Quantum kinetic equation and thermal conductivity tensor for bosons

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We systematically derive the quantum kinetic equation in full phase space for any quadratic Hamiltonian of bosonic fields, including in the absence of translational invariance. This enables the treatment of boundaries, inhomogeneous systems, and states with nontrivial textures, such as skyrmions in the context of magnetic bosons. We relate the evolution of the distribution of bosons in phase space to single-electron, band-diagonal physical quantities such as Berry curvature and energy magnetization by providing a procedure to "diagonalize" the Hamiltonian in phase space, using the formalism of the Moyal product. We obtain *exact* equations, which can be expanded order by order, for example, in the "smallness" of the spatial gradients, providing a "semiclassical" approximation. In turn, at first order, we recover the usual full Boltzmann equation and give a self-contained and exact derivation of the intrinsic thermal Hall effect of bosons. The formulation clarifies the contribution from "energy magnetization" in a natural manner, and does not require the inclusion of Luttinger's pseudogravitational field to obtain thermal transport quantities.

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I. INTRODUCTION

It is now widely recognized that the dynamics of electrons in energy bands is fundamentally affected by topological and geometric aspects of the Bloch eigenstates. This is described semiclassically by Berry phases and curvatures, which enter the equations of motion for a wave packet [1]. The same ideas apply to any smooth collection of nondegenerate single-particle states indexed by *continuous* parameters, and in particular to bosonic elementary excitations like phonons or magnons [2,3]. These concepts are particularly powerful in describing out of, but near, equilibrium phenomena such as charge and energy transport: incorporating the semiclassical topological dynamics into a Boltzmann description provides the most compact and intuitive understanding of the anomalous Hall effect, as well as many other transport properties [4–9]. Being semiclassical, i.e., allowing simultaneous consideration of position and momentum, such a method also makes it possible to treat inhomogeneous systems, including boundaries, textures, etc., in a compact and natural manner. It sometimes offers advantages even for calculation of bulk transport properties like thermal conductivity because it allows to transparently isolate spurious contributions due to bound "magnetization" currents, which have in the past obscured correct results [10].

The semiclassical approach is known to reproduce the exact result for the intrinsic anomalous Hall effect [6], as is often the case for transport coefficients. This is because the semiclassical approximation is controlled by the smallness of spatial and temporal gradients, of the Hamiltonian and of the distribution function, which are indeed small in a perturbative response to uniform DC fields. Theoretically, the smallness of gradients is required *both* to justify the semiclassical equations of motion, which describe the motion of *one* particle in phase space, the Boltzmann equation itself, which describes the evolution of the *distribution* of particles, and which is an approximation to the fully quantum evolution of the density matrix. The derivation of the Boltzmann equation semiclassically from the full quantum kinetic equation (QKE) is an old problem studied extensively *prior* to the widespread incorporation of Berry phase effects into band dynamics [11–13]. Despite the common practice of combining these two semiclassical approximations, they are almost always treated independently, with attention focusing primarily on the single-particle equations of motion and the Boltzmann equation adopted without justification.

A unified derivation for the electronic case was provided by Wickles and Belzig [14,15]. They showed that the equation of motion for the "one-particle density matrix" of a multiband electron system can be systematically reduced to the Boltzmann one, with the renormalized single-particle equations of motion emerging in the same treatment via a single semiclassical expansion. Here we extend this treatment to bosons, which, contrary to electrons, do not have a conserved number or charge, but only energy. The difference of statistics, but more importantly the latter lack of conserved charge, leads to some significant differences from the Wickles and Belzig treatment. Nevertheless, we obtain a full derivation of the leading semiclassical kinetic equation and associated observable quantities such as energy and current densities, and the formulation allows a clear route to extend to higher orders in the semiclassical expansion.

The derivation we present assumes noninteracting bosons, and hence neglects any scattering (but this can of course be

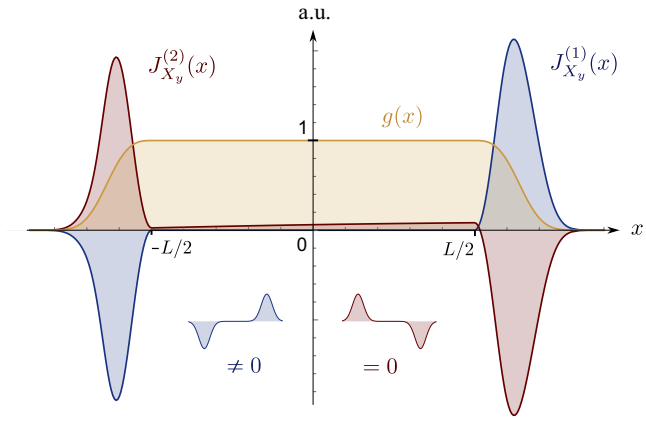

FIG. 1. Local momentum-integrated energy currents $J^{(1)}_{X_y}(x) \equiv \int_p \mathcal{J}^{(1)}_{X_y}$ [first term in Eq. (18), blue] and $J^{(2)}_{X_y}(x) \equiv \int_p \mathcal{J}^{(2)}_{X_y}$ [magnetization current, second term in Eq. (18), red] in arbitrary units for a system of chiral phonons described by the Lagrangian (67), calculated for $g(x) = \exp\{-\frac{1}{4}\Theta[|x| - L/2](|x| - L/2)^3/\xi^3\}$ (yellow, mimicking the existence of a boundary) in the presence of a constant temperature gradient $\delta T/L$, i.e., for a temperature profile $T(x) = T_0 + (x/L)\delta T$, using the numerical values given in Table I. The insets emphasize the fact that $\int dx\, J^{(2)}_{X_y}(x) = 0$ while $\int dx\, J^{(1)}_{X_y}(x) \neq 0$ (*a priori*).

added). Aside from the neglect of interactions, the derived quantum kinetic equation is asymptotically exact in the limit in which the parameters of the theory are slowly varying in space. In particular, we obtain exactly the thermal conductivity tensor, which describes the heat current induced by a small temperature gradient. Our approach therefore provides a self-contained and exact derivation of the intrinsic thermal Hall effect of noninteracting bosons such as phonons and magnons, and in the last section we show that the quantum kinetic equation (QKE) indeed is in agreement with Kubo formula calculations [2,3]. The advantage is that the QKE calculation is considerably more intuitive, and naturally avoids subtleties associated with magnetization currents [16–18], which plagued early Kubo calculations [10], as mentioned above. For concreteness we further compute and plot (see Fig. 1) the local (i.e., position-resolved) energy currents for a system of chiral phonons (excitations of linear elasticity theory with a phonon Hall viscosity term) in a finite geometry in the presence of a temperature gradient.

II. OVERVIEW

The principal result of this paper, the derivation of the semiclassical QKE and that of observables, is somewhat technical and subtle. Hence, we provide in this section an overview of the derivation, highlighting the key steps, leaving full details to Sec. III.

TABLE I. Values used for the parameters defined in Eqs. (67) and (69) and in the caption of Fig. 1.

T_0	δT	ρ	c_1	c_2	c_3	c_4	c_5	η_1	η_2	ξ	L
1.0	0.8	1.0	1.5	1.2	0.8	0.4	0.0	−0.25	−0.15	1.0	10

A. Basic formulation

Our starting point is a general free-boson system described by $2N$ free Hermitian fields $\Phi_a(r)$, $r \in \mathbb{R}^d$, where d is the dimension of our system, $a = 1, \ldots, 2N$. For free fields the commutator

$$[\Phi_a(r_1), \Phi_b(r_2)] \equiv \hbar\, \hat{\Gamma}_{ab}(r_1, r_2) \tag{1}$$

is a *c-number* $2N \times 2N$ matrix. For example, in the context of three-dimensional elasticity, $\Phi_{1,2,3}$ might be the displacement fields $u_{x,y,z}$, and $\Phi_{4,5,6}$ represent the conjugate momenta $\Pi_{x,y,z}$. Since the commutator is antisymmetric and the fields are Hermitian, we have $\hat{\Gamma}_{ab}(r_1, r_2) = -\hat{\Gamma}_{ba}(r_2, r_1)$ and $\hat{\Gamma}_{ba}(r_2, r_1) = (\hat{\Gamma}_{ab}(r_1, r_2))^*$.

We take an arbitrary quadratic Hamiltonian

$$\mathrm{H} = \frac{1}{2}\int_{r_1,r_2} \sum_{a,b} \hat{\mathsf{H}}_{ab}(r_1, r_2)\Phi_a(r_1)\Phi_b(r_2), \tag{2}$$

where $\hat{\mathsf{H}}(r_1, r_2)$ is a *c-number* $2N \times 2N$ matrix. The Hermiticity of H requires $\hat{\mathsf{H}}(r_2, r_1) = (\hat{\mathsf{H}}(r_1, r_2))^*$, and to remove any redundancy we impose $\hat{\mathsf{H}}_{ab}(r_1, r_2) = \hat{\mathsf{H}}_{ba}(r_2, r_1)$. Both $\hat{\mathsf{H}}$ and $\hat{\Gamma}$, regarded as matrices in the index $(a, b) \times$ coordinate space (r_1, r_2), are Hermitian matrices.

Apart from Hermiticity and the symmetry and antisymmetry of $\hat{\mathsf{H}}$ and $\hat{\Gamma}$, respectively, we need only assume that $\hat{\mathsf{H}}$ has a spectrum which is bounded below, so that the system is stable.

The expectation value of any quadratic observable can be obtained as a linear combination of the expectation values

$$\hat{\mathsf{F}}_{ab}(r_1, r_2) \equiv \tfrac{1}{2}\langle\{\Phi_a(r_1), \Phi_b(r_2)\}\rangle. \tag{3}$$

Here the expectation value is the expectation value with respect to an initial many-body density matrix ρ_0, $\langle A(t)\rangle = \mathrm{Tr}[A(t)\rho_0]$, with $A(t)$ an operator in the Heisenberg picture. The corresponding c-number matrix $\hat{\mathsf{F}}$, which captures any two-point expectation value of any two observables, is sometimes called "density matrix," and is symmetric and Hermitian in the index $\times$ coordinate space.

B. Transition to phase space

From Eqs. (1) and (2) the unitary time evolution of any operator is determined. To frame this in phase space, we introduce the Wigner transform

$$f(X, p) = [\hat{f}]^W(X, p) \equiv \int_x e^{-i\frac{px}{\hbar}} \hat{f}\left(X + \frac{x}{2}, X - \frac{x}{2}\right), \tag{4}$$

where $p, X, x \in \mathbb{R}^d$ [we can identify $X = \frac{1}{2}(r_1 + r_2)$ and $x = r_1 - r_2$] and $\hbar$ is Planck's constant divided by 2π. We denote $\int_x = \int d^d x$ for spatial integrals, and later we will use $\int_p = \int d^d p/(2\pi\hbar)^d$ for momentum integrals (then $\int_{x,p}$ is dimensionless). We will denote Wigner transformed matrices by removing the hats, e.g., $[\hat{\mathsf{F}}]^W = \mathsf{F}$. These objects, F, H, Γ, etc., are functions on phase space (X, p), and are all Hermitian as matrices at fixed X, p. Note that we will use the convention that p denotes momentum while k denotes wave number, i.e., $p = \hbar k$. This is useful when we keep track of factors of $\hbar$ and to obtain the classical limit.

In terms of Wigner transformed objects, one can then derive the equation of motion. Namely, the equation of motion for the density matrix in phase space is

$$\partial_t \mathsf{F}(X,p) = -\frac{i}{\hbar}(\mathsf{K}\star\mathsf{F} - \mathsf{F}\star\mathsf{K}^\dagger), \tag{5}$$

where

$$\mathsf{K}(X,p) = \hbar\,\Gamma\star\mathsf{H} \tag{6}$$

is sometimes called the "dynamical matrix." Note that K is *not* Hermitian.

Here the $\star$ represents the "star" or "Moyal" product, defined as, for any two matrix functions $\mathsf{O}_{1,2}$,

$$\mathsf{O}_1^\dagger\mathsf{O}_2 \equiv \mathsf{O}_1 \exp\left(i\frac{\hbar}{2}\epsilon_{\alpha\beta}\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\right)\mathsf{O}_2, \tag{7}$$

where repeated indices are summed, $\alpha,\beta = 1,\ldots,2d$, $\partial_\alpha \equiv \partial_{q_\alpha}$, where $q_\mu = X_\mu$ and $q_{d+\mu} = p_\mu$ for $\mu = 1,\ldots,d$, and where we have defined $\epsilon_{\alpha\beta}$ to be such that

$$\epsilon_{X_\mu p_\nu} = -\epsilon_{p_\mu X_\nu} = \delta_{\mu\nu}, \tag{8}$$

with $\mu = 1,\ldots,d$. Note that $\hbar = \hbar$ but we are formally distinguishing the two here to treat $\hbar$ as an expansion parameter when it stems from an expansion of Eq. (7) [15]. Notably, when $\mathsf{o}_{1,2}$ are scalar functions,

$$\lim_{\hbar\to 0}\frac{1}{i\hbar}[\mathsf{o}_1 \stackrel{\star}{,} \mathsf{o}_2] = \{\mathsf{o}_1,\mathsf{o}_2\}_{\text{p.b.}} \tag{9}$$

with the star (Moyal) bracket defined as $[\mathsf{A}\stackrel{\star}{,}\mathsf{B}] \equiv \mathsf{A}\star\mathsf{B} - \mathsf{B}\star\mathsf{A}$, and $\{\cdot,\cdot\}_{\text{p.b.}}$ the Poisson bracket, which illustrates the role of the $\hbar$ expansion as a semiclassical expansion. We provide more details in Appendix A. See also Refs. [13,15,19].

C. Semiclassical solution

A proper semiclassical limit is obtained by first formally diagonalizing the dynamical matrix and distribution function, i.e., reducing them to scalars within each band, and then performing a small $\hbar$ expansion, which expands the star products (7). Diagonalization is accomplished via a similarity transformation S such that $\mathsf{S}^{-1}\star\mathsf{K}\star\mathsf{S} = \mathsf{K}_d$ is a diagonal matrix of mode frequencies. Similarly, $\mathsf{F} = \mathsf{S}\star\mathsf{F}_d\star\mathsf{S}^\dagger$, with F_d a diagonal distribution function and the inverse S^{-1} is defined with respect to the star product ($\mathsf{S}^{-1}\star\mathsf{S} = \mathsf{S}\star\mathsf{S}^{-1} = 1$, see Sec. III B for details). Note that F_d is real by definition, and we show later that K_d is also real.

There is some phase ambiguity in the diagonalization, which can be regarded as a gauge freedom since the diagonalization exists at every point in phase space. Physical quantities must be gauge invariant. We find that the *diagonal* gauge-invariant quantities $\underline{\mathsf{O}}_d$ take the form

$$\underline{\mathsf{O}}_d(q) \equiv \mathsf{O}_d[\underline{q}], \tag{10}$$

with $\mathsf{O}_d = \mathsf{K}_d, \mathsf{F}_d$, and

$$\underline{q}_\alpha = q_\alpha + \hbar\epsilon_{\alpha\beta}\mathsf{A}_\beta + O(\hbar^2), \tag{11}$$

where A_α is a Berry gauge field:

$$\mathsf{A}_\alpha = \mathrm{Im}\,(\Lambda_\alpha)^{(\mathrm{d})},$$
$$\Lambda_\alpha = \mathsf{S}^{-1}\star\partial_\alpha\mathsf{S}. \tag{12}$$

Here (d) denotes the diagonal part and Im the imaginary part. Note that A is diagonal so these equations, e.g., Eq. (10), are unambiguous: see Eq. (45) for an explicit expression for a diagonal matrix function of a diagonal matrix.

Applying this prescription we obtain the gauge-invariant kinetic equation to first order in $\hbar$,

$$\partial_t\underline{\mathsf{F}}_d + \upsilon\,\partial_t\mathsf{q}_\alpha\,\partial_\alpha\underline{\mathsf{F}}_d = 0 + O(\upsilon\hbar^2), \tag{13}$$

where we work in the collisionless limit of free particles without short-distance scatterers although scattering effects could be straightforwardly included on the right-hand side of Eq. (13) in the form of a "collision integral" $\mathcal{I}_{\text{coll}}$ [20,21]. Here $\upsilon = \hbar/\hbar$ is a symbolic parameter that counts the order of the semiclassical expansion, shorn of any dimension: taken literally, $\upsilon = 1$ since $\hbar = \hbar$. The total number of powers of $\hbar$ and υ in an expression gives the semiclassical order of the corresponding term in the kinetic equation, and is equal to the number of spatial gradients.

In Eq. (13) we have defined the object

$$\partial_t\mathsf{q}_\alpha \equiv \epsilon_{\alpha\beta}(\partial_\beta\underline{\mathsf{K}}_d + \hbar\epsilon_{\gamma\sigma}\Omega_{\beta\gamma}\partial_\sigma\underline{\mathsf{K}}_d) + O(\hbar^2). \tag{14}$$

Mathematically $\partial_t\mathsf{q}_\alpha$ is just a function of phase space given by the right-hand side of Eq. (14), i.e., there is no need to solve a dynamical equation for $\mathsf{q}_\alpha(t)$ to solve the Boltzmann equation. However, we can identify physically Eq. (14) as the single-particle equation of motion in phase space of a particle with phase-space coordinates q. We also introduced

$$\Omega_{\alpha\beta} = \partial_\alpha\mathsf{A}_\beta - \partial_\beta\mathsf{A}_\alpha \tag{15}$$

which is the Berry curvature in phase space. Note that this quantity is band diagonal by definition and the relation $\Omega_{\alpha\beta} = \mathrm{Im}[\Lambda_\beta\stackrel{\star}{,}\Lambda_\alpha]^{(\mathrm{d})}$ also holds. This result, which is derived systematically using only the assumptions in Sec. II A, reproduces the expected form from the heuristic wave-packet theory [1].

By reexpressing physical quantities in the gauge-invariant diagonal variables, we analogously obtain explicitly gauge-invariant expressions for them. The simplest is the energy density in phase space,

$$\mathcal{H}(q) = \tfrac{1}{2}\mathrm{Tr}[\,\mathfrak{J}\,\underline{\mathsf{F}}_d\,] + O(\hbar^2), \tag{16}$$

where

$$\mathfrak{J} = 1 + \hbar\,\Omega_{X_\mu p_\mu} + O(\hbar^2) \tag{17}$$

is a Jacobian in phase space (see Appendix D). It has the physical implication of inducing on the density of states a dependence on Berry curvature [22].

Similarly, we obtain the current density in phase space

$$\mathcal{J}_\alpha(q) = \frac{\upsilon}{2}\mathrm{Tr}[\,\mathfrak{J}\,\partial_t\mathsf{q}_\alpha\,\underline{\mathsf{F}}_d + \hbar\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma(\mathsf{M}_{\beta\lambda}\underline{\mathsf{F}}_d)] + O(\upsilon\hbar^2). \tag{18}$$

One should keep in mind that α here is a general phase-space index, so that this is more general than the usual current density. Specifically, $\mathcal{J}_{X_\mu}(X,p)$ is the contribution of states at momentum p to the energy current density at position X, while $\mathcal{J}_{p_\mu}(X,p)$ gives the analogous contribution to the "force density."

In Eq. (18), we introduced

$$\mathsf{M}_{\alpha\beta} = \tfrac{1}{2}\mathrm{Im}\{\Lambda_\beta,[\Lambda_\alpha,\mathsf{K}_d]\}^{(\mathrm{d})}, \tag{19}$$

which has the interpretation of an energy magnetization [17,23].

D. Practical remarks

Note that in Eq. (18) we have separated out two contributions to the current density, in which the second is manifestly a total derivative (phase-space curl). The curl term gives zero when integrated over phase space, and most importantly gives zero net (momentum integrated) flux through a surface in real space for which the magnetization $\mathsf{M}_{\beta\lambda}$ vanishes on the boundary (i.e., a surface which cuts the entire sample volume) (see also Appendix E). Thus, the energy magnetization does not contribute to the total "transport current," and for transport purposes only the first term (containing $\partial_t \mathsf{q}_\alpha$) needs be considered. This distinguishes the kinetic equation approach from previous calculations based on the Kubo formalism, where independently evaluating the correct energy magnetization is required to obtain the proper transport current by subtraction.

To use the above results in a specific problem, one needs to specify the diagonalized band energy K_d and the connection Λ_α. These are obtained by solving the diagonalization problem perturbatively in $\hbar$. For the energy, there is a simple general result to first order,

$$\begin{aligned}\underline{\mathsf{K}}_d(q) &= \mathsf{K}_d[q] \\ &= \mathsf{K}_{0,d} - \hbar\frac{\epsilon_{\alpha\beta}}{2}\mathsf{M}_{\alpha\beta} + \hbar\,\mathrm{Re}\big[\big(\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0\big)^{(\mathrm{d})}\big] + O(\hbar^2),\end{aligned} \tag{20}$$

where $\mathsf{K}_{0,d}$ is a diagonal matrix containing the conventional eigenvalues of the matrix $\mathsf{K}_0 = \hbar\Gamma\mathsf{H}$, and the connection Λ_α may be approximated at the same accuracy by the connection obtained from the eigenvalue problem at zeroth order, i.e., $\Lambda_\alpha \approx \mathsf{S}_0^{-1}\partial_\alpha\mathsf{S}_0$, with $\mathsf{S}_0^{-1}\mathsf{K}_0\mathsf{S}_0 = \mathsf{K}_{0,d}$. The final term in the square brackets in Eq. (20) is a correction arising from the semiclassical expansion of K itself, with $\mathsf{K} = \mathsf{K}_0 + \hbar\mathsf{K}_1 + O(\hbar^2)$ and $\mathsf{K}_1 = i\hbar\frac{\epsilon_{\alpha\beta}}{2}\partial_\alpha\Gamma\partial_\beta\mathsf{H}$ (see Appendix B 2). We will show that the spectrum of K is symmetric with positive and negative energy eigenstates related by complex conjugation and momentum reversal (see Appendix B 3).

It can be useful both for intuition and as a check to consider the dimensions of the various objects encountered in this treatment. This is somewhat problem dependent, owing to different possible choices of the fundamental Φ_a fields, which might be position and momentum densities for phonons, or transverse magnetization densities in a ferromagnet, for example. Thus, different components might even have different dimensions. However, by appropriate rescaling, it is generally possible to bring them to a form in which all Φ_a fields have dimension of the square root of density $[\Phi_a] = L^{-d/2}$. Then one obtains

$$\begin{aligned}&[X_\mu] = L, \quad [k_\mu] = 1/L, \quad [p_\mu] = Et/L, \quad [\mathfrak{J}] = 1,\\ &[\mathcal{H}] = [\mathsf{K}] = [\mathsf{K}_d] = [\mathsf{F}_d] = E,\\ &[\mathcal{J}_X] = EL/t, \quad [\mathcal{J}_p] = E^2/L,\\ &[\Lambda_\alpha] = [\mathsf{A}_\alpha] = 1/[\alpha],\\ &[\Omega_{\alpha\beta}] = 1/([\alpha][\beta]), \quad [\mathsf{M}_{\alpha\beta}] = E/([\alpha][\beta]),\\ &[\mathsf{H}][\mathsf{F}] = E, \quad [\mathsf{H}][\Gamma] = 1/t, \quad [\mathsf{S}]^2 = 1/E,\end{aligned} \tag{21}$$

where E, L, and t stand for energy, length, and time, respectively. Note that in Eq. (21), all quantities from $\mathfrak{J}$ onward are defined in phase space. This brings a factor of L^d to many quantities owing to the Wigner transform, which contains a spatial integral, for example, $[\hat{\mathsf{H}}] = E/L^d$ is an energy density while $[\mathsf{H}] = E$ is an energy (if $[\Phi_a] = L^{-d/2}$). Note that all the quantities in Eq. (21) are of order $O(\hbar^0)$. This is irrespective of our choice of definitions for p_μ and K, for example, which we chose to be momentum $p_\mu = \hbar k_\mu = O(\hbar^0)$ and energy $\mathsf{K} = \hbar\,\Gamma \star \mathsf{H} = O(\hbar^0)$ as opposed to wave vector and frequency, respectively.

III. DETAILED DERIVATION

Here we provide the derivation of the results derived in Sec. II.

A. Dynamics and continuity equation

Together, Eqs. (1) and (2) determine the dynamics of the fields according to the Heisenberg equation of motion,

$$\partial_t \Phi_a = -\frac{i}{\hbar}[\Phi_a, \mathrm{H}] = -\frac{i}{\hbar}(\hat{\mathsf{K}} \otimes \Phi)_a, \tag{22}$$

where convolution $\otimes$ is defined as

$$(\hat{\mathsf{O}}_1 \otimes \hat{\mathsf{O}}_2)_{ab}(r_1, r_2) \equiv \int_r \sum_c \hat{\mathsf{O}}_{1,ac}(r_1, r)\hat{\mathsf{O}}_{2,cb}(r, r_2), \tag{23}$$

and the dynamical matrix is

$$\hat{\mathsf{K}} \equiv \hbar\,\hat{\Gamma} \otimes \hat{\mathsf{H}}. \tag{24}$$

In turn, this implies that

$$\partial_t \hat{\mathsf{F}} = -\frac{i}{\hbar}(\hat{\mathsf{K}} \otimes \hat{\mathsf{F}} - \hat{\mathsf{F}} \otimes \hat{\mathsf{K}}^\dagger), \tag{25}$$

where $(\hat{\mathsf{K}}^\dagger)_{ab}(r_1, r_2) \equiv (\hat{\mathsf{K}}_{ba}(r_2, r_1))^*$. In the same notation the total energy is

$$H \equiv \langle \mathrm{H} \rangle = \frac{1}{2}\int_r \mathrm{Tr}[(\hat{\mathsf{H}} \otimes \hat{\mathsf{F}})(r, r)], \tag{26}$$

where Tr is the usual matrix trace, i.e., $\mathrm{Tr}[\hat{\mathsf{O}}] = \sum_a \hat{\mathsf{O}}_{aa}$.

Now we pass to the Wigner transform, and use the fact that the Wigner transform of a convolution is the star (or Moyal) product of the Wigner transforms, i.e., $[\hat{f}_1 \otimes \hat{f}_2]^W(X, p) = f_1(X, p) \star f_2(X, p)$ (see Appendix A). This leads directly to Eqs. (5) and (6). In this way, the total energy can also be written as

$$\langle \mathrm{H} \rangle \equiv H = \frac{1}{2}\int_{X,p} \mathrm{Tr}[\mathsf{H} \star \mathsf{F}], \tag{27}$$

where $\mathrm{Tr}[\mathsf{O}] = \sum_a \mathsf{O}_{aa}$ for any matrix O is the regular matrix trace.

Conservation of energy is ensured by Heisenberg evolution, Eq. (22), which is unitary, and when the Hamiltonian is local, this gives rise to a continuity equation. To obtain it we first define from Eq. (27) a phase-space energy density

$$\mathcal{H}(X, p) = \tfrac{1}{4}\mathrm{Tr}(\mathsf{H} \star \mathsf{F} + \mathsf{F} \star \mathsf{H}) \equiv \tfrac{1}{4}\mathrm{Tr}\{\mathsf{H} \overset{\star}{,} \mathsf{F}\}. \tag{28}$$

Here we symmetrized by hand the argument of the trace, so that the energy density is real $\mathcal{H} = \mathcal{H}^*$ (note that the choice of *local* energy density is not unique [24,25]).

Taking the time derivative of the energy density, we find

$$\partial_t \mathcal{H} = \frac{1}{2\hbar}\mathrm{Im}\mathrm{Tr}([\mathsf{K} \overset{\star}{,} \mathsf{F} \star \mathsf{H}]). \tag{29}$$

Very generally, the phase-space integral of the trace of a Moyal bracket vanishes (see Appendix A). Hence, the right-hand side of Eq. (29) can be written as a *phase-space* divergence, i.e., there exists $\mathcal{J}(X, p)$ such that, in the collisionless limit,

$$\partial_t \mathcal{H} + \partial_\alpha \mathcal{J}_\alpha = 0. \tag{30}$$

Up to this point, all the above formulas are *exact*.

Expanding the "center" of the Moyal bracket (i.e., "$\overset{\star}{,}$") in $\hbar$ in Eq. (29), we find

$$\mathcal{J}_\alpha(X, p) = \frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re}\,\mathrm{Tr}[\partial_\beta \mathsf{K}(\mathsf{F} \star \mathsf{H})] + O(\upsilon\hbar^2), \tag{31}$$

where Re is the real part. We note that the local phase-space current density $\mathcal{J}$ depends on the precise choice of local energy density and enjoys an additional freedom in the form of an additive phase-space curl $\mathcal{J}_\alpha \to \mathcal{J}_\alpha + \partial_\beta \mathcal{M}_{\alpha\beta}$ for any differentiable phase-space function $\mathcal{M}_{\alpha\beta}$ odd under $\alpha \leftrightarrow \beta$, provided that the curl is *gauge invariant*.

To summarize, the energy density is defined by Eq. (28), and the current density by Eq. (31). Now by expressing these quantities in the diagonal band basis, we will eventually arrive at Eqs. (16) and (18).

B. Diagonalization

The effects of curvature and geometry arise from the projection into the manifold of a band [26]. Formally, the projection is carried out in the phase-space representation by diagonalizing the dynamical matrix using a similarity transformation S, introduced in Sec. II C:

$$\mathsf{S}^{-1} \star \mathsf{K} \star \mathsf{S} = \mathsf{K}_d, \tag{32}$$

where K_d is a diagonal matrix (whose entries are *real*, as shown in Appendix B 1).

Because K is not Hermitian, S cannot be chosen star unitary, but instead can be chosen to satisfy (see Appendix B 1)

$$\mathsf{S}^\dagger \star \mathsf{H} \star \mathsf{S} = 1, \tag{33}$$

where 1 is the identity matrix. Equation (33) is the analog of the star-unitarity condition for a nonunitary matrix. Note that we define the inverse here in the sense of the star product, i.e., $\mathsf{S}^{-1} \star \mathsf{S} = \mathsf{S} \star \mathsf{S}^{-1} = 1$, as mentioned in Sec. II C.

Equations (32) and (33) are somewhat formal, due to the nontrivial nature of the star product, but can be solved order by order in $\hbar$, starting from a zeroth-order solution which is a standard matrix diagonalization problem, and which is guaranteed to exist by the properties of K.

The similarity transformation S is therefore determined (up to a gauge freedom we will return to below) by K, and once it is found, we can, as mentioned in Sec. II C, define the transformed density matrix F_d as well via

$$\mathsf{F} = \mathsf{S} \star \mathsf{F}_d \star \mathsf{S}^\dagger. \tag{34}$$

Note that because S is not unitary but instead satisfies Eq. (33), F_d differs dimensionally from F by a factor of energy. In this "diagonal frame," and in the collisionless limit, one has

$$\partial_t \mathsf{F}_d + \frac{i}{\hbar}[\mathsf{K}_d \overset{\star}{,} \mathsf{F}_d] = 0. \tag{35}$$

This equation, which is exact, allows a diagonal and time-independent solution for F_d. This is the standard form of the distribution function at and near equilibrium in the following sense. There is nothing *prohibiting* off-diagonal terms in F_d, but such solutions necessarily oscillate on the scale of the differences between elements in $\mathsf{K}_d/\hbar$, i.e., the mode frequencies. This means that slowly time-varying solutions are predominantly diagonal. Moreover, when weak scattering is included, such that the associated relaxation time τ is longer than these oscillation periods, the effects of such oscillations largely average out. In any case, in the present formulation without scattering, there is no mixing of the diagonal and off-diagonal parts of F_d, so we can focus on the former consistently.

C. Gauge transformations

1. General

As mentioned above, Eqs. (32) and (33) leave some freedom in the choice of S. In particular, if S satisfies these equations in the sense that it produces a diagonal K_d, then so too does

$$\mathsf{S}' = \mathsf{S} \star \Theta \tag{36}$$

produce a diagonal K'_d, where Θ is a diagonal (star-) unitary matrix

$$\Theta^\dagger \star \Theta = \Theta \star \Theta^\dagger = 1. \tag{37}$$

Note that the diagonalized matrix K'_d is not generally equal to K_d. Rather, if we view the change from S to S' as a transformation

$$\mathsf{S} \to \mathsf{S}' = \mathsf{S} \star \Theta, \tag{38}$$

then

$$\mathsf{K}_d \to \mathsf{K}'_d = \Theta^\dagger \star \mathsf{K}_d \star \Theta, \tag{39}$$

which is manifestly also diagonal and Hermitian. The same transformation law holds for the density matrix

$$\mathsf{F}_d \to \Theta^\dagger \star \mathsf{F}_d \star \Theta. \tag{40}$$

Physical quantities defined in terms of H, Γ, F must, by construction, be invariant under these gauge transformations.

We will now show, perturbatively in $\hbar$, that such gauge transformations can be interpreted as coordinate changes in phase space.

2. First order in $\hbar$

We can express explicitly

$$\Theta = \mathrm{diag}\,(e^{i\theta_1}, \ldots, e^{i\theta_{2N}}), \tag{41}$$

where the condition of unitarity (37) perturbatively reduces to

$$\mathrm{Im}\,\theta_a = O(\hbar^2). \tag{42}$$

Hence, to $O(\hbar)$, we can use Eq. (41) with real-valued $\theta_a(q) = \theta_a(X, p)$, which in general are arbitary functions on phase space.

Using the explicit form of Θ in Eq. (41), Eqs. (39) and (40) reduce to

$$\begin{aligned}\mathsf{O}_d(q) \to \mathsf{O}'_d(q) &= \mathsf{O}_d(q) - \hbar\epsilon_{\alpha\beta}\partial_\beta\theta\partial_\alpha\mathsf{O}_d(q) + O(\hbar^2) \\ &= \mathsf{O}_d[q'] + O(\hbar^2),\end{aligned} \tag{43}$$

for $\mathsf{O}_d = \mathsf{K}_d, \mathsf{F}_d$, where $\theta = \mathrm{diag}\,(\theta_1, \ldots, \theta_{2N})$, and

$$q'_\alpha = q_\alpha - \hbar\epsilon_{\alpha\beta}\partial_\beta\theta. \tag{44}$$

In Eqs. (43) and (44), we have noticed that the change of O_d at order $\hbar$ under a gauge transformation is proportional to a derivative of O_d, and hence can be absorbed into an equivalent shift of its argument.

Note that q'_α in Eq. (44) contains θ which is a diagonal matrix, i.e., a band-dependent quantity, and so Eq. (43) defines a matrix function of a matrix. This is not ambiguous because both the matrix and the matrix argument are diagonal, and so can simply be evaluated for each band. More explicitly, the diagonal matrix function O of a diagonal matrix argument C is defined as

$$\mathsf{O}[\mathsf{C}] \equiv \mathrm{diag}(\{\mathsf{O}_{aa}(\mathsf{C}_{aa})\}_a). \tag{45}$$

D. Gauge connection and gauge-invariant quantities

From Eq. (43) it is evident that $\mathsf{O}_d = \mathsf{F}_d, \mathsf{K}_d$ are not gauge invariant. This ultimately reflects the presence of Berry phases. In particular, we see that the effect of a gauge transformation is to shift the "canonical" variable q_α within the argument of O_d by a quantity proportional to $\partial_\beta\theta$. This motivates a procedure to restore gauge invariance by modifying the argument of O_d to compensate for this shift.

This compensation is obtained from the Berry gauge field introduced in Eq. (12). Under the gauge transformation (38), to zeroth order

$$\mathsf{A}_\alpha \to \mathsf{A}_\alpha + \partial_\alpha\theta + O(\hbar). \tag{46}$$

We will see that A_α as well as Λ_α emerge naturally in calculations of physical quantities. Comparing Eqs. (46) and (44) shows how to compensate the transformation of O_d by changing its argument.

By combining Eqs. (10), (11) and (43), (46), we can see that $\underline{\mathsf{O}}_d(q)$ is gauge invariant to first order,

$$\underline{\mathsf{O}}'_d(q) = \underline{\mathsf{O}}_d(q) + O(\hbar^2). \tag{47}$$

It is expected that physical quantities can be expressed as manifestly gauge-invariant expressions when written in terms of such functions, i.e., $\underline{\mathsf{K}}_d, \underline{\mathsf{F}}_d$.

E. Gauge-invariant kinetic equation

To derive the gauge-invariant band-diagonal kinetic equation, we semiclassically expand the Moyal bracket in Eq. (35):

$$\partial_t\mathsf{F}_d = \upsilon\epsilon_{\alpha\beta}\partial_\alpha\mathsf{K}_d\partial_\beta\mathsf{F}_d + O(\upsilon\hbar^2). \tag{48}$$

Next we take the *time* derivative of Eq. (10) with $\mathsf{O}_d = \mathsf{F}_d$, which gives

$$\partial_t\underline{\mathsf{F}}_d(q) = \partial_t\mathsf{F}_d[\underline{q}] \tag{49}$$

because A_β is time independent by assumption since we did not include any time dependence in H. Hence, from Eq. (48)

$$\partial_t\underline{\mathsf{F}}_d(q) = \upsilon\epsilon_{\alpha\beta}\partial_\alpha\mathsf{K}_d(q)\partial_\beta\mathsf{F}_d(q)|_{q\to\underline{q}} + O(\upsilon\hbar^2). \tag{50}$$

Now we need to convert the functions on the right-hand side above to underlined quantities. To do so, we take the *phase-space* derivative of Eq. (10):

$$\begin{aligned}\partial_\alpha\underline{\mathsf{O}}_d(q) &= \partial_\alpha\mathsf{O}_d[\underline{q}] + \hbar\epsilon_{\gamma\lambda}\partial_\gamma\mathsf{O}_d\,\partial_\alpha\mathsf{A}_\lambda + O(\hbar^2) \\ \Rightarrow \partial_\alpha\mathsf{O}_d[\underline{q}] &= \partial_\alpha\underline{\mathsf{O}}_d(q) - \hbar\epsilon_{\gamma\lambda}\partial_\gamma\underline{\mathsf{O}}_d\,\partial_\alpha\mathsf{A}_\lambda + O(\hbar^2).\end{aligned} \tag{51}$$

Using Eq. (51) for both K_d and F_d in the right-hand side of Eq. (50), we arrange terms to obtain

$$\begin{aligned}\partial_t\underline{\mathsf{F}}_d = \upsilon[&\epsilon_{\alpha\beta}\partial_\alpha\underline{\mathsf{K}}_d\partial_\beta\mathsf{F}_d - \hbar\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma\underline{\mathsf{K}}_d\partial_\beta\underline{\mathsf{F}}_d\partial_\alpha\mathsf{A}_\lambda \\ &- \hbar\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\alpha\underline{\mathsf{K}}_d\partial_\gamma\underline{\mathsf{F}}_d\partial_\beta\mathsf{A}_\lambda + O(\hbar^2)].\end{aligned} \tag{52}$$

Now relabeling dummy indices in the last term above according to $\alpha \to \gamma$, $\beta \to \lambda$, $\gamma \to \beta$, $\lambda \to \alpha$, one obtains Eq. (13) with $\partial_t\mathsf{q}_\alpha$ as in Eq. (14).

F. Gauge-invariant energy and current density

Now we proceed to derive diagonal and gauge-invariant expressions for the energy density (28) and energy current density (31).

1. Energy density

Let us begin with the energy density. Using properties of the trace and others recapped in Appendix A, we write Eq. (28) as

$$\begin{aligned}\mathcal{H} &= \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,(\mathsf{F}\star\mathsf{H}) \\ &= \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,(\mathsf{S}\star\mathsf{F}_d\star\mathsf{S}^{-1}),\end{aligned} \tag{53}$$

using Eqs. (32))–(34).

To simplify this, we use the identity, valid for any operator O,

$$\mathrm{Tr}\,(\mathsf{S}\star\mathsf{O}\star\mathsf{S}^{-1}) = \mathrm{Tr}\,\mathsf{O} + i\hbar\epsilon_{\alpha\beta}\partial_\beta\mathrm{Tr}\,(\Lambda_\alpha\mathsf{O}) + O(\hbar^2), \tag{54}$$

which is derived straightforwardly by the expansion of the star product. Taking $\mathsf{O} = \mathsf{F}_d$ and applying this to Eq. (53), we then obtain

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}(\mathrm{Tr}\mathsf{F}_d - \hbar\epsilon_{\alpha\beta}\partial_\beta\mathrm{Tr}[\mathsf{A}_\alpha\mathsf{F}_d]) + O(\hbar^2) \\ &= \frac{1}{2}\mathrm{Tr}\left[\left(1 + \frac{\hbar}{2}\epsilon_{\gamma\lambda}\Omega_{\gamma\lambda}\right)(\mathsf{F}_d + \hbar\epsilon_{\alpha\beta}\mathsf{A}_\beta\partial_\alpha\mathsf{F}_d)\right] + O(\hbar^2),\end{aligned} \tag{55}$$

where we defined (the band-diagonal quantity) $\Omega_{\alpha\beta}$ in Eq. (15). This expression is diagonalized and gauge invariant but the latter gauge invariance is not explicit. To make it so, we use the definition in Eq. (10) and Taylor expand in $\hbar$ *at fixed* q_α. This gives

$$\underline{\mathsf{O}}_d(q) = \mathsf{O}_d(q) + \hbar\epsilon_{\alpha\beta}\mathsf{A}_\beta\partial_\alpha\mathsf{O}_d(q) + O(\hbar^2). \tag{56}$$

Applying Eq. (56) with $\mathsf{O}_d = \mathsf{F}_d$ immediately gives Eqs. (16) and (17) in Sec. II. The fact that $\mathfrak{J}$, defined in Eq. (17), identifies with the Jacobian of the $q \mapsto \underline{q}$ transformation is shown in Appendix D.

2. Energy current

Now we turn to the energy current, written in Eq. (31). It contains two factors inside the trace, $\partial_\beta \mathsf{K}$, and $\mathsf{F}\star\mathsf{H}$, which we want to write in diagonal form. To express the former, we note the useful identity

$$\partial_\alpha(\mathsf{S}\star\mathsf{O}\star\mathsf{S}^{-1}) = \mathsf{S}\star(\partial_\alpha\mathsf{O} + [\Lambda_\alpha \overset{\star}{,} \mathsf{O}])\star\mathsf{S}^{-1}. \tag{57}$$

Applying Eq. (57) with $\mathsf{O} = \mathsf{K}_d = \mathsf{S}^{-1}\star\mathsf{K}\star\mathsf{S}$ gives

$$\partial_\alpha \mathsf{K} = \mathsf{S}\star\mathsf{Q}_\alpha\star\mathsf{S}^{-1}, \tag{58}$$

where we defined for convenience

$$\mathsf{Q}_\alpha \equiv \partial_\alpha \mathsf{K}_d + [\mathsf{S}^{-1}\star\partial_\alpha\mathsf{S} \overset{\star}{,} \mathsf{K}_d]. \tag{59}$$

The second multiplicative factor in the trace in Eq. (31) is, using Eqs. (33) and (34),

$$\mathsf{F}\star\mathsf{H} = \mathsf{S}\star\mathsf{F}_d\star\mathsf{S}^{-1}. \tag{60}$$

Using Eqs. (58) and (60), one can convert Eq. (31) to diagonal form using manipulations similar to those for the energy density, though with more involved algebra. The details are left to Appendix C. One obtains in this way

$$\mathcal{J}_\alpha = \mathcal{J}_\alpha^{(1)} + \mathcal{J}_\alpha^{(2)}, \tag{61}$$

in which each term is separately gauge invariant:

$$\begin{aligned}\mathcal{J}_\alpha^{(1)} = \frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Tr}\Big[&\Big(1+\frac{\hbar}{2}\epsilon_{\gamma\lambda}\Omega_{\gamma\lambda}\Big)\\ &\times(\partial_\beta(\mathsf{K}_d + \hbar\epsilon_{\rho\sigma}\mathsf{A}_\sigma\partial_\rho\mathsf{K}_d) + \hbar\epsilon_{\rho\sigma}\Omega_{\beta\rho}\partial_\sigma\mathsf{K}_d)\\ &\times(\mathsf{F}_d + \hbar\epsilon_{\xi\eta}\mathsf{A}_\eta\partial_\xi\mathsf{F}_d)\Big] + O(\upsilon\hbar^2),\end{aligned} \tag{62a}$$

$$\mathcal{J}_\alpha^{(2)} = \frac{\upsilon\hbar}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma\mathrm{Tr}[\mathsf{M}_{\beta\lambda}\mathsf{F}_d] + O(\upsilon\hbar^2), \tag{62b}$$

where $\mathsf{M}_{\alpha\beta}$ was defined in Eq. (19). Now we can again use Eq. (56) to identify gauge-invariant functions $\underline{\mathsf{K}}_d$ and $\underline{\mathsf{F}}_d$, and we directly thereby obtain Eq. (18) in Sec. II.

Let us discuss the physical meaning and relevance of $\mathcal{J}_\alpha^{(2)}$. This term satisfies the continuity equation *by definition*, as it is a phase-space "curl." Such terms appear to be ambiguous from the introduction of the heat current, which occurred in passing from Eq. (29) to (31). That ambiguity arises because the quantity which enters the continuity equation is only the divergence of the phase-space current, so that a shift $\mathcal{J}_\alpha \to \mathcal{J}_\alpha + \partial_\beta\mathcal{M}_{\alpha\beta}$ with antisymmetric $\mathcal{M}$ leaves the continuity equation unchanged.

The above derivation certainly provides an unambiguous result for $\mathcal{J}_\alpha^{(2)}$ given the definition in Eq. (31). However, we would like to ask whether the result in Eq. (62b) can be considered to resolve the intrinsic ambiguity of defining *the* magnetization current. To define the latter unambiguously, clearly some other conditions than the continuity equation must be imposed. Related conditions are discussed for the energy magnetization itself in Ref. [23]. For the energy magnetization current, we are not sure of the full set of conditions that should be imposed, but we do note that the above definition satisfies many reasonable ones:

(1) The quantity $\mathsf{M}_{\alpha\beta}$ itself is a gauge-invariant band-diagonal quantity which can be identified as an energy magnetization. Then, $\mathcal{J}_\alpha^{(2)}$ is naturally the gauge-invariant divergenceless current which can be built from it, and it indeed directly appears when extracting the divergence from the time derivative of the local energy density $\partial_t\mathcal{H}$.

(2) The magnetization contribution $\mathcal{J}_\alpha^{(2)}$ vanishes in equilibrium when the system is locally uniform. We may define equilibrium to be such that the energy density F_d depends only on the band energy K_d, i.e., $\mathsf{F}_d(q) = f[\mathsf{K}_d(q)]$, then $\partial_\gamma\mathsf{F}_d = \partial_\gamma\mathsf{K}_d f'[\mathsf{K}_d(q)]$. Then $\mathcal{J}_\alpha^{(2)}$ vanishes if $\partial_X\mathsf{K}_d = 0$ and $\partial_X\mathsf{S} = 0$. It does not, however, *a priori* vanish otherwise, even in equilibrium. A local nonvanishing current in equilibrium is in fact how the magnetization current has been defined in the literature [17].

(3) One might expect that, in equilibrium, even the local thermal current $J_{X_\mu} = \int_p(\mathcal{J}_{X_\mu}^{(1)} + \mathcal{J}_{X_\mu}^{(2)})$, vanishes at zero temperature. This would seem natural since the entropy vanishes at zero temperature. We are able to show that this is the case in the special but important case of a (X, p)-separable problem such as we describe below and in Appendix F.

(4) As noted in Refs. [2,23], the momentum integral defining J_{X_μ} is rid of a UV divergence when evaluated in a continuum theory, in the (X, p)-separable case. This is not the case for the integrands $\mathcal{J}_{X_\mu}^{(1)}$ and $\mathcal{J}_{X_\mu}^{(2)}$ considered separately. Thus, Eq. (62b) is "physical" in the sense that it provides a compensating term to an otherwise unphysical divergence in Eq. (62a).

These properties, satisfied by $\mathcal{J}_\alpha^{(2)}$, are necessary (though *a priori* not sufficient) conditions for defining the magnetization current: they partially, if not fully in some cases, lift the ambiguity.

IV. APPLICATION: THERMAL HALL CURRENT IN A FINITE SYSTEM OF CHIRAL BOSONS

We now show how our QKE formalism can be applied to any general system with a boundary but homogeneous away from the boundary, and further specialize to a system inhomogeneous linear elasticity theory with a phonon Hall viscosity term.

A. Uniform finite system with a boundary: Formal result for thermal conductivity

One natural application of our formalism is to a finite system, in which the presence of a boundary is included via variations of the Hamiltonian in space. Specifically, we assume that the (2D) system is constrained within the interval $I_x = \{x \in D[-\frac{L}{2}, +\frac{L}{2}]\}$, and we consider a Hamiltonian of the form $\mathsf{H}(X, p) = g(X)\mathsf{H}^{\mathrm{h}}(p)$, with g a c-number function equal to 1 in the I_x interval and decaying over distances $O(L^0)$ to 0 outside, mimicking the presence or absence of physical degrees of freedom in these regions. This is just a specific instance of the more general problem of a (X, p)-separable theory, which we consider in detail in Appendix F.

There, we show that solving the separable problem amounts to solving the homogeneous one [i.e., $g(X) = 1$ everywhere], which we label with h superscripts:

$$(\mathsf{S}^{\mathrm{h}})^{-1}\mathsf{K}^{\mathrm{h}}\mathsf{S}^{\mathrm{h}} = \mathsf{K}_d^{\mathrm{h}}, \tag{63a}$$

$$(\mathsf{S}^{\mathrm{h}})^{\dagger}\mathsf{H}^{\mathrm{h}}\mathsf{S}^{\mathrm{h}} = 1, \tag{63b}$$

where the $\star$ product was replaced by the standard matrix product in this X-independent problem, and $\mathsf{K}^{\mathrm{h}} = \hbar\,\Gamma^{\mathrm{h}}\mathsf{H}^{\mathrm{h}}$. This is now a standard linear algebra problem, yielding in particular the homogeneous energies $\mathsf{K}^{\mathrm{h}}_d$ and Berry curvatures $\Omega^{\mathrm{h}}_{p_y p_x}$.

We assume translational invariance along y and consider a temperature gradient $\partial_x T$ along the x direction. The ("transport") thermal conductivity relates the latter to the net energy flux flowing in response, via Fourier's law $\boldsymbol{J}_{\mathrm{tot}} = -\boldsymbol{\kappa}^{\mathrm{tr}} \cdot \nabla T$, which is obtained by integrating the energy current along the x direction and taking the thermodynamic limit:

$$\kappa^{\mathrm{tr}}_{xy} = \frac{1}{\partial_x T} \lim_{L\to\infty} \frac{1}{L} \int_{-\infty}^{+\infty} dx \int_p \mathcal{J}_{X_y}(X, p). \tag{64}$$

Note that we denote here $(X_x, X_y) \equiv (x, y)$, and we dropped the $O(\hbar^2)$ correction since $\kappa^{\mathrm{tr}}_{xy}$ is by definition a linear response quantity. The explicit form of both currents from the decomposition Eq. (61) is derived in Appendix F 3.

Because $\mathcal{J}^{(2)}_{X_y}$ is a total real-space derivative, it does not contribute to the transport current. Similarly, the zeroth-order term $g\partial_{p_y}\mathsf{K}^{\mathrm{h}}_d$ in $\mathcal{J}^{(1)}_{X_y}$ drops out as it is a total momentum-space derivative. The only contribution which does not vanish after performing the momentum integral is

$$\mathcal{J}^{(\mathrm{eff})}_{X_y} \equiv -\frac{\upsilon\hbar}{2}\psi_x \mathrm{Tr}\big[\Omega^{\mathrm{h}}_{p_y p_x} g \mathsf{K}^{\mathrm{h}}_d f\big(g\mathsf{K}^{\mathrm{h}}_d, T\big)\big] + O(\upsilon\hbar^2), \tag{65}$$

where $\psi_x \equiv \partial_x g/g$ and $f(\varepsilon, T) = \varepsilon[n_{\mathrm{B}}(\varepsilon, T) + \frac{1}{2}]$, with $n_{\mathrm{B}}(\varepsilon, T) = 1/(e^{\varepsilon/(k_{\mathrm{B}}T)} - 1)$ the Bose function (k_{B} is Boltzmann's constant), is the equilibrium energy distribution evaluated at ε of a system of bosons at temperature T. Further decomposing explicitly $\mathrm{Tr} = \mathrm{Tr}_+ + \mathrm{Tr}_-$ where Tr_+ (respectively Tr_-) sums over positive (respectively negative) energy bands (see Appendix B 3), and the fact that $\Omega^{\mathrm{h}}_{p_y p_x}, \mathsf{K}^{\mathrm{h}}_d$ are odd upon spectrum reflection while f is even [note that $f(-\varepsilon, T) = f(\varepsilon, T)$ and see Appendix B 3] allows one to replace $\mathrm{Tr} \mapsto 2\,\mathrm{Tr}_+$ in Eq. (65).

Importantly, the gradient of temperature is not introduced "artificially" in the form of a gravitational field like in Luttinger's trick [27] (in particular it is unrelated to ψ_x), but simply arises from the fact that the function f depends implicitly on x since the Bose function must be computed at the *local equilibrium* temperature $T(x)$ [17].

Equations (64) and (65) are our solution to the problem, which is readily amenable to numerical evaluation. It is also possible to reconcile it with existing literature with just one extra step. Performing the spatial integration with a few simple tricks to turn it into an integral over energies ε (cf. Appendix G), we arrive at the result

$$\kappa^{\mathrm{tr}}_{xy} = -\frac{\upsilon\hbar}{T}\mathrm{Tr}_+ \int_p \Omega^{\mathrm{h}}_{p_x p_y} \int d\varepsilon\, \Theta\big(\varepsilon - \mathsf{K}^{\mathrm{h}}_d\big)\, \varepsilon^2 \partial_\varepsilon n_{\mathrm{B}}(\varepsilon, T), \tag{66}$$

where Θ is the step function and T the mean temperature within I_x. This formula (66) agrees with prior results [2,3] which were obtained using the Kubo formalism.

B. Local energy current in a chiral phonon system

We now consider a concrete example. The model is that of elasticity with a time-reversal-breaking term, described by the (Euclidian) Lagrangian

$$\mathcal{L}_{\mathrm{ph}} = \frac{\rho}{2}(\partial_\tau \boldsymbol{u})^2 + \frac{1}{2} c_{ij\mu\nu}\partial_\mu u^i \partial_\nu u^j + i\eta_{ij\mu\nu}\partial_\tau u^i \partial^2_{\mu\nu} u^j, \tag{67}$$

where $\boldsymbol{u} = (u^i)_i$ is the displacement field ($i = x, y, z$), ρ is the mass density, $c_{ij\mu\nu}$ is the bulk modulus ($j, \mu, \nu = x, y, z$), and $\eta_{ij\mu\nu}$ is the lattice (or phonon) Hall viscosity. To remove ambiguity we take $c_{ij\nu\mu} = c_{ij\mu\nu} = c_{ji\mu\nu}$ and $\eta_{ij\nu\mu} = \eta_{ij\mu\nu} = -\eta_{ji\mu\nu}$.

The Hall viscosity term breaks time-reversal symmetry explicitly, and accounts for the chiral nature of bosonic excitations in the system (chiral phonons). The physical origin of this term has been discussed elsewhere in various contexts such as quantum Hall systems [28], magnetic insulators [29], ionic crystals [30], etc. Here we simply use Eq. (67) as our starting point, as an effective theory for chiral phonons.

The full inhomogeneous problem, where all parameters $\rho, c_{ij\mu\nu}, \eta_{ij\mu\nu}$ are arbitrary functions of position, can be recast into the form of Eqs. (1) and (2) and can be solved explicitly following the procedure of Secs. II and III. We leave the details of calculations and explicit expressions to Appendix H.

Here in the main text, we restrict ourselves to the spatial dependence described in the previous subsection. The details are left to Appendix F, but it suffices to then study the homogeneous problem, with position-independent parameters and matrices $\mathsf{H}^{\mathrm{h}}, \Gamma^{\mathrm{h}}$ [$\Phi = (u, \Pi/\rho)$, with Π_i the canonically conjugate momentum to u_i, see Eq. (H2)] which in the case of the Lagrangian (67) take the simple forms

$$\Gamma^{\mathrm{h}} = \frac{i}{\hbar\rho}\begin{bmatrix} 0 & 1 \\ -1 & -\frac{2\eta_{\mu\nu}}{\hbar^2\rho} p_\mu p_\nu \end{bmatrix}, \quad \mathsf{H}^{\mathrm{h}} = \begin{bmatrix} \frac{1}{\hbar^2} c_{\mu\nu} p_\mu p_\nu & 0 \\ 0 & \rho \end{bmatrix}, \tag{68}$$

where $(c_{\mu\nu})_{ij} \equiv c_{ij\mu\nu}$, $(\eta_{\mu\nu})_{ij} \equiv \eta_{ij\mu\nu}$, and so each block in Eq. (68) is a 3×3 matrix, and $\mathsf{K}^{\mathrm{h}} = \hbar\,\Gamma^{\mathrm{h}}\mathsf{H}^{\mathrm{h}}$. For concreteness, we choose the following expressions for the Hall viscosity and bulk modulus:

$$\begin{aligned} \eta_{ij\mu\nu} &= \epsilon^{ijz}\delta_{\mu\nu}(\eta_1 \overline{\delta}_{\nu z} + \eta_2 \delta_{\nu z}), \\ c_{ij\mu\nu} &= c_1 \delta_{ij}\delta_{\mu\nu} + \delta_{i\mu}\delta_{j\nu}(c_2 \overline{\delta}_{iz}\overline{\delta}_{jz} + c_3 \delta_{iz}\delta_{jz}) \\ &\quad + \delta_{\mu\nu}(c_4 \overline{\delta}_{\nu z}\delta_{iz}\delta_{jz} + c_5 \delta_{\nu z}\overline{\delta}_{iz}\overline{\delta}_{jz}), \end{aligned} \tag{69}$$

where ϵ^{ijl} is the three-dimensional (3D) Levi-Civita tensor, $\overline{\delta}_{ab} \equiv 1 - \delta_{ab}$, and $\eta_1, \eta_2, c_a, a = 1..5$, are real parameters. This makes $\mathcal{L}_{\mathrm{ph}}$ the most general possible Lagrangian preserving $O(2)$ symmetry around the $\hat{z}$ axis, except that, for simplicity, we do *not* include terms which couple the u^z to the (u^x, u^y) components, even if they preserve $O(2)$ symmetry. This allows us to reduce the problem to be effectively a 4×4 problem in matrix space.

We leave details of the solution of the diagonalization problem to the Appendix F, and show in Fig. 1 the position-resolved momentum-integrated energy currents for a given profile of g and the parameter values provided in Table I.

As is evident in Fig. 1, the current $\mathcal{J}^{(2)}_{X_y}(x)$ is not zero in the bulk (the exaggerated temperature gradient we used is made to render this visible), but its integral over a slice of material vanishes exactly since it is a magnetization current. Meanwhile, the current $\mathcal{J}^{(1)}_{X_y}(x)$ is exactly zero in the bulk, but

the counterpropagating edge densities yield a finite integral, i.e., a nonzero Hall conductivity.

V. CONCLUSION

In this paper, we presented a systematic derivation of the quantum kinetic equation in the full phase space for any quadratic Hamiltonian of bosonic fields. Translation invariance is not assumed, and boundaries may be included. The treatment provides a derivation of how the quasiclassical distribution function descends from the full quantum density matrix, and how the evolution of these quantities is related to single-particle characteristics such as the spectrum, Berry curvature, and energy magnetization. To obtain these results, we follow the exact statistical formulation of quantum dynamics using the Moyal product in phase space, and thereby obtain exact equations that can be expanded systematically order by order in a semiclassical parameter such as the smallness of spatial gradients.

From this procedure, in the leading approximation we recover the full Boltzmann equation and provide a self-contained and exact derivation of the intrinsic thermal Hall effect of bosons. Moreover, the procedure allows a spatially resolved examination of the associated thermal currents, treating boundaries and the spatial profile they generate. For the intrinsic thermal Hall effect, this method clarifies the separation of transport and "magnetization" currents transparently, and without the need to introduce any artificial gravitational field which is required in a Kubo formulation following Luttinger [27].

Aside from pedagogical value, we believe this formalism will be useful in simplifying and clarifying future calculations. It allows a direct application to intrinsically inhomogeneous systems such as skyrmionic textures in chiral magnets, which will be addressed in a forthcoming publication. It also offers a natural extension to higher orders in the semiclassical expansion, in which deviations from Boltzmann transport become evident. This is important, for example, in nonlinear transport effects (i.e., beyond linear order in applied electric fields and thermal gradients) since such effects are indeed higher order in gradients. Existing theory of such effects reveals a role for geometric properties of bands such as Berry curvature and quantum metric, but because it is based on Boltzmann transport which is only formally exact up to *first* order in phase-space gradients, the theory is incomplete and should be revisited.

A natural important future direction is the inclusion of scattering and interactions into the present formalism. Scattering can be straightforwardly added in an *ad hoc* manner, through a collision integral, such as that provided in Refs. [20,21]. It would also be interesting to derive it rigorously from the Keldysh formalism, which is a natural extension with some precedents [11,31–33]. Another relevant extension is to time-dependent Hamiltonians, such as would occur in a sliding texture like a driven spin density wave or skyrmion lattice, which leads to Berry phase effects in space-time. Moreover, the same methods can be applied to fermions, as in Ref. [15] for normal electrons, and extended to Bogoliubov–de Gennes quasiparticles for the superconducting case. This potentially provides a new way to address phenomena in topological superconductors.

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APPENDIX A: PROPERTIES OF THE MOYAL PRODUCT

The Moyal, or star, product is widely used in the phase-space formulation of quantum mechanics, and is particularly convenient for semiclassical analysis. It is defined in the main text [Eq. (7)] and can be applied to functions, or, componentwise, to tensor multiplication. We review some key useful properties of the Moyal product here.

Even for scalar functions, it is noncommutative, reflecting the noncommutativity of operators in quantum mechanics. It is, however, associative,

$$\mathsf{O}_1 \star \mathsf{O}_2 \star \mathsf{O}_3 = (\mathsf{O}_1 \star \mathsf{O}_2) \star \mathsf{O}_3 = \mathsf{O}_1 \star (\mathsf{O}_2 \star \mathsf{O}_3). \quad \text{(A1)}$$

Under Hermitian conjugation, we have

$$(\mathsf{O}_1 \star \mathsf{O}_2)^\dagger = \mathsf{O}_2^\dagger \star \mathsf{O}_1^\dagger, \quad \text{(A2)}$$

where the Hermitian conjugate † is defined in the usual matrix sense, and the phase-space argument is unchanged.

The star product is not cyclic in the trace, but the combination of the trace and phase space ($q \in \mathbb{R}^{2d}$) integral is cyclic if $\mathrm{Tr}[\mathsf{O}_1 \star \mathsf{O}_2]$ is integrable:

$$\int_q \mathrm{Tr}[\mathsf{O}_1 \star \mathsf{O}_2] = \int_q \mathrm{Tr}[\mathsf{O}_2 \star \mathsf{O}_1]. \quad \text{(A3)}$$

This implies that the phase-space integral of a Moyal bracket vanishes

$$\int_q \mathrm{Tr}[\mathsf{O}_1 \overset{\star}{,} \mathsf{O}_2] = 0. \quad \text{(A4)}$$

If the functions $\mathsf{O}_{1,2}$ are differentiable, the chain rule applies, i.e., for $\alpha = 1, \ldots, 2d$,

$$\partial_{q_\alpha}(\mathsf{O}_1 \star \mathsf{O}_2) = \partial_{q_\alpha}\mathsf{O}_1 \star \mathsf{O}_2 + \mathsf{O}_1 \star \partial_{q_\alpha}\mathsf{O}_2. \quad \text{(A5)}$$

A matrix function O is star invertible if there exists O^{-1} such that

$$\mathsf{O} \star \mathsf{O}^{-1} = \mathsf{O}^{-1} \star \mathsf{O} = 1. \quad \text{(A6)}$$

Note that, as for ordinary matrix multiplication, the left and right star inverses are identical. This follows from viewing the convolution as a matrix product in the combinated coordinate-index space, and then using the fact that the Wigner transform of a convolution is the star product of Wigner transforms.

If O_1 and O_2 are star invertible,

$$(\mathsf{O}_1 \star \mathsf{O}_2)^{-1} = \mathsf{O}_2^{-1} \star \mathsf{O}_1^{-1}. \quad \text{(A7)}$$

Up to first order in $\hbar$, we have

$$\mathsf{O}_1 \star \mathsf{O}_2 = \mathsf{O}_1\mathsf{O}_2 + i\hbar\frac{\epsilon_{\alpha\beta}}{2}\partial_\alpha \mathsf{O}_1 \partial_\beta \mathsf{O}_2 + O(\hbar^2). \tag{A8}$$

The zeroth-order term is just standard matrix multiplication while the first-order one is related to the Poisson bracket when $\mathsf{O}_{1,2}$ are scalar functions: $\epsilon_{\alpha\beta}\partial_\alpha \mathsf{O}_1 \partial_\beta \mathsf{O}_2 = \partial_{X_\mu}\mathsf{O}_1 \partial_{p_\mu}\mathsf{O}_2 - \partial_{p_\mu}\mathsf{O}_1\partial_{X_\mu}\mathsf{O}_2 \equiv \{\mathsf{O}_1, \mathsf{O}_2\}_{\text{p.b.}}$. Equation (A8) can be rewritten, up to first order,

$$\mathsf{O}_1\mathsf{O}_2 = \mathsf{O}_1 \star \mathsf{O}_2 - i\frac{\hbar}{2}\epsilon_{\alpha\beta}\partial_\alpha \mathsf{O}_1 \partial_\beta \mathsf{O}_2 + O(\hbar^2). \tag{A9}$$

From the above, we can find the ordinary product of a matrix with its star inverse

$$\mathsf{O}^{-1}\mathsf{O} = 1 - i\frac{\hbar}{2}\epsilon_{\alpha\beta}\partial_\alpha \mathsf{O}^{-1}\partial_\beta \mathsf{O} + O(\hbar^2), \tag{A10}$$

which notably is not the identity.

APPENDIX B: DIAGONALIZATION OF K

1. Formal relation to a Hermitian problem

Here we show that the star diagonalization of K is related to a Hermitian star-eigenvalue problem, provided the original physical problem is well defined. In particular, this requires that H is positive definite. The latter implies that there exists a positive-definite matrix $\mathsf{H}^{1/2}$ such that

$$\mathsf{H}^{1/2} \star \mathsf{H}^{1/2} = \mathsf{H}. \tag{B1}$$

We define the matrix $\check{\mathsf{K}}$:

$$\check{\mathsf{K}} \equiv \hbar\, \mathsf{H}^{1/2} \star \Gamma \star \mathsf{H}^{1/2} = \check{\mathsf{K}}^\dagger. \tag{B2}$$

Since $\check{\mathsf{K}}$ is Hermitian, one can find a star-unitary matrix U, i.e.,

$$\mathsf{U}^\dagger \star \mathsf{U} = \mathsf{U} \star \mathsf{U}^\dagger = 1, \tag{B3}$$

which star diagonalizes $\check{\mathsf{K}}$ according to

$$\mathsf{U}^\dagger \star \check{\mathsf{K}} \star \mathsf{U} = \check{\mathsf{K}}_d, \tag{B4}$$

where $\check{\mathsf{K}}_d$ is a diagonal matrix with real entries (the "star eigenvalues").

Plugging in Eq. (B2) into (B4), we obtain

$$\begin{aligned}\check{\mathsf{K}}_d &= \hbar\, \mathsf{U}^\dagger \star \mathsf{H}^{1/2} \star \Gamma \star \mathsf{H}^{1/2} \star \mathsf{U} \\ &= \hbar\, (\mathsf{U}^\dagger \star \mathsf{H}^{1/2}) \star \Gamma \star \mathsf{H} \star (\mathsf{H}^{-1/2} \star \mathsf{U}).\end{aligned} \tag{B5}$$

Using $\mathsf{K} = \hbar\,\Gamma \star \mathsf{H}$ we see that this has the form of Eq. (32) if we identify

$$\check{\mathsf{K}}_d = \mathsf{K}_d, \tag{B6}$$

which, in particular, means that K_d is real, and

$$\mathsf{S} = \mathsf{H}^{-1/2} \star \mathsf{U}, \quad \mathsf{S}^{-1} = \mathsf{U}^\dagger \star \mathsf{H}^{1/2}, \tag{B7}$$

up to a gauge choice, as discussed in the main text.

From Eq. (B7) we can immediately show Eq. (33) by direct calculation. Equations (33) and (B7) can be considered a choice of normalization of S. Independent of that normalization, we note the identity

$$\mathsf{K}^\dagger = \mathsf{H} \star \mathsf{K} \star \mathsf{H}^{-1}. \tag{B8}$$

2. Diagonalization to $O(\hbar)$

Now we show how to convert the formidable star-diagonalization problem defined by Eqs. (32) and (33) into conventional linear algebra in the semiclassical expansion. We aim here to obtain K_d and S up to first order in $\hbar$ (S up to gauge freedom, of course).

To do so, we specify the expansions of K and S. The former is unambiguously determined by expanding the Moyal product, regarding H and Γ as $\hbar$ independent. We have

$$\mathsf{K} = \hbar\left(\Gamma\mathsf{H} + i\hbar\frac{\epsilon_{\alpha\beta}}{2}\partial_\alpha\Gamma\partial_\beta\mathsf{H}\right) + O(\hbar^2), \tag{B9}$$

and so

$$\mathsf{K} = \mathsf{K}_0 + \hbar\mathsf{K}_1 + O(\hbar^2), \tag{B10}$$

with

$$\begin{aligned}\mathsf{K}_0 &= \hbar\,\Gamma\mathsf{H}, \\ \mathsf{K}_1 &= i\hbar\frac{\epsilon_{\alpha\beta}}{2}\partial_\alpha\Gamma\partial_\beta\mathsf{H}.\end{aligned} \tag{B11}$$

Next, we write the expansion of S according to the convenient form

$$\mathsf{S} = \mathsf{S}_0(1 + \hbar\tilde{\mathsf{S}}_1) + O(\hbar^2). \tag{B12}$$

Here the first two terms in the expansion are specified by S_0 and $\tilde{\mathsf{S}}_1$, both $O(\hbar^0)$ *a priori*, which are to be determined by enforcing the star-diagonalization conditions up to first order in $\hbar$. The H conjugate gives

$$\mathsf{S}^\dagger = (1 + \hbar\tilde{\mathsf{S}}_1^\dagger)\mathsf{S}_0^\dagger + O(\hbar^2). \tag{B13}$$

Inserting Eqs. (B12) and (B13) into (33), and equating terms at zeroth and first order in $\hbar$ gives the conditions

$$\mathsf{S}_0^\dagger\mathsf{H}\mathsf{S}_0 = 1, \tag{B14a}$$

$$\tilde{\mathsf{S}}_1^\dagger + \tilde{\mathsf{S}}_1 = -\frac{i\epsilon_{\alpha\beta}}{2}(\partial_\alpha\mathsf{S}_0^\dagger\mathsf{H}\partial_\beta\mathsf{S}_0 + \partial_\alpha\mathsf{S}_0^\dagger\partial_\beta\mathsf{H}\mathsf{S}_0 + \mathsf{S}_0^\dagger\partial_\alpha\mathsf{H}\partial_\beta\mathsf{S}_0). \tag{B14b}$$

We see that the Hermitian part of $\tilde{\mathsf{S}}_1$ is fixed by the second condition, once the zeroth-order term is known.

To proceed, we also need S^{-1}, which we obtain using Eqs. (33) and (B13) and expanding consistently to $O(\hbar)$:

$$\begin{aligned}\mathsf{S}^{-1} &= \mathsf{S}^\dagger \star \mathsf{H} \\ &= \left[1 + \hbar\left(\tilde{\mathsf{S}}_1^\dagger + \frac{i\epsilon_{\alpha\beta}}{2}\partial_\alpha\mathsf{S}_0^\dagger\partial_\beta\mathsf{H}\mathsf{S}_0\right)\right]\mathsf{S}_0^{-1} + O(\hbar^2)\end{aligned} \tag{B15}$$

(take care that S^{-1} is the star inverse not the ordinary matrix inverse). We also used the zeroth-order condition

$$\mathsf{S}_0^{-1} = \mathsf{S}_0^\dagger\mathsf{H}, \tag{B16}$$

which follows from Eq. (B14a) [here S_0^{-1} is the $O(\hbar^0)$ term in the expansion of S^{-1} as well as the inverse of S_0 for the *usual* matrix product]. Then one can use Eq. (B14b) to eliminate $\tilde{\mathsf{S}}_1^\dagger$

in favor of $\tilde{\mathsf{S}}_1$ within Eq. (B15), which yields

$$\begin{aligned}\mathsf{S}^{-1} &= \left[1-\hbar\left(\tilde{\mathsf{S}}_1+\frac{i\epsilon_{\alpha\beta}}{2}\partial_\alpha\mathsf{S}_0^{-1}\partial_\beta\mathsf{S}_0\right)\right]\mathsf{S}_0^{-1}+O(\hbar^2)\\ &= \left[1-\hbar\left(\tilde{\mathsf{S}}_1-\frac{i\epsilon_{\alpha\beta}}{2}\Lambda_\alpha\Lambda_\beta\right)\right]\mathsf{S}_0^{-1}+O(\hbar^2). \qquad \text{(B17)}\end{aligned}$$

Equation (B17) follows from the constraint on S from the normalization condition (33), but so far we have not used the actual *diagonalization* condition in Eq. (32). Using Eqs. (B12) and (B17) in (32) and collecting terms to $O(\hbar)$ we find

$$\begin{aligned}\mathsf{K}_d =& \mathsf{K}_{0,d}+\hbar\Big(\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0+[\mathsf{K}_{0,d},\tilde{\mathsf{S}}_1]\\ &-i\frac{\epsilon_{\alpha\beta}}{2}\{\Lambda_\alpha,\partial_\beta\mathsf{K}_{0,d}-\mathsf{K}_{0,d}\Lambda_\beta\}\Big)+O(\hbar^2), \qquad \text{(B18)}\end{aligned}$$

where

$$\mathsf{K}_{0,d}=\mathsf{S}_0^{-1}\mathsf{K}_0\mathsf{S}_0. \qquad \text{(B19)}$$

The $O(\hbar^0)$ conditions (B19) and (B14a), taken together, define a standard (but non-Hermitian) eigenvalue problem, where we demand that $\mathsf{K}_{0,d}$ is diagonal. By a classical limit, i.e., a regular matrix product version, of the argument in Appendix B 1, it is straightforward to show that a solution always exists and that $\mathsf{K}_{0,d}$ is not only diagonal but Hermitian, i.e., real diagonal.

To achieve the star diagonalization to $O(\hbar)$, we must require that the term in parentheses in Eq. (B18) be diagonal, i.e., that

$$\Big(\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0+[\mathsf{K}_{0,d},\tilde{\mathsf{S}}_1] -i\frac{\epsilon_{\alpha\beta}}{2}\{\Lambda_\alpha,\partial_\beta\mathsf{K}_{0,d}-\mathsf{K}_{0,d}\Lambda_\beta\}\Big)_{ab}=0,\quad a\neq b. \qquad \text{(B20)}$$

This is a condition on $\tilde{\mathsf{S}}_1$, and while an explicit form for the latter can be found, we will actually not need it for our calculations, so we suffice to say that Eq. (B20) is well defined and consistent with Eq. (B14b) and the Hermiticity of K_d.

Since K_d is diagonal and real, we can take the real part of the diagonal projection of Eq. (B18), which gives

$$\begin{aligned}\mathsf{K}_d =& \mathsf{K}_{0,d}+\hbar\,\mathrm{Re}\big[\big(\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0\big)^{(\mathrm{d})}\big]\\ &+\hbar\epsilon_{\alpha\beta}\big(\mathsf{A}_\alpha\partial_\beta\mathsf{K}_{0,d}-\tfrac{1}{4}\mathrm{Im}\{\Lambda_\beta,[\Lambda_\alpha,\mathsf{K}_{0,d}]\}^{(\mathrm{d})}\big)+O(\hbar^2). \qquad \text{(B21)}\end{aligned}$$

Transforming to the gauge-invariant function $\underline{\mathsf{K}}_d$ then immediately gives Eq. (20) in the main text, noting that $\mathrm{Im}\{\Lambda_\beta,[\Lambda_\alpha,\mathsf{K}_{0,d}]\}^{(\mathrm{d})}$ is antisymmetric in $\alpha\leftrightarrow\beta$, and so $\mathsf{M}_{\alpha\beta}=-\mathsf{M}_{\beta\alpha}$.

3. Symmetry of the spectrum

From their definitions, we know that $\Gamma(X,p)=-\Gamma(X,-p)^*$ and $\mathsf{H}(X,p)=\mathsf{H}(X,-p)^*$. This implies that the dynamical matrix obeys $\mathsf{K}(X,p)=-\mathsf{K}(X,-p)^*$. The latter induces relations among the star eigenvalues of K_d at reversed momenta. Taking the complex conjugate of Eq. (32) evaluated at momentum $-p$, and using the fact that K_d is real, we deduce that

$$\mathsf{K}_d(X,-p)=-[\mathsf{S}^{-1}(X,-p)]^*\star\mathsf{K}(X,p)\star[\mathsf{S}(X,-p)]^*. \qquad \text{(B22)}$$

Taking the complex conjugate of Eq. (33) at momentum $-p$ gives

$$\mathsf{S}(X,-p)^\top\star\mathsf{H}(X,p)\star\mathsf{S}(X,-p)^*=1. \qquad \text{(B23)}$$

These two equations are solved by using again Eqs. (32) and (33), by

$$\mathsf{S}(X,-p)=\mathsf{S}(X,p)^*,\quad \mathsf{K}_d(X,-p)=-\mathsf{K}_d(X,p), \qquad \text{(B24)}$$

up to a gauge transformation, and up to permutations of the diagonal entries of K_d. We see that the negative energies at momentum p appear, with the opposite sign, as positive energies at momentum $-p$, and vice versa. Thus, the modes of the spectrum of K come in positive (the "physical" ones) and negative energy pairs $(\varepsilon_a(X,p),\varepsilon_{\bar{a}}(X,p)\equiv-\varepsilon_a(X,-p))$. This defines the notation $\bar{a}$. Moreover, we can deduce that

$$\begin{aligned}[\Omega_{\alpha\beta}(X,-p)]_{\bar{a}\bar{a}} &= -[\Omega_{\alpha\beta}(X,p)]_{aa},\\ [\mathsf{M}_{\alpha\beta}(X,-p)]_{\bar{a}\bar{a}} &= +[\mathsf{M}_{\alpha\beta}(X,p)]_{aa}. \qquad \text{(B25)}\end{aligned}$$

We make use of these last two properties in Sec. IV A and Appendix F 4.

APPENDIX C: DIAGONALIZATION OF $\mathcal{J}_\alpha$

Here we provide details of the calculations leading from Eq. (31) to Eqs. (61), (62a), and (62b) in Sec. III F 2. Beginning with Eq. (31), we use Eqs. (58) and (60) to obtain

$$\mathcal{J}_\alpha=\frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re}\,\mathrm{Tr}[(\mathsf{S}\star\mathsf{Q}_\beta\star\mathsf{S}^{-1})(\mathsf{S}\star\mathsf{F}_d\star\mathsf{S}^{-1})]+O(\upsilon\hbar^2). \qquad \text{(C1)}$$

We now use Eq. (A9) with $\mathsf{O}_1=\mathsf{S}\star\mathsf{Q}_\beta\star\mathsf{S}^{-1}$ and $\mathsf{O}_2=\mathsf{S}\star\mathsf{F}_d\star\mathsf{S}^{-1}$ to "restore" a star product and as a second step keep only the zeroth-order term of the expansion of the star products in terms which are already of order $\hbar$ (which amounts to turning the star products into regular matrix products):

$$\mathcal{J}_\alpha=\frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re}\,\mathrm{Tr}[\mathsf{S}\star\mathsf{Q}_\beta\star\mathsf{F}_d\star\mathsf{S}^{-1}]+\frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\mathrm{Im}\,\mathrm{Tr}[\partial_\gamma(\mathsf{S}\mathsf{Q}_\beta\mathsf{S}^{-1})\partial_\lambda(\mathsf{S}\mathsf{F}_d\mathsf{S}^{-1})]+O(\upsilon\hbar^2). \qquad \text{(C2)}$$

Now we use Eq. (54) with $\mathsf{O} = \mathsf{Q}_\beta \star \mathsf{F}_d$, extract a ∂_γ derivative using $\epsilon_{\gamma\lambda}\partial^2_{\gamma\lambda} = 0$, expand the ∂_λ derivative using the chain rule, and use the cyclicity of the trace:

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re\,Tr}[\mathsf{Q}_\beta \star \mathsf{F}_d] - \frac{\upsilon}{2}\hbar\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\lambda \mathrm{Im\,Tr}[\mathsf{S}^{-1}\partial_\gamma \mathsf{S}(\mathsf{Q}_\beta \mathsf{F}_d)] \\ &\quad + \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma \mathrm{Im\,Tr}[\mathsf{S}\mathsf{Q}_\beta\mathsf{S}^{-1}(\partial_\lambda \mathsf{S}\mathsf{F}_d\mathsf{S}^{-1} + \mathsf{S}\partial_\lambda\mathsf{F}_d\mathsf{S}^{-1} + \mathsf{S}\mathsf{F}_d\partial_\lambda\mathsf{S}^{-1})] + O(\upsilon\hbar^2) \\ &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re\,Tr}[\mathsf{Q}_\beta \star \mathsf{F}_d] + \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma \mathrm{Im\,Tr}[\mathsf{Q}_\beta(\{\mathsf{S}^{-1}\partial_\lambda\mathsf{S}, \mathsf{F}_d\} + \partial_\lambda \mathsf{F}_d)] + O(\upsilon\hbar^2). \end{aligned} \tag{C3}$$

Then, plugging in the expression for Q_β in Eq. (59), we get

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}(\mathrm{Re\,Tr}[\partial_\beta \mathsf{K}_d \star \mathsf{F}_d] + \mathrm{Re\,Tr}[[\mathsf{S}^{-1} \star \partial_\beta \mathsf{S} \overset{\star}{,} \mathsf{K}_d] \star \mathsf{F}_d]) \\ &\quad + \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma \mathrm{Im\,Tr}[(\partial_\beta \mathsf{K}_d + [\mathsf{S}^{-1}\partial_\beta\mathsf{S}, \mathsf{K}_d])(\partial_\lambda \mathsf{F}_d + \{\mathsf{S}^{-1}\partial_\lambda\mathsf{S}, \mathsf{F}_d\})] + O(\upsilon\hbar^2). \end{aligned} \tag{C4}$$

In turn, expanding the star products yields

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}(\mathrm{Re\,Tr}[\partial_\beta \mathsf{K}_d \mathsf{F}_d] + \mathrm{Re\,Tr}[[\mathsf{S}^{-1}\partial_\beta\mathsf{S}, \mathsf{K}_d]\mathsf{F}_d]) \\ &\quad - \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\mathrm{Im\,Tr}\big[\partial^2_{\beta\gamma}\mathsf{K}_d\partial_\lambda\mathsf{F}_d + \partial_\gamma[\mathsf{S}^{-1}\partial_\beta\mathsf{S}, \mathsf{K}_d]\partial_\lambda\mathsf{F}_d + \{\partial_\gamma(\mathsf{S}^{-1}\partial_\beta\mathsf{S}), \partial_\lambda\mathsf{K}_d\}\mathsf{F}_d + \big[\partial_\gamma\mathsf{S}^{-1}\partial^2_{\beta\lambda}\mathsf{S}, \mathsf{K}_d\big]\mathsf{F}_d\big] \\ &\quad + \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma \mathrm{Im\,Tr}[(\partial_\beta \mathsf{K}_d + [\mathsf{S}^{-1}\partial_\beta\mathsf{S}, \mathsf{K}_d])(\partial_\lambda \mathsf{F}_d + \{\mathsf{S}^{-1}\partial_\lambda\mathsf{S}, \mathsf{F}_d\})] + O(\upsilon\hbar^2). \end{aligned} \tag{C5}$$

The contributions from $\partial^2_{\beta\gamma}\mathsf{K}_d\partial_\lambda\mathsf{F}_d$ and $\partial_\beta\mathsf{K}_d\partial_\lambda\mathsf{F}_d$ vanish because the latter are real, and those involving commutators, except for $[\mathsf{S}^{-1}\partial_\beta\mathsf{S}, \mathsf{K}_d]\{\mathsf{S}^{-1}\partial_\lambda\mathsf{S}, \mathsf{F}_d\}$, also vanish because the trace selects only the diagonal elements. One is then left with

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}\mathrm{Re\,Tr}[\partial_\beta \mathsf{K}_d \mathsf{F}_d] - \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\mathrm{Im\,Tr}[\{\partial_\gamma\Lambda_\beta, \partial_\lambda\mathsf{K}_d\}\mathsf{F}_d] \\ &\quad + \frac{\upsilon\hbar}{4}\epsilon_{\alpha\beta}\epsilon_{\gamma\lambda}\partial_\gamma \mathrm{Im\,Tr}[[\Lambda_\beta, \mathsf{K}_d]\{\Lambda_\lambda, \mathsf{F}_d\} + \partial_\beta\mathsf{K}_d\{\Lambda_\lambda, \mathsf{F}_d\}] + O(\upsilon\hbar^2). \end{aligned} \tag{C6}$$

Here we replaced $\mathsf{S}^{-1}\partial_\rho\mathsf{S} \to \Lambda_\rho$ which is correct to this order.

In almost all terms in Eq. (C6), the imaginary diagonal part of Λ_ρ, which is simply A_ρ, is selected by the trace against other diagonal real functions. The one term not of this form can be rewritten using the cyclicity of the trace as

$$\mathrm{Tr}([\Lambda_\alpha, \mathsf{K}_d]\{\Lambda_\beta, \mathsf{F}_d\}) = \mathrm{Tr}(\{\Lambda_\beta, [\Lambda_\alpha, \mathsf{K}_d]\}\mathsf{F}_d) = 2\,\mathrm{Tr}(\mathsf{M}_{\alpha\beta}\mathsf{F}_d), \tag{C7}$$

where $\mathsf{M}_{\alpha\beta}$ defined in Eq. (19). Consequently,

$$\begin{aligned}\mathcal{J}_\alpha &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}(\mathrm{Tr}[\partial_\beta \mathsf{K}_d \mathsf{F}_d] + \frac{\hbar}{2}\epsilon_{\gamma\lambda}(\mathrm{Tr}[-\{\partial_\gamma\mathsf{A}_\beta, \partial_\lambda\mathsf{K}_d\}\mathsf{F}_d + \partial_\gamma(\partial_\beta\mathsf{K}_d\{\mathsf{A}_\lambda, \mathsf{F}_d\})] + 2\partial_\gamma\mathrm{Tr}[\mathsf{M}_{\beta\lambda}\mathsf{F}_d])) + O(\upsilon\hbar^2) \\ &= \frac{\upsilon}{2}\epsilon_{\alpha\beta}(\mathrm{Tr}[\partial_\beta \mathsf{K}_d \mathsf{F}_d] + \hbar\epsilon_{\gamma\lambda}(\mathrm{Tr}[\partial_\gamma(\mathsf{A}_\lambda\partial_\beta\mathsf{K}_d - \mathsf{A}_\beta\partial_\lambda\mathsf{K}_d)\mathsf{F}_d + \mathsf{A}_\lambda\partial_\beta\mathsf{K}_d\partial_\gamma\mathsf{F}_d] + \partial_\gamma\mathrm{Tr}[\mathsf{M}_{\beta\lambda}\mathsf{F}_d])) + O(\upsilon\hbar^2). \end{aligned} \tag{C8}$$

From here, it is straightforward to separate the term containing $\mathsf{M}_{\beta\lambda}$ from the others, and with some rearrangement to first order in $\hbar$, obtain Eqs. (61), (62a), and (62b) in Sec. III F 2.

APPENDIX D: JACOBIAN

When expressed in gauge-invariant band-diagonal form, physical quantities like the energy and current density display explicit dependence on the "Jacobian" factor defined in Eq. (17). Here we show that this can indeed be regarded as a Jacobian, i.e., a determinant defining the measure for a change of variables in phase space.

Consider the change of variables in Eq. (11) from $\underline{q}$ to q, which induces the change in measure

$$d^D\underline{q} = d^D q \left|\det_D\left[\left(\frac{\partial \underline{q}_\alpha}{\partial q_\beta}\right)_{\alpha\beta}\right]\right|. \tag{D1}$$

Here the measure is defined in phase space, with total dimension $D = 2d$ twice the spatial dimension, and the subscript D on the determinant is meant to indicate that the determinant is in this space. One should avoid confusion with band space, which plays a trivial role here because the change of variables is diagonal [since A in Eq. (11) is diagonal]. The determinant can be regarded as a scalar for a given band, or a diagonal matrix in band space, evaluated separately for each band.

The Jacobian of the transformation is just this determinant. Using the explicit form of Eq. (11),

$$\begin{aligned}\mathfrak{J} &= \det{}_D\left[\left(\frac{\partial q_\alpha}{\partial q_\beta}\right)_{\alpha\beta}\right] \\ &= \det{}_D[(\delta_{\alpha\beta} + \hbar\epsilon_{\alpha\gamma}\partial_\beta \mathsf{A}_\gamma)_{\alpha\beta}] + O(\hbar^2). \end{aligned} \tag{D2}$$

Note that we have dropped the absolute value of the determinant because the Jacobian is perturbatively close to 1 in the semiclassical expansion and so has definite sign. We have, with Tr_D the trace in phase space,

$$\begin{aligned}\mathfrak{J} &= e^{\mathrm{Tr}_D\{\ln[(\delta_{\alpha\beta}+\hbar\epsilon_{\alpha\gamma}\partial_\beta \mathsf{A}_\gamma)_{\alpha\beta}]\}} + O(\hbar^2) \\ &= e^{\mathrm{Tr}_D[(\hbar\epsilon_{\alpha\gamma}\partial_\beta \mathsf{A}_\gamma)_{\alpha\beta}]} + O(\hbar^2) \\ &= 1 + \mathrm{Tr}_D[(\hbar\epsilon_{\alpha\gamma}\partial_\beta \mathsf{A}_\gamma)_{\alpha\beta}] + O(\hbar^2) \\ &= 1 + \hbar\epsilon_{\alpha\beta}\partial_\alpha \mathsf{A}_\beta + O(\hbar^2). \end{aligned} \tag{D3}$$

This is precisely the definition in Eq. (17).

APPENDIX E: EXPLICIT SEPARATION INTO POSITION AND MOMENTUM COORDINATES

In the main text, we presented the quasiparticle equations of motion and current density in compact forms combining positions and momenta into a single phase-space coordinate. To connect to more familiar forms, we expand these explicitly here.

Writing $q_{X_\mu} = X_\mu$ and $q_{p_\mu} = p_\mu$, we obtain from Eq. (14) the two analogs of Hamilton's equations

$$\partial_t \mathsf{X}_\mu = \partial_{p_\mu}\underline{\mathsf{K}}_d - \hbar\big(\Omega_{p_\mu p_\nu}\partial_{X_\nu}\underline{\mathsf{K}}_d - \Omega_{p_\mu X_\nu}\partial_{p_\nu}\underline{\mathsf{K}}_d\big) + O(\hbar^2), \tag{E1a}$$

$$\partial_t \mathsf{p}_\mu = -\partial_{X_\mu}\underline{\mathsf{K}}_d + \hbar\big(\Omega_{X_\mu p_\nu}\partial_{X_\nu}\underline{\mathsf{K}}_d - \Omega_{X_\mu X_\nu}\partial_{p_\nu}\underline{\mathsf{K}}_d\big) + O(\hbar^2). \tag{E1b}$$

This defines a velocity $\mathsf{v}_\mu \equiv \partial_t \mathsf{X}_\mu$ and a force $\mathsf{f}_\mu \equiv \partial_t \mathsf{p}_\mu$. We can similarly inspect the real-space and momentum-space components of $\mathcal{J}_\alpha$. Taking $\alpha = X_\mu$, we obtain the quantity describing the flow of energy in the X_μ direction. This is just the momentum-resolved energy current. We have

$$\begin{aligned}\mathcal{J}_{X_\mu} &= \frac{\upsilon}{2}\mathrm{Tr}\big\{\mathfrak{J}\,\partial_t \mathsf{X}_\mu\,\underline{\mathsf{F}}_d + \hbar\big[\partial_{X_\nu}\big(\mathsf{M}_{p_\mu p_\nu}\underline{\mathsf{F}}_d\big) - \partial_{p_\nu}\big(\mathsf{M}_{p_\mu X_\nu}\underline{\mathsf{F}}_d\big)\big]\big\} + O(\upsilon\hbar^2) \\ &= \frac{\upsilon}{2}\mathrm{Tr}\big[\mathfrak{J}\,\partial_t \mathsf{X}_\mu\,\underline{\mathsf{F}}_d + \hbar\epsilon^{\mu\nu\lambda}\big(\partial_{X_\nu}\mathfrak{M}^\lambda_{pp} - \partial_{p_\nu}\mathfrak{M}^\lambda_{pX}\big)\big] + O(\upsilon\hbar^2), \end{aligned} \tag{E2}$$

where $\epsilon^{\mu\nu\lambda}$ (note here we use superscripts and not subscripts for the indices) is the usual 3D Levi-Civita tensor, and

$$\mathfrak{M}^\lambda_{pp} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\mathsf{M}_{p_\mu p_\nu}\underline{\mathsf{F}}_d, \quad \mathfrak{M}^\lambda_{pX} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\mathsf{M}_{p_\mu X_\nu}\underline{\mathsf{F}}_d, \tag{E3}$$

because $\mathsf{M}_{\alpha\beta} = -\mathsf{M}_{\beta\alpha}$. The total energy current at position X_μ is obtained by integrating $\mathcal{J}_{X_\mu}(X_\mu, p_\mu)$ over p. Then clearly the $\partial_{p_\nu}\mathfrak{M}^\lambda_{pX}$ term drops. Moreover, if one computes the flux of $\mathcal{J}_{X_\mu}$ through an open 2D surface at whose boundary $\mathfrak{M}^\lambda_{pp}$ vanishes, the $\partial_{X_\nu}\mathfrak{M}^\lambda_{pp}$ term drops.

Taking now $\alpha = p_\mu$ describes the flow of energy in momentum space. This is something like an "energy force" density. It is

$$\begin{aligned}\mathcal{J}_{p_\mu} &= \frac{\upsilon}{2}\mathrm{Tr}\big\{\mathfrak{J}\,\partial_t \mathsf{p}_\mu\,\underline{\mathsf{F}}_d - \hbar\big[\partial_{X_\nu}\big(\mathsf{M}_{X_\mu p_\nu}\underline{\mathsf{F}}_d\big) - \partial_{p_\nu}\big(\mathsf{M}_{X_\mu X_\nu}\underline{\mathsf{F}}_d\big)\big]\big\} + O(\upsilon\hbar^2) \\ &= \frac{\upsilon}{2}\mathrm{Tr}\big[\mathfrak{J}\,\partial_t \mathsf{p}_\mu\,\underline{\mathsf{F}}_d - \hbar\epsilon^{\mu\nu\lambda}\big(\partial_{X_\nu}\mathfrak{M}^\lambda_{Xp} - \partial_{p_\nu}\mathfrak{M}^\lambda_{XX}\big)\big] + O(\upsilon\hbar^2), \end{aligned} \tag{E4}$$

where

$$\mathfrak{M}^\lambda_{Xp} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\mathsf{M}_{X_\mu p_\nu}\underline{\mathsf{F}}_d, \quad \mathfrak{M}^\lambda_{XX} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\mathsf{M}_{X_\mu X_\nu}\underline{\mathsf{F}}_d. \tag{E5}$$

APPENDIX F: THE CASE OF A SEPARABLE POSITION AND MOMENTUM DEPENDENCE

Here we specialize our theory to the case where the dependence on X and p of H is separable, i.e., we can write $\mathsf{H}(X, p) = g(X)\mathsf{H}^{\mathrm{h}}(p)$, where g is a c-number real function, and the superscript h stands for "homogeneous," and we additionally require $\Gamma(X, p) = \Gamma(p) = \Gamma^{\mathrm{h}}(p)$. In other words, all the spatial dependence is encoded in the function $g(X)$.

1. Solving the inhomogeneous problem

In this section, we show how the solution to the inhomogeneous problem [Eqs. (32) and (33)] can be deduced from that of the homogeneous one [Eqs. (63a) and (63b)] in the main text.

This can be worked out perturbatively in $\hbar$. To zeroth order, comparing Eq. (63b) to (33) yields immediately

$$\mathsf{S}_0 = \mathsf{S}^{\mathrm{h}}/\sqrt{g}, \tag{F1}$$

hence

$$\Lambda_\alpha = -\frac{\partial_\alpha g}{2g} + (\mathsf{S}^{\mathrm{h}})^{-1}\partial_\alpha \mathsf{S}^{\mathrm{h}} + O(\hbar). \tag{F2}$$

Explicitly in terms of position and momentum components, and up to corrections $O(\hbar)$, this reads as $\Lambda_{p_\mu} = \Lambda^{\mathrm{h}}_{p_\mu}$ so that $\mathsf{A}_{p_\mu} = \mathsf{A}^{\mathrm{h}}_{p_\mu}$, and $\Lambda_{X_\mu} = -\frac{1}{2}\psi_\mu$ and in turn $\mathsf{A}_{X_\mu} = 0$. Here we defined $\psi_\mu = \partial_{X_\mu} g/g$ which is sometimes called a "gravitational field" [2,3,17,27].

The decomposition (B11), $\mathsf{K} = \mathsf{K}_0 + \hbar\mathsf{K}_1 + O(\hbar^2)$, becomes here

$$\begin{aligned}\mathsf{K}_0 &= g\mathsf{K}^{\mathrm{h}},\\ \mathsf{K}_1 &= \hbar\,\frac{i}{2}\partial_{X_\mu} g \partial_{p_\mu}\Gamma^{\mathrm{h}}\mathsf{H}^{\mathrm{h}}.\end{aligned} \tag{F3}$$

Thus, $\mathsf{K}_{0,d} = g\mathsf{K}^{\mathrm{h}}_d$. Then one just needs to plug Eqs. (F1) and (F3) into (B21). Because $\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0$ is purely anti-Hermitian, its real diagonal part vanishes, so the first $O(\hbar)$ term in Eq. (B21), $\mathrm{Re}[(\mathsf{S}_0^{-1}\mathsf{K}_1\mathsf{S}_0)^{(\mathrm{d})}]$, is zero. Besides, because of the $\alpha \leftrightarrow \beta$ antisymmetry, for the last term it is sufficient to look at $\alpha = X_\mu$, $\beta = p_\mu$, where clearly the commutator vanishes because Λ_{X_μ} is diagonal (actually $\propto 1$), so the third $O(\hbar)$ term, $\epsilon_{\alpha\beta}\mathrm{Im}\{\Lambda_\beta, [\Lambda_\alpha, \mathsf{K}_{0,d}]\}^{(\mathrm{d})}$, is zero as well. Consequently, the only $O(\hbar^1)$ contribution in Eq. (B21) comes from the second $O(\hbar^1)$ term, and

$$\mathsf{K}_d = g\big(1 + \hbar\psi_\mu \mathsf{A}^{\mathrm{h}}_{p_\mu}\big)\mathsf{K}^{\mathrm{h}}_d + O(\hbar^2). \tag{F4}$$

This solves the problem to order $O(\hbar^1)$.

2. Distribution function

The distribution F_d close to equilibrium is a function of the energies K_d only,

$$\begin{aligned}\mathsf{F}_d &= f(\mathsf{K}_d, T)\\ &= f\big(g\mathsf{K}^{\mathrm{h}}_d, T\big) + g\psi_\mu \mathsf{A}^{\mathrm{h}}_{p_\mu}\mathsf{K}^{\mathrm{h}}_d f'\big(g\mathsf{K}^{\mathrm{h}}_d, T\big) + O(\hbar^2),\end{aligned} \tag{F5}$$

where a function of a diagonal matrix is transparently defined [cf. Eq. (45)]. It should be understood that f is a real function which depends solely on *local equilibrium* properties, such as the local temperature [17,23], and f' is the derivative of f with respect to its first variable. Because f does not depend on the profile of $g(X)$, it is the same function as in the homogeneous case $\mathsf{F}^{\mathrm{h}}_d = f(\mathsf{K}^{\mathrm{h}}_d, T)$. It is then not difficult to show that

$$f(\varepsilon, T) = \varepsilon\big[n_{\mathrm{B}}(\varepsilon, T) + \tfrac{1}{2}\big], \tag{F6}$$

where $n_{\mathrm{B}}(\varepsilon, T) = [\exp(\varepsilon/k_{\mathrm{B}}T) - 1]^{-1}$ is the Bose function, T is the local temperature, and the extra ε factor comes from the normalization of eigenvectors, Eq. (63b).

3. Energy current

We are now in a position to compute the energy current (61). First consider Eq. (62a). Because the only nonvanishing curvature here is $\Omega_{p_\mu p_\nu}$ since $\mathsf{A}_\mu = 0$, the Jacobian factor reduces to $\mathfrak{J} = 1 + O(\hbar^2)$. We specialize to $\beta = p_\mu$, and only $\sigma = X_\nu$ in the second factor can contribute [cf. Eq. (E1a)], yielding

$$\begin{aligned}\partial_t \mathsf{X}_\mu &= \partial_{p_\mu}\underline{\mathsf{K}}_d - \hbar\Omega_{p_\mu p_\nu}\partial_{X_\nu}\underline{\mathsf{K}}_d + O(\hbar^2)\\ &= g\big(\partial_{p_\mu}\mathsf{K}^{\mathrm{h}}_d - \hbar\Omega^{\mathrm{h}}_{p_\mu p_\nu}\psi_\nu \mathsf{K}^{\mathrm{h}}_d\big) + O(\hbar^2),\end{aligned} \tag{F7}$$

where the $O(\hbar)$ terms generated by going from $\underline{\mathsf{K}}_d$ to K_d canceled against each other. Finally, plugging Eq. (F5) into the third factor and expanding, it is straightforward to show that the latter reduces to $f(g\mathsf{K}^{\mathrm{h}}_d, T)$. The first contribution to the current is thus

$$\mathcal{J}^{(1)}_{X_\mu} = \frac{\upsilon}{2}\mathrm{Tr}\big[g\big(\partial_{p_\mu}\mathsf{K}^{\mathrm{h}}_d - \hbar\Omega^{\mathrm{h}}_{p_\mu p_\nu}\psi_\nu \mathsf{K}^{\mathrm{h}}_d\big) f(g\mathsf{K}^{\mathrm{h}}_d, T)\big] + O(\upsilon\hbar^2). \tag{F8}$$

Note that the first [order $O(\hbar^0)$] contribution is a total momentum derivative, therefore, it always vanishes after momentum integration.

Now consider Eq. (62b). Because $\mathsf{M}_{\alpha\beta}$ is antisymmetric and Λ_{X_μ} is diagonal, the only [to $O(\hbar^0)$] nonvanishing component of M is

$$\mathsf{M}_{p_\mu p_\nu} = g\tfrac{1}{2}\mathrm{Im}\big\{\Lambda^{\mathrm{h}}_{p_\nu}, \big[\Lambda^{\mathrm{h}}_{p_\mu}, \mathsf{K}^{\mathrm{h}}_d\big]\big\}^{(\mathrm{d})} + O(\hbar), \tag{F9}$$

hence the second contribution to the current

$$\begin{aligned}\mathcal{J}^{(2)}_{X_\mu} &= \frac{\upsilon\hbar}{2}\partial_{X_\nu}\mathrm{Tr}\big[\mathsf{M}_{p_\mu p_\nu}\mathsf{F}_d\big] + O(\upsilon\hbar^2)\\ &= \frac{\upsilon\hbar}{2}\mathrm{Tr}\big[g\mathsf{M}^{\mathrm{h}}_{p_\mu p_\nu}\big(f\big(g\mathsf{K}^{\mathrm{h}}_d, T\big)(\psi_\nu + \partial_\nu T/T)\\ &\quad + g\mathsf{K}^{\mathrm{h}}_d f'\big(g\mathsf{K}^{\mathrm{h}}_d, T\big)(\psi_\nu - \partial_\nu T/T)\big)\big] + O(\upsilon\hbar^2).\end{aligned} \tag{F10}$$

Equations (F8) and (F10) provide the total local, momentum-resolved energy current.

4. Discussion

Here we restrict ourselves to the equilibrium case $\partial_{X_\nu} T = 0$. Then $\mathcal{J}^{(1)}_{X_\mu}$ besides $\mathcal{J}^{(2)}_{X_\mu}$ is also a total ∂_{X_ν} derivative, so that the net energy current $\boldsymbol{J}_{\mathrm{tot}}$ vanishes, as it should in equilibrium. Meanwhile, neither the *local* momentum-integrated currents

$$\begin{aligned}J^{(1)}_{X_\mu} &= -\upsilon\hbar g\,\psi_\nu \int_p \mathrm{Tr}_+\big[\Omega^{\mathrm{h}}_{p_\mu p_\nu}\mathsf{K}^{\mathrm{h}}_d\, f\big(g\mathsf{K}^{\mathrm{h}}_d, T\big)\big] + O(\upsilon\hbar^2),\\ J^{(2)}_{X_\mu} &= \upsilon\hbar\,\partial_{X_\nu}\int_p \mathrm{Tr}_+\big[g\mathsf{M}^{\mathrm{h}}_{p_\mu p_\nu} f\big(g\mathsf{K}^{\mathrm{h}}_d, T\big)\big] + O(\upsilon\hbar^2)\end{aligned} \tag{F11}$$

nor their sum $J_{X_\mu}(X) \equiv J^{(1)}_{X_\mu} + J^{(2)}_{X_\mu}$ need vanish even at equilibrium because the energy magnetization is not zero *a priori*. Note that in Eq. (F11) the trace Tr_+ runs over positive en-

ergy eigenstates only, following the argument developed in Sec. IV A and Appendix B 3.

There is, however, a physical constraint that the energy magnetization should obey the third law of thermodynamics, so that $J_{X_\mu}(X)$ should at least vanish in the limit of zero temperature. Let us show that Eqs. (F8) and (F10) indeed satisfy this property. We will resort to the fact that $f(\varepsilon, T) \overset{T\to 0}{=} \frac{1}{2}\varepsilon$ for any $\varepsilon > 0$, in other words, only the zero-point fluctuations contribute to the density matrix. Taking this limit within Eq. (F11), expanding the definition (19), using cyclicity of the trace and eventually identifying $\Omega^{\rm h}_{p_\mu p_\nu} = {\rm Im}[\Lambda^{\rm h}_{p_\nu}, \Lambda^{\rm h}_{p_\mu}]^{\rm (d)}$, it is then straightforward to show that (in equilibrium)

$$J^{(2)}_{X_\mu} \overset{T\to 0}{=} \frac{\upsilon\hbar}{2} g^2 \psi_\nu \int_p {\rm Tr}_+\big(\mathsf{K}^{\rm h}_d \Omega^{\rm h}_{p_\mu p_\nu} \mathsf{K}^{\rm h}_d\big) + O(\upsilon\hbar^2)$$
$$\overset{T\to 0}{=} -J^{(1)}_{X_\mu} + O(\upsilon\hbar^2). \tag{F12}$$

This shows, to $O(\upsilon\hbar^1)$, the cancellation of local magnetization currents in the zero-temperature limit (this is in fact already true for the momentum-resolved current). More generally, the above shows that the zero-point fluctuations (the "$+\frac{1}{2}$" in the definition of f) cancel exactly when adding $\mathcal{J}^{(1)}_{X_\mu}$ to $\mathcal{J}^{(2)}_{X_\mu}$.

APPENDIX G: EXPLICIT SPATIAL INTEGRATION FOR A HOMOGENEOUS SYSTEM WITH A BOUNDARY

We now perform the $\int dx$ (recall $x \equiv X_x$) integration in Eqs. (64) and (65). We have

$$\kappa^{\rm tr}_{xy} = -\frac{\upsilon\hbar}{\partial_x T} {\rm Tr}_+ \int_p \lim_{L\to\infty} \frac{1}{L} \int dx\, \Omega^{\rm h}_{p_y p_x} \partial_x[\mathsf{K}_d(x)] \mathsf{K}_d(x) \bigg(n_{\rm B}(\mathsf{K}_d(x), T)|_{T(x)} + \frac{1}{2} \bigg), \tag{G1}$$

where we recall $\mathsf{K}_d(x) = g(x)\mathsf{K}^{\rm h}_d$. Clearly, the integrand decays over distances $O(L^0)$ outside of I_x, therefore, one does not change the result by multiplying the former by any cutoff function $\Lambda(x)$ strictly equal to 1 wherever the integrand is nonzero, and decaying slowly to zero at $x \to \pm\infty$. In addition, we introduce $1 = \int d\varepsilon\, \delta(\varepsilon - \mathsf{K}_d(x))$ in order to perform a change of variables from the spatial coordinate x to energy ε:

$$\begin{aligned}
\kappa^{\rm tr}_{xy} &= -\frac{\upsilon\hbar}{\partial_x T} {\rm Tr}_+ \int_p \Omega^{\rm h}_{p_y p_x} \lim_{L\to\infty} \frac{1}{L} \int dx\, \partial_x[\mathsf{K}_d(x)] \int d\varepsilon\, \delta(\varepsilon - \mathsf{K}_d(x))\, \varepsilon \left(n_{\rm B}(\varepsilon, T)|_{T(x)} + \frac{1}{2} \right) \Lambda(x) \\
&= \frac{\upsilon\hbar}{\partial_x T} {\rm Tr}_+ \int_p \Omega^{\rm h}_{p_y p_x} \lim_{L\to\infty} \frac{1}{L} \int dx \int d\varepsilon\, \partial_x \Theta(\varepsilon - \mathsf{K}_d(x))\, \varepsilon \left(n_{\rm B}(\varepsilon, T)|_{T(x)} + \frac{1}{2} \right) \Lambda(x) \\
&= -\frac{\upsilon\hbar}{\partial_x T} {\rm Tr}_+ \int_p \Omega^{\rm h}_{p_y p_x} \lim_{L\to\infty} \frac{1}{L} \int dx \int d\varepsilon\, \Theta(\varepsilon - \mathsf{K}_d(x))\, \varepsilon\, \partial_x n_{\rm B}(\varepsilon, T)|_{T(x)}.
\end{aligned} \tag{G2}$$

In going to the second line we used the chain rule, and in going to the third line we integrated by parts: no boundary term is generated because of the cutoff function $\Lambda(x)$, and the extra term proportional to $\partial_x\Lambda$ vanishes like $O(1/L)$.
We use finally that

$$\partial_x n_{\rm B}(\varepsilon, T)|_{T(x)} = -[\varepsilon/T(x)] \partial_x T(x) \partial_\varepsilon n_{\rm B}(\varepsilon, T)|_{T(x)}, \tag{G3}$$

and do not include $T(x)$ position dependence anywhere else, and assume that $\partial_x T(x)$ is uniform within I_x and zero outside, i.e., $\partial_x T(x) = {\rm const} \neq 0$ for $x \in I_x$ and $\partial_x T(x) = 0$ for $x \notin I_x$. This is enough for the linear response regime up to $O(1/L)$ corrections thanks to the decay of $g(x)$ over distances $O(L^0)$. Since then $g(x) = 1$ in the whole support of the integrand, one can replace $\mathsf{K}_d(x) \to \mathsf{K}_d$ as well as $\frac{1}{L}\int_{I_x} dx \to 1$, which yields exactly Eq. (66), where we also used $\Omega^{\rm h}_{p_x p_y} = -\Omega^{\rm h}_{p_y p_x}$.

APPENDIX H: EXAMPLE OF INHOMOGENEOUS ELASTICITY WITH A TIME-REVERSAL-BREAKING TERM

1. General

The Lagrangian density (67) is equivalent to the Hamiltonian density

$$\mathcal{H}_{\rm ph}(r) = \frac{1}{2\rho(r)} \Pi_i \Pi_i + \frac{c_{ij\mu\nu}(r)}{2} \partial_\mu u_i \partial_\nu u_j, \tag{H1}$$

upon introducing the conjugate lattice momentum

$$\Pi_i = -\frac{i}{\hbar} \frac{\delta}{\delta u_i} - A_i[u], \quad A_i[u] \equiv \eta_{ij\mu\nu}(r) \partial^2_{\mu\nu} u_j. \tag{H2}$$

Thus, the viscosity term introduces an effective vector potential which couples to the lattice momentum and entails a nontrivial $[\Pi_i(r), \Pi_j(r')]$ commutator. For convenience, we now define the "reduced" momentum $\pi_i(r) = \frac{\Pi_i(r)}{\rho(r)}$. To connect with the general formulation of the main text, we define the six-dimensional vector

$$\Phi(r) = [u_1(r), u_2(r), u_3(r), \pi_1(r), \pi_2(r), \pi_3(r)]^\top, \tag{H3}$$

and by comparing Eq. (H1) and the commutation relations of the $u_i(r)$ and $\pi_j(r')$ fields to Eqs. (1), and (2) in the main text, one can readily identify Γ and H. Performing the Wigner transform to phase-space coordinates, one then obtains the dynamical

matrix $\mathsf{K} = \hbar\,\Gamma \star \mathsf{H}$:

$$\Gamma = \frac{i}{\hbar\rho}\begin{bmatrix} 0 & 1 \\ -1 & [\Gamma_{22}] \end{bmatrix}, \quad \mathsf{H} = \begin{bmatrix} [\mathsf{H}_{11}] & 0 \\ 0 & \rho \end{bmatrix}, \quad \mathsf{K} = i\begin{bmatrix} 0 & 1 \\ [\mathsf{K}_{21}] & [\mathsf{K}_{22}] \end{bmatrix}, \tag{H4}$$

where each block is a 3×3 matrix indexed by $i, j = 1 \ldots 3$, and

$$\begin{aligned}
[\mathsf{H}_{11}] &= \frac{1}{4}\partial^2_{X_\mu X_\nu} c_{\mu\nu} + \frac{1}{\hbar^2} c_{\mu\nu} p_\mu p_\nu, \\
[\Gamma_{22}] &= \frac{1}{2\hbar\rho}\left[\partial^2_{X_\mu X_\nu}\eta_{\mu\nu} + 2\eta_{\mu\nu}\left(\frac{\partial_{X_\mu}\rho\partial_{X_\nu}\rho}{\rho^2} - \frac{\partial^2_{X_\mu X_\nu}\rho}{\rho} - \frac{2}{\hbar^2}p_\mu p_\nu\right)\right], \\
[\mathsf{K}_{21}] &= -\frac{1}{\rho}\left(\frac{1}{4}\partial^2_{X_\mu X_\nu} c_{\mu\nu} + \frac{1}{\hbar^2} c_{\mu\nu} p_\mu p_\nu\right) - \upsilon^2 \frac{c_{\mu\nu}}{4\rho}\left(\frac{\partial^2_{X_\mu X_\nu}\rho}{\rho} - 2\frac{\partial_{X_\mu}\rho\partial_{X_\nu}\rho}{\rho^2}\right) + \frac{i\upsilon}{\hbar\rho} c_{\mu\nu} p_\nu \frac{\partial_{X_\mu}\rho}{\rho}, \\
[\mathsf{K}_{22}] &= \frac{1}{2\hbar\rho}\left[\partial^2_{X_\mu X_\nu}\eta_{\mu\nu} + 2\eta_{\mu\nu}\left(\frac{\partial_{X_\mu}\rho\partial_{X_\nu}\rho}{\rho^2} - \left(1 - \frac{\upsilon^2}{2}\right)\frac{\partial^2_{X_\mu X_\nu}\rho}{\rho} - \frac{2}{\hbar^2}p_\mu p_\nu\right)\right] + \frac{i\upsilon}{\hbar\rho}\frac{2}{\hbar}\eta_{\mu\nu} p_\nu \frac{\partial_{X_\mu}\rho}{\rho}.
\end{aligned} \tag{H5}$$

Equations (H4) and (H5) involve parameters ρ, $(c_{\mu\nu})_{ij}$, $(\eta_{\mu\nu})_{ij}$ whose dependence on position X can be chosen to be any arbitrary smooth functions of position X, so that Eqs. (H4) and (H5) are the general expression, in phase space, of inhomogeneous (linear) elasticity with a Hall viscosity term.

2. Specific

Now we focus on the special case of the simple spatial dependence from Appendix F. For this application we need only consider the spatially independent version of the above, Eq. (68) in the main text. We also choose the elasticity parameters to have the specific form (69), corresponding to the Lagrangian density

$$\mathcal{L}_{\text{ph}} = \frac{\rho}{2}(\partial_\tau \boldsymbol{u})^2 + \frac{1}{2}(-c_1 \boldsymbol{u}\cdot(\nabla^2\boldsymbol{u}) + c_2(\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}_\perp)^2 + c_3(\partial_z u^z)^2 + c_4(\boldsymbol{\nabla}_\perp u^z)^2 + c_5(\partial_z \boldsymbol{u}_\perp)^2) + i\big[\partial_\tau\boldsymbol{u}\times\big(\eta_1\nabla_\perp^2 + \eta_2\partial_{zz}^2\big)\boldsymbol{u}\big]\cdot\hat{\boldsymbol{z}}, \tag{H6}$$

where $\boldsymbol{\nabla}_\perp = (\partial_x, \partial_y, 0)$ and $\boldsymbol{u}_\perp = (u^x, u^y, 0)$.

This problem can be solved analytically for arbitrary values of $\eta_1, \eta_2, c_a, a = 1 \ldots 5$. In the following, we expand the solution in powers of the viscosity coefficients $\eta_{1,2}$, which we assume to be small with respect to $c_{1,2}\sqrt{\rho}$; this assumption is largely valid in known relevant cases [29]. We furthermore take $c_5 = 0$, which has little physical consequence but makes analytical expressions considerably shorter. We also define $\tilde{c}_a = c_a/\rho$ for $a = 1 \ldots 5$ and $\tilde{\eta}_b = \eta_b/\rho$ for $b = 1, 2$.

We find the energy bands

$$\begin{aligned}
\mathsf{K}^{\text{h}}_d &= \text{diag}(\varepsilon_1, -\varepsilon_1, \varepsilon_2, -\varepsilon_2, \varepsilon_3, -\varepsilon_3), \\
\varepsilon_1 &= [\tilde{c}_1\boldsymbol{p}^2]^{\frac{1}{2}}, \qquad \varepsilon_2 = [\tilde{c}_1\boldsymbol{p}^2 + \tilde{c}_2 p_\perp^2]^{\frac{1}{2}}, \\
\varepsilon_3 &= \big[\tilde{c}_1\boldsymbol{p}^2 + \tilde{c}_3 p_z^2 + \tilde{c}_4 p_\perp^2\big]^{\frac{1}{2}},
\end{aligned} \tag{H7}$$

where $p_\perp^2 = p_x^2 + p_y^2$, and the "normalized" eigenvectors [normalized according to Eq. (33)] Ψ_i, Ψ_i^* such that

$$\mathsf{S}^{\text{h}} = [\Psi_1 \mid \Psi_1^* \mid \Psi_2 \mid \Psi_2^* \mid \Psi_3 \mid \Psi_3^*]. \tag{H8}$$

In writing the explicit expressions for the eigenvectors, we use the shorthand notations $\zeta^{(\text{I})}_{\boldsymbol{p}} \equiv \tilde{\eta}_1 p_\perp^2 + \tilde{\eta}_2\boldsymbol{p}^2$, $\zeta^{(\text{II})}_{\boldsymbol{p}} \equiv p_\perp^2(\tilde{\eta}_1 p_\perp^2 - \tilde{\eta}_2 p_z^2)$, and $\zeta^{(\text{III})}_{\boldsymbol{p}} \equiv \tilde{\eta}_1 p_\perp^4 - 2\tilde{\eta}_2 p_z^4 - \tilde{\eta}_2 p_z^2 p_\perp^2$, so that a choice of properly normalized eigenvectors can be

$$\begin{aligned}
\Psi_1 &= \frac{1}{\sqrt{2\rho}}[1 + (p_y/p_x)^2]^{-\frac{1}{2}}\left(\frac{i}{\varepsilon_1}\frac{p_y}{p_x} + \frac{2\zeta^{(\text{I})}_{\boldsymbol{p}}}{\tilde{c}_2 p_x^2}, -\frac{i}{\varepsilon_1}, 0, \frac{p_y}{p_x} - i\varepsilon_1\frac{2\zeta^{(\text{I})}_{\boldsymbol{p}}}{\tilde{c}_2 p_x^2}, -1, 0\right)^\top + O(\eta^2), \\
\Psi_2 &= \frac{1}{\sqrt{2\rho}}[1 + (p_x/p_y)^2]^{-\frac{1}{2}}\left(\frac{i}{\varepsilon_2}\frac{p_x}{p_y} + \frac{2\zeta^{(\text{I})}_{\boldsymbol{p}}}{\tilde{c}_2 p_y^2}, \frac{i}{\varepsilon_2}, 0, \frac{p_x}{p_y} - i\varepsilon_2\frac{2\zeta^{(\text{I})}_{\boldsymbol{p}}}{\tilde{c}_2 p_y^2}, 1, 0\right)^\top + O(\eta^2),
\end{aligned} \tag{H9}$$

and $\Psi_3 = \frac{1}{\sqrt{2\rho}}(0, 0, \frac{i}{\varepsilon_3}, 0, 0, 1)^\top$.

Here we provide Ψ_1 and Ψ_2 to linear order in $\eta_{1,2}$ only. This is sufficient to compute the leading contribution to the magnetization $\mathsf{M}_{p_x p_y}$ and Berry curvature $\Omega_{p_x p_y}$, which are odd under time reversal, and thus odd in powers of $\eta_{1,2}$. In particular, Ψ_3 does not depend on $\eta_{1,2}$, therefore, the magnetization and Berry curvature are zero in the third band. In the other two bands

we find, explicitly,

$$\mathsf{M}^{(1)}_{p_x p_y} = \frac{1}{\varepsilon_1^2 p_\perp^2}\tilde{c}_1 \zeta_{\boldsymbol{p}}^{(\mathrm{III})} \quad + O(\eta^3),$$
$$\mathsf{M}^{(2)}_{p_x p_y} = \frac{1}{\varepsilon_2^2 p_\perp^2}\left(\tilde{c}_2 \zeta_{\boldsymbol{p}}^{(\mathrm{II})} + \tilde{c}_1 \zeta_{\boldsymbol{p}}^{(\mathrm{III})}\right) \quad + O(\eta^3), \tag{H10}$$

$$\Omega^{(1)}_{p_x p_y} = \frac{1}{\tilde{c}_2 p_\perp^4}\frac{\tilde{c}_1}{\varepsilon_1^3}\left(\tilde{c}_2 p_\perp^2 \zeta_{\boldsymbol{p}}^{(\mathrm{II})} + 2\tilde{c}_2 \eta_1 p_\perp^4 p_z^2 + 4\tilde{c}_1 \boldsymbol{p}^2 \zeta_{\boldsymbol{p}}^{(\mathrm{III})}\right) \quad + O(\eta^3),$$
$$\Omega^{(2)}_{p_x p_y} = -\frac{1}{\tilde{c}_2 p_\perp^4}\frac{1}{\varepsilon_2^3}\left(3\tilde{c}_2^2 p_\perp^2 \zeta_{\boldsymbol{p}}^{(\mathrm{II})} + 4\tilde{c}_1^2 \boldsymbol{p}^2 \zeta_{\boldsymbol{p}}^{(\mathrm{III})} + \tilde{c}_1 \tilde{c}_2 p_\perp^2 \left[2p_z^2\left(\tilde{\eta}_1 p_\perp^2 + \tilde{\eta}_2 p_z^2\right) + 7\zeta_{\boldsymbol{p}}^{(\mathrm{III})}\right]\right) \quad + O(\eta^3). \tag{H11}$$

We make a final remark about momentum integration. The integrand in Eqs. (F8) and (F10) does not vanish at large energies due to the zero-point contribution to the boson density (see details in Appendix F 4 and discussion in Sec. III F 2). While this does not have physical consequences for the total current $\mathcal{J}_{X_y} = \mathcal{J}^{(1)}_{X_y} + \mathcal{J}^{(2)}_{X_y}$, it makes both integrals diverge independently in a continuum model. Thus, when evaluating momentum integrals we restore a conventional Brillouin zone $\mathrm{BZ} \equiv [-\pi/\xi, \pi/\xi]^3$.

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