain with renormalized hopping.

Slow dynamics in the disordered noninteracting system. We now consider the disordered case, which leads to Anderson localization for any nonzero value of the disorder strength in the undriven model [85]. We study the stroboscopic evolution

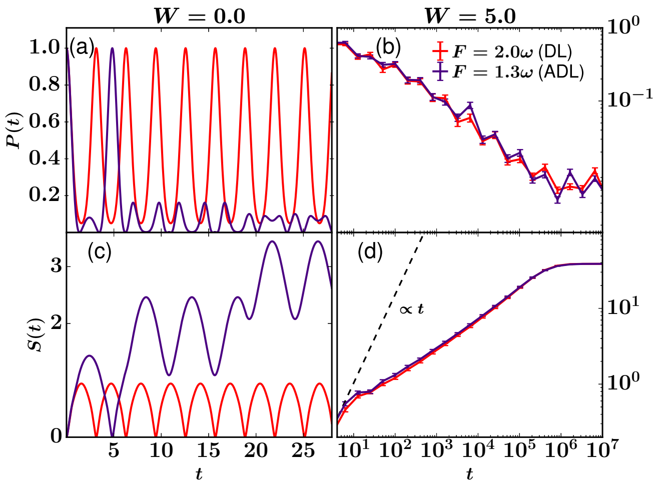

FIG. 1. (a), (b) Return probability $P(t)$, and (c), (d) entanglement entropy $S(t)$ for a Thue-Morse driven clean ($W = 0.0$) and disordered ($W = 5.0$) system at dynamical localization (DL) ($F = 2.0\omega$) and away from it (ADL) ($F = 1.3\omega$). The other parameters are $\Delta = 4.0$, $\omega = 1.0$, $L = 200$. We take the time discretization $dt = 0.001$ for (a) and (c). The data are averaged over 100 realizations of disorder for (b) and (d). The black dashed line provides a guide to the ballistic growth.

of the return probability and the entanglement entropy for disorder strength $W = 5.0$. From Fig. 1(b), we see no sign of either dynamical or Anderson localization; instead, we observe that Thue-Morse driving leads to a slow decay of the return probability accompanied by a sublinear growth of the entropy $S(t) \propto t^{\gamma}$ where $\gamma < 1$ [101]. This is in stark contrast with the effect of a high-frequency periodic drive, where Anderson localization remains stable [88,102].

An understanding of the slow dynamics of a quasiperiodically driven system can be obtained by performing a high-frequency expansion for the first two cycles, for which the effective Hamiltonian can be written as

$$H_{\text{eff}} = \Delta_{\text{eff}}\{\hat{K}e^{-iFT/4} + \hat{K}^{\dagger}e^{iFT/4}\} + D_0 + H_{\text{LRH}}, \quad (7)$$

where D_0 is the static disorder term and H_{LRH} contains the longer-range hopping terms. A derivation of Eq. (7) and the form of H_{LRH} is provided in Ref. [97]. When dealing with higher-order terms and a higher level of the quasiperiodic sequence, the calculations become more complex. However, valuable information can still be extracted from H_{eff} at this level. For frequencies larger than the local bandwidth Δ_{ξ} of a system of size of the order of the localization length ξ, H_{LRH} can be neglected since it contains factors involving powers of the time period T which becomes very small. Moreover, due to the renormalization Δ_{eff}, the effective disorder increases, i.e., $H_{\text{eff}}(J, W) \approx H(J, W/J_{\text{eff}})$, and hence a stronger localization would be expected. Thus H_{eff} obtained for two cycles suggests the stability of Anderson localization in the presence of a high-frequency drive. For a periodic drive, these two cycles repeat indefinitely; therefore, indeed this robustness of Anderson localization in the presence of a high-frequency periodic drive has been reported [88,102]. For our drive protocol, however, we must incorporate higher levels of the quasiperiodic sequence. This introduces a mixture of low- and high-frequency components. While frequencies larger than the local bandwidth $\omega \gg \Delta_{\xi}$ do not influence localization, lower frequencies $\omega \ll \Delta_{\xi}$ initiate transitions between the localized states. This competition gives rise to a gradual relaxation of $P(t)$ and a sublinear growth of $S(t)$, ultimately leading to delocalization.

Slow dynamics in the disordered interacting system. Having discussed the emergence of dynamical localization under quasiperiodic driving, we now explore the interplay of quasiperiodic driving, many-body interactions, and disorder. We employ exact diagonalization for a system of size $L = 16$ at half filling and focus on the dynamics of entanglement entropy and the autocorrelation function. For driven quantum systems, the entanglement entropy typically saturates to the Page value, $S_{\text{Page}} = \frac{1}{2}(L\ln 2 - 1)$, which corresponds to the infinite-temperature state [103]. The autocorrelation function is defined as [87,104–106]

$$C(t) = 4\langle \hat{S}^z_{L/2}(t)\hat{S}^z_{L/2}(0)\rangle, \quad (8)$$

where $\hat{S}^z_i = \hat{n}_i - \frac{1}{2}$. While in the localized phase, the autocorrelation function $C(t)$ saturates to a nonzero value, in the ergodic phase, it rapidly goes to zero [68,87].

We first start with the clean limit $W = 0$, and observe that dynamical localization is destroyed in the presence of many-body interactions. However, for Fibonacci driving (and also Thue-Morse [97]), the dynamics is found to be very slow if the interactions are turned on at the dynamical localization point as opposed to any arbitrary choice of the parameters. In Figs. 2(a) and 2(d), we plot the dynamics of autocorrelation $C(t)$, and entanglement entropy $S(t)$ for $\frac{F}{\omega} = 2$ (at DL) and $\frac{F}{\omega} = 1.3$ (away from DL) and a range of driving frequencies. We average the quantities over 50 different initial product states close to the Néel state. The entanglement entropy rapidly reaches the Page value while the autocorrelation function quickly decays to zero for $\frac{F}{\omega} = 1.3$ for all the driving frequencies considered. On the other hand, for $\frac{F}{\omega} = 2$, $C(t)$ exhibits a slow decay accompanied by a slow relaxation of $S(t)$ to the Page value [Figs. 2(a) and 2(d)]. On increasing the driving frequency, the growth further slows down, suggesting slow heating.

We next consider the disordered case with $W = 1.0$, which in the undriven model lies in the ergodic region where $C(t)$ shows fast decay [87]. We plot the autocorrelation and the entanglement entropy for Fibonacci driving in Figs. 2(b) and 2(e) for different driving frequencies and with the parameters tuned at the dynamical localization point. Although the undriven system lies in the ergodic regime, we see that the quasiperiodic drive induces logarithmically slow relaxation of the autocorrelation and a slow growth of the entropy. Earlier work has reported a logarithmic relaxation in the MBL phase alone [68].

These findings can be understood through a high-frequency expansion analysis. Analogous to the noninteracting case, by considering the first two cycles, we get $H_{\text{eff}}(\Delta, W, U) \approx H(\Delta_{\text{eff}}, W, U)$ [97]. This implies a suppression of hopping or, conversely, an augmented effective disorder strength W/Δ_{eff}, driving the system towards the MBL regime [49,62]. However, when we incorporate higher terms of the quasiperiodic sequence, low-frequency components come into play,

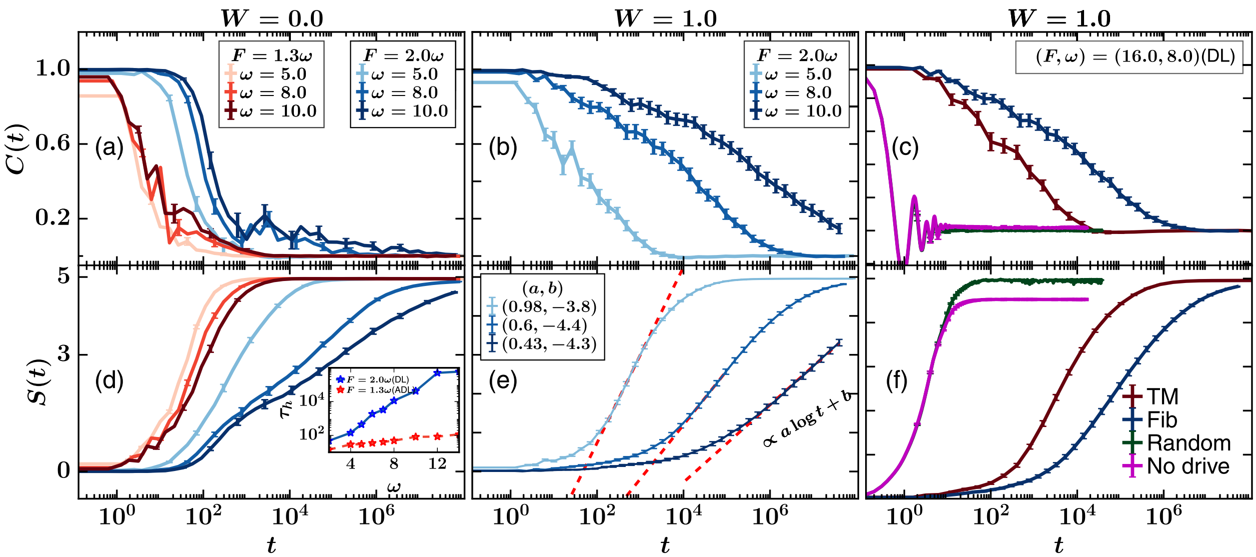

FIG. 2. Dynamics of the autocorrelation function $C(t)$ and the half-chain entanglement entropy $S(t)$ for the Fibonacci driven interacting fermionic system. (a), (d) For $W = 0.0$. The red shades correspond to a tuning of the parameters away from dynamical localization ($F = 1.3\omega$, ADL), and blue shades correspond to a tuning at dynamical localization ($F = 2.0\omega$, DL). The inset shows the heating time τ_h as a function of ω. (b), (e) For $W = 1.0$. When disorder is present, we have averaged over 100 disorder realizations. In (e), the red dashed lines are fits to the function $a \log t + b$. (c), (f) Comparison of dynamics of differently driven and undriven interacting disordered systems. The other parameters are $\Delta = 4.0$, $U = 1.0$, $F = 2.0\omega$, and $L = 16$.

prohibiting the possibility of MBL. Nevertheless, a competition between the low- and high-frequency components emerges, leading to the slow relaxation of $C(t)$ and a logarithmic growth of $S(t)$. The interplay between these components is also reflected in the lifetime of the slowly relaxing phase observed in the different driving protocols, as shown in Figs. 2(c) and 2(f). While a random drive with numerous low-frequency components yields fast relaxation, quasiperiodic driving results in a prolonged relaxation phase, with Fibonacci driving exhibiting even slower relaxation compared to Thue-Morse driving. This distinction can be attributed to the additional low-frequency components present in the Fourier spectrum of the Thue-Morse drive. A similar argument holds for the clean limit, where due to the hopping renormalization, we see a slow heating at dynamical localization as compared to away from it.

To investigate the impact of driving frequency on slow dynamics, we examined the heating time τ_h defined as the time when $S(t)$ attains half of the Page value [107], at different driving frequencies tuned at the dynamical localization point. We see that at the dynamical localization point, the heating time is several orders of magnitude greater than when the parameters are tuned away from it [Fig. 2(d) inset]. Figure 3 illustrates the relationship between τ_h and ω for both Fibonacci and Thue-Morse driving, considering cases with $W = 0.0$ and $W = 1.0$. For $W = 0.0$ [Figs. 3(a) and 3(b)], we observe an exponential dependence of τ_h on ω for Fibonacci driving, whereas Thue-Morse driving follows a subexponential ($\tau_h \sim e^{[\ln \omega]^2}$) trend consistent with the theoretical bound [24]. On the other hand, for $W = 1.0$, where drive induces logarithmic relaxation, an extended heating time is evident. Fibonacci driving now displays a superexponential dependence ($\tau_h \sim e^{\omega^\beta}$, $\beta > 1$), while Thue-Morse driving exhibits an exponential dependence.

Conclusions. We investigate the dynamics of fermions in a disordered potential under the influence of a time-dependent electric field generated from Fibonacci and Thue-Morse sequences. In the absence of interactions and disorder, we demonstrate that dynamical localization can be achieved through quasiperiodic driving and identify the conditions for its realization. When disorder is introduced, the quasiperiodic drive disrupts Anderson localization but leads to a slow relaxation of observables. By introducing interactions to a system near the dynamical localization point, we show a significant suppression of heating compared to cases where parameters deviate from it. Furthermore, utilizing the concept of hopping suppression, we show a logarithmic relaxation induced by quasiperiodic driving in the ergodic regime of an interacting system. This drive-induced slow relaxation alters the dependence of heating time on the driving frequency, resulting in a superexponential dependence for Fibonacci driving and an exponential dependence for Thue-Morse driving, contrasting the expected exponential and subexponential dependencies [68,75].

We look forward to our work inspiring ways to host nonequilibrium phases of matter such as time quasicrystals [68,74]. In general, it would be intriguing to explore the possibility of using a quasiperiodic electric-field drive to manipulate the properties of quantum systems in the same spirit as a periodic drive. In the future, it will be interesting to see features such as coherent/incoherent destruction of Wannier-Stark localization [47,108], super-Bloch oscillations [50,51], and the effect of long-range interactions in such systems.

Acknowledgments. We are grateful to the High Performance Computing (HPC) facility at IISER Bhopal, where large-scale calculations in this project were run. We acknowledge the QUSPINpackage [109,110] for help in the numerical

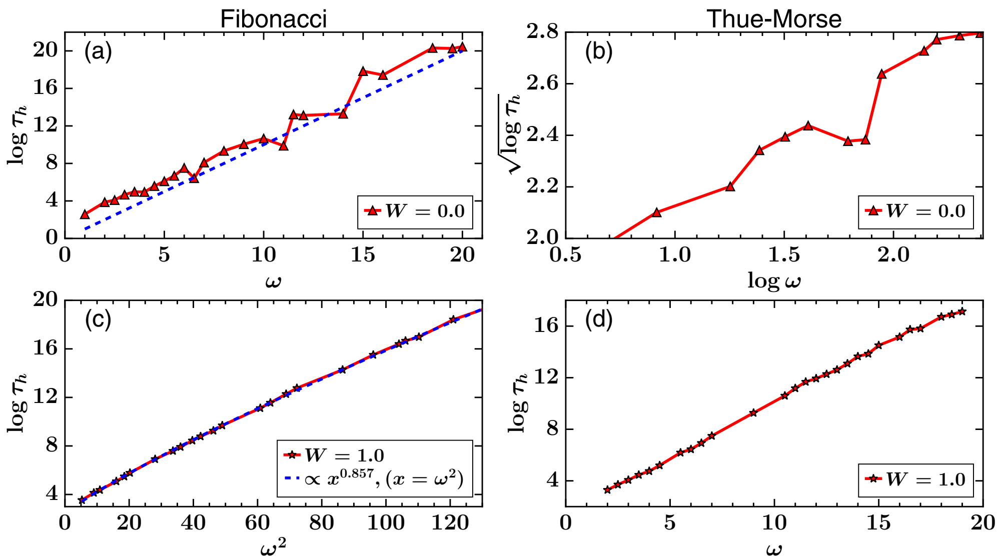

FIG. 3. Heating time as a function of frequency for Fibonacci and Thue-Morse driving. (a), (b) $W = 0$. (c), (d) $W = 1$. The other model parameters are $J = 1$, $U = 1$, $F = 2\omega$.

exact diagonalization. V.T. is grateful to DST-INSPIRE for support from a Ph.D. fellowship. A.S acknowledges financial support from SERB via Grant No. CRG/2019/003447, and from DST via the DST-INSPIRE Faculty Award No. DST/INSPIRE/04/2014/002461. We thank Arnab Das for helpful comments.

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