Letter

Giant nonreciprocity of current-voltage characteristics of noncentrosymmetric superconductor–normal metal–superconductor junctions

T. Liu,[1] M. Smith,[2] A. V. Andreev,[1] and B. Z. Spivak[1]

[1]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*

[2]*Materials Science Division, Argonne National Laboratory, Lemont, Illinois 60439, USA*

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We develop a theory of nonreciprocal current-voltage (I-U) characteristics in noncentrosymmetric superconductor-normal metal-superconductor junctions. We show that at small voltages the nonreciprocal features of the I-U characteristics can be expressed entirely in terms of the dependence of the nonreciprocal part of the quasiparticle density of states in the normal metal part of the junction on the order parameter phase difference χ across the junction. The amplitude of the nonreciprocity in this regime is proportional to the inelastic quasiparticle relaxation time τ_{in}, and can be much larger than that in normal materials, where it is proportional to the elastic relaxation time τ_{el}. At low bias the I-U characteristics possess additional symmetry not present in normal conductors; they remain invariant under simultaneous reversal of current, voltage, and the magnetic field.

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Due to Onsager's principle, the linear two-terminal conductance of time-reversal invariant systems, $G_l(\mathbf{H})$, must be an even function of the magnetic field $\mathbf{H}$ [1,2]:

$$G_l(\mathbf{H}) = G_l(-\mathbf{H}). \tag{1}$$

Beyond the linear in U regime, the current-voltage (I-U) characteristics in noncentrosymmetric systems can be nonreciprocal: $J(U, \mathbf{H}) \neq J(U, -\mathbf{H})$.

In noncentrosymmetric normal conductors this phenomenon has been investigated in several articles, see for example Refs. [3–10]. In this case, at small $\mathbf{H}$ and U, the degree of nonreciprocity is proportional to the elastic relaxation time τ_{el},

$$\delta J = J(\mathbf{H}) - J(-\mathbf{H}) \sim \eta \tau_{el} U^2 \mathbf{H}. \tag{2}$$

Here η is a material dependent parameter.

The shapes of the current-voltage (I-U) characteristics in superconductor-normal metal-superconductor (SNS) junctions are more complicated than those of normal metals. The reciprocal part of the I-U characteristics of SNS junctions have been studied in many articles, see for example Refs. [11–15]. It was shown in Ref. [15] that the shape of I-U characteristics at small and large bias are qualitatively different, as illustrated in Figs. 1 and 2. At relatively large bias, the I-U characteristics are controlled by the elastic mean free time τ_{el}. As a result, the degree of nonreciprocity in this regime is expected to be of order of that of the normal part of the junction, which is relatively small and featureless. These parts of the I-U characteristics are shown in Figs. 1 and 2 by dashed black lines. At small bias the I-U characteristics are controlled by the long quasiparticle inelastic relaxation time $\tau_{in} \gg \tau_{el}$, which is typically much longer than the elastic relaxation time. Therefore, the nonlinear conductance of the junctions turns out to be much larger than the normal state conductance. These parts of the I-U characteristics are shown in Figs. 1 and 2 by green and blue lines.

There is extended literature about the nonreciprocity of supercurrent in superconductors in general, and the nonreciprocity of the critical current of Josephson junctions in particular (see for example Refs. [8,16–31], and references therein). On the other hand, the nonreciprocity of the dissipative part of nonlinear I-U characteristics has attracted much less attention. The goal of this article is to develop a theory of nonreciprocity of I-U characteristics of noncentrosymmetric SNS junctions in this regime. We will show that at small voltages the degree of the nonreciprocity of the I-U characteristics of SNS junctions turns out to be much larger than in normal metals.

At small voltages $eU \ll E_T$, the quasiparticle spectrum, and the I-U characteristics of the junction can be calculated in the adiabatic approximation, treating the phase difference $\chi(t)$ as a slowly varying parameter [15]. Here E_T is the Thouless energy, which is inversely proportional to the characteristic time required for quasiparticles to travel between the two superconducting banks of the junction. For simplicity we assume that the transmission coefficient of superconductor-normal metal boundary is of order one and focus on the case of long diffusive junctions, where $E_T \ll \Delta$. In this case the charge transport through SNS junction can be expressed entirely in terms of the dependence of the density of states $\nu(\epsilon, \chi)$ on χ and the quasiparticle energy ϵ.

In the presence of the voltage, the phase difference across the junction $\chi(t)$ evolves in time according to the Josephson relation,

$$\dot{\chi} \equiv \frac{d\chi}{dt} = 2eU(t). \tag{3}$$

Due to Andreev reflection at the normal metal-superconductor boundaries the low energy quasiparticles ($\epsilon < \Delta$) are trapped inside the normal region, and the spectrum of these quasiparticles depends on the phase difference χ. At nonzero temperature T, the quasiparticles occupying these levels move

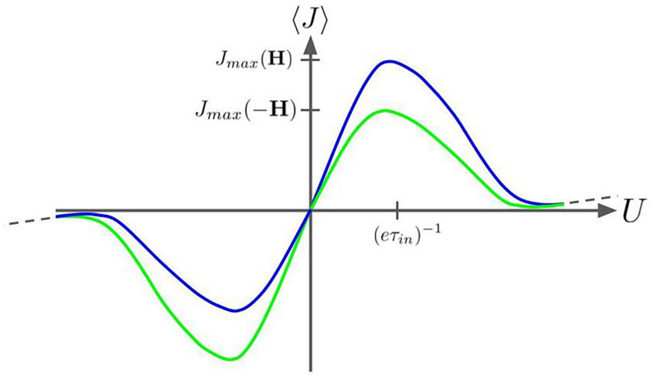

FIG. 1. The *I-U* characteristics of a nonreciprocal SNS junction at low voltage bias for opposite signs of the magnetic field are sketched in blue and green. The dashed lines correspond to the high voltage regime.

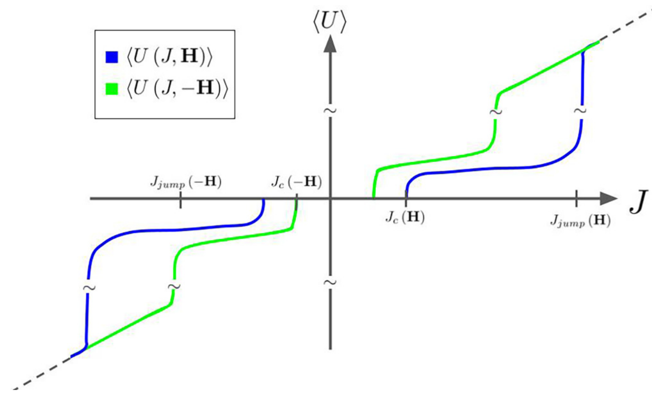

FIG. 2. A qualitative picture of the *I-U* characteristics of a nonreciprocal SNS junction at fixed current. The blue and green curves correspond to the low voltage regime of the *I-U* characteristics for opposite signs of magnetic field. The dashed lines correspond to the high voltage regime.

in energy space together with the levels. This motion creates a nonequilibrium quasiparticle distribution, which relaxes via inelastic scattering producing a dissipative contribution to the current. The physical mechanism of this contribution is similar to the Debye mechanism of microwave absorption in gases [32], the Mandelstam-Leontovich mechanism of the second viscosity in liquids [33], the Pollak-Geballe mechanism of microwave absorption in the hopping conductivity regime [34], and the mechanism of low frequency microwave absorption in superconductors [35–37].

A quantitative description of the current in the low bias regime can be obtained as follows (see for example [15] and references therein). If $eU \ll 1/\tau_{el}, E_T$ the quasiparticle distribution function depends only on the energy. In the adiabatic approximation, contributions to the current related to transitions between energy levels can be neglected, and the current may be expressed as

$$J(t) = -2e\int_0^\infty d\epsilon\,\nu(\epsilon,\chi(t))V_\nu(\epsilon,\chi(t))[n(\epsilon,t)-n_F(\epsilon)] + J_s(\chi(t)). \quad (4)$$

Here $\nu(\epsilon,\chi)$ is the total density of states which includes spin degree of freedom, $n(\epsilon,t)$ is the nonequilibrium occupancy of quasiparticle levels with energy ϵ, $n_F(\epsilon)$ is the Fermi distribution function at temperature T (which we will assume to be much smaller than the superconducting gap in the banks of the junction, $T \ll \Delta$), and $V_\nu(\epsilon,\chi(t))$ is the sensitivity of quasiparticle energy levels to changes in χ. The latter can be expressed in terms of the density of states as

$$V_\nu(\epsilon,\chi) \equiv -\frac{1}{\nu(\epsilon,\chi)}\int_0^\epsilon d\tilde{\epsilon}\,\frac{\partial\nu(\tilde{\epsilon},\chi)}{\partial\chi}. \quad (5)$$

Finally the supercurrent $J_s(\chi)$, which contains contributions from both the ground state and the equilibrium quasiparticle excitations, can be written as

$$J_s(\chi) = -2e\int_0^\infty d\epsilon\,\tanh\left(\frac{\epsilon}{2T}\right)\nu(\epsilon,\chi)V_\nu(\epsilon,\chi), \quad (6)$$

(see for example Ref. [38]).

The time evolution of the distribution function is described by the kinetic equation,

$$\partial_t n(\epsilon,t) + \dot{\chi}\cdot V_\nu(\epsilon,\chi(t))\,\partial_\epsilon n(\epsilon,t) = \frac{n_F(\epsilon)-n(\epsilon,t)}{\tau_{in}}, \quad (7)$$

where τ_{in} is the inelastic relaxation time. Equation (7) can be derived both phenomenologically or more rigorously using Green's functions [15].

Equations (3)–(7), together with the expression for the density of states, provide a complete description of charge transport through SNS junctions provided the phase evolution rate is sufficiently slow ($eU \ll E_T$). Thus, in this regime the *I-U* characteristics of the junctions are determined by ϵ and χ dependence of the density of states.

In general, both the current and voltage across junctions exhibit oscillations in time. We will be interested in the shape of the *I-U* characteristics averaged over the period of oscillations, and will indicate the time-averaged quantities by overline, e.g., $\bar{J}$, and $\bar{U}$.

In nonmagnetic systems the density of states is invariant under the change of sign of both χ and $\boldsymbol{H}$, $\nu(\epsilon,\chi,\mathbf{H}) = \nu(\epsilon,-\chi,-\mathbf{H})$, which implies

$$V_\nu(\epsilon,\chi,\mathbf{H}) = -V_\nu(\epsilon,-\chi,-\mathbf{H}) \quad (8)$$

for the level sensitivity.

It follows from Eqs. (3)–(8) that in the low bias regime considered here the *I-U* characteristics in SNS junctions possesses a symmetry, which is not present in devices based on normal conductors (see e.g., Ref. [39]), and is independent of the device geometry. Namely, it is invariant under the simultaneous reversal of the magnetic field, current, and voltage,

$$\bar{J}(-\bar{U},-\mathbf{H}) = -\bar{J}(\bar{U},\mathbf{H}). \quad (9)$$

Thus, at low bias one can equivalently define nonreciprocity as part of the *I-U* characteristic, which is odd under the reversal of current and voltage for a fixed magnetic field, $\delta\bar{J} = \bar{J}(\bar{U},\mathbf{H}) - J(\bar{U},-\mathbf{H}) = \bar{J}(\bar{U},\mathbf{H}) + J(-\bar{U},\mathbf{H})$.

The nonreciprocity of the *I-U* characteristics in the low bias regime described by Eqs. (3)–(8) arises from the odd in $\mathbf{H}$ dependence of the density of states, $\delta\nu(\epsilon,\chi,\mathbf{H}) = \nu(\epsilon,\chi,\mathbf{H}) -$

$\nu(\epsilon, \chi, -\mathbf{H})$. The latter depends on the orientation of the $\mathbf{H}$ relative to the current. It is important to note that in the special case where the influence of $\mathbf{H}$ on $\nu(\epsilon, \chi, \mathbf{H})$ reduces a constant phase shift $\phi(\mathbf{H})$,

$$\nu(\epsilon, \chi, \mathbf{H}) = \nu_0(\epsilon, \chi + \phi(\mathbf{H})), \tag{10}$$

the *I-U* characteristics remain reciprocal. Indeed, the phase shift $\chi + \phi(\mathbf{H}) \to \chi$ removes the magnetic field from the problem and does not change the voltage in Eq. (3). Furthermore, it follows from Eq. (6) that in this case the critical current of the junction is also reciprocal [26]. We note however, that if different junctions obeying Eq. (10) are connected in parallel [40,41], the critical current and resistance at current bias are nonreciprocal.

The shapes of *I-U* characteristics of SNS junctions depends on the external circuits; we will consider below two limiting cases of voltage and current biased junctions.

Voltage bias: We begin the consideration of nonreciprocity with the voltage bias setup. The *I-U* characteristic in this case has an *N* shape [15], sketched in Fig. 1. At low bias, where the rate of phase evolution obeys the inequality $\dot{\chi}(t)\tau_{in} \ll 1$, the solution to Eq. (7) may be expressed as a series in the small parameter $\dot{\chi}(t)\tau_{in}$. To second order, we obtain for the instantaneous current,

$$\begin{aligned} J(t, T, \mathbf{H}) = J_s(\chi(t), T, \mathbf{H}) + g_1(\chi(t), \mathbf{H})(\dot{\chi}(t)\tau_{in}) \\ + g_2(\chi(t), \mathbf{H})(\dot{\chi}(t)\tau_{in})^2, \end{aligned} \tag{11}$$

where

$$g_1(\chi, \mathbf{H}) = -2e\int_0^\infty d\epsilon\,\partial_\epsilon n_F(\epsilon)\nu(\epsilon, \chi, \mathbf{H})V_\nu^2(\epsilon, \chi, \mathbf{H}), \tag{12a}$$

$$\begin{aligned} g_2(\chi, \mathbf{H}) = e\int_0^\infty d\epsilon\big\{\partial_\epsilon n_F(\epsilon)\partial_\chi\big[\nu(\epsilon, \chi, \mathbf{H})V_\nu^2(\epsilon, \chi, \mathbf{H})\big] \\ + \partial_\epsilon^2 n_F(\epsilon)\nu(\epsilon, \chi, \mathbf{H})V_\nu^3(\epsilon, \chi, \mathbf{H})\big\}. \end{aligned} \tag{12b}$$

At $\dot{\chi}(t) = 2eU = \text{const}$, the average current can be obtained by averaging Eq. (11) over the phase χ. The supercurrent averages to zero. The linear conductance arises from the second term on the RHS and has the form

$$G_l(\mathbf{H}) = 2e\tau_{in}\langle g_1(\chi, \mathbf{H})\rangle, \tag{13}$$

where $\langle\ldots\rangle$ denotes averaging over the phase χ. Using Eqs. (8) and (12a) it is easy to see that the linear conductance is an even function of $\mathbf{H}$ in accordance with the Onsager symmetry principle.

The nonreciprocal part of the dc current $\delta\bar{J}(U, \mathbf{H}) = \bar{J}(U, \mathbf{H}) - \bar{J}(U, -\mathbf{H})$ arises from the third term on the RHS of Eq. (11), and has the form

$$\delta\bar{J}(U, \mathbf{H}) = 2\,\langle g_2(\chi, \mathbf{H})\rangle(2eU\tau_{in})^2. \tag{14}$$

It follows from Eq. (8) that

$$\langle g_2(\chi, -\mathbf{H})\rangle = -\langle g_2(\chi, \mathbf{H})\rangle$$

is an odd function of $\mathbf{H}$, in agreement with Eq. (9). It is interesting that the linear conductance in Eq. (13) is proportional to τ_{in}, while the nonreciprocal current in Eq. (14) is proportional to τ_{in}^2.

In the voltage bias setup it is possible to obtain closed-form expressions for the *I-U* characteristic for arbitrary values of $2eU\tau_{in}$. The results are presented in the Supplemental Material [42]. Here we summarize their main features. The magnitude of the current reaches its maximum at $eU\tau_{in} \sim 1$, and the values of the maximum current $J_{\max}(\mathbf{H})$ and its nonreciprocity $\delta J_{\max}(\mathbf{H}) = J_{\max}(\mathbf{H}) - J_{\max}(-\mathbf{H})$ may be estimated as

$$J_{\max}(\mathbf{H}) \sim \langle g_1(\chi, \mathbf{H})\rangle, \quad \delta J_{\max}(\mathbf{H}) \sim \langle g_2(\chi, \mathbf{H})\rangle. \tag{15}$$

We note that at sufficiently large temperatures the critical current becomes exponentially small in T, while according to Eqs. (15) and (12a), the maximal current $J_{\max}$ does not have such strong temperature dependence. Therefore, $J_{\max}$ can be much larger than J_c [15].

At relatively large voltages $eU\tau_{in} \gg 1$ the solutions of Eqs. (3)–(7) yield an expression for the current which decreases with voltage,

$$\bar{J}(U, \mathbf{H}) \sim \frac{C_1(\mathbf{H})}{eU\tau_{in}} + \frac{C_2(\mathbf{H})}{(eU\tau_{in})^2}. \tag{16}$$

Here to within a factor of order unity, $C_1 \sim \langle g_1(\chi, \mathbf{H})\rangle$ and $C_2 \sim \langle g_2(\chi, \mathbf{H})\rangle$. In this high frequency regime, the period of oscillations of the energy levels $1/eU$ is much smaller than the typical relaxation time τ_{in}. As a result, quasiparticles excited by the motion of the energy levels are unable to undergo inelastic scattering, and the dissipative current is suppressed.

Eventually, at large voltages the *I-U* characteristics are controlled by the contribution to the current which is beyond the adiabatic approximation and is of order $G_N U$. Here G_N is the conductance of the normal metal part of the junction. As the voltage approaches this regime, the current reaches a minimum at $U = U_{\min}$. At this point, the contribution to the conductance proportional to τ_{in} and τ_{el} are of the same order. The position of the minimum is approximately given by

$$U_{\min}(\mathbf{H}) \sim \frac{1}{\tau_{in}}\sqrt{\frac{G_l(\mathbf{H})}{G_N} + \frac{\langle g_2(\mathbf{H})\rangle}{\langle g_1(\mathbf{H}))\rangle}\frac{G_l(\mathbf{H})}{G_N}}. \tag{17}$$

At $U \gg U_{\min}$ the *I-U* characteristics of the system (including its nonreciprocity) is roughly the same as that of the normal part of the SNS junction.

Equations (14)–(17) show that the general key features of nonreciprocity of the *I-U* characteristics of voltage-biased junctions are determined by the parameters $\langle g_1(\chi, \mathbf{H})\rangle$ and $\langle g_2(\chi, \mathbf{H},)\rangle$. Note that the first term on the RHS of Eq. (12b) is a total derivative and averages to zero. In many physical situations, the relevant quasiparticle energies are of order of the Thouless energy E_T. For example, in the diffusive case the level sensitivity $V(\epsilon, \chi, \mathbf{H})$ decays exponentially at energies larger than E_T [15]. In this case, for $\Delta \gg T \gg E_T$ Eq. (12) simplifies to

$$\langle g_1(\chi, \mathbf{H})\rangle = \frac{e}{2T}\int_0^\infty d\epsilon\big\langle\nu(\epsilon, \chi, \mathbf{H})V_\nu^2(\epsilon, \chi, \mathbf{H})\big\rangle, \tag{18a}$$

$$\langle g_2(\chi, \mathbf{H})\rangle = \frac{e}{32\pi T^3}\int_0^\infty d\epsilon\,\epsilon\,\big\langle\nu(\epsilon, \chi, \mathbf{H})V_\nu^3(\epsilon, \chi, \mathbf{H})\big\rangle, \tag{18b}$$

and we get

$$\frac{\langle g_2(\chi, \mathbf{H}, T)\rangle}{\langle g_1(\chi, T)\rangle} = \gamma(\mathbf{H})\frac{E_T^2}{T^2}, \tag{19}$$

where $\gamma(\mathbf{H})$ is an odd function of $\mathbf{H}$, which is determined by the sensitivity of the phase-dependent quasiparticle spectrum on the magnetic field. The different temperature dependence of $\langle g_1(\chi, \mathbf{H})\rangle$ and $\langle g_2(\chi, \mathbf{H})\rangle$ in Eq. (18) arises because the second term in Eq. (12b) requires going to higher order in the Sommerfeld expansion.

Current bias: Let us now turn to the consideration of nonreciprocity in the current-bias setup. The shape of the *I-U* characteristic in this case [15] is illustrated in Fig. 2. The nonreciprocity of the critical current J_c of SNS junctions has been studied in several articles [8,16–26,28–30]. Here we study the *I-U* characteristics $\bar{U}(J)$ at currents larger than the critical current.

Similarly to the voltage bias case, the dissipative part of the *I-U* characteristics at current bias can be separated into two regimes: (i) At low bias currents J, where the average voltage is small, $\bar{U} < 1/(e\tau_{in})$, the adiabatic approximation is valid, and dissipation is controlled by the inelastic relaxation time; (ii) As J is increased past a threshold value which was denoted by $J_{\rm jump}$ in Ref. [15], the system transitions into a different regime where the adiabatic approximation becomes invalid and the dissipation is determined by the elastic relaxation time. This transition is accompanied by a sharp jumplike increase of the bias voltage over a small range of bias current $J \sim J_{\rm jump}$. The degree of nonreciprocity in the high bias regime (ii) is expected to be relatively small, roughly the same as in the normal state. Below we consider nonreciprocity in the low bias regime (i).

In the current bias setup the phase difference $\chi(t)$ increases monotonically with time, but at a varying rate. Using the Josephson relation Eq. (3) we can relate the average voltage across the junction to the duration t_p of the time interval during which χ increases by 2π,

$$\bar{U} = \frac{\pi}{et_p}. \tag{20}$$

To evaluate t_p, we must determine the dynamics of $\chi(t)$.

When the current is close to the critical current, $J - J_c(\mathbf{H}) \ll J_c(\mathbf{H})$, the dominant contribution to the period t_P comes from the interval of phase where $\chi(t)$ is near the phase $\chi_m(\mathbf{H})$, at which the supercurrent reaches its maximum value (see [15]). In this interval $\dot{\chi}(t) \ll 1/\tau_{in}$, and Eqs. (11) and (12) can be used, leading to

$$t_p \approx \tau_{in}\int_{-\pi}^{\pi}\frac{g_1(\chi_m(\mathbf{H}), \mathbf{H})d\chi}{J - J_s(\chi, \mathbf{H}, T)}. \tag{21}$$

Using the quadratic phase dependence of $J_s(\chi, \mathbf{H}, T)$ near $\chi = \chi_m(\mathbf{H})$, $J_s(\chi, \mathbf{H}, T) \approx J_c(\mathbf{H}, T) + \frac{(\chi-\chi_m(\mathbf{H}))^2}{2}\partial_\chi^2 J_s(\chi, \mathbf{H}, T)|_{\chi=\chi_m(\mathbf{H})}$ we can express the average voltage as

$$\bar{U}(J, \mathbf{H}) \approx A(\mathbf{H})\sqrt{(J - J_c(\mathbf{H}, T))}, \tag{22}$$

where

$$A(\mathbf{H}) = \frac{\sqrt{-\partial_\chi^2 J_s(\chi, \mathbf{H}, T)|_{\chi=\chi_m(\mathbf{H})}}}{2^{5/2}e\tau_{in}g_1(\chi_m(\mathbf{H}), \mathbf{H})}. \tag{23}$$

The voltage in Eq. (22) has the standard square root dependence on the excess current $J - J_c(\mathbf{H}, T)$. The nonreciprocity in this regime is characterized by the nonreciprocity of the critical current $J_c(\mathbf{H}, T)$ and the coefficient $A(\mathbf{H})$.

Equation (23) shows that nonreciprocity of $A(\mathbf{H})$ is determined not only by the sensitivity of the phase-dependent supercurrent to the magnetic field but also by the dissipative parameter $g_1(\chi)$. Indeed, the phase $\chi_m(\mathbf{H})$ at which the supercurrent attains the maximal value in the direction of the bias shifts linearly with $\mathbf{H}$. Since $g_1(\chi)$ does not have an extremum at $\chi_m(\mathbf{H})$ the denominator in Eq. (22) contributes to nonreciprocity of $A(\mathbf{H})$.

It was shown in Ref. [15], that

$$J_{\rm jump} \sim J_{\rm max}, \tag{24}$$

where $J_{\rm max}$ was determined for the voltage bias case. Interestingly, it follows from our equations that at $T > E_T$ the degree of nonreciprocity for current bias, $\delta J_{\rm jump} = J_{\rm jump}(\mathbf{H}) - J_{\rm jump}(-\mathbf{H})$, is enhanced with respect to the voltage bias case, Eqs. (15) and (19), by the factor T^2/E_T^2

$$\frac{\delta J_{\rm jump}}{\delta J_{\rm max}} \sim \begin{cases} 1, & T \lesssim E_T, \\ \frac{T^2}{E_T^2}, & T > E_T. \end{cases} \tag{25}$$

The reason for this can be traced to the nonconstant phase evolution rate $\dot{\chi}(t)$. Because of this the leading term in the Sommerfeld expansion of $g_2(\chi, \mathbf{H})$ [first term in Eq. (12b)] can no longer be written as a total derivative of some function of χ, and its contribution to the average voltage does not vanish.

The results presented above assume that the low-energy quasiparticles are trapped inside the normal region of the junction by insulating boundaries and Andreev reflection from the S-N boundaries, and the only channel of quasiparticle relaxation is inelastic scattering. In situations where escape of quasiparticles from the normal region is possible, τ_{in} should be replaced by the characteristic time of quasiparticle escape from the normal region.

The above consideration can also be extended to the cases where the nonreciprocity of the *I-U* characteristics is associated with existence of a spontaneous magnetization $\mathbf{M}$ (or spontaneous valley symmetry breaking in twisted graphene [18,22,43]) in the normal region of the junction at $\mathbf{H} = 0$. In this case the nonreciprocal part of the *I-U* characteristics can be expressed in terms of $\delta\nu = \nu(\epsilon, \chi, \mathbf{M}) - \nu(\epsilon, \chi, -\mathbf{M})$.

To estimate the magnitude of the effect, below we apply the general results obtained above to a planar junction of length L and width L_1(shown in Fig. 3), in which the normal region is described by the following Hamiltonian:

$$H = \mathbf{p}^2/2m - E_F + \beta^{\alpha i}p^i\sigma^\alpha + V_{\rm imp}(\mathbf{r}) + g\mu_0\mathbf{H}\cdot\boldsymbol{\sigma}. \tag{26}$$

Here E_F is the Fermi energy, m is the electron mass, σ_i are the Pauli matrices in spin space, g is the g factor, μ_0 is the Bohr magneton, and $V_{\rm imp}(\mathbf{r})$ is the random impurity potential. For Rashba spin-orbit coupling $\beta^{\alpha i} = \alpha_R\epsilon^{\alpha ij}\hat{n}^j$, where $\hat{n}$ is a unit polar vector, and for Dresselhaus spin-orbit coupling $\beta^{\alpha i} = \alpha_D\delta^{\alpha i}$.

The direction of the magnetic field is chosen to be parallel to the film, as depicted in Fig. 3. Therefore, it enters the Hamiltonian, Eq. (26), only via the Zeeman term.

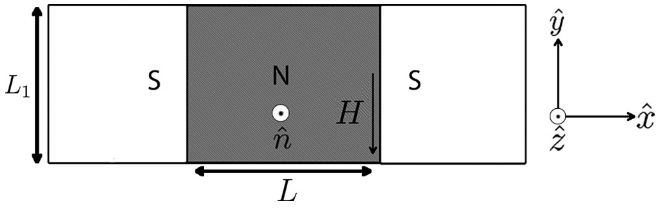

FIG. 3. Top down view of a planar SNS junction. The junction is aligned along the $\hat{\mathbf{x}}$ direction, there is a parallel magnetic field $\mathbf{H}$ directed in the $\hat{\mathbf{y}}$ direction, and there is an out of plane vector $\hat{\mathbf{n}}$ pointing in the $\hat{\mathbf{z}}$ direction which breaks inversion symmetry.

Below we will focus on linear in $\mathbf{H}$ contribution to the nonreciprocity of the I-U characteristics. We consider a case of weak spin-orbit coupling $\beta p_F \ll \tau_{el}^{-1}$ and focus on the diffusive regime, $L \gg \sqrt{D\tau_{so}}$. Here $D = v_F^2\tau_{el}/2$ is the typical value of the electron diffusion coefficient in the normal region, τ_{so} is the spin relaxation time, and v_F is the Fermi velocity. We also assume that the distance between the superconductors, L, is much larger than the coherence length in the superconductors, and therefore the order parameter $\Delta(\mathbf{r})$ has a constant modulus Δ in the superconducting leads, and vanishes in the normal region, see Fig. 3.

After averaging over the random impurity potential, the density of states in the SNS junction $\nu(\epsilon, \chi, \mathbf{H})$ is obtained by solving Usadel's equation in the presence of spin-orbit coupling and a magnetic field [10,23,44]. Here we present the main results leaving the details of the calculations to Supplemental Material [42].

The main feature of the density of states of a diffusive SNS junction is the existence of a mini gap at $\chi = 0$ of order $E_T \sim D/L^2$ [45–48] (for simplicity we restrict ourselves to the case where the S-N boundaries of the junction are transparent). The density of states $\nu(\epsilon, \chi, \mathbf{H})$ exhibits a significant χ dependence only for energies of the order of the mini gap. This means that the level sensitivity, Eq. (5), is peaked in the energy interval $\epsilon \gtrsim E_T$.

We show in Supplemental Material [42] that if the diffusion coefficient $D(x)$ and the strength of spin-orbit coupling $\beta^{\alpha x}(x)$ depend only on the x coordinate, the density of states at $\mathbf{H} \neq 0$ can be written in the form of Eq. (10) with

$$\phi(\mathbf{H}) = \int_{-L/2}^{L/2} dx \frac{2\tau_{so} g\mu_0 \beta^{\alpha x}(x)\mathbf{H}^{\alpha}}{D(x)}. \tag{27}$$

Therefore, in this idealized 1D model the I-U characteristics are reciprocal. However, in the general case where $D(\mathbf{r})$ and $\beta(\mathbf{r})$ are functions of the two coordinates, x and y, or the shape of the normal metal part of the junction is not rectangular, the I-U characteristics of the junction are nonreciprocal.

Below we estimate the degree of nonreciprocity of the I-U characteristics in the case where $L \lesssim L_1$, the amplitude of fluctuations the diffusion coefficient in the y -direction is of order δD, and the correlation length of such fluctuations is of order L_1. In this case we get the following estimates for $\langle g_1\rangle$ and $\langle g_2\rangle$:

$$\langle g_1\rangle \sim e\nu_N \frac{E_T^3}{T}, \quad \langle g_2\rangle \sim \langle g_1\rangle \frac{\beta(g\mu_0 H)\tau_{so}E_T}{LT^2}\left(\frac{\delta D}{D}\right)^2, \tag{28}$$

where $\tau_{so}^{-1} = 4p_F^2\beta^2\tau_{el}$. For the nonreciprocal part $\delta A \equiv [A(\mathbf{H}) - A(-\mathbf{H})]$ of the coefficient $A(\mathbf{H})$ in Eq. (23), we find

$$\delta A \sim \frac{\sqrt{J_c(0,T)}}{e\tau_{in}\langle g_1\rangle} \frac{\tau_{so}\beta(g\mu_0 H)}{LE_T}\left(\frac{\delta D}{D}\right)^2. \tag{29}$$

The parameters in Eqs. (28) and (29), together with Eqs. (14) and (22), characterize the degree of nonreciprocity of I-U characteristics of SNS junctions with weak spin-orbit coupling in both the voltage and current bias cases.

We note that transport in diffusive junctions in which the normal region is formed by a surface of a topological insulator can also be analyzed using Usadel's equation. This is done in Supplemental Material [42]. The magnitude of nonreciprocity in this case is obtained by setting in Eqs. (28) and (29) $\tau_{so} \to \tau_{el}$ and $\beta \to v$, where v is the velocity of the relativistic dispersion.

We conclude by summarizing our main results. We have shown that at low bias ($eU < E_T$) the nonreciprocal features of the I-U characteristics in SNS junctions can be expressed in terms of the nonreciprocal part of the density of states, $\delta\nu = \nu(\epsilon, \chi, \mathbf{H}) - \nu(\epsilon, \chi, -\mathbf{H})$. This leads to a symmetry not present for normal conductors (9); the I-U characteristics remain invariant under a simultaneous reversal of current, voltage, and magnetic field. At low bias all features of I-U characteristics are controlled by the inelastic relaxation time. This leads to a much stronger nonreciprocity in comparison to normal conductors. Although the maximal current $J_{\max}$ in the voltage bias setup and the threshold current J_{jump}, at which the voltage rapidly rises in the current bias setup, are of the same order (24), their nonreciprocity turns out to be parametrically different at $T > E_T$ [see Eq. (25)].

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