Letter

# Diagnosing non-Hermitian many-body localization and quantum chaos via singular value decomposition

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Strong local disorder in interacting quantum spin chains can turn delocalized eigenmodes into localized eigenstates, giving rise to many-body localized phases. This is accompanied by distinct spectral statistics: chaotic for the delocalized phase and integrable for the localized phase. In isolated systems, localization and chaos are defined through a web of relations among eigenvalues, eigenvectors, and real-time dynamics. These may change as the system is made open. We ask whether random dissipation (without random disorder) can induce chaotic or localized behavior in an otherwise integrable system. The dissipation is described using non-Hermitian Hamiltonians, which can effectively be obtained from Markovian dynamics conditioned on null measurement. In this non-Hermitian setting, we argue in favor of the use of the singular value decomposition. We complement the singular value statistics with different diagnostic tools, namely, the singular form factor and the inverse participation ratio and entanglement entropy of singular vectors. We thus identify a crossover of the singular values from chaotic to integrable spectral features and of the singular vectors from delocalization to localization. Our method is illustrated in an XXZ Hamiltonian with random local dissipation.

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Since the early days of quantum mechanics, understanding the dynamics of many-body quantum systems continues to be a hard challenge. One of the chief questions on the behavior of interacting quantum systems concerns the presence of quantum chaos [1–4]. Additionally, the extension of quantum localization [5] to interacting systems [6,7] has led to postulating the existence of a robust, nonergodic phase of matter known as *many-body localization* (MBL). The competition between localized and chaotic quantum dynamics has been studied extensively in spin Hamiltonians [8–12], with relevant implications for applications, including quantum annealing [13,14], as well as fundamental questions, such as the lack of thermalization [15–21]. As a result, localization has become central for understanding complex quantum dynamics, with connections to quantum simulation experiments [22–24], topological phases of matter [25–27], and Floquet time crystals [28].

The peculiarities in the dynamics of many-body quantum systems are not limited to the ideal situation where the system is isolated from the environment and the dynamics is unitary [29–33]. In the last few years, the conventional understanding of renormalization group approaches—by which the coupling to a thermal bath would render quantum fluctuations irrelevant [34]—has been shown to be incomplete. Indeed, evidence is being accumulated that open quantum systems may host unusual phases that would exist neither in a quantum unitary setting nor at equilibrium [35–38].

One of the many intriguing features of open quantum system dynamics is the phenomenon of dissipative localization [39–42] and dissipative quantum chaos [43–46]. For the unitary counterpart, both localization and chaos are defined through a web of relations among eigenvalues, eigenvectors, and real-time dynamics [4,18,20,21,47]. As the constraints set by unitarity are lifted, it is natural to expect that such relations may change in nature. In particular, open dynamics conditioned to no jumps, described by effective *non-Hermitian* (NH) Hamiltonians, are being thoroughly investigated. While NH localization is fairly well understood in the single-particle case, which admits exact solutions [48] and a clear renormalization group treatment in one dimension [41], its many-body version has been the object of several numerical studies, suggesting the presence of a stable, localized phase [49–54]. These studies mostly relied on the *eigendecomposition* of large NH matrices. However, because of non-Hermiticity, the indicators of Hermitian localization had to be generalized, causing some ambiguity given the complex nature of eigenvalues [55,56] and the nonorthonormality of right and left eigenvectors [57]. Recent works put forward the idea that using the *singular value decomposition*, one can circumvent certain problems set by the eigendecomposition of NH Hamiltonians [58–61] since the left and right singular vectors are always orthonormal and the singular values are always real. This approach was benchmarked against standard random matrix ensembles [62] and has not yet been used to study the localization transition of many-body open quantum systems in finite dimensions.

In this work, we fill the gap by studying NH many-body localization via the singular value decomposition. Our objective is twofold. First, we show that the singular value decomposition clearly distinguishes between the chaotic and localized regimes in NH models, providing cleaner and more robust numerical indicators than those obtained from the spectrum. Second, we show that random local dissipation in an

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otherwise integrable XXZ Hamiltonian induces quantum chaos for small dissipation strength, followed by localization for large dissipation. These crossovers are similar to the ones caused by a purely Hermitian disorder, so our results provide one more point of contact between Hermitian and NH MBL.

*Model.* The tools we introduce to diagnose NH quantum chaos and MBL are illustrated in a model made of an integrable interacting Hermitian term and a disordered non-Hermitian contribution describing random site-dependent losses, namely,

$$\hat{H} = \hat{H}_{\mathrm{XXZ}} - i\hat{\Gamma}/2, \tag{1}$$

with

$$\hat{H}_{\mathrm{XXZ}} = J\sum_{i=1}^{N}\left(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y + \Delta\hat{S}_i^z\hat{S}_{i+1}^z\right), \tag{2a}$$

$$\hat{\Gamma} = \sum_{i=1}^{N}\gamma_i\left(\hat{S}_i^z + 1/2\right). \tag{2b}$$

Above, $\hat{S}_i^{x,y,z} = \mathbb{1}_2^{\otimes(i-1)} \otimes \frac{1}{2}\hat{\sigma}^{x,y,z} \otimes \mathbb{1}_2^{\otimes(N-i)}$ are spin-1/2 operators acting on site $i$ ($\hat{\sigma}^{x,y,z}$ are the Pauli matrices). The coefficient $J$ is set to unity, fixing the energy scale, and we take $\Delta = 1$. The rates $\gamma_i$ are independently sampled from a uniform distribution over the interval $[0, \gamma]$. We assume periodic boundary conditions; this does not influence the system's behavior since the hoppings are symmetric (contrary to the Hatano-Nelson model [49], which we also analyze in the Supplemental Material [63]). As the magnetization is conserved, we choose to work in the zero-magnetization sector of dimension $D = \binom{N}{N/2}$.

The XXZ Hamiltonian (2a) is an integrable many-body system that has been extensively studied when complemented with random local magnetic fields $\sum_i h_i\hat{S}_i^z$, where $h_i$ are random variables, e.g., uniformly distributed over $[-h, h]$ [9,64]. As such, it has been used to probe a transition between chaos and integrability, occurring as a function of the disorder strength [65,66]. The XXZ chain with weak disorder exhibits chaotic spectral properties, described by the *Gaussian orthogonal ensemble* (GOE), and delocalized eigenstates, while in the presence of strong disorder, it shows integrable spectral properties and localized eigenstates, at least for the system sizes accessible by numerics. By contrast, the existence of a finite-disorder MBL phase with local integrals of motion [15–17] in the thermodynamic limit is still debated [67–73].

The NH Hamiltonian $\hat{H}$ we consider here can be obtained from the full Lindblad master equation with coherent dynamics driven by $\hat{H}_{\mathrm{XXZ}}$ and dissipative dynamics dictated by the quantum jump operators $\sqrt{\gamma_i}\hat{S}_i^-$, where $\hat{S}_i^{\pm} = \hat{S}_i^x \pm i\hat{S}_i^y$. The Lindblad equation can be regarded as the unconditional evolution of the system, that is, averaged over a large number of trajectories [74–77]. Focusingonly on the no-jump trajectories (null-measurement condition), the open system dynamics is described through the effective NH Hamiltonian $\hat{H}$ [63]. In turn, this physical origin of $\hat{H}$ leads to non-negative jump rates $\gamma_i \geqslant 0$ [78].

Our investigation uses the Hamiltonian (1) as a toy model, but our methods are not model dependent. Similar results were found for another commonly considered NH model, the interacting Hatano-Nelson model [49], as detailed in [63].

*Eigenvalue vs singular value decomposition.* Hermitian Hamiltonians have real eigenvalues and orthonormal eigenstates, whose physical meanings are the possible energy measurement outcomes and the corresponding quantum states, respectively. For this reason, the *eigendecomposition* (ED) of a Hermitian Hamiltonian, $\hat{H} = \sum_n E_n|w_n\rangle\langle w_n|$, has a fundamental role in quantum mechanics. Importantly, the ED is realized with a single unitary operator [63].

In turn, NH Hamiltonians have generally complex eigenvalues and nonorthogonal eigenvectors. For a diagonalizable NH Hamiltonian $\hat{H}$, the ED can be generalized with the tools of *biorthogonal quantum mechanics* [58,79]. This approach has been used to (attempt to) generalize many known results from the Hermitian setting [80]: using the eigenvectors of $\hat{H}^\dagger$, which are orthogonal to those of $\hat{H}$, it is possible to resolve the identity and diagonalize the Hamiltonian using two different nonunitary operators, namely, $\hat{H} = \sum_n E_n|R_n\rangle\langle L_n|$.

Regardless, a complex spectrum and the use of both right and left eigenvectors (those of $\hat{H}$ and $\hat{H}^\dagger$, respectively) pose a challenge to the generalization of certain quantities that are well defined in the Hermitian case, as they rely on a real spectrum and orthonormal states. For instance, we show that complex spectral gap ratios [56] provide a less clear distinction between chaotic or integrable dissipative models [63]. Also, the ambiguity in the definition of the density matrix corresponding to a pure state $|R_n\rangle$ ($|R_n\rangle\langle R_n|$ or $|R_n\rangle\langle L_n|$) is reflected in the different choices made in the definitions of topological invariants, for which both right and left eigenvectors are typically used [81]; in the definition of the entanglement entropy, commonly used in NH MBL works [49] and based on only the right (or left) eigenvectors; and even in the definition of the inverse participation ratio, for which all combinations have been considered [48] without reaching a consensus [63].

Recently, the attention on the *singular value decomposition* (SVD) [82], which can be regarded as a generalized version of the ED, has been growing. Note that, for a (non-)Hermitian Hamiltonian $\hat{H}$, its ED and the SVD are (not) related [63].

The SVD, namely, $\hat{H} = \sum_n \sigma_n|u_n\rangle\langle v_n|$, has the advantage, compared to the ED, of providing real (non-negative) singular values $\{\sigma_n\}$ and two sets of (independently) orthonormal singular vectors, $\{|u_n\rangle\}$ and $\{|v_n\rangle\}$ [63]. Although the biorthogonal left and right eigenvectors of a NH Hamiltonian can be made *biorthonormal*, $\langle L_n|R_m\rangle = \delta_{nm}$, it is not possible to normalize both simultaneously. In contrast, the left and right singular vectors are automatically normalized, therefore corresponding to physical states. This represents strong motivation to use the SVD for NH Hamiltonians to generalize and study well-established Hermitian phenomena. Indeed, the SVD has been shown to be instrumental in describing the bulk-boundary correspondence in NH topological models [59–61,83,84] and in studying the statistics of NH random matrices as a measure of dissipative quantum chaos [62].

For these reasons, we use the SVD to study the model (1). Its Hermitian version exhibits MBL and chaotic behavior for strong and weak disorder, respectively. Here, we investigate whether random non-Hermiticity, which physically corresponds to random losses [see Eq. (2b)], can induce a chaotic to integrable crossover. We do so by using singular value statistics [62] and the *singular form factor*, a measure of correlations we define below. Furthermore, we study the localization transition of the singular vectors. Our results further motivate the use of the SVD as a sensitive tool to generalize Hermitian phenomena, such as MBL, to the NH setting.

*Dissipative quantum chaos: The singular form factor.* One of the defining features of quantum chaos is the level spacing distribution, which is well known in random Hermitian Hamiltonians taken from Gaussian ensembles (orthogonal, unitary, and symplectic) and Hamiltonians whose eigenvalues are not correlated (the Poisson ensemble) [4,85]. Indeed, the spectrum of a chaotic Hamiltonian is conjectured to have a level spacing distribution that follows random matrix behavior [1], while the spectrum of an integrable Hamiltonian is uncorrelated, and its level spacing is expected to follow an exponential (but usually referred to as Poisson) distribution [86]. However, to be able to make such statements for a specific system, the spectrum must be unfolded before one computes the level spacing distribution to remove the global energy dependence of the eigenvalue density [4]. Two alternative measures to extract information about the onset of chaos while circumventing the unfolding procedure are the *spectral form factor* (SFF) [87–95] and the spectral ratio statistics [8,96].

For a standard Hermitian Hamiltonian $\hat{H}$ with eigendecomposition $(\hat{H}-E_n)|w_n\rangle = 0$, we recall that the spectral form factor is defined as $\mathrm{SFF}(t) = |\sum_n e^{-iE_nt}/D|^2$, where $D$ is the Hilbert space dimension. One may also express it as $\mathrm{SFF}(t) = |\langle\psi|e^{-i\hat{H}t}|\psi\rangle|^2$, that is, as the return probability of the infinite-temperature *coherent Gibbs state*, $|\psi\rangle = \sum_n |w_n\rangle/\sqrt{D}$. This form is particularly handy and has been used to generalize the SFF to dissipative and non-Hermitian dynamics [45,97]. The SFF is a time-dependent quantity with distinct features at different timescales. In both integrable and chaotic systems, the ensemble-averaged SFF decays at early times and saturates to a plateau of value $1/D$ at very late times [98]. Its behavior in between differentiates integrable systems, which go directly from decay to plateau, from chaotic systems, in which the SFF exhibits a correlation hole followed by a linear growth before the plateau. This additional “ramp” stems from correlations between eigenvalues; it is visible whether or not the spectrum is unfolded [91,99].

We generalize this return probability to a NH Hamiltonian $\hat{H}$ by introducing the *singular form factor* ($\sigma$FF):

$$\sigma\mathrm{FF}(t) = \left|\frac{1}{D}\sum_n e^{-i\sigma_n t}\right|^2 = |\langle\psi_\mathrm{R}|e^{-i\sqrt{\hat{H}^\dagger\hat{H}}t}|\psi_\mathrm{R}\rangle|^2. \quad (3)$$

This extends the SFF to the NH case via the SVD: $\sigma_n$ are the singular values of $\hat{H}$, and $|\psi_\mathrm{R}\rangle = \sum_n |v_n\rangle/\sqrt{D}$ is the *right* infinite-temperature coherent Gibbs state, built from its right singular vectors $|v_n\rangle$ [63]. Note that Eq. (3) can also be written in terms of the left singular vectors $|u_n\rangle$, replacing $|\psi_\mathrm{R}\rangle$ with $|\psi_\mathrm{L}\rangle = \sum_n |u_n\rangle/\sqrt{D}$ and $\hat{H}$ with $\hat{H}^\dagger$.

We argue that the $\sigma$FF is a good indicator of quantum chaos in a NH setting, being able to detect the presence of correlations among singular values. Figure 1(a) shows the $\sigma$FF for various disorder strengths in the considered model, Eq. (1). The $\sigma$FF exhibits a correlation hole before the plateau for small disorder. This correlation hole closes as the disorder strength gets larger, indicating the loss of correlations between the singular values.

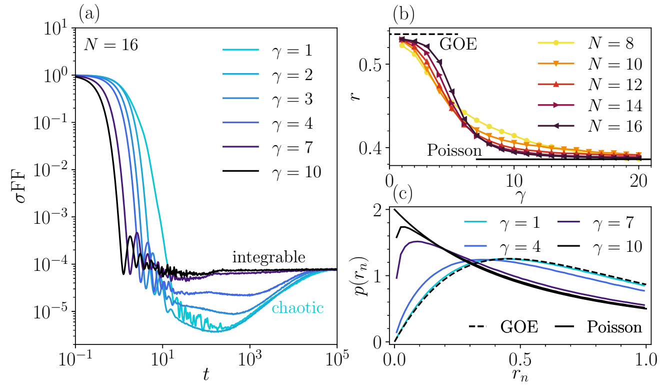


FIG. 1. Dissipative quantum chaos through the SVD. (a) Singular form factor $\sigma$FF, Eq. (3), for the XXZ model with random losses (1). For small dissipation strength a ramp is present, signaling repulsion between singular values, while for stronger dissipation, the correlation hole disappears, leaving only a plateau, as in the integrable case (black). (b) Ratio statistics for the singular values, computed using a portion of the smallest singular values. As the dissipation is made stronger, the average of the ratio distribution signals a crossover from chaos (GOE value) to integrability (Poisson value). Error bars are smaller than symbols. (c) The crossover is also apparent in the full probability distribution of the ratios (shown for $N = 16$). All the data are averaged over at least 7000 disorder realizations.

In parallel to the form factor, the distribution of the spectral ratios $r_n = \min(s_{n+1}, s_n)/\max(s_{n+1}, s_n)$, where $s_n = E_{n+1} - E_n$ are the level spacings between ordered eigenvalues, is also used as a spectral probe of chaos vs integrability [8,96]. Because it involves ratios of level spacings, the density of states cancels out, removing the need for unfolding to compare systems with different global densities. Distributions of $r_n$ are known for the Gaussian and Poisson ensembles [96]. In our case, the relevant ensembles are the GOE for low disorder and the Poissonian ensemble for high disorder. The probability density distributions $p(r_n)$ and their average values $r$ can be found in Ref. [96].

The statistics of the singular values can also be studied via the spectral ratios defined above, replacing $E_n$ with $\sigma_n$ in the definition of the level spacing $s_n$ [62]. This idea was recently used to classify the singular value statistics of NH random matrices [62], and we extend it to study the chaotic to integrable crossover. In this respect, Figs. 1(b) and 1(c) clearly display a crossover from GOE to Poisson statistics as the dissipation strength $\gamma$ is ramped up. In particular, Fig. 1(b) suggests the presence of a finite-size crossover around $\gamma_c/J \gtrsim 9$. In [63], we further show that, in the NH case, the results for the generalizations of the ratio statistics to complex eigenvalues [56] are rather vague compared with the results for the singular values. The $\sigma$FF we introduce, together with the singular value spacing statistics, show how the SVD is an appropriate tool to detect a chaotic to integrable crossover in NH quantum systems. Furthermore, our results point to the occurrence of a localized regime, as detailed below.

*Dissipative localization of singular vectors.* The singular value indicators presented in Fig. 1 support the presence of a localized regime in the XXZ model with random dissipation, Eq. (1). The localization induced by disorder, however, is better understood from a real-space perspective, as the name itself suggests. It is thus interesting to see whether the singular vectors of NH models display the same signatures of localization as their Hermitian counterparts (or even as NH eigenvectors).

For a single particle, the eigenstates of a Hermitian and localized Hamiltonian have a well-understood real-space structure. Each eigenstate $|w_n\rangle$ is concentrated around its localization center $\mathbf{x}_n$, and its decaying profile is characterized by a localization length $\xi$, namely, $\langle \mathbf{x}|w_n\rangle \sim e^{-|\mathbf{x}-\mathbf{x}_n|/\xi}$. A similar situation takes place in single-particle, NH, localized Hamiltonians, in which the disorder-induced localization competes with the localization yielded by the non-Hermitian skin effect [39].

In the many-body case, the situation is more complicated. Even in the Hermitian setup, there seems to be no simple localization in Hilbert space [100]. Rather, the eigenstates of MBL Hamiltonians are believed to be eigenstates of *local integrals of motion* [15–18] as well and to obey the area law of entanglement [21].

Previous works studied some aspects of NH, localized, many-body eigenvectors, e.g., identifying a crossover from volume to area law for the entanglement entropy [49]. Here, we perform a fundamentally different analysis, studying the localization of singular vectors of NH models. These vectors, having all the properties of physical states, do not suffer from the ambiguities of right and left eigenvectors [63]. We use the SVD to study localization and show it can discern between the localized and ergodic regimes. As such, it extends the use of SVD beyond diagnosing dissipative chaos [62].

For simplicity, we use two commonly considered indicators: the *inverse participation ratio* (IPR) and the entanglement entropy across a bipartition. Our analysis is based on the right singular vectors; using the left ones yields similar results.

The IPR of singular vectors is defined as the ensemble average of $\sum_{k=1}^{D} |\langle e_k|v_n\rangle|^4/D$, where $\{|e_k\rangle\}$ is the computational basis and $|v_n\rangle$ are the (right) singular vectors. It is expected that IPR $= O(1/D)$ for delocalized states (as obtained for $|v_n\rangle$ uniformly spread over the computational basis), while IPR $= O(1)$ if $|v_n\rangle$ is localized on a single Fock state. Figure 2(a) presents the logarithm of the IPR of singular vectors, scaled with system size: our data show the presence of a finite-size crossover from delocalized to localized singular vectors around a value $\gamma_c/J \approx 9$, consistent with the picture extracted from the average gap ratio ($r$-parameter) statistics.

We further present our results for the entanglement entropy across a bipartition in Fig. 2(b). Recall that, for a generic state $|\phi\rangle$, the entanglement entropy is defined as $S_\mathrm{E} = -\mathrm{tr}\rho_A \ln \rho_A$, where $\rho_A = \mathrm{Tr}_B(|\phi\rangle\langle\phi|)$ and $A \cup B$ form a bipartition of the chain in two intervals.

As for the IPR, the entanglement entropy supports the presence of a localized regime: it crosses over from a volume law at small dissipation to an area law at large dissipation, again consistently indicating a critical value $\gamma_c/J \approx 9$ for the system sizes considered. Remarkably, the same analysis of the IPR and entanglement entropy with the eigenvectors does not display such a clear crossover [63], thus strongly motivating the use of the SVD. While a weak, inhomogeneous dissipation

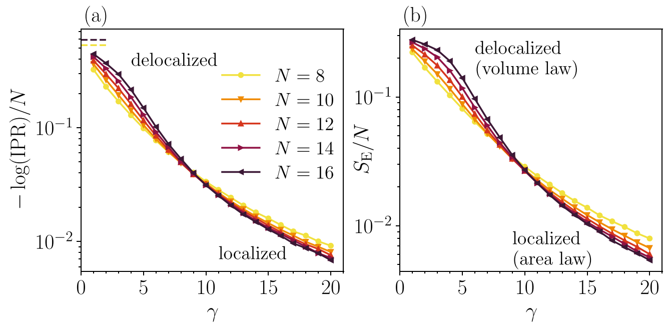


FIG. 2. Dissipative localization of singular vectors. (a) The inverse participation ratio (IPR) quantifies how much a state is localized in the many-body Fock space, passing from IPR $= O(1/D)$ (value illustrated with the dashed lines; delocalized regime) for weak dissipation to IPR $= O(1)$ (localized) for strong dissipation. Our data show a crossover between the two regimes as the disorder in the dissipator is ramped up. (b) Entanglement entropy $S_\mathrm{E}$ across a bipartition of the chain in two halves. For small dissipation $\gamma$, $S_\mathrm{E}$ increases at least with the system size $N$ (volume law, delocalized states), while at larger $\gamma$ the entanglement entropy decreases with system size when divided by $N$, thus suggesting an area law of entanglement (localized states). The data are averaged over at least 7000 disorder realizations, and error bars are smaller than symbols.

breaks the integrability of the XXZ chain, making it (dissipatively) chaotic, more disordered losses localize it again and restore integrability.

*Conclusions.* We investigated the role of a disordered dissipative term on an otherwise integrable, interacting quantum system. Adding such a term makes the system evolve under an effective non-Hermitian Hamiltonian, physically representing the average evolution of quantum trajectories conditioned to no quantum jumps. The eigendecomposition of NH Hamiltonians yields complex eigenvalues and nonorthogonal (left and right) eigenvectors. We argued in favor of using the singular value decomposition and showed that, indeed, the singular values can be used to detect a crossover from chaotic to integrable spectral features and the singular vectors can be used to probe a crossover from delocalization to localization. We introduced the singular form factor; it features a correlation hole when the dissipative disorder is weak, eventually closing for large dissipative disorder. In this setting, random dissipation-induced localization points to a quantum dynamics that is highly sensitive to the effect of inhomogeneities. This contrasts with a homogeneous dissipation which, in our case, does not induce localization.

A crucial point in the Hermitian setting is how the chaotic/localized crossover scales with system size. This point has been highly debated in the last few years, and consensus has yet to be reached [67–73]. For the dissipative case, this is not as relevant because the NH evolution describes an exponentially small (in system size and time) fraction of trajectories. In turn, our findings are meaningful, especially for systems with small sizes: only in these cases can the chaotic/localized behaviors actually be observed in experiments. Our results are thus relevant for *noisy intermediate-scale quantum* devices [101] since we show that the Hamiltonian properties are significantly altered by disordered dissipation.

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