

Longitudinal (curvature) couplings of an N -level qudit to a superconducting resonator at the adiabatic limit and beyond

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Understanding how and to what magnitude solid-state qubits couple to metallic wires is crucial to the design of quantum systems such as quantum computers. Here, we investigate the coupling between a multilevel system, or qudit, and a superconducting (SC) resonator's electromagnetic field, focusing on the interaction involving both the transition and diagonal dipole moments of the qudit. Specifically, we explore the effective dynamical (time-dependent) longitudinal coupling that arises when a solid-state qudit is adiabatically modulated at small gate frequencies and amplitudes, in addition to a static dispersive interaction with the SC resonator. We derive Hamiltonians describing the longitudinal multilevel interactions in a general dispersive regime, encompassing both dynamical longitudinal and dispersive interactions. These Hamiltonians smoothly transition between their adiabatic values, where the couplings of the n th level are proportional to the level's energy curvature concerning a qudit gate voltage, and the substantially larger dispersive values, which occur due to a resonant form factor. We provide several examples illustrating the transition from adiabatic to dispersive coupling in different qubit systems, including the charge ($1e$ double quantum dot) qubit, the transmon, the double-quantum-dot singlet-triplet qubit, and the triple-quantum-dot exchange-only qubit. In some of these qubits, higher-energy levels play a critical role, particularly when their qubit's dipole moment is minimal or zero. For an experimentally relevant scenario involving a spin-charge qubit with magnetic field gradient coupled capacitively to a SC resonator, we showcase the potential of the dynamical longitudinal coupling. It enables close-to-quantum-limited quantum nondemolition measurements and remote geometric phase gates, demonstrating its practical utility in quantum information processing.

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I. INTRODUCTION

Effective coupling of a multi-quantum-dot encoded spin qubit to a superconducting (SC) resonator would be a major step towards establishing high-fidelity qubit quantum measurement as well as long-range spin-spin interactions on an electronic chip. Following the success with superconducting qubits [1–4], the usual approach is to establish a transverse (dipole) coupling g_{\perp} of the qubit to the quantized electromagnetic (EM) field of the resonator. In a dispersive regime (at a qubit-resonator frequency detuning Δ) an excitation exchange between qubit and resonator is suppressed by a small probability $(\frac{g_{\perp}}{\Delta})^2$. Thus, avoiding a direct excitation of the qubit by the resonator in this limit, a quantum nondemolition (QND) measurement is possible, with non-QND effects of the order of $(\frac{g_{\perp}}{\Delta})^2$.

Despite the smaller dipole strength and in the presence of stronger charge noise, such a dispersive coupling was also successfully introduced in the quantum dot (QD) spin-qubit architectures [5–12]. Aside from the non-QND effects, this approach may suffer from two main issues (both for SC qubits and QD spin qubits): (1) qubit relaxation Purcell enhancement via the resonator due to relative closeness of the qubit and

resonator frequencies in the dispersive regime; (2) typically, a large transverse (dipole) coupling would couple to nearby two-level fluctuators, leading to charge noise. Also, for QD spin qubits a large dipole coupling arises at regions of a “charge degeneracy points” (c.d.p.), where the gate charge noise is enhanced as well.

Motivated to avoid these issues, we have proposed [13,14] qubit-resonator energy curvature couplings in the adiabatic regime for encoded multidot spin qubits (in which the qubit dipole moment is zero or suppressed), such as the triple quantum dot (TQD) always on exchange only (AEON) qubit and the double-quantum-dot singlet-triplet (DQD S-T) qubit at their symmetric operating point (SOP). (Similar proposals already existed for superconducting qubits [15–18].) The nonzero energy curvature (with respect to a gate voltage), arising from higher levels, is essentially the qubit quantum capacitance $C_q \propto \frac{\partial^2 E_q}{\partial V_m^2}$, and it introduces two effective QND interactions to the resonator:

$$\mathcal{H}_{\delta\omega}/\hbar = \delta\omega \sigma_z \hat{a}^{\dagger} \hat{a}, \quad (1)$$

$$\mathcal{H}_{\parallel}/\hbar = [\tilde{g}_{\parallel} \sigma_z + \tilde{g}_{av}](\hat{a}^{\dagger} + \hat{a}) \cos(\omega_m t). \quad (2)$$

The first one is an always on dispersive coupling with strength $\delta\omega \propto g_0^2 C_q$, and the second one is a dynamical longitudinal coupling with strength $\tilde{g}_{\parallel} \propto g_0 C_q \tilde{V}_m$, that arises when a suitable qubit gate voltage is time modulated, $\sim \tilde{V}_m \cos(\omega_m t)$,

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at or around the resonator frequency ω_r , in an adiabatic regime of small frequencies and modulation amplitudes.¹ Both couplings are proportional to the qubit energy curvature C_q , while \tilde{g}_{\parallel} can be switched on or off by the external gate voltage modulation. In the above we have introduced qubit-resonator “bare” dipole coupling, $g_0 \equiv \alpha_c \frac{\omega_r}{2} \sqrt{\frac{Z_r}{\hbar/e^2}} \ll \omega_r$ (with α_c and Z_r being the resonator-to-dot lever arm and resonator impedance, respectively) (see Sec. II and Ref. [19]). These couplings were derived in a “soft-field” approach [13,14] at the SOP, essentially in the adiabatic regime

$$\omega_r \ll E_{\text{gap}} \sim U_{\text{charge}}, \quad (3)$$

since the typical energy gap to compare for these encoded qubits is the dot’s charging energy $U_{\text{charge}} \geq 200$ GHz (in the absence of a qubit transverse dipole moment at the SOP, see Secs. VI C and VI D and Ref. [14]). While at the SOP it was proposed to use these couplings to perform a QND quantum measurement [14] as well as N -qubit geometric phase gates [20,21], the high-performance regime of these operations (reaching infidelities of 10^{-3}) was shown to be limited either by small quantum capacitances (for realistic experimental parameters) or by a necessity to work at a small ratio [21] $\frac{\delta\omega}{\tilde{g}_{\parallel}}$.

Despite the SOP regime, it is worth to study also the regime of dots’ detunings closer to the c.d.p. Indeed, the ratio of the curvatures at c.d.p. vs SOP for encoded spin qubits can reach high values [14,21]:

$$\frac{C_q^{\text{c.d.p.}}}{C_q^{\text{SOP}}} \sim \frac{U_{\text{charge}}^3}{16t_c^3} \sim 10^2\text{--}10^4 \quad (4)$$

for typical charging energies, $U_{\text{charge}} \sim 200\text{--}300$ GHz, and tunnelings $t_c \sim 5\text{--}10$ GHz (in frequency units). At the c.d.p. a typical qubit level splitting (with an attached to it transition dipole moment) is of the order of $\hbar\omega_q \sim 2t_c \ll U_{\text{charge}}$, and the adiabatic regime $\hbar\omega_r \ll E_{\text{gap}} \sim 2t_c$ requires much smaller resonator frequencies. Recently, both the dispersive and dynamical longitudinal couplings have been observed in the adiabatic regime for a charge qubit coupled to SC resonator [11]. As the qubit and resonator are highly detuned ($\Delta \approx \omega_q$) the Purcell effect will be strongly suppressed. In this adiabatic regime the relevant dispersive coupling $\delta\omega \propto \frac{g_{\perp}^2}{\omega_q}$ (see Sec. VC), will be also suppressed with respect to the usual dispersive coupling (since $\omega_q \gg \Delta \approx 10g_{\perp}$). However, this suppression may be compensated via much stronger dynamical longitudinal coupling. Estimations give for their ratio (this ratio is independent of the qubit frequency detuning, Sec. IV D)

$$\frac{\tilde{g}_{\parallel}}{\delta\omega} = \frac{e\tilde{V}_m}{2\hbar g_0} \equiv \frac{\tilde{V}_m}{\alpha_c V_{\text{vac}}} \sim 12\text{--}120, \quad (5)$$

where $V_{\text{vac}} = \frac{\hbar\omega_r}{e} \sqrt{\frac{Z_r}{\hbar/e^2}}$ is the amplitude of zero-point (vacuum) voltage fluctuations of the resonator. Then, the overall

quantum measurement rate (with modulation) [14] in the adiabatic regime can be of the order or stronger than the usual measurement rate of the dispersive regime.

Since the relevant energy gap around the c.d.p., $E_{\text{gap}} \sim 2t_c$, is relatively small, the adiabatic regime may not be fully satisfied (e.g., when the qubit detuning is in the usual dispersive regime). It is then desirable to derive effective Hamiltonians of the type of Eqs. (1) and (2), which would be relevant both in the adiabatic regime and in the much less detuned dispersive regime.

In this paper we perform the task for a general N -level system (qudit), such that the above effective Hamiltonians obtain contributions from many levels, not just from the qubit levels, in a general dispersive regime. First, in Sec. II we introduce the usual dipole interaction of an N -level atom to a quantized or classical electromagnetic field. In Sec. III we derive the corresponding effective Hamiltonians in the adiabatic regime, using the “soft-field” approach. Then, in Sec. IV A similar effective Hamiltonians are derived in an approach based on a time-dependent quantum mechanical (QM) perturbation theory, extending them to a regime when resonator frequency dependence is not neglected. In Sec. IV B we obtained more general result, when the modulation frequency is different from the resonator frequency. In Sec. IV C and in Appendix A the equal frequency case is rederived in a formal time-dependent Schrieffer-Wolff transformation approach. In Sec. IV D we further extend these results by adding an atom’s polarizability contribution to the effective Hamiltonians $\mathcal{H}_{\delta\omega}$ and \mathcal{H}_{\parallel} , which is of the same order ($\sim g_0^2$) as the above second-order PT contributions. This provides the most general expressions for the effective Hamiltonians $\mathcal{H}_{\delta\omega}$ and \mathcal{H}_{\parallel} of Eqs. (1) and (2). The polarizability contribution for each level is recast to the atom’s energy level curvatures plus a sum over atom’s transition dipole matrix elements via a “low-energy QM sum rule” (Appendix B). This allows us to perform the adiabatic limit in the general effective Hamiltonians (Sec. VA), confirming the “soft-field” results. We also consider the case of zero polarizability in Sec. VB, relevant to QD spin qubits, as well as the dispersive regime in Sec. VC. Out of the adiabatic regime, the second approach will provide an ω_r dependence of the effective Hamiltonian strengths $\delta\omega$, \tilde{g}_{\parallel} , which approaches that of the usual dispersive regime in the appropriate limit.

In Secs. VIA–VID, we apply the general theory to several cases of interest, including a DQD charge qubit, a transmon, a DQD S-T qubit, and TQD exchange-only qubit (in Appendix C we calculated the dipole matrix elements for the TQD system). Finally, in Sec. VIE, we consider implications of the theory for (continuous) quantum measurements on the example of a charge qubit [11] and a spin-charge qubit with magnetic field gradient [7,22]. Implications for geometric quantum gates [20,21] are briefly mentioned in Sec. VIF.

II. DIPOLE INTERACTIONS OF AN N -LEVEL SYSTEM WITH EM FIELDS

One starts with a qudit plus resonator Hamiltonian including the dipole interaction with the EM field of the resonator

¹In Eq. (2) \tilde{g}_{av} is proportional to the average curvature of the qubit levels (see Refs. [14,21] for discussion).

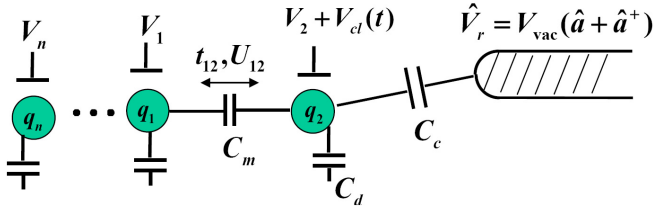


FIG. 1. A multi-quantum-dot system is coupled capacitively to a SC resonator via dot 2. The voltage variations at the dot 2 can be quantum, $\delta V_2 = \alpha_c \hat{V}_r$, or classical, $\delta V_2 = V_{cl}(t) = \tilde{V}_m \cos(\omega_m t)$; $\alpha_c \simeq \frac{C_c}{C_c + C_d}$ is the resonator-to-dot lever arm, where C_c and C_d are dot's coupling capacitances.

$\hat{H}_{\text{dipole}}(t) \equiv -\vec{d} \cdot \delta \vec{E}_\lambda(t)$ (see Fig. 1 for a multi-quantum-dot system):

$$\mathcal{H}_{\text{tot}} = \sum_k E_k |k\rangle\langle k| + \hbar\omega_r^{(\lambda)} \hat{a}^\dagger \hat{a} + \hat{H}_{\text{dipole}}(t), \quad (6)$$

$$\hat{H}_{\text{dipole}}(t) = -\sum_{k,j} \vec{d}_{kj} \cdot \delta \vec{E}_\lambda(t). \quad (7)$$

Here E_k is the energy of the k th level, $\omega_r^{(\lambda)}$ is the frequency of a single-resonator mode (λ), and the EM field applied to the system has quantum and classical components:

$$\delta \vec{E}_\lambda(t) = \vec{E}_{\lambda,\text{vac}}(\hat{a} + \hat{a}^\dagger) + 2\vec{E}_{\text{cl}} \cos(\omega_m t). \quad (8)$$

The quantum field of the resonator has a normalized amplitude $\vec{E}_{\lambda,\text{vac}}$ [chosen real, $\vec{E}_{\lambda,\text{vac}} = \vec{E}_{\lambda,\text{vac}}^*$ (see, e.g., [23]). The classical field amplitude \vec{E}_{cl} may come from an applied gate voltage to the dot system (see Fig. 1) which is generally time dependent. The dipole matrix elements \vec{d}_{kj} are either off diagonal (transition dipole matrix elements) or diagonal ($j = k$). The latter are generally nonzero for a multidot system where space parity is not a good quantum number. In what follows we skip the index of the resonator mode so that $\omega_r^{(\lambda)} \equiv \omega_r$ (the results can be easily extended to a multimode case).

It is often more convenient to deal with applied voltages to the multidot system instead of the corresponding electric fields. As an example, consider a DQD with a gate voltage at the right dot V_2 . Then $E_x = \frac{V_2}{l_x}$, where l_x is the distance between dots 1 and 2 (Fig. 1). If δV_2 is induced by the quantized voltage of the resonator [19], $\hat{V}_r = V_{\text{vac}}(a + a^\dagger)$ then the corresponding field amplitude is

$$E_{x,\text{vac}} = \alpha_c \frac{V_{\text{vac}}}{l_x} \equiv \frac{2\hbar}{l_x e} g_0, \quad (9)$$

where $\alpha_c \simeq \frac{C_c}{C_c + C_d}$ is the lever arm of dot 2 to the resonator, and $g_0 \equiv \alpha_c \frac{eV_{\text{vac}}}{2\hbar}$ is the DQD ‘‘bare’’ coupling to the resonator, introduced above.

The dipole coupling in Eq. (7) is then $g_{kj} \equiv d_{x,kj} E_{x,\text{vac}} = \frac{(\hat{x})_{kj}}{l_x} 2g_0$. By introducing also a classical voltage variation to dot 2 [$\delta V_2 = V_{cl}(t) \equiv \tilde{V}_m \cos(\omega_m t)$], Eqs. (7) and (8) can be

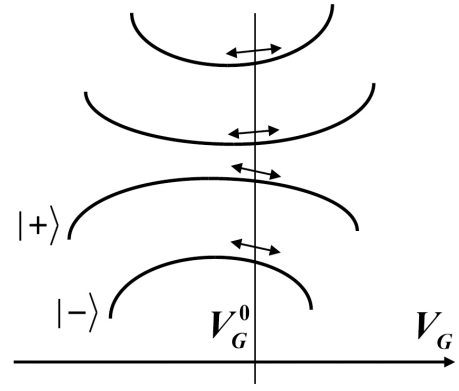


FIG. 2. A generic multilevel system where the energy levels depend on a voltage parameter V_G . By (Taylor) expanding the energy levels to second order around a working point V_G^0 , where the voltage variation $\delta V_G = V_{\text{vac}}(\hat{a} + \hat{a}^\dagger) + V_{cl}(t)$, obtains quantum and classical part, allows to derive effective adiabatic interactions of the N -level system to a superconducting resonator, Eqs. (13) and (14) (Sec. III). These interactions are proportional to the levels' energy curvatures (quantum capacitances), shown with \leftrightarrow for each level. The effective interactions imply adiabaticity condition $\hbar\omega_r \ll E_{\text{gap}}$, where a typical energy gap E_{gap} is either the qubit splitting or some higher-energy transition [see Eq. (3)]. The adiabatic interactions (13) and (14) are rederived in Secs. IV and V in a perturbation theory that also allows to go beyond the adiabatic regime.

rewritten as

$$\hat{H}_{\text{dipole}}(t) = \sum_{j,k} \hbar g_{kj} |k\rangle\langle j| \{[\hat{a} + b_{cl}(t)] + [\hat{a}^\dagger + b_{cl}^*(t)]\}, \quad (10)$$

where $b_{cl}(t) \equiv \bar{b}_{cl} e^{-i\omega_m t}$ is a classical field with amplitude

$$\bar{b}_{cl} = \frac{e\tilde{V}_m}{4\hbar g_0} = \frac{\tilde{V}_m}{\alpha_c V_{\text{vac}}}. \quad (11)$$

From Eqs. (10) and (11) it is clear that the small parameters of the problem are the energies associated with the quantized and classical voltage variations eV_{vac} and $e\tilde{V}_m$ (relative to the qubit energy scale E_q).

III. N -LEVEL EFFECTIVE HAMILTONIANS IN THE ‘‘SOFT-FIELD’’ (ADIABATIC) LIMIT

Consider an adiabatic limit where $\omega_r \ll |\omega_{jk}| \equiv |\omega_j - \omega_k|$ ($\omega_j \equiv E_j/\hbar$) for any energy levels E_j , and also that the voltage variations at the dot 2 (quantum and classical), $\alpha_c \hat{V}_r$, $V_{cl}(t) \equiv \tilde{V}_m \cos(\omega_m t)$ are considered small (or soft field) so that photon excitations of the qudit are highly suppressed (Fig. 2). (Also, in this limit it is assumed that phonon-assisted excitations are suppressed, and therefore neglected, compare with Refs. [24,25].) Assuming that the qudit energy levels depend on the dot gate voltage, one can Taylor expand the energy levels to second order in the gate voltage variation $\delta V_G = V_{\text{vac}}(\hat{a} + \hat{a}^\dagger) + V_{cl}(t)$, that contains a quantum and classical

part²:

$$\begin{aligned} E_k(V_G) &= E_k(V_G^0) + \frac{\partial E_k}{\partial V_G} \delta V_G + \frac{1}{2} \frac{\partial^2 E_k}{\partial V_G^2} \delta V_G^2 \\ &= E_k(V_G^0) + Q_d^{(k)} \delta V_G + \frac{1}{2} C_d^{(k)} \delta V_G^2. \end{aligned} \quad (12)$$

In the second row of Eq. (12), $Q_d^{(k)} \equiv \frac{\partial E_k}{\partial V_G}$ is a (quasi)charge.³ It generates a static longitudinal interaction [13,14,16,18] [see also Eq. (36) below]. The coefficient in the second-order term $C_d^{(k)} \equiv \frac{\partial^2 E_k}{\partial V_G^2}$ is the quantum capacitance of the k th level. Substituting for δV_G in the second order one obtains the following effective qudit-resonator adiabatic Hamiltonians:

$$\mathcal{H}_{\delta\omega}/\hbar = 4\hbar g_0^2 \frac{\partial^2 \hat{\Sigma}}{\partial V_G^2} \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (13)$$

and

$$\mathcal{H}_{\parallel}/\hbar = 2g_0(e\tilde{V}_m) \frac{\partial^2 \hat{\Sigma}}{\partial V_G^2} (\hat{a} + \hat{a}^\dagger) \cos(\omega_m t), \quad (14)$$

which replace the original dipole interactions [Eq. (10)] in the adiabatic regime. Here, $\hat{\Sigma} \equiv \text{diag}\{E_1, E_2, \dots, E_k, \dots\}$ is the qudit eigenenergy matrix. For a qubit the above expressions coincide with that of Refs. [13,14].

The effective interactions (13) and (14) can be used to perform (continuous) quantum measurements on a qudit via measuring the photocurrent of a nearby superconducting (SC) resonator [13,14] (see also recent paper, Ref. [26]). The simplest example could be a DQD singlet-triplet qubit, which has three relevant levels and the qubit subspace has no dipole moment for any DQD detuning (see Sec. VIC below). Another interesting example relevant to the experiment is a spin-charge qubit where a DQD charge qubit (1e) with a micromagnet-induced gradient magnetic field between the two dots creates effective dipole coupling to the resonator [7,22] (see Sec. VIE below). Since the adiabatic conditions may not be fulfilled in the relevant experiments, we develop an alternative derivation of the effective interactions of the type of (13) and (14), extending their range of applicability.

IV. N-LEVEL EFFECTIVE HAMILTONIANS FROM A PERTURBATION THEORY

A. Derivation via second-order time-dependent perturbation theory

The dipole interaction (10) is a time-dependent perturbation. One (heuristic) way to deal with it is to apply a time-dependent perturbation theory (PT). By treating the quantized field in the Heisenberg picture one arrives at time-dependent field operators $\hat{a}(t) = \hat{a} e^{-i\omega t}$, that will be treated semiclassically, i.e., on the same ground as the classical fields [27,28] $b_{cl}(t) \equiv \bar{b}_{cl} e^{-i\omega_m t}$, in Eq. (10). The Hamiltonian is

then $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ with a time-dependent perturbation matrix element of the form

$$V_{kl}(t) = \hbar g_{kl} (\hat{a} e^{-i\omega t} + \bar{b}_{cl} e^{-i\omega_m t}) + \text{H.c.} \quad (15)$$

Working in the interaction picture the Hamiltonian becomes $\bar{V}(t) = e^{i\hat{H}_0 t} \hat{V}(t) e^{-i\hat{H}_0 t}$. To obtain an effective Hamiltonian in a nonresonant (dispersive) regime $|\omega_k - \omega_l - \omega_r| \gg g_{kl}$ for any k, l , and to effectively eliminate the fast-oscillating components of $\bar{V}(t)$, one performs a unitary transform to the system's state $\tilde{\rho} = e^{-iQ(t)} \rho e^{iQ(t)}$, and a corresponding Hamiltonian transform

$$\tilde{V}(t) = e^{-iQ(t)} \bar{V}(t) e^{iQ(t)} - i\hbar e^{-iQ(t)} \frac{\partial}{\partial t} e^{iQ(t)}, \quad (16)$$

requiring that the fast-oscillating components of $\bar{V}(t)$ to be removed. Assuming that $\bar{V}(t) \propto \lambda$ has a small parameter (e.g., the field amplitudes), one can expand $Q(t)$ and $\tilde{V}(t)$ in a perturbation series:

$$Q(t) = Q_1(t) + Q_2(t) + \dots, \quad (17)$$

$$\tilde{V}(t) = \tilde{V}_0(t) + \tilde{V}_1(t) + \tilde{V}_2(t) + \dots, \quad (18)$$

where $Q_n(t), \tilde{V}_n(t) \propto \lambda^n$. Using the Baker-Hausdorff expansion $e^{-iQ} \bar{V} e^{iQ} = \bar{V}(t) - i[Q, \bar{V}(t)] - \frac{1}{2}[Q, [Q, \bar{V}]] + \dots$, one substitutes in it the perturbative series to obtain the following equations collected at a given power of λ :

$$\tilde{V}_0(t) = 0, \quad (19)$$

$$\tilde{V}_1(t) = \bar{V}(t) + \hbar \frac{\partial Q_1(t)}{\partial t}, \quad (20)$$

$$\tilde{V}_2(t) = -\frac{i}{2}[Q_1, \bar{V}(t)] - \frac{i}{2}[Q_1, \tilde{V}_1(t)] + \hbar \frac{\partial Q_2(t)}{\partial t} \dots \quad (21)$$

One requires that $\tilde{V}_1(t) = 0$ in the nonresonant dispersive case [in the original frame $\bar{V}(t)$ is highly oscillating, so one effectively remove it here]. Then, the solution of Eq. (20) takes the standard form of a time-dependent PT [29]:

$$Q_{1,kl}(t) = -\frac{1}{\hbar} \int_{-\infty}^t dt' e^{i\omega_{kl}t'} V_{kl}(t'), \quad (22)$$

where $\omega_{kl} \equiv (E_k - E_l)/\hbar$. The effective Hamiltonian can be obtained from the second-order diagonal term $\tilde{V}_{2,kk}(t)$, thus requiring that the off-diagonal elements are zero, eliminating highly oscillating terms.⁴

In the lowest order, from Eq. (21) with $\tilde{V}_1(t) = 0$ and requiring⁵ $\frac{\partial Q_{2,kk}(t)}{\partial t} = 0$ one then obtains

$$\tilde{V}_{2,kk}(t) \equiv U_{kk}^{\text{eff}}(t) = -\frac{i}{2} \sum_l [Q_{1,kl}(t) \tilde{V}_{lk}(t) - \bar{V}_{kl}(t) Q_{1,lk}(t)]. \quad (23)$$

⁴Based on Eq. (21), to eliminate $\tilde{V}_{2,kl}(t)$ for $k \neq l$, one requires that $\hbar \frac{\partial Q_{2,kl}(t)}{\partial t} = \frac{i}{2}[Q_1, \bar{V}(t)]_{kl}$, which can be used for higher-order calculations, e.g., of $\tilde{V}_3(t)$ (not shown).

⁵The condition $\frac{\partial Q_{2,kk}(t)}{\partial t} = 0$ allows us to obtain the effective Hamiltonian in a closed form, Eq. (24). Since the higher-order contribution $\tilde{V}_3(t)$ contains only commutators of Q_2 , its diagonal part $Q_{2,kk}(t)$ does not contribute.

²In an early work [69] a Taylor expansion in light Higgs fields allowed to obtain effective interactions to light hadrons without calculating Feynman diagrams.

³For a DQD charge qubit, $Q^{(k)}$ can be recast to the excess charge on the right dot (see Ref. [70]).

Calculating the commutator for equal frequencies, $\omega_m = \omega_r$ [in this paper we mainly focus on this resonant case (see Sec. VI), with some exceptions], and neglecting the fast-oscillating (contrarotating) terms⁶ in a rotating-wave approximation (RWA), one obtains an effective dispersive Hamiltonian⁷ (diagonal in the atomic index)

$$\begin{aligned} \mathcal{H}_{\text{eff}}/\hbar &= \sum_k U_{kk}^{\text{eff}}/\hbar |k\rangle\langle k| \\ &= \sum_{k,l} \{[r_{kl}(\omega_r) - r_{lk}(\omega_r)] \\ &\quad \times (\hat{a}^\dagger \hat{a} + \hat{a} b_{\text{cl}}^*(t) + \hat{a}^\dagger b_{\text{cl}}(t)) \\ &\quad + [r_{kl}(\omega_r) - r_{lk}(\omega_r)] |\bar{b}_{\text{cl}}|^2 + r_{kl}(\omega_r)\} |k\rangle\langle k|, \end{aligned} \quad (24)$$

where

$$r_{kl}(\omega_r) \equiv \frac{|g_{kl}|^2}{\omega_{kl} - \omega_r}. \quad (25)$$

The effective Hamiltonian (24) contains a ‘‘dispersivelike’’ coupling $D_k(\omega_r) \hat{a}^\dagger \hat{a} |k\rangle\langle k|$, and a dynamical longitudinal coupling $L_k(\omega_r) (\hat{a} + \hat{a}^\dagger) |k\rangle\langle k| \cos(\omega_r t)$, of the k th level to the resonator, where

$$D_k(\omega_r) \equiv \sum_{l \neq k} [r_{kl}(\omega_r) - r_{lk}(\omega_r)], \quad (26)$$

$$L_k(\omega_r) \equiv \sum_{l \neq k} 2\bar{b}_{\text{cl}} [r_{kl}(\omega_r) - r_{lk}(\omega_r)] \quad (27)$$

are given by a sum of terms $r_{kl}(\omega_r) \sim g_0^2$ over dipole transitions $k \rightarrow l$, $l \neq k$. Introducing a drive-independent frequency shift $\delta\omega_k^0(\omega_r)$ and a drive-dependent one $\delta\omega_k^{\bar{b}_{\text{cl}}}(\omega_r)$,

$$\delta\omega_k^0(\omega_r) \equiv \sum_{l \neq k} r_{kl}(\omega_r), \quad (28)$$

$$\delta\omega_k^{\bar{b}_{\text{cl}}}(\omega_r) \equiv \sum_{l \neq k} \bar{b}_{\text{cl}}^2 [r_{kl}(\omega_r) - r_{lk}(\omega_r)], \quad (29)$$

one gets from Eq. (24)

$$\begin{aligned} \mathcal{H}_{\text{eff}}/\hbar &= \sum_k \{D_k(\omega_r) \hat{a}^\dagger \hat{a} + L_k(\omega_r) (\hat{a} + \hat{a}^\dagger) \cos(\omega_r t) \\ &\quad + \delta\omega_k^0(\omega_r) + \delta\omega_k^{\bar{b}_{\text{cl}}}(\omega_r)\} |k\rangle\langle k|. \end{aligned} \quad (30)$$

The second-order PT effective Hamiltonian (24) [or Eq. (30)] is one of the main results of this paper, particularly relevant for QD spin qubits. In Sec. IV D we will show, however, that in general, \mathcal{H}_{eff} may obtain additional atom’s polarizability contributions of the same order $\sim g_0^2$. The latter contributions

⁶In this approach the counter-rotating (fast-oscillating) terms are that of the first order $\sim g_0$ [Eq. (15)], as well as the off-diagonal in atomic index terms of the second order $\sim g_0^2$, in Eq. (21).

⁷This Hamiltonian is justified in two ways. First, for no-modulation ($\bar{b}_{\text{cl}} = 0$) it coincides with the result of Ref. [32]; note also that in our semiclassical approach, quantum and classical fields are treated on the same ground [see Eq. (15)]. Second, the Hamiltonian (24) is obtained in an independent derivation in Sec. IV C and Appendix A via explicit time-dependent Schrieffer-Wolff transformation.

are ω_r independent and are important to obtain the correct adiabatic limit (Sec. V A).

B. Case of different frequencies, $\omega_m \neq \omega_r$

While in this paper we consider mostly the resonant case of equal frequencies, we provide here the result for different frequencies for the sake of further reference. The above effective Hamiltonian (30) is then transformed to

$$\begin{aligned} \tilde{\mathcal{H}}_{\text{eff}}/\hbar &= \sum_k \{D_k(\omega_r) \hat{a}^\dagger \hat{a} + \tilde{L}_k(\omega_r, \omega_m) (\hat{a} + \hat{a}^\dagger) \cos(\omega_m t) \\ &\quad + \delta\omega_k^0(\omega_r) + \delta\omega_k^{\bar{b}_{\text{cl}}}(\omega_m)\} |k\rangle\langle k|, \end{aligned} \quad (31)$$

where

$$\tilde{L}_k(\omega_r, \omega_m) \equiv \sum_{l \neq k} \bar{b}_{\text{cl}} [r_{kl}(\omega_r) - r_{lk}(\omega_r) + r_{kl}(\omega_m) - r_{lk}(\omega_m)]. \quad (32)$$

C. Alternative derivation via Schrieffer-Wolff transformation

Alternatively (see Appendix A), Eq. (24) can be obtained via a time-dependent Schrieffer-Wolff transformation (see, e.g., Refs. [30,31]) such that

$$\mathcal{H}_{\text{eff}} = U^\dagger(t) \mathcal{H}(t) U(t) - i\hbar \frac{\partial U^\dagger(t)}{\partial t} U(t)$$

with $U(t) = e^{S_1(t)}$ and $S_1(t)^\dagger = -S_1(t)$:

$$\begin{aligned} S_1(t) &= \sum_{lk} \left\{ -\frac{g_{kl}}{\omega_{lk} - \omega_r} |k\rangle\langle l| (\hat{a}^\dagger + b_{\text{cl}}^*(t)) \right. \\ &\quad \left. + \frac{g_{lk}}{\omega_{lk} - \omega_r} |l\rangle\langle k| (\hat{a} + b_{\text{cl}}(t)) \right\}. \end{aligned} \quad (33)$$

This particular transformation is a generalization of the time-independent case [32].

D. Generalization for systems with nonlinear voltage dependence

In the models of quantum circuits that include QD qubits, the qubit charging energy is a linear function of the dots’ gate voltages in a charge basis where each charge state corresponds to certain QD’s occupation. (The charge basis states are assumed voltage independent.) Thus, in a simplified model the total qubit Hamiltonian is a sum of a charging energy part (linear in gate voltages) and a (voltage-independent) tunneling part (see, e.g., Refs. [33–35]). In a more elaborated QD models (see, e.g., Refs. [36,37]) the qubit Hamiltonian could acquire a voltage nonlinearity. Since our derivations are general, it is worth to mention other systems. For example, a Cooper pair box or transmon would possess a quadratic dependence on a gate voltage [38] (see Sec. VI B below).

One can now expand a voltage-dependent atom Hamiltonian [10,13,14,20] near a working point V_G^{08} :

$$\mathcal{H}_{\text{qb}}(V_G^0 + \delta V_G) = \mathcal{H}_{\text{qb}}(V_G^0) + \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V_G} \delta V_G + \frac{1}{2} \frac{\partial^2 \mathcal{H}_{\text{qb}}}{\partial V_G^2} \delta V_G^2, \quad (34)$$

⁸Similar expansion was provided in Ref. [40].

where the gate voltage variation contains a quantum and classical part,

$$\begin{aligned}\delta V_G &= v_q(\hat{a} + \hat{a}^\dagger) + \tilde{V}_m(t) \\ &= v_q(\hat{a} + \hat{a}^\dagger + \bar{b}_{cl}e^{-i\omega_m t} + \bar{b}_{cl}e^{i\omega_m t}),\end{aligned}\quad (35)$$

$v_q \equiv \alpha_c V_{vac}$ is the amplitude of quantum voltage fluctuations imposed on the QD system, and $\bar{b}_{cl} = \frac{\tilde{V}_m}{2v_q}$, as in Eq. (11) [v_q is a small parameter and $\bar{b}_{cl} \sim 1$ or $\gg 1$].

The linear term in voltage variation leads to the usual static longitudinal and transverse couplings. Indeed, the operator $\frac{\partial \mathcal{H}_{qb}}{\partial V_G}$ has dimension of charge and in the absence of modulation the atom-resonator interaction can be written in the atom's eigenbasis $\{|\psi_i\rangle\}$ as

$$\begin{aligned}\frac{\partial \mathcal{H}_{qb}}{\partial V_G} \delta V_G &= \sum_{i,j} \langle \psi_i | \frac{\partial \mathcal{H}_{qb}}{\partial V_G} | \psi_j \rangle | \psi_i \rangle \langle \psi_j | v_q (\hat{a} + \hat{a}^\dagger) \\ &= \sum_i \hbar g_{ii} | \psi_i \rangle \langle \psi_i | (\hat{a} + \hat{a}^\dagger) \\ &\quad + \sum_{i \neq j} \hbar g_{ij} | \psi_i \rangle \langle \psi_j | (\hat{a} + \hat{a}^\dagger),\end{aligned}\quad (36)$$

where $\hbar g_{ij} \equiv v_q \langle \psi_i | \frac{\partial \mathcal{H}_{qb}}{\partial V_G} | \psi_j \rangle$ plays the role of a dipole matrix element. In Eq. (36) the single sum is the static longitudinal interaction of the N -level atom. Note that using the Feynman-Hellman theorem, $\langle \psi_i | \frac{\partial \mathcal{H}_{qb}}{\partial V_G} | \psi_i \rangle = \frac{\partial E_i}{\partial V_G}$. Thus, it is reduced to the form presented in Ref. [14], and the double sum (with $i \neq j$) is the usual transverse interaction. In RWA, transverse interaction is reduced to the ‘‘energy-conserving’’ Jaynes-Cummings form, while the static longitudinal interaction is suppressed. Still, for two qubits, for example, the static longitudinal interaction can lead to qubits' entanglement (see, e.g. [16,21]), which amounts to interesting observable effects [16,18,39].

Consider now the quadratic term in the Taylor expansion in Eq. (34), which has the interpretation of atom's polarizability (see, e.g., Ref. [29]) since $\delta^{(2)} \mathcal{H}_{qb} \sim \tilde{\alpha} \delta V_G^2$ (also note that $\frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2}$ has dimension of capacitance). Expanding in the atom's eigenbasis one obtains

$$\begin{aligned}\frac{1}{2} \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} \delta V_G^2 &= \sum_{i,j} \langle \psi_i | \frac{1}{2} \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_j \rangle | \psi_i \rangle \langle \psi_j | \delta V_G^2 \\ &\simeq \sum_i \langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle | \psi_i \rangle \langle \psi_i | \\ &\quad \times v_q^2 \left[\hat{a}^\dagger \hat{a} + \frac{1}{2} + \bar{b}_{cl} (\hat{a} e^{-i\omega_m t} + \hat{a}^\dagger e^{i\omega_m t}) \right],\end{aligned}\quad (37)$$

where the last equation follows in the RWA.

For the dispersive interaction $\sim |\psi_i\rangle \langle \psi_i | \hat{a}^\dagger \hat{a}$, one combines the polarizability contribution from Eq. (37) and the second-order PT contribution from Eq. (30) [both are of order $\sim v_q^2$]

to obtain

$$\begin{aligned}\mathcal{H}_{\delta\omega} &= \sum_i | \psi_i \rangle \langle \psi_i | \hat{a}^\dagger \hat{a} \left\{ v_q^2 \langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle + D_i(\omega_r) \right\} \\ &\equiv \sum_i \delta\omega_i | \psi_i \rangle \langle \psi_i | \hat{a}^\dagger \hat{a},\end{aligned}\quad (38)$$

where $D_i(\omega_r)$ is given by the sum over dipole transitions of Eq. (26) and

$$\delta\omega_i \equiv v_q^2 \langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle + D_i(\omega_r)\quad (39)$$

is the effective dispersive interaction for the i th level.

Combining the corresponding polarizability and second-order PT contributions, for the dynamical longitudinal interaction $\sim |\psi_i\rangle \langle \psi_i | (\hat{a} + \hat{a}^\dagger) \cos(\omega_m t)$, one obtains

$$\begin{aligned}\mathcal{H}_{\parallel} &= \sum_i | \psi_i \rangle \langle \psi_i | (\hat{a} + \hat{a}^\dagger) \cos(\omega_m t) \\ &\quad \times \left\{ v_q^2 (2\bar{b}_{cl}) \langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle + \tilde{L}_i(\omega_r, \omega_m) \right\} \\ &\equiv \sum_i \tilde{g}_{\parallel,i}(\omega_r, \omega_m) | \psi_i \rangle \langle \psi_i | (\hat{a} + \hat{a}^\dagger) \cos(\omega_m t),\end{aligned}\quad (40)$$

where $\tilde{L}_i(\omega_r, \omega_m) = \bar{b}_{cl} [D_i(\omega_r) + D_i(\omega_m)]$ is given by Eq. (32), and

$$\tilde{g}_{\parallel,i}(\omega_r, \omega_m) = v_q^2 (2\bar{b}_{cl}) \langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle + \tilde{L}_i(\omega_r, \omega_m)\quad (41)$$

is the effective dynamical longitudinal coupling of the i th level in the general case of $\omega_m \neq \omega_r$.

We will use now a quantum-mechanical sum rule [40] relating the polarizability matrix element and the energy curvature (quantum capacitance) of the level via the dipole matrix elements (for completeness, it is rederived in Appendix B):

$$\langle \psi_i | \frac{\partial^2 \mathcal{H}_{qb}}{\partial V_G^2} | \psi_i \rangle = \frac{\partial^2 E_i}{\partial V_G^2} + 2 \sum_{j \neq i} \frac{|\langle \psi_i | \frac{\partial \mathcal{H}_{qb}}{\partial V_G} | \psi_j \rangle|^2}{E_j - E_i}.\quad (42)$$

The ‘‘dispersivelike’’ Hamiltonian $\mathcal{H}_{\delta\omega}$ [Eqs. (38) and (39)] then obtains the dispersive couplings

$$\begin{aligned}\delta\omega_i &= \left(v_q^2 \frac{\partial^2 E_i}{\partial V_G^2} + 2 \sum_{j \neq i} \frac{|g_{ij}|^2}{\omega_{ji}} \right) + D_i(\omega_r) \\ &= v_q^2 \frac{\partial^2 E_i}{\partial V_G^2} + \sum_{j \neq i} |g_{ij}|^2 \left(\frac{2}{\omega_{ji}} - \frac{1}{\omega_{ji} + \omega_r} - \frac{1}{\omega_{ji} - \omega_r} \right)\end{aligned}\quad (43)$$

(compare with Ref. [40]).

In the resonant case $\omega_m = \omega_r$, the dynamical longitudinal Hamiltonian \mathcal{H}_{\parallel} [Eq. (40)] obtains the couplings $\tilde{g}_{\parallel,i}$, expressed in a similar manner as a function of ω_r :

$$\tilde{g}_{\parallel,i}(\omega_r) = \frac{\tilde{V}_m}{v_q} \delta\omega_i(\omega_r),\quad (44)$$

so that their ratio is independent of ω_r . Since typically the voltage modulation can be made much larger than the amplitude of vacuum fluctuations $\tilde{V}_m \gg v_q \equiv \alpha_c V_{vac}$, one gets

the corresponding enhancement of the dynamical longitudinal coupling vs the dispersive coupling:

$$\tilde{g}_{\parallel,i} \gg \delta\omega_i. \quad (45)$$

The dispersive Hamiltonian $\mathcal{H}_{\delta\omega}$ [Eq. (38)] and the dynamical longitudinal Hamiltonian \mathcal{H}_{\parallel} [Eq. (40)], with their partial interaction strengths $\delta\omega_i$, $\tilde{g}_{\parallel,i}$ of Eqs. (43) and (41) [or Eq. (44)], respectively, are the main results of this paper.

V. LIMITING REGIMES: ADIABATIC VS DISPERSIVE

A. Adiabatic regime

In the adiabatic limit when $\omega_r, \omega_m \ll |\omega_{kl}| \equiv |\omega_k - \omega_l|$ for any k, l , and ω_r, ω_m can be neglected in Eqs. (43) and (41). Then, for any polarizability, the corresponding strengths are expressed in terms of the energy curvature of the levels as a result of the low-energy QM sum rule (42):

$$\delta\omega_i(\omega_r = 0) = v_q^2 \frac{\partial^2 E_i}{\partial V_G^2}, \quad (46)$$

$$\tilde{g}_{\parallel,i}(\omega_r = 0, \omega_m = 0) = v_q \tilde{V}_m \frac{\partial^2 E_i}{\partial V_G^2}. \quad (47)$$

Thus, we recover the N -level adiabatic expressions (13) and (14), derived via Taylor expansion of small and slow voltage variations as in Refs. [13,14].

B. QD spin qubits: Zero polarizability

In the case of QD spin qubits the qubit Hamiltonian $\mathcal{H}_{\text{qb}}^{\text{QDs}}$ is linear in the applied QDs' gate voltages $V_{G,i}$ (in a simple model when interdot tunnelings t_i are not affected by $V_{G,i}$). Then, the polarizability matrix element is zero in Eqs. (42) and (43),

$$v_q^2 \langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{qb}}}{\partial V_G^2} | \psi_i \rangle \equiv v_q^2 \frac{\partial^2 E_i}{\partial V_G^2} + 2 \sum_{j \neq i} \frac{|g_{ij}|^2}{\omega_{ji}} = 0, \quad (48)$$

and one ends up with a simplified expression (26) for the effective dispersive coupling $\delta\omega_i$:

$$\delta\omega_i = - \sum_{j \neq i} |g_{ij}|^2 \left(\frac{1}{\omega_{ji} + \omega_r} + \frac{1}{\omega_{ji} - \omega_r} \right) \equiv D_i(\omega_r), \quad (49)$$

which coincides with the result of Ref. [32]. As per Eq. (44), the dynamical longitudinal couplings in the resonant case are simplified to

$$\tilde{g}_{\parallel,i} = L_i(\omega_r) \equiv \frac{\tilde{V}_m}{\alpha_c V_{\text{vac}}} D_i(\omega_r). \quad (50)$$

C. Dispersive regime for QD spin qubits

By definition, in the dispersive regime the resonator frequency is of the order of a particular qubit's energy difference $\omega_r \approx |\omega_l - \omega_{l'}|$, while the corresponding detuning is large $|\omega_l - \omega_{l'} - \omega_r| \gg g_{ll'}$ for any l, l' . Then, the effective Hamiltonian $\mathcal{H}_{\delta\omega}$ recovers the well-known dispersive interaction.

For example, in a two-level approximation, and neglecting a small term $\sim \frac{g_{12}^2}{\omega_q + \omega_r} \ll \frac{g_{12}^2}{\omega_q - \omega_r}$, one obtains from Eq. (49)

$$\mathcal{H}_{\text{disp}}/\hbar = \frac{g_{12}^2}{\omega_q - \omega_r} \sigma_z \hat{a}^\dagger \hat{a} \equiv \delta\omega^{\text{disp}} \sigma_z \hat{a}^\dagger \hat{a}, \quad (51)$$

where $\omega_q \equiv \omega_2 - \omega_1$ and $\delta\omega^{\text{disp}} \equiv \chi = \frac{g_{12}^2}{\omega_q - \omega_r}$. It should be noted that in the adiabatic regime, which is formally similar to the dispersive regime (with $\Delta = \omega_q$), one should keep both terms in Eq. (49) which are exactly equal to each other. Then, $\mathcal{H}_{\delta\omega}/\hbar \approx 2 \frac{g_{12}^2}{\omega_q} \sigma_z \hat{a}^\dagger \hat{a}$, i.e., the dispersive coupling in the adiabatic regime is twice the standard dispersive coupling of the dispersive regime.

By considering the dynamical longitudinal coupling in the dispersive limit, one obtains in a two-level approximation

$$\mathcal{H}_{\parallel, \text{disp}}/\hbar = \frac{e\tilde{V}_m}{2\hbar g_0} \frac{g_{12}^2}{\omega_q - \omega_r} \sigma_z (\hat{a}^\dagger + \hat{a}) \cos(\omega_m t). \quad (52)$$

The enhancement of both couplings going from the adiabatic regime to the dispersive regime can be estimated as (assuming $|\omega_q - \omega_r| \approx 10g_{12}$)

$$\frac{\tilde{g}_{\parallel}^{\text{disp}}}{\tilde{g}_{\parallel}^{\text{adiabat}}} = \frac{\delta\omega^{\text{disp}}}{\delta\omega^{\text{adiabat}}} \approx \frac{\omega_q}{20g_{12}} \approx 12.5\text{--}50, \quad (53)$$

for $\omega_q = 10$ GHz and $g_{12} \approx 10\text{--}40$ MHz.

VI. ILLUSTRATIVE EXAMPLES OF QUBITS

A. DQD 1e charge qubit

For a charge qubit one considers a double quantum dot (DQD) with a single electron. The Hamiltonian in the charge basis of left (dot 1) and right (dot 2) localized states is (cf. Fig. 1)

$$\mathcal{H}_q = \frac{\varepsilon}{2} \tilde{\sigma}_z + t_c \tilde{\sigma}_x, \quad (54)$$

where $\varepsilon = e(V_1 - V_2)$ is the double-dot energy detuning parameter, V_1, V_2 are the gate voltages applied to the dots, t_c is the tunneling, and $\tilde{\sigma}_z, \tilde{\sigma}_x$ are the Pauli matrices in this basis. The qubit-resonator dipole coupling ($\vec{d} \cdot \vec{E}$) arises via the resonator-quantized voltage \hat{V}_r [see Eq. (9)].

The resonator is coupled to dot 2 so that the quantum voltage change $\delta\hat{V}_2 = \alpha_c \hat{V}_r$ at the dot 2 is given by the lever arm $\alpha_c \simeq \frac{C_c}{C_c + C_d}$, where C_c is the dot 2 to resonator capacitance, and C_d is the dot to ground capacitance. The corresponding energy change is given by

$$\delta\mathcal{H}_q = \frac{\partial \mathcal{H}_q}{\partial V_2} \delta\hat{V}_2 = -\frac{1}{2} (e \tilde{\sigma}_z) (\alpha_c \hat{V}_r), \quad (55)$$

and results in a DQD-resonator dipole interaction

$$\delta\mathcal{H}_q = \hbar g_0 \tilde{\sigma}_z (\hat{a} + \hat{a}^\dagger). \quad (56)$$

By diagonalizing the qubit Hamiltonian \mathcal{H}_q at a fixed detuning ε , one obtains the total DQD plus resonator Hamiltonian:

$$\mathcal{H}_{\text{tot}}/\hbar = \frac{\omega_q}{2} \sigma_z + \omega_r \hat{a}^\dagger \hat{a} + (g_{\parallel}^{\text{static}} \sigma_z + g_{\perp} \sigma_x) (\hat{a} + \hat{a}^\dagger), \quad (57)$$

where σ_z, σ_x are the Pauli matrices in the qubit eigenbasis $|+\rangle, |-\rangle$, with energies $E_{q0,\pm} = \pm \frac{E_q}{2}$ with $E_q = \hbar\omega_q = \sqrt{\varepsilon^2 + 4t_c^2}$. In Eq. (57) the $g_{\parallel}^{\text{static}}$ and g_{\perp} are the static longitudinal and transverse coupling, respectively:

$$\begin{aligned} g_{\parallel}^{\text{static}} &= g_0 \frac{\varepsilon}{\sqrt{\varepsilon^2 + 4t_c^2}}, \\ g_{\perp} &= g_0 \frac{2t_c}{\sqrt{\varepsilon^2 + 4t_c^2}}, \end{aligned} \quad (58)$$

that correspond to diagonal ($\sim g_{\parallel}^{\text{static}}$) and transition ($\sim g_{\perp}$) dipole matrix elements of Eq. (36). In a rotating-wave approximation the former is suppressed while the latter obtains the energy-conserving Jaynes-Cummings form $\propto g_{\perp}(\hat{a}\sigma_+ + \hat{a}^{\dagger}\sigma_-)$.

In the adiabatic limit one can obtain the effective curvature (quantum capacitance) couplings from the ‘‘soft-field’’ formulas (13) and (14) of Sec. III, using the energy level dependence of $E_q(\varepsilon)$. One then obtains the dispersivelike and dynamical longitudinal (curvature) Hamiltonians of Eqs. (1) and (2), with strengths

$$\delta\omega = 2\hbar g_0^2 \frac{\partial^2 E_q}{\partial \varepsilon^2} = \hbar g_0^2 \frac{8t_c^2}{[\varepsilon^2 + 4t_c^2]^{3/2}} \quad (59)$$

and

$$\tilde{g}_{\parallel} = g_0 \frac{\partial^2 E_q}{\partial \varepsilon^2} e\tilde{V}_m = g_0 \frac{4t_c^2}{[\varepsilon^2 + 4t_c^2]^{3/2}} e\tilde{V}_m \quad (60)$$

[same result can be obtained by modulating Eq. (57), compare with Ref. [17]]. The above dynamical longitudinal coupling \tilde{g}_{\parallel} is on or off together with the dot’s gate voltage modulation.

By using the perturbative method of Sec. IV A one can obtain the more general effective couplings that depend on the resonator frequency. By taking into account the zero polarizability condition for QD Hamiltonians (48), one obtains for a charge qubit system (in a two-level approximation) the effective couplings

$$\delta\omega_{\text{eff}} = g_{\perp}^2 \left(\frac{1}{\omega_q - \omega_r} + \frac{1}{\omega_q + \omega_r} \right), \quad (61)$$

while $\tilde{g}_{\parallel,\text{eff}} = \frac{\tilde{V}_m}{v_q} \delta\omega_{\text{eff}}$. In Eq. (61), in the limit $\omega_r \ll \omega_q$, one obtains $\delta\omega_{\text{eff}} \approx 2\frac{g_{\perp}^2}{\omega_q}$, which coincides with the soft-field expression (59), as has been shown in general in Sec. V.

We note that, recently, to reveal the curvature couplings in adiabatic limit, an experiment was performed on a hybrid spin qubit [11] (approximated as a two-level charge qubit), showing a clear signature of the adiabatic longitudinal interactions (59) and (60).

B. Transmon qubit

The transmon is the capacitively shunted (by C_S , see Fig. 3) Josephson junction with energy E_J . In the charge basis (of

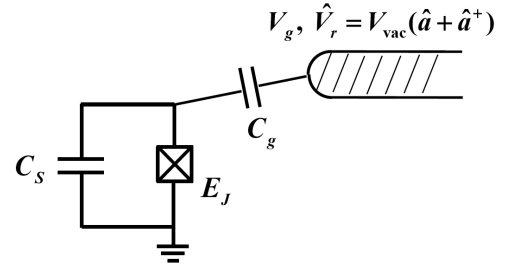


FIG. 3. A schematic of transmon, capacitively coupled via C_g to a voltage source V_g (or \hat{V}_r) (compare with Ref. [41]). C_S is the shunting capacitance of the Josephson junction with JJ energy E_J .

states with definite number (N) of Cooper pairs ($2e$) on the transmon island one has the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{tran}} &= 4E_C \sum_N (N - N_g)^2 |N\rangle \langle N| \\ &\quad - \frac{E_J}{2} (|N\rangle \langle N+1| + |N+1\rangle \langle N|), \end{aligned} \quad (62)$$

where $N_g = \frac{C_g V_g}{2e}$ is the induced charge on the island by a capacitively coupled voltage source V_g , $E_C \simeq \frac{(2e)^2}{2C_S}$ is the island charging energy scale ($C_S \gg C_J, C_g$), and E_J plays the role of charge tunneling matrix element.

The derivatives of $\mathcal{H}_{\text{tran}}$ are defined with respect to the voltage at the island, $\tilde{V}_g = \frac{C_g}{C_g + C_S} V_g \equiv \alpha_g V_g$, and the matrix element of the first derivative defines the corresponding transverse coupling g_{ij} [see Eq. (36)]:

$$\hbar g_{ij} = \alpha_g V_{\text{vac}} \langle \psi_i | \frac{\partial \mathcal{H}_{\text{tran}}}{\partial \tilde{V}_g} | \psi_j \rangle = -2\alpha_g (eV_{\text{vac}}) \langle \psi_i | \hat{N} | \psi_j \rangle \quad (63)$$

(see Ref. [41]). The polarizability contribution to the i th level dispersive shift in Eq. (39) is then obtained

$$\begin{aligned} \delta\omega_i^P &= v_q^2 \langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{tran}}}{\partial \tilde{V}_g^2} | \psi_i \rangle \\ &= \alpha_g^2 V_{\text{vac}}^2 (C_S + C_g) \\ &= \frac{C_g^2}{C_S + C_g} V_{\text{vac}}^2, \end{aligned} \quad (64)$$

that is independent of the energy level. Since in the transmon limit $E_J \gg E_C$, the curvature of the levels is zero, $\frac{\partial^2 E_i}{\partial \tilde{V}_g^2} \simeq 0$, then the polarizability contribution is entirely due to the sum of transition dipole matrix elements contributions in Eq. (42). It is convenient to denote these matrix elements as $m_{ij} \equiv \langle \psi_i | \frac{\partial \mathcal{H}_{\text{tran}}}{\partial \tilde{V}_g} | \psi_j \rangle$. By using the transmon energy levels $E_i \simeq -E_J + \sqrt{8E_C E_J} - \frac{E_C}{12} (6i^2 + 6i + 3)$, and that for given ‘‘ i ’’ only the nearest-neighbor dipole matrix elements survive in the transmon limit [41] $|\langle i+1 | \hat{N} | i \rangle| \simeq \sqrt{\frac{i+1}{2}} \left(\frac{E_J}{8E_C} \right)^{1/4}$, one can show that for any energy level i the right-hand side of Eq. (42) provides the same expression as the direct second-order derivative $\langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{tran}}}{\partial \tilde{V}_g^2} | \psi_i \rangle$ [cf. Eq. (64)], as

expected:

$$\begin{aligned} & \frac{\partial^2 E_i}{\partial \tilde{V}_g^2} + 2 \sum_{j \neq i} \frac{|m_{ij}|^2}{E_j - E_i} \\ & \simeq \dots + 2 \frac{|m_{i-1,i}|^2}{E_{i-1} - E_i} + 2 \frac{|m_{i+1,i}|^2}{E_{i+1} - E_i} + \dots \\ & \simeq 4e^2 \left(\frac{E_J}{8E_C} \right)^{1/2} \frac{1}{\sqrt{8E_C E_J}} = C_S + C_g, \end{aligned} \quad (65)$$

where the second and third rows are written in the transmon limit.

While in the adiabatic limit $\omega_r \ll |\omega_{l'}|$, all couplings are zero (in the transmon limit the energy levels are flat), $\delta\omega_i = 0$, $\tilde{g}_{\parallel} = 0$, in the dispersive regime one recovers the known expressions for the dispersive coupling (restricting to three levels; see, e.g., Ref. [41]):

$$\delta\omega_0^{\text{disp}} \approx -\frac{|g_{01}|^2}{\omega_{10} - \omega_r} \equiv -\chi_{01}, \quad (66)$$

$$\delta\omega_1^{\text{disp}} \approx \frac{|g_{01}|^2}{\omega_{10} - \omega_r} - \frac{|g_{12}|^2}{\omega_{21} - \omega_r} \equiv \chi_{01} - \chi_{12}, \quad (67)$$

$$\chi_{\text{eff}}^{\text{disp}} \approx \frac{\delta\omega_1^{\text{disp}} - \delta\omega_0^{\text{disp}}}{2} = \chi_{10} - \frac{1}{2}\chi_{12}, \quad (68)$$

$$\mathcal{H}_{\text{disp}} = \chi_{\text{eff}}^{\text{disp}} \sigma_z a^\dagger a. \quad (69)$$

For the dynamical longitudinal coupling in the dispersive regime, one then obtains

$$\tilde{g}_{\parallel, \text{eff}}^{\text{disp}} = \frac{\tilde{V}_m}{\alpha_g V_{\text{vac}}} \left(\chi_{10} - \frac{1}{2}\chi_{12} \right), \quad (70)$$

$$\mathcal{H}_{\parallel, \text{disp}} = \tilde{g}_{\parallel, \text{eff}}^{\text{disp}} \sigma_z (a + a^\dagger) \cos(\omega_r t). \quad (71)$$

We note that the first term in Eq. (70) coincides with a similar longitudinal coupling derived in a different approach in Ref. [42].

When one deviates from the dispersive regime (by considering smaller ω_r) one obtains the full expression for χ_{eff} (in the transmon limit), which interpolates between $\chi_{\text{eff}}^{\text{disp}}$ of Eq. (68) and zero:

$$\begin{aligned} \chi_{\text{eff}} & \simeq \frac{\delta\omega_1 - \delta\omega_0}{2} \\ & = -|g_{01}|^2 \left(\frac{2}{\omega_{10}} - \frac{1}{\omega_{10} + \omega_r} - \frac{1}{\omega_{10} - \omega_r} \right) \\ & \quad + \frac{1}{2} |g_{12}|^2 \left(\frac{2}{\omega_{21}} - \frac{1}{\omega_{21} + \omega_r} - \frac{1}{\omega_{21} - \omega_r} \right). \end{aligned} \quad (72)$$

Out of the transmon limit one has to take into account the whole sum over transition dipole matrix elements in Eqs. (43) and (44) [see also Eq. (65)], which we did not show here explicitly. The expression for the dynamical longitudinal coupling has the same functional dependence since the proportionality of the two couplings $\tilde{g}_{\parallel, \text{eff}} = \frac{\tilde{V}_m}{\alpha_g V_{\text{vac}}} \chi_{\text{eff}}$ (in the resonance case).

One notes that for large detunings, when $\Delta_{10} = \omega_{10} - \omega_r$ becomes comparable to the transmon splitting, the relative difference $\frac{\chi_{\text{eff}} - \chi_{\text{eff}}^{\text{disp}}}{\chi_{\text{eff}}}$ can reach $\lesssim 100\%$.

C. DQD singlet-triplet qubit

Consider now a DQD singlet-triplet (S-T) system in which the qubit states are given mainly by the charge configurations $(1,1)$ with some admixture of the higher charge states [14] $|S(0, 2)\rangle$ and $|S(2, 0)\rangle$. Thus, the qubit ground state and excited state are approximated as $|-\rangle \approx |S(1, 1)\rangle$ and $|+\rangle \approx |T_0(1, 1)\rangle$, respectively, both having zero spin projection $S_z = 0$. Due to tunneling (with amplitude t_c) between singlet-charge configurations, the ground state $|-\rangle$ obtains curvature as a function of the DQD energy detuning $\varepsilon = e(V_1 - V_2)$ (compare with the charge qubit, Sec. VI A). The excited state $|+\rangle$ remains flat, essentially due to Pauli spin blockade.

In what follows, we will consider detuning regimes when one of the charge states [e.g., $|S(2, 0)\rangle$] is highly gapped by the dot's charging energy ($E_{\text{gap}} \sim U_{\text{charge}} \sim \text{hundreds GHz}$), and can be neglected. In the remaining three-level system, the upper charge state $|S(0, 2)\rangle$ is gapped from the qubit ground state $|-\rangle$ by several tens of GHz, reaching a minimum of $2t_c$ at the charge degeneracy point (c.d.p.). By measuring the detuning ε from the c.d.p. the three-level Hamiltonian in the charge basis $\{T_0(1, 1), S(1, 1), S(0, 2)\}$ reads as

$$\mathcal{H}_{\text{DQD}}^{\text{c.d.p.}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t_c \\ 0 & t_c & -\varepsilon \end{pmatrix}. \quad (73)$$

For the eigenenergies one gets

$$E_-(\varepsilon) = -\frac{\varepsilon}{2} - \frac{1}{2}\sqrt{\varepsilon^2 + 4t_c^2}, \quad (74)$$

$$E_+(\varepsilon) = 0, \quad (75)$$

$$E_{S(0,2)}(\varepsilon) = -\frac{\varepsilon}{2} + \frac{1}{2}\sqrt{\varepsilon^2 + 4t_c^2}. \quad (76)$$

By calculating the curvatures of the levels one obtains from Eqs. (13) and (14) the effective dispersivelike and dynamical longitudinal interactions (projected on the qubit space):

$$\delta\omega = 2\hbar g_0^2 \frac{\partial^2 E_q}{\partial \varepsilon^2}, \quad \tilde{g}_{\parallel} = \frac{e\tilde{V}_m}{2\hbar g_0} \delta\omega, \quad (77)$$

$$\frac{\partial^2 E_q}{\partial \varepsilon^2} = \frac{1}{2} \frac{4t_c^2}{[\varepsilon^2 + 4t_c^2]^{3/2}}. \quad (78)$$

The qubit energy curvature is $\frac{1}{2}$ of that for a charge qubit since the contribution from the ("flat") state $|T_0(1, 1)\rangle$ is zero.

To obtain the effective interactions to the resonator in a nonadiabatic regime one needs the dipole interactions of the levels. Similar to the charge qubit case above, one writes the dipole interaction using the linear response approach [14]. In the system's charge basis the dipole interaction is diagonal:

$$\mathcal{H}_{\text{DQD, dipole}}^{\text{c.d.p.}} \simeq 2\hbar g_0 (a + a^\dagger) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad (79)$$

In the eigenbasis one then obtains a diagonal dipole part (corresponding to a static longitudinal interaction)

$$\mathcal{H}_{\text{DQD},\parallel}^{\text{stat}} \simeq 2\hbar g_0 (a + a^\dagger) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} + \frac{\varepsilon}{2\sqrt{\varepsilon^2 + 4t^2}} & 0 \\ 0 & 0 & \frac{3}{2} - \frac{\varepsilon}{2\sqrt{\varepsilon^2 + 4t^2}} \end{pmatrix}, \quad (80)$$

and transverse dipole part

$$\mathcal{H}_{\text{DQD},\perp} \simeq 2\hbar g_0 (a + a^\dagger) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{t_c}{\sqrt{\varepsilon^2 + 4t^2}} \\ 0 & -\frac{t_c}{\sqrt{\varepsilon^2 + 4t^2}} & 0 \end{pmatrix}, \quad (81)$$

according to the general structure of Eq. (36). We would stress the following points. First, while in a RWA the (static) longitudinal dipole part is suppressed, a time modulation of the detuning survives in RWA and will lead exactly to the expression for a qubit dynamical longitudinal coupling Hamiltonian of Eq. (14) (for a specially designed superconducting qubit such interaction was derived in Ref. [17]).

More important for our further study is the transverse dipole part (here and below we use the enumeration of the states: $|0\rangle \equiv |- \rangle \approx |S(1, 1)\rangle$, $|1\rangle \equiv |+\rangle \approx |T_0(1, 1)\rangle$, $|2\rangle \equiv |S(0, 2)\rangle$): A nonzero transition dipole matrix element exists only between the states $|- \rangle$ and $|S(0, 2)\rangle$,

$$g_{02} = -2g_0 \frac{t_c}{\sqrt{\varepsilon^2 + 4t^2}}, \quad (82)$$

while the other two dipole couplings are zero, $g_{21} = 0$, $g_{01} = 0$. This selection rule is essentially due to Pauli spin blockade. With these dipole matrix elements it is straightforward to show that the effective dispersive and longitudinal couplings of the S-T qubit are

$$\delta\omega_{\text{eff}} = \frac{|g_{02}|^2}{2} \left(\frac{1}{\omega_{20} + \omega_r} + \frac{1}{\omega_{20} - \omega_r} \right), \quad (83)$$

$$\tilde{g}_{\parallel, \text{eff}} = \frac{e\tilde{V}_m}{2\hbar g_0} \delta\omega_{\text{eff}}, \quad (84)$$

where we take into account that for QDs' systems the polarizability contribution (42) is zero (see Sec. VIA). For the adiabatic limit we take $\omega_r \ll \omega_{20}$ since only the states $|0\rangle$, $|2\rangle$ are dipole coupled. Then,

$$\delta\omega_{\text{eff}}^{\text{adiabat}} \simeq \frac{|g_{02}|^2}{\omega_{20}} = 2\hbar g_0^2 \frac{\partial^2 E_q}{\partial \varepsilon^2}, \quad (85)$$

where the last equation reproduces the general low-energy sum rule [Eqs. (46) and (47)]. We note that $\delta\omega_{\text{eff}}$ is $\frac{1}{2}$ of the expression for a charge qubit [Eq. (61)] since only the curvature of the ground state contributes.

While the adiabatic limit implies $\omega_r \ll \omega_{20}$, the resonator frequency can be comparable or even larger than the qubit frequency ω_{10} . So, the dispersive regime in Eq. (83) would correspond to $\omega_r \approx \omega_{20}$ and $|\omega_{20} - \omega_r| \gg g_{02}$.

From these considerations, and from the general expressions for the quantum measurement rate $\Gamma_{\text{meas}}(\tilde{g}_{\parallel, \text{eff}}, \delta\omega_{\text{eff}})$, considered in Ref. [14], it follows that the general strategy to perform a S-T qubit strong quantum measurement is to be in

the dispersive regime with respect to the third level $|S(0, 2)\rangle$, and to use the enhancement of the coupling strength via the dynamical longitudinal coupling.

D. TQD exchange-only qubit

Similar analysis can be performed for the TQD exchange-only qubit. The system consists of three dots filled with three electrons, mostly in the charge configuration (1,1,1), that are gated by voltages V_1 , V_2 , V_3 , respectively; the dots are tunnel coupled, with tunneling amplitudes $t_{12} \equiv t_l$ and $t_{23} \equiv t_r$. (For an introduction to the system, energy levels, study of the relevant experimental parameters and couplings, see, e.g. Refs. [14,21].) The relevant gate voltage (energy) detunings $\varepsilon_v \equiv e(V_3 - V_1)/2$ and $\varepsilon_m \equiv e[(V_3 + V_1)/2 - V_2] \equiv eV_m$ define the qubit energy levels which obtain energy curvature due to the highly gapped states (see Appendix C). The SOP for this system is a double sweet spot [14,43], where the gate voltage fluctuations vanish to first order: $\frac{\partial E_q}{\partial \varepsilon_v} = \frac{\partial E_q}{\partial \varepsilon_m} = 0$.

At and around the SOP the sum over dipole matrix elements for the dispersivelike and dynamical longitudinal couplings $\delta\omega$ and \tilde{g}_{\parallel} of Eqs. (39) and (41) are dominated by the dipole transitions from the qubit states $|- \rangle$, $|+\rangle$ [of charge configuration (1,1,1)], to the four highly gapped ($E_{\text{gap}} \approx U_{\text{charge}}$) charge states, all of spin $S_z = \frac{1}{2}$, i.e., $|3\rangle \equiv (2, 0, 1) \equiv |S(2_1, 0_2) \uparrow_3\rangle$, $|4\rangle \equiv (1, 0, 2) \equiv |\uparrow_1 S(0_2, 2_3)\rangle$, and similar for $|5\rangle \equiv (1, 2, 0)$, $|6\rangle \equiv (1, 0, 2)$. These dipole couplings are denoted as $g_{-,l}$ and $g_{+,l}$, $l = 3, 4, 5, 6$, and calculated in Appendix C, using a linear response approach [14]. At the SOP (or out of it) the qubit dipole element $g_{-,+}$ is of zero (or of small nonzero) value [14].

The system is in the adiabatic limit with respect to the higher gapped levels (as $\omega_r \lesssim 10 \text{ GHz} \ll U_{\text{charge}} \lesssim 200 \text{ GHz}$), while ω_r could be of the order of the qubit splitting $\omega_{+,-}$. In a two-level approximation, the effective dispersive coupling obtains qubit's and higher level contributions:

$$\delta\omega_{\text{eff}} = \delta\omega_{\text{qb}} + \delta\omega_{\text{higher levels}}, \quad (86)$$

$$\delta\omega_{\text{qb}} = |g_{-,+}|^2 \left(\frac{1}{\omega_{+,-} + \omega_r} + \frac{1}{\omega_{+,-} - \omega_r} \right), \quad (87)$$

$$\delta\omega_{\text{higher levels}} = \sum_{l=3}^6 \left(\frac{|g_{-,l}|^2}{\omega_{l,-}} - \frac{|g_{+,l}|^2}{\omega_{l,+}} \right). \quad (88)$$

Using the dipole matrix elements $g_{-,l}$ and $g_{+,l}$, one can recover the curvature (quantum capacitance) couplings at SOP, that is, the dynamical longitudinal and dispersivelike couplings [14] $\tilde{g}_{\parallel, \text{eff}}^{\text{adiabat}}$, $\delta\omega_{\text{eff}}^{\text{adiabat}} \propto \frac{\partial^2 E_q}{\partial V_m^2}$ in the adiabatic limit.

Out of SOP, but still in the deep (1,1,1) region, the dispersive regime would correspond to $\omega_r \sim \omega_{+,-}$. In this case the qubit level dispersive coupling $\delta\omega_{\text{qb}} \simeq \chi = \frac{|g_{-,+}|^2}{\omega_{+,-} - \omega_r}$ can become comparable to the dispersivelike coupling $\delta\omega_{\text{higher levels}}$ induced by the higher levels [14].

E. Implications for quantum measurements

Below we consider implications for reaching quantum limited regime of continuous quantum measurements on the promising example of a spin-charge qubit [7,12], i.e., a charge

qubit with magnetic field gradient between the two dots [22]. While the spin-charge qubit is essentially a Λ system [44] (with levels 0, 1, 2), the relevant dynamical longitudinal coupling $\tilde{g}_{\parallel}(0, 1, 2)$ can be made large with respect to the dispersive coupling $\delta\omega(0, 1, 2)$, by the possibly large ratio (44) since $\frac{\tilde{g}_{\parallel}(\omega_r)}{\delta\omega(\omega_r)} = \frac{\tilde{V}_m}{\alpha_c V_{\text{vac}}} \simeq 10\text{--}100$. For (continuous) quantum measurements in general, both curvature Hamiltonians $\mathcal{H}_{\delta\omega}$ and \mathcal{H}_{\parallel} contribute to the measurement rate [13,14]: $\Gamma_{\text{meas}} = \Gamma_{\delta\omega} + \Gamma_{\parallel}$. However, in what follows, in the estimations we will focus on the dynamical longitudinal coupling only since it can dominate the measurement rate.

Indeed, in the so-called ‘‘bad-cavity limit’’ (see, e.g. Refs. [45,46]) one requires that the resonator damping rate is much faster $\kappa \gg \Gamma_{\text{meas}}$, so that one is measuring the qubit alone (and not the combined system of qubit plus resonator) (see, e.g., Ref. [47]). Thus, one has the conditions $\kappa^2 \gg 2\tilde{g}_{\parallel}^2$, $\delta\omega^2$, and therefore

$$\frac{\Gamma_{\parallel}}{\Gamma_{\delta\omega}} \approx \frac{\tilde{g}_{\parallel}^2}{4\delta\omega^2} \frac{\kappa^2/4}{|\varepsilon_d^2 - \tilde{g}_{\parallel}^2/4|} \gg 1, \quad (89)$$

where ε_d is the resonator driving strength. With this, one gets approximately $\Gamma_{\text{meas}} \approx \frac{\tilde{g}_{\parallel}^2}{\kappa/2}$, so the measurement rate scales quadratically with the dynamical longitudinal coupling (see also Ref. [17]).

By considering the charge-qubit example above (Sec. VIA) (this is also relevant for the DQD S-T qubit and for the spin-charge qubit [7,12]), the scaling with the resonator frequency ω_r is

$$\frac{\tilde{g}_{\parallel}^{\text{disp}}}{\tilde{g}_{\parallel}^{\text{adiabat}}} = \frac{\omega_q}{2|\omega_q - \omega_r|}. \quad (90)$$

The enhancement of the measurement rate then scales quadratically with detuning

$$\frac{\Gamma_{\text{meas}}^{\text{disp}}}{\Gamma_{\text{meas}}^{\text{adiabat}}} = \frac{\omega_q^2}{4(\omega_q - \omega_r)^2} \approx (1.6\text{--}2.5) 10^2, \quad (91)$$

where the estimated numbers are for the same conditions as in Eq. (53).

It is now instructive to compare the measurement rate to the Purcell relaxation rate. For the Purcell relaxation of a qubit into the resonator (for $\kappa \ll \sqrt{(\omega_q - \omega_r)^2 + 4g_{\perp}^2}$, while $\kappa \gtrsim g_{\perp}$) one has [48] $\Gamma_P \simeq \frac{\kappa g_{\perp}^2}{(\omega_q - \omega_r)^2}$, i.e., the same scaling $\sim 1/\Delta^2$ as for the measurement rate. Therefore, for the purpose of quantum measurements there is no profit of going to the adiabatic regime as to the Purcell rate suppression. Fortunately, their ratio can be suppressed

$$\frac{\Gamma_P}{\Gamma_{\text{meas}}} = \frac{\kappa^2}{2g_{\perp}^2} \left(\frac{\alpha_c V_{\text{vac}}}{\tilde{V}_m} \right)^2 \approx \frac{\kappa^2}{2g_{\perp}^2} \times (10^{-2}\text{--}10^{-4}) \quad (92)$$

since the external voltage modulation can be made much larger than the amplitude of vacuum voltage fluctuations.

It is worth now to compare to the charge dephasing (due to voltage fluctuations only). Applying a theoretical model for $1/f$ charge noise due to gate voltage fluctuations at the dots or tunnel barrier [21] (see also Refs. [49,50]), one gets for a

charge qubit at the charge degeneracy point ($\varepsilon = 0$) the rate

$$\tilde{\Gamma}_{\phi} |_{\varepsilon=0} \approx \frac{1}{\hbar} \left[3 \log \left(\frac{\omega_{\text{UV}}}{\omega_{\text{IF}}} \right) S_{t_c} \right]^{1/2} \simeq \frac{1}{\hbar} 0.6 \mu\text{eV}, \quad (93)$$

where the ratio of ultraviolet to infrared cutoff parameters for the $1/f$ spectrum is $\frac{\omega_{\text{UV}}}{\omega_{\text{IF}}} \approx 10^6$, and $S_{t_c} \simeq 10^{-2} S_{\varepsilon}$, $S_{\varepsilon} \simeq (1 \mu\text{eV})^2$, are the spectral density constants of the $1/f$ noise associated with the dot gates (S_{ε}) and tunneling barrier (S_{t_c}), extracted from the experiment (see Ref. [21]). For the above parameters one estimates dephasing time $1/\tilde{\Gamma}_{\phi} \simeq 1.1$ ns which is comparable with the experimentally measured value [7].

Out of the charge degeneracy point ($\varepsilon \gg 2t_c$) the rate is dominated by the dot gates fluctuations:

$$\tilde{\Gamma}_{\phi} |_{\varepsilon \gg 2t_c} \approx \frac{1}{\hbar} \left[\frac{1}{2} \log \left(\frac{\omega_{\text{UV}}}{\omega_{\text{IF}}} \right) S_{\varepsilon} \right]^{1/2} \simeq \frac{1}{\hbar} 2.8 \mu\text{eV}, \quad (94)$$

which corresponds to even (≈ 5 times) shorter dephasing time.

This situation considerably improves for a spin-charge qubit [7,12,22]. For the parameters of the experiment, the measured dephasing rate of the spin-charge qubit is $\gamma_s \simeq 0.4$ MHz at $\varepsilon = 0$ (c.d.p. at $2t_c \simeq 11$ GHz), which corresponds to a 400 times longer dephasing time $T_s = 0.4 \mu\text{s}$. The physical reason for this is that the qubit levels have opposite spin [7], $|0\rangle \equiv |-, \downarrow\rangle$ and $|1\rangle \equiv |-, \uparrow\rangle$, while an emission of acoustic phonons cannot flip the spin (see, e.g., Ref. [51]).

For the measured parameters of the experiment of Ref. [7] they have $\kappa \simeq 1.4$ MHz, $\omega_r \simeq 5.85$ GHz, spin-charge dipole coupling $g_s \simeq 1.4$ MHz, charge noise dephasing rate $\gamma_s \simeq 0.4$ MHz, and relaxation rate $\gamma_1(\varepsilon = 0) = 0.05$ MHz. One can calculate the measurement rate due to dispersive coupling $\delta\omega \simeq \chi_s \equiv \frac{g_s^2}{\Delta}$ and no modulation [14]:

$$\Gamma_{\text{meas}}(\tilde{g}_{\parallel} = 0) = \frac{(2\delta\omega)^2 \kappa/2}{[\delta\omega^2 + \kappa^2/4]^2} \varepsilon_d^2 \approx 0.1 \text{ MHz} \approx 2\gamma_1, \quad (95)$$

i.e., this measurement rate is not fast enough to reach a quantum-limited measurement regime, as it is comparable both to the charge dephasing rate γ_s and to the relaxation rate γ_1 .

One can show, however, that with a qubit (gate) modulation with even moderate coupling enhancement ratio of $\frac{\tilde{g}_{\parallel}}{\delta\omega} \approx 15$ (which is within reach, see, e.g., Ref. [11]) one can have a measurement rate

$$\Gamma_{\text{meas}}(\tilde{g}_{\parallel} \neq 0) \approx \frac{\tilde{g}_{\parallel}^2}{\kappa/2} \approx 50 \Gamma_{\text{meas}}(\tilde{g}_{\parallel} = 0), \quad (96)$$

which is considerably stronger. Thus, with the use of the dynamical longitudinal coupling $\tilde{g}_{\parallel}^{\text{disp}}(\omega_r)$ in a dispersive regime [Eq. (52)], one can perform a close to quantum limited continuous measurement of a spin-charge qubit.

F. Implications for geometric quantum gates

While there are other means of parametric driving [52] aiming to obtain entanglement gate between remote spin qubits, here we perform an estimation based on the geometric phase gates proposed in Refs. [21,53]. As shown in Ref. [21], in order to obtain high-gate fidelity one needs to

suppress dispersivelike coupling $\delta\omega$ with respect to the dynamical longitudinal one \tilde{g}_{\parallel} . For example, to reach infidelity of 10^{-2} – 10^{-3} in the presence of $\delta\omega$ one needs the ratio small, $\frac{\delta\omega}{\tilde{g}_{\parallel}} = 0.011$ – 0.035 , which is within reach [see Eq. (5)].

Assuming this ratio is suppressed, the main source of infidelity in the QD spin system is that due to the charge noise (see Ref. [21]). For a two-qubit controlled π -phase gate the infidelity reads as [21]

$$\delta\varepsilon_{\phi,1/f}^{2Qb} = \frac{8}{10}(\tilde{\Gamma}_{\phi} t_{\pi})^2, \quad (97)$$

where the π -phase gate time $t_{\pi} = \frac{\pi\sqrt{2}}{\tilde{g}_{\parallel}}$ (for equal couplings, $\tilde{g}_{\parallel}^{(1)} = \tilde{g}_{\parallel}^{(2)}$) is inversely proportional to the dynamical longitudinal coupling \tilde{g}_{\parallel} . Assuming the experimental charge dephasing rate for a spin-charge qubit [7] [$\tilde{\Gamma}_{\phi}^{\text{exp}} \equiv \gamma_s \simeq 0.4$ MHz, such that $(\tilde{\Gamma}_{\phi}^{\text{exp}})^{-1} \simeq 400$ ns], to get infidelity of the level of 10^{-1} – 10^{-3} one needs $\tilde{g}_{\parallel}/2\pi \approx 5$ – 50 MHz, which is reachable in the current experiments [7,12] due to an enhancement at the charge degeneracy point [Eq. (4)]. Indeed, in the experiment [7] the dispersive coupling can reach $\chi_s \simeq 0.14$ – 0.55 MHz and an enhancement factor, Eq. (5), can make the dynamical longitudinal coupling in a dispersive regime as large as $\tilde{g}_{\parallel} \approx 2$ – 66 MHz.

The entangling rate enhancement of a geometric phase gate vs usual dispersive gate [12,21] can be estimated as $\approx \Delta \sqrt{\tilde{g}_{\parallel}^{\text{disp,(1)}} \tilde{g}_{\parallel}^{\text{disp,(2)}}} / (g_{\perp}^{(1)} g_{\perp}^{(2)}) \simeq \tilde{g}_{\parallel}^{\text{disp}} / \delta\omega^{\text{disp}} = \tilde{V}_m / \alpha_c V_{\text{vac}} \simeq 10$ – 100 , as of Eqs. (5) and (44). In the current remote spin entangling experiment of Dijkema *et al.* [12], using similar spin-charge qubits, they provided $T_2^* \approx 40$ – 80 ns $\lesssim T_1$, with an entangling gate time of $\simeq 42$ ns which is comparable to T_2^* . Thus, by Eq. (97), enhancing the entangling rate by 10 times will allow to reach an infidelity of $\sim 10^{-2}$.

VII. CONCLUSION

In this paper we have derived effective interaction Hamiltonians, generically called “dispersivelike” and dynamical longitudinal one, for an N -level atom coupled to a superconducting resonator. These interaction Hamiltonians replace the “original” electric dipole interaction $-\vec{d} \cdot \delta\vec{E}_{\lambda}(t)$ in a situation when the frequency of the EM field $\vec{E}_{\lambda}(t)$ is relatively small to create excitations in the N -level system. These Hamiltonians are diagonal in the system eigenlevels. The dispersivelike Hamiltonian (time independent) is of “energy-energy” type, $\mathcal{H}_{\delta\omega} \sim \delta\omega_i |i\rangle\langle i| a^{\dagger} a$. The dynamical longitudinal (time-dependent) Hamiltonian is of “energy-field” type, $\mathcal{H}_{\parallel} \sim \tilde{g}_{\parallel,i} |i\rangle\langle i| (a + a^{\dagger}) \cos(\omega_m t)$, and appears due to periodic voltage modulation of a qubit gate at $\omega_m \approx \omega_r$.

It is worth mentioning some analogy between the interactions discussed in this paper (in case of a transmon) and the cross-resonance gate between two transmons (see, e.g., Refs. [54,55]). Since the second transmon can be approximated as an oscillator (neglecting the small anharmonicity, cf. Ref. [42]), one can make the replacements $\sigma_z^{\text{target}} \rightarrow \hat{a}^{\dagger} \hat{a}$ and $\sigma_x^{\text{target}} \rightarrow (\hat{a} + \hat{a}^{\dagger})/2$, such that the dispersivelike interaction resembles that of a residual ZZ two-qubit term $\sim \sigma_z \sigma_z^{\text{target}}$, and the dynamical longitudinal one resembles that of a

cross-resonance qubit-qubit interaction $\sim \sigma_z \sigma_x^{\text{target}} \cos(\omega_m t)$, driven at the frequency of the “target” qubit.

The derivation of these effective Hamiltonians is presented in two ways. First, in a more heuristic way, we treat the quantized EM field of the resonator as a time-dependent classical field (with frequency ω_r). Thus, both classical and quantum field perturbations are treated in a kind of time-dependent perturbation theory in second order, to derive the effective interactions. Second, in a more formal way, we consider a time-dependent Schrieffer-Wolff transformation based on the laboratory frame (time-independent) photon operators \hat{a} , \hat{a}^{\dagger} to reproduce the same results.

The effective interactions reproduce previous results in limiting cases. For example, in the absence of qubit gate modulation (and dropping off a polarizability contribution), the dispersivelike Hamiltonian is that of Zhu *et al.* [32]. With qubit gate modulation, we reproduce the dynamical longitudinal interaction of a transmon [42] in the dispersive regime, for example.

As a side note, we mention that our derivations can be equally applied to a qubit that is coupled inductively to a SC resonator, e.g., in a flux qubit [56,57], a charge-flux qubit [58], or Andreev qubit [59]. In this case one should consider the system’s energy curvatures vs magnetic flux, which constitute the quantum (Josephson) reverse inductances $L_{k,j}^{-1} \propto \frac{\partial^2 E_k}{\partial \Phi^2}$ (compare with, e.g., [58,59]), and consider the quantized flux $\hat{\Phi}_r$ of the resonator instead of the quantized voltage \hat{V}_r .

We have considered in this paper both the dispersive and adiabatic (ultradispersive) regimes. In the latter, the modulation frequencies are much smaller than any qubit transition frequencies ω_{kl} . For both effective interactions we have shown that in this limit ($\omega_r, \omega_m \ll \omega_{kl}$) the effective Hamiltonians are expressed through the energy curvatures of the levels $\frac{\partial^2 E_i}{\partial V_G^2}$, which is a nontrivial consequence of a (low-energy) quantum-mechanical sum rule [40]. Thus, we exactly reproduce in a new way the effective Hamiltonians $\mathcal{H}_{\delta\omega}, \mathcal{H}_{\parallel}$ in the adiabatic limit, that was previously derived via Taylor expansion of the energy levels with respect to a voltage parameter [13,14].

As an application of the general theory, we consider several examples of quantum dot qubits including a charge qubit, a DQD singlet-triplet qubit, and a transmon. The charge-qubit example is relevant to a recent experiment [11] demonstrating detailed observation of the curvature couplings $\delta\omega, \tilde{g}_{\parallel}$ in the adiabatic regime. The DQD S-T qubit example will be relevant to a recent experiment [10] on parametric longitudinal coupling or its extension. It is also worth mentioning that the proposed strategy in Sec. VIC to measure the S-T qubit via the strongly dipole coupled third level $|S(0, 2)\rangle$ using the dynamical longitudinal coupling can be used for enhancement of the measurement strength of a capacitively coupled fluxonium qubit, where the SC resonator is coupled dispersively to a high-gapped plasmon level [32,60,61] possessing much higher dipole moment.

Finally, we have also performed crude estimations relevant to the spin-charge qubit of the Petta’s group [7,22] and Vandersypen’s group [12], showing, in particular, that using the dynamical longitudinal coupling (via gate modulation) can significantly increase the quantum measurement rate (see, e.g., Refs. [13,14]) so that the system can approach

a quantum-limited measurement regime, a prerequisite for interesting quantum measurement experiments, e.g., performing continuous quantum feedback control [62,63], entanglement-by-continuous joint measurement [47,64,65], and others, however, with spin qubits.

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APPENDIX A: DERIVATION OF THE EFFECTIVE HAMILTONIAN (24) VIA TIME-DEPENDENT SCHRIEFFER-WOLFF TRANSFORMATION

An N -level atom and the EM field mode of frequency ω_r are represented by the Hamiltonian \mathcal{H}_0 , and interact via the (time-dependent) interaction $\hat{V}(t)$, $\mathcal{H}(t) = \mathcal{H}_0 + \hat{V}(t)$ (below $\hbar = 1$). In the case of a dipole interaction with external gate voltage modulation one gets $\hat{V}(t) \equiv \mathcal{H}_{\text{dipole}}$ [Eq. (10)]. In what follows, we consider only the nondiagonal contributions $\hat{V}_{\text{nd}}(t)$ as the diagonal contributions are suppressed in the rotating-wave approximation (RWA):

$$\mathcal{H}_0 = \sum_k \omega_k |k\rangle \langle k| + \omega_r \hat{a}^\dagger \hat{a}, \quad (\text{A1})$$

$$\hat{V}_{\text{nd}}(t) = \sum_{k,l,k \neq l} g_{kl} |k\rangle \langle l| (\hat{a}^\dagger + b^*(t) + \hat{a} + b(t)) \quad (\text{A2})$$

(note that $g_{kl} = g_{lk}^*$). One can obtain an effective Hamiltonian $\mathcal{H}_{\text{eff}}(t)$, diagonal in the atom index, by applying a time-dependent unitary transformation $U_1(t) = \exp[-S_1(t)]$ (time-dependent Schrieffer-Wolff transformation [30,31]). It can be shown that to eliminate the off-diagonal terms in the atom index in the next order in perturbation theory (PT) the operator $S_1(t)$ needs to satisfy the equation [31]

$$i \frac{\partial S_1(t)}{\partial t} + [S_1(t), \mathcal{H}_0] + V_{\text{nd}}(t) = 0. \quad (\text{A3})$$

The operator $S_1^\dagger(t) = -S_1(t)$ is anti-Hermitian and will be searched in the form

$$S_1(t) = \sum_{l,l'} S_{l'l}(t) |l'\rangle \langle l| (\hat{a}^\dagger + b^*(t)) - \text{H.c.}, \quad (\text{A4})$$

where $S_1(t)$ is off diagonal, i.e., $S_{ll} = 0$.

The commutator $[S_1(t), \mathcal{H}_0]$ is obtained using the standard relations $[|l'\rangle \langle l|, |m\rangle \langle m'|] = |l'\rangle \langle m'| \delta_{lm} - |m\rangle \langle l| \delta_{m'l'}$ and $[\hat{a}, \hat{a}^\dagger] = 1$. One gets

$$\begin{aligned} [S_1(t), \mathcal{H}_0] &= \sum_{l,l'} \{ S_{l'l}(t) |l'\rangle \langle l| [\omega_{l'l} (\hat{a}^\dagger + b^*(t)) - \omega_r \hat{a}^\dagger] \\ &\quad + S_{l'l}^*(t) |l\rangle \langle l'| [\omega_{l'l'} (\hat{a} + b(t)) - \omega_r \hat{a}] \}, \quad (\text{A5}) \end{aligned}$$

where $\omega_{l'l} \equiv \omega_l - \omega_{l'}$. One now substitutes Eq. (A4) for $S_1(t)$ into Eq. (A3) and obtains equations at the different operator structures. For example, at the structures $|l'\rangle \langle l| \hat{a}^\dagger$ and $|l'\rangle \langle l| \hat{a}$

one gets, respectively,

$$i \frac{\partial S_{l'l}(t)}{\partial t} + S_{l'l}(\omega_{l'l} - \omega_r) + g_{l'l} = 0, \quad (\text{A6})$$

$$-i \frac{\partial S_{l'l}^*(t)}{\partial t} + S_{l'l}^*(\omega_{l'l} - \omega_r) + g_{l'l} = 0. \quad (\text{A7})$$

One should note that the equation obtained at the structure $|l'\rangle \langle l|$ is not independent, but is a linear combination of Eqs. (A6) and (A7). For time-independent dipole couplings $g_{l'l}$, one gets the solutions

$$S_{l'l} = -\frac{g_{l'l}}{\omega_{l'l} - \omega_r}, \quad (\text{A8})$$

$$S_{l'l}^* = -\frac{g_{l'l}}{\omega_{l'l} - \omega_r}. \quad (\text{A9})$$

Substituting these results into Eq. (A4) one obtains $S_1(t)$ [Eq. (33)] of the main text.

Using the result for $S_1(t)$, the effective Hamiltonian in the dispersive and adiabatic regimes is calculated from the commutator

$$\mathcal{H}_{\text{eff}}|_{\text{RWA}} = \frac{1}{2} [S_1(t), \hat{V}_{\text{nd}}(t)] \quad (\text{A10})$$

in the RWA. Calculating the commutator we have neglected terms of the form $[\hat{a}^\dagger + b^*(t)]^2$, $[\hat{a} + b(t)]^2$, in a RWA. Thus, we recover the effective Hamiltonian (24) of the main text, that was obtained via a time-dependent PT.

APPENDIX B: QUANTUM-MECHANICAL SUM RULE FOR THE POLARIZABILITY MATRIX ELEMENT

For completeness, here we present a detailed derivation of the quantum-mechanical sum rule (42) of the main text, essentially following Ref. [40].

To express the diagonal matrix element of the polarizability in the energy eigenbasis $\langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{qb}}}{\partial V^2} | \psi_i \rangle$, one differentiates the Hellmann-Feynman relation $\frac{\partial E_i}{\partial V} = \langle \psi_i | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_i \rangle$, with respect to a suitable voltage parameter V to obtain

$$\begin{aligned} \frac{\partial^2 E_i}{\partial V^2} &= \left\langle \frac{\partial \psi_i}{\partial V} \left| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \right| \psi_i \right\rangle + \langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{qb}}}{\partial V^2} | \psi_i \rangle \\ &\quad + \left\langle \psi_i \left| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \right| \frac{\partial \psi_i}{\partial V} \right\rangle. \quad (\text{B1}) \end{aligned}$$

It is convenient to introduce the i th level Green's function $G_i \equiv \frac{1}{E_i - \mathcal{H}_{\text{qb}}}$. Differentiating the relation $G_i^{-1} | \psi_i \rangle \equiv (E_i - \mathcal{H}_{\text{qb}}) | \psi_i \rangle = 0$ one gets

$$\left| \frac{\partial \psi_i}{\partial V} \right\rangle = -G_i \frac{\partial (G_i^{-1})}{\partial V} | \psi_i \rangle. \quad (\text{B2})$$

To evaluate the third term in Eq. (B1) one substitutes in it Eq. (B2) to obtain

$$\begin{aligned} \left\langle \psi_i \left| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \right| \frac{\partial \psi_i}{\partial V} \right\rangle &= -\sum_{j,k} \langle \psi_i | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_j \rangle \langle \psi_j | G_i | \psi_k \rangle \\ &\quad \times \langle \psi_k | \frac{\partial (G_i^{-1})}{\partial V} | \psi_i \rangle, \quad (\text{B3}) \end{aligned}$$

where we have inserted the completeness condition $\sum_j | \psi_j \rangle \langle \psi_j | = I$. For the last multiplier of Eq. (B3) one

gets

$$\langle \psi_k | \frac{\partial(G_i^{-1})}{\partial V} | \psi_i \rangle = \delta_{ki} \frac{\partial E_i}{\partial V} - \langle \psi_k | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_i \rangle. \quad (\text{B4})$$

By differentiating the identities $\langle \psi_k | \mathcal{H}_{\text{qb}} | \psi_i \rangle = \delta_{ki} E_i$ and $\langle \psi_k | \psi_i \rangle = \delta_{ki}$, one gets the simple relations

$$\langle \psi_k | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_i \rangle = \delta_{ki} \frac{\partial E_i}{\partial V} - \left\langle \frac{\partial \psi_k}{\partial V} \middle| \psi_i \right\rangle E_i - \left\langle \psi_k \middle| \frac{\partial \psi_i}{\partial V} \right\rangle E_k, \quad (\text{B5})$$

$$\left\langle \frac{\partial \psi_k}{\partial V} \middle| \psi_i \right\rangle + \left\langle \psi_k \middle| \frac{\partial \psi_i}{\partial V} \right\rangle = 0, \quad (\text{B6})$$

and combining them one gets

$$\langle \psi_k | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_i \rangle = \delta_{ki} \frac{\partial E_i}{\partial V} + (E_i - E_k) \left\langle \psi_k \middle| \frac{\partial \psi_i}{\partial V} \right\rangle. \quad (\text{B7})$$

Substituting Eqs. (B7) and (B4) into Eq. (B3) and taking into account that $\langle \psi_j | G_i | \psi_k \rangle = \frac{1}{E_i - E_k} \delta_{jk}$ one finally obtains for the third term of Eq. (B1):

$$(\text{B3}) \equiv \left\langle \psi_i \middle| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \middle| \frac{\partial \psi_i}{\partial V} \right\rangle = - \sum_{j \neq i} \frac{|\langle \psi_i | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} | \psi_j \rangle|^2}{E_j - E_i}. \quad (\text{B8})$$

The first term in Eq. (B1) is expressed via the third one (B3) since $\frac{\partial \mathcal{H}_{\text{qb}}}{\partial V}$ is Hermitian:

$$\left\langle \frac{\partial \psi_i}{\partial V} \middle| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \middle| \psi_i \right\rangle = \left(\left\langle \psi_i \middle| \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V} \middle| \frac{\partial \psi_i}{\partial V} \right\rangle \right)^*. \quad (\text{B9})$$

Substituting this in Eq. (B1) one recovers the quantum-mechanical sum rule (42) for the polarizability matrix element:

$$\langle \psi_i | \frac{\partial^2 \mathcal{H}_{\text{qb}}}{\partial V_G^2} | \psi_i \rangle = \frac{\partial^2 E_i}{\partial V_G^2} + 2 \sum_{j \neq i} \frac{|\langle \psi_i | \frac{\partial \mathcal{H}_{\text{qb}}}{\partial V_G} | \psi_j \rangle|^2}{E_j - E_i}. \quad (\text{B10})$$

The mere purpose of this sum rule is to establish the proper limit for $\mathcal{H}_{\delta\omega}$ and \mathcal{H}_{\parallel} [Eqs. (38) and (40) in the adiabatic regime] when they are expressed via the energy curvature with respect to voltage (i.e., quantum capacitance) effective Hamiltonians (13), (14), and (46), (47).

APPENDIX C: DIPOLE MATRIX ELEMENTS OF THE TQD EXCHANGE-ONLY QUBIT AT OR AROUND THE FULL SWEET SPOT

Here we briefly sketch the calculation of the transition dipole moments of the TQD system, where the higher excited (doubly occupied) states contributions will dominate in the effective adiabatic interactions at or around the full sweet spot (SOP). Some details of the calculations can be found in Ref. [14]. We repeat some of the results and definitions from that reference for the sake of completeness.

The TQD Hamiltonian and the dipole interaction are formulated [14] in the charge basis of the six states of spin projection $S_z = +\frac{1}{2}$, namely, $|1\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1 \uparrow_2 \downarrow_3\rangle - |\downarrow_1 \uparrow_2 \uparrow_3\rangle)$, $|2\rangle = -\frac{1}{\sqrt{6}}(|\uparrow_1 \uparrow_2 \downarrow_3\rangle + |\downarrow_1 \uparrow_2 \uparrow_3\rangle - 2|\uparrow_1 \downarrow_2 \uparrow_3\rangle)$ [qubit subspace with charge configuration (1,1,1)], and the four

highly gapped charge states ($E_{\text{gap}} \approx U_{\text{charge}}$), i.e., $|3\rangle = (201) = |S(2_1, 0_2)\rangle |\uparrow_3\rangle$, $|4\rangle = (102) = |\uparrow_1\rangle |S(0_2, 2_3)\rangle$, $|5\rangle = (120) = |\uparrow_1\rangle |S(2_2, 0_3)\rangle$, $|6\rangle = (021) = |S(0_1, 2_2)\rangle |\uparrow_3\rangle$. The TQD system Hamiltonian has diagonal energies $E_1 = E_2 = 0$, $E_3 = \varepsilon_v - \varepsilon_m + \tilde{U}_1$, $E_4 = -\varepsilon_v - \varepsilon_m + \tilde{U}_3$, $E_5 = \varepsilon_v + \varepsilon_m + \tilde{U}_2$, $E_6 = -\varepsilon_v + \varepsilon_m + \tilde{U}'_2$, and off-diagonal tunneling matrix elements that are linear in the left or right tunneling amplitudes t_l , t_r that couple only the qubit subspace $\{|1\rangle, |2\rangle\}$ to the upper gapped states [14]. The dipole interaction in the charge basis is diagonal and assuming the coupling to the SC resonator is through the middle dot 2 via the V_2 gate voltage (we neglect small corrections due to capacitance coupling between the dots), one gets

$$\mathcal{H}_{\text{dipole}} = 2\hbar g_0 (\hat{a} + \hat{a}^\dagger) \hat{D}, \quad \hat{D} \equiv \text{diag}[1, 1, 0, 0, 2, 2]. \quad (\text{C1})$$

This is obtained using the linear response approach [14]. In the above energies of the excited states we have used the notations of the two gate voltage (energy) detunings $\varepsilon_v \equiv e(V_3 - V_1)/2$ and $\varepsilon_m \equiv e[(V_3 + V_1)/2 - V_2] \equiv eV_m$. Also, the charging energies $\tilde{U}_i \approx U_{\text{charge}}$ are defined as the energy costs to go from the (1,1,1) configuration to a configuration where the i th dot is doubly occupied, e.g., \tilde{U}_1 is the energy cost for transition from (1,1,1) to (2,0,1), etc.

One first performs a (static) Schrieffer-Wolff transformation [66,67] that brings the TQD Hamiltonian to a block-diagonal form, decoupling the qubit subspace from the highly gapped four states. The qubit block (Hamiltonian) takes the well-known form in the transformed basis

$$\mathcal{H}_q^{\text{TQD}} = -J(\varepsilon_v, \varepsilon_m) + \frac{J(\varepsilon_v, \varepsilon_m)}{2} \tilde{\sigma}_z - \frac{\sqrt{3}}{2} j(\varepsilon_v, \varepsilon_m) \tilde{\sigma}_x \quad (\text{C2})$$

$$= -J(\varepsilon_v, \varepsilon_m) - \frac{E_q(\varepsilon_v, \varepsilon_m)}{2} \sigma_z, \quad (\text{C3})$$

with the exchange energies $J \equiv (J_l + J_r)/2$, $j \equiv (J_l - J_r)/2$ and a qubit splitting $E_q = \sqrt{J_l^2 + J_r^2} - J_l J_r$, where the left and right exchange energies are given as

$$J_l(\varepsilon_v, \varepsilon_m) = 2t_l^2 \left[\frac{1}{\varepsilon_v - \varepsilon_m + \tilde{U}_1} + \frac{1}{-\varepsilon_v + \varepsilon_m + \tilde{U}'_2} \right], \quad (\text{C4})$$

$$J_r(\varepsilon_v, \varepsilon_m) = 2t_r^2 \left[\frac{1}{\varepsilon_v + \varepsilon_m + \tilde{U}_2} + \frac{1}{-\varepsilon_v - \varepsilon_m + \tilde{U}_3} \right]. \quad (\text{C5})$$

In Eq. (C2) the diagonalization is further performed by a unitary transformation

$$U_{\text{qb}} = \begin{pmatrix} \cos(\eta/2) & \sin(\eta/2) \\ -\sin(\eta/2) & \cos(\eta/2) \end{pmatrix}, \quad (\text{C6})$$

where

$$\eta/2 = \arccos \left[\frac{1}{\sqrt{2}} \left(1 - \frac{J}{\sqrt{J^2 + 3j^2}} \right)^{1/2} \right]. \quad (\text{C7})$$

Thus, the qubit eigenstates $|+\rangle$, $|-\rangle$ obtain energy curvature with respect to voltage detunings ϵ_v , ϵ_m due to the higher levels. The other block of the four highly gapped states (at or around the full sweet spot) remains approximately diagonal since the corrections to the diagonal energies $E_l \approx U_{\text{charge}}$, $l = 3, 4, 5, 6$, are of the order of $\sim t_{l,r}^2/U_{\text{charge}}$, i.e., highly suppressed for $t_{l,r} \sim 5\text{--}10$ GHz and $U_{\text{charge}} \sim 200\text{--}300$ GHz.

By performing the same transformations as above to the dipole Hamiltonian, one obtains the qubit dipole coupling $g_{-,+}$ as [14]

$$g_{-,+} \equiv g_{\perp} = -g_0 \frac{\sqrt{3}}{4E_q} \left[\frac{\partial J_r}{\partial \epsilon_m} J_l - \frac{\partial J_l}{\partial \epsilon_m} J_r \right]. \quad (\text{C8})$$

Note that at the full sweet spot (SOP) when the detunings take the values [43]

$$\epsilon_v^0 = \frac{1}{4}(-\tilde{U}_1 + \tilde{U}'_2 - \tilde{U}_2 + \tilde{U}_3), \quad \epsilon_m^0 = \frac{1}{4}(\tilde{U}_1 - \tilde{U}'_2 - \tilde{U}_2 + \tilde{U}_3), \quad (\text{C9})$$

the qubit dipole coupling is zero, $g_{-,+} = 0$ [14]. It is also zero in the symmetric situation $t_l = t_r$, $\tilde{U}_1 = \tilde{U}_3$, $\tilde{U}'_2 = \tilde{U}_2$, and $\epsilon_v = \epsilon_v^0 = 0$ is at sweet spot value while ϵ_m is arbitrary [14].

For the dipole couplings of the qubit levels to the upper highly gapped levels, one obtains (up to a factor of $2g_0$)

$$g_{-,3} = \frac{\sqrt{2}t_l \sin(\alpha_+)}{\epsilon_v - \epsilon_m + \tilde{U}_1}, \quad g_{-,4} = \frac{\sqrt{2}t_r \sin(\alpha_-)}{-\epsilon_v - \epsilon_m + \tilde{U}_3}, \quad (\text{C10})$$

$$g_{-,5} = -\frac{\sqrt{2}t_r \sin(\alpha_-)}{\epsilon_v + \epsilon_m + \tilde{U}_2}, \quad g_{-,6} = -\frac{\sqrt{2}t_l \sin(\alpha_+)}{-\epsilon_v + \epsilon_m + \tilde{U}'_2} \quad (\text{C11})$$

and

$$g_{+,3} = \frac{\sqrt{2}t_l \cos(\alpha_+)}{\epsilon_v - \epsilon_m + \tilde{U}_1}, \quad g_{+,4} = \frac{\sqrt{2}t_r \cos(\alpha_-)}{-\epsilon_v - \epsilon_m + \tilde{U}_3}, \quad (\text{C12})$$

$$g_{+,5} = -\frac{\sqrt{2}t_r \cos(\alpha_-)}{\epsilon_v + \epsilon_m + \tilde{U}_2}, \quad g_{+,6} = -\frac{\sqrt{2}t_l \cos(\alpha_+)}{-\epsilon_v + \epsilon_m + \tilde{U}'_2}, \quad (\text{C13})$$

where $\alpha_{\pm} \equiv \frac{\pi}{6} \pm \frac{\eta}{2}$. These dipole couplings are used in the main text (Sec. VID). They are relevant in a range of detunings $\epsilon_v \approx \epsilon_v^0$, $\epsilon_m \approx \epsilon_m^0$ at or around the full sweet spot.

Calculations for the resonant exchange (RX) regime [68] are analogous to the above, but not performed here. In the RX regime (approaching a c.d.p.) some of the energy denominators become small and their contributions are enhanced. Then the block of four upper states need to be diagonalized as well, in order to obtain the dipole elements to the upper states $g_{-,l}$, $g_{+,l}$.

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