Dipolar background field theory and dipolar braiding statistics

Jung Hoon Han●

Department of Physics, Sungkyunkwan University, Suwon 16419, South Korea

We analyze the recently proposed dipolar background field (BF) theory with couplings to charge and dipole currents. The quasiparticles of the theory are either chargelike or dipolelike, and the mutual braiding statistics between chargelike and dipolelike quasiparticles are dipolar, meaning that it depends on the position of the quasiparticle being encircled. The braiding statistics between two dipolelike quasiparticles is that of ordinary anyons. We further prove that the dipolar BF theory is equivalent to the rank-2 tensor BF theory developed earlier as an effective theory for the rank-2 toric code. Although the two theories are equivalent, the dipolar BF formulation embodies the dipole symmetry explicitly and gives a clean insight into the way the dipole symmetry manifests itself in various conservation laws and the dipolar braiding statistics.

I. INTRODUCTION

Extending the horizon of existing many-body models by incorporating dipolar, multipolar, or general modulated symmetries has been a fruitful theme of condensed matter theory in recent years. It was driven in part by a desire to incorporate fractonic behavior [1,2] into field-theoretic formulations [3–5], and was further fueled by the experimental realization of tilted optical lattice and some interesting phenomena found in such systems [6–15]. Now the converging view is that these extensions can be systematically categorized by the order of the multipole symmetry [16–19] being incorporated in the theory. Examples of extensions include rank-2 gauge theory coupled to fractonic quasiparticles [3,4,20], Hubbard models with pair-hopping fermions or bosons [12–14], Chern-Simons theory with dipolar symmetry [20–23], and a 1D SPT model protected by multipolar symmetry [24], to name a few.

Following an unrelated line of reasoning, we have written down a variant of the toric code called the rank-2 toric code (R2TC) [25] by building on some earlier works [26,27]. Anyonic excitations in the R2TC have mobility restrictions not unlike the fractons and are forced to hop only by N sites at a time in a $\mathbb{Z}_N$ stabilizer model of R2TC [25,28]. Several other stabilizer models with similar mobility restrictions on anyonic excitations were discovered [29–33]. The braiding statistics among the anyons in R2TC displayed unusual position dependence, distinct from the ordinary anyon braiding statistics that are purely topological. Attempts to explain the position-dependent braiding in the field theory framework were made by several authors [28,34]. In particular, the dipolar BF theory proposed in [34] had rank-2 tensor gauge fields coupled to rank-2 tensor matter currents, and embodied dipole symmetry in its symmetry structure although the R2TC model had no such symmetry to begin with. In that regard the dipole

symmetry appeared to be "emergent" rather than imprinted on the lattice Hamiltonian of R2TC from the outset.

Quite recently, topological Chern-Simons and BF theories incorporating multipolar symmetries explicitly at the level of the effective action have been put forward [35,36]. They are indeed lucid representations of topological systems embodying both charge and dipole symmetries and are claimed to be equivalent to certain rank-2 Chern-Simons or BF theories after suitable transformations [20–22]. Here we look into this issue further and prove that the dipolar BF theory put forward in [36] is in fact equivalent to the effective theory of the R2TC previously suggested in [34], which was also dubbed the dipolar BF theory. To avoid the confusion, here we dub the new action the dipolar BF (dBF) theory, and rename the rank-2 BF theory of [34] as the rank-2 BF (R2BF) theory. The origin of dipolar braiding statistics, first discussed field-theoretically in [28,34], is given a more transparent interpretation in the framework of dBF theory thanks to the dipole symmetry being embedded in the action from the outset.

In Sec. II we revisit the dBF action [36] and analyze it in greater depth than [36] with additional coupling to matter fields. A complete set of conservation laws related to charge and dipole symmetries are derived. In Sec. III we show that the dBF action can be transformed into the R2BF action and prove their equivalence. Some aspect of this analysis has been done in [36], but we perform a more complete analysis with the full set of gauge-matter coupling allowed by the theory. Then in Sec. IV we revisit the issue of dipolar braiding statistics and rederive the results on the basis of insights gained from the analysis of dBF action. The paper is summarized in Sec. V.

II. DIPOLAR BF THEORY

The purpose of this section is to introduce the dipolar BF theory and outline its basic properties. Some outstanding features of dBF theory is shared by the dipolar Chern-Simons

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*Contact author: hanjemme@gmail.com

(dCS) theory [35], which is simpler and discussed first as a way of illustrating the main principle.

The dCS theory consists of two parts $\mathcal{L} = \mathcal{L}_{\mathrm{dCS}} + \mathcal{L}_{AJ}$, where

$$\mathcal{L}_{\mathrm{dCS}} = \frac{N}{4\pi} \varepsilon_{\lambda\mu\nu} A_\lambda^a \partial_\mu A_\nu^a,$$

$$\mathcal{L}_{AJ} = -A_\mu J_\mu - A_\mu^a J_\mu^a. \tag{2.1}$$

For each spatial direction $a = x, y$ we have the gauge field A_μ^a described by the Chern-Simons (CS) action with level N. Two kinds of matter fields (J_μ, J_μ^a) are introduced, corresponding to the charge and the dipole current associated with each dipole moment orientation a and coupled to the gauge fields A_μ and A_μ^a, respectively. The summation over the spacetime indices μ, $\nu = t, x, y$ and the spatial index $a = x, y$ is implicit.

The gauge fields A_μ, A_μ^a are not totally independent. Their gauge transformations are linked by the dipole conservation, and follow [23,35,36]

$$A_\mu^\nu \to A_\mu^\nu + \partial_\mu \beta^\nu,$$

$$A_\mu \to A_\mu + \partial_\mu \alpha + \beta^\mu. \tag{2.2}$$

Here β^μ is nonzero only for the spatial components $\mu = x, y$. There is no temporal component in the dipolar gauge fields, i.e., $A_\mu^t = 0$. The dCS action given above is invariant under the transformation (2.2). There can be no CS term for A_μ or a mixture of A_μ and A_μ^a since such terms fail to remain invariant under (2.2). Although A_μ does not participate in the CS action, it can still couple to the charge current J_μ and have a nontrivial effect. Throughout the paper we refer to quantities without the superscript a as "charge" and those with the superscript as "dipole" or "dipolar."

Requiring the invariance of the dCS action under the transformation (2.2) gives the continuity equations

$$\partial_\mu J_\mu = 0, \quad \partial_\mu J_\mu^x = J_x, \quad \partial_\mu J_\mu^y = J_y. \tag{2.3}$$

While the charge current satisfies the usual continuity equation, the dipole currents J_μ^x, J_μ^y satisfy modified continuity equations with source terms coming from the charge current. Physically, it means that when a charge moves in a certain direction it will violate the dipole moment conservation in that direction and must act as a source term in the dipole continuity equation. One can rewrite (2.3) in a more suggestive form

$$\partial_\mu \big[J_\mu^x - (x - x_0) J_\mu \big] = \partial_\mu \big[J_\mu^y - (y - y_0) J_\mu \big] = 0, \tag{2.4}$$

where x_0, y_0 are some constants. We will make heavy use of this form when we discuss dipolar braiding in Sec. IV.

We move to examine the dipolar BF theory given by [36]

$$\mathcal{L} = \mathcal{L}_{\mathrm{dBF}} + \mathcal{L}_{AJ} + \mathcal{L}_{bK}, \tag{2.5}$$

where

$$\mathcal{L}_{\mathrm{dBF}} = \frac{N}{2\pi} \varepsilon_{\lambda\mu\nu} b_\lambda \big(\partial_\mu A_\nu + A_\nu^\mu \big) + \frac{N}{2\pi} \varepsilon_{\lambda\mu\nu} b_\lambda^a \partial_\mu A_\nu^a,$$

$$\mathcal{L}_{AJ} = -A_\mu J_\mu - A_\mu^a J_\mu^a,$$

$$\mathcal{L}_{bK} = -b_\mu K_\mu - b_\mu^a K_\mu^a. \tag{2.6}$$

Quantization of the coefficient to N in the BF theory follows from the invariance of the action under the large gauge transformation. Though the dipolar BF action in the first line looks

different from the standard BF action, the gauge transformation argument applies exactly in the same manner, leading to the same quantization of the coefficient to N. Two species of charge and dipole matter currents, (J_μ, J_μ^a) and (K_μ, K_μ^a), are introduced. The last term $\mathcal{L}_{bK}$ was not included in [36] but is necessary for a complete analysis. The gauge transformation properties of the dBF action are as follows:

$$A_\mu^\nu \to A_\mu^\nu + \partial_\mu \beta^\nu$$

$$A_\mu \to A_\mu + \partial_\mu \alpha + \beta^\mu$$

$$b_\mu \to b_\mu + \partial_\mu \theta$$

$$b_\mu^a \to b_\mu^a + \partial_\mu \phi^a - \theta \delta_{\mu a}. \tag{2.7}$$

It is again assumed that $\beta^t = 0 = A_\mu^t$.

Requiring the invariance of the dBF action under the transformation (2.7) gives several continuity equations:

$$\partial_\mu J_\mu = 0, \quad \partial_\mu K_\mu^a = 0,$$

$$\partial_\mu \big[J_\mu^x - (x - x_0) J_\mu \big] = \partial_\mu \big[J_\mu^y - (y - y_0) J_\mu \big] = 0,$$

$$\partial_\mu \big[K_\mu + (x - x_0) K_\mu^x + (y - y_0) K_\mu^y \big] = 0. \tag{2.8}$$

These equations play vital roles in deducing braiding statistics among charge and dipoles, as discussed in Sec. IV.

III. EQUIVALENCE OF DIPOLAR BF THEORY TO RANK-2 BF THEORY

We show how to gradually morph the dBF action in (2.6) into the R2BF theory, written in terms of rank-2 tensor gauge fields and rank-2 tensor currents. In the end, we find that dBF action is in complete agreement with the R2BF action constructed in [34] as an effective field theory of R2TC. Some aspect of this transformation was worked out in [36] for the pure gauge part $\mathcal{L}_{\mathrm{dBF}}$ without the coupling terms $\mathcal{L}_{AJ} + \mathcal{L}_{bK}$.

The proof proceeds by first integrating out b_t from the dBF action (2.6) where it appears linearly. The resulting constraint can be conveniently expressed if we first introduce rank-2 gauge fields A_{ab},

$$A_{ab} = \partial_a A_b - A_a^b, \tag{3.1}$$

which transforms under (2.7) as

$$A_{ab} \to A_{ab} + \partial_a \partial_b \alpha. \tag{3.2}$$

The constraint can be conveniently expressed now as

$$A_{xy} - A_{yx} = (2\pi/N) K_t. \tag{3.3}$$

Note that only the charge sector shows up in the transformation of A_{ab}. This is the well-known transformation associated with the tensor gauge fields previously used in fractonic field theories [3,20–22]. The new insight here is that the tensor fields A_{ab} are not the fundamental objects themselves, but are constructed from the underlying vector gauge fields A_μ^a and A_μ. Although both A_{xy}, A_{yx} transform in the same way, they are not the same fields when $K_t \neq 0$. The definition of A_{ab} in (3.1) does not guarantee such symmetry either. We will proceed for the time being by treating A_{xy} and A_{yx} as two distinct fields.

Among the remaining terms in the action $\mathcal{L}$ (after integrating out b_t), terms containing A_t^a can be collected into

$$\frac{N}{2\pi}\left[A_t^a\big(\partial_x b_y^a - \partial_y b_x^a\big) - b_y A_t^x + b_x A_t^y\right] - A_t^a J_t^a. \quad (3.4)$$

Integrating out A_t^a from (3.4) enforces two relations:

$$\partial_x b_y^x - \partial_y b_x^x - b_y = (2\pi/N)J_t^x,$$
$$\partial_x b_y^y - \partial_y b_x^y + b_x = (2\pi/N)J_t^y. \quad (3.5)$$

Since b_t has been integrated out already, $\varepsilon_{\lambda\mu\nu}b_\lambda\partial_\mu A_\nu$ becomes $b_x(\partial_y A_t - \partial_t A_y) + b_y(\partial_t A_x - \partial_x A_t)$. We can use (3.5) to write b_x, b_y in this action in terms of other fields:

$$\frac{N}{2\pi}[b_x(\partial_y A_t - \partial_t A_y) + b_y(\partial_t A_x - \partial_x A_t)]$$
$$\rightarrow \frac{N}{2\pi}\big[b_x^x(\partial_y\partial_t A_x - \partial_x\partial_y A_t) - b_y^y(\partial_x\partial_t A_y - \partial_x\partial_y A_t)$$
$$- b_x^y\big(\partial_y^2 A_t - \partial_y\partial_t A_y\big) + b_y^x\big(\partial_x^2 A_t - \partial_x\partial_t A_x\big)\big]$$
$$- A_t\big(\partial_x J_t^x + \partial_y J_t^y\big) + A_x\partial_t J_t^x + A_y\partial_t J_t^y. \quad (3.6)$$

In the meantime, the coupling $-J_x^a A_x^a - J_y^a A_y^a$ is rewritten, using $A_{ab} = \partial_a A_b - A_a^b$ as

$$-\big(J_x^a A_x^a + J_y^a A_y^a\big) \rightarrow A_x\big(\partial_x J_x^x + \partial_y J_y^x\big) + A_y\big(\partial_x J_x^y + \partial_y J_y^y\big)$$
$$+ A_{xx}J_x^x + A_{yy}J_y^y + A_{xy}J_x^y + A_{yx}J_y^x. \quad (3.7)$$

(Note that A_t^a is already integrated out and no longer present in the action). Collecting terms proportional to A_μ from the last line of (3.6) and the first line of (3.7) and combining them with $-A_\mu J_\mu$, we conclude that $A_\mu J_\mu$ is modified to $A_\mu\tilde{J}_\mu$ where

$$\tilde{J}_t = J_t + \partial_x J_t^x + \partial_y J_t^y,$$
$$\tilde{J}_x = J_x - \partial_\mu J_\mu^x = 0,$$
$$\tilde{J}_y = J_y - \partial_\mu J_\mu^y = 0. \quad (3.8)$$

The last two lines come from invoking the dipole current conservation (2.3). We thus arrive at the very simple result that only the temporal coupling $-A_t\tilde{J}_t$ survives in the charge sector of $\mathcal{L}_{AJ}$ after integrating out some fields. In the end we find

$$\mathcal{L}_{AJ} \rightarrow -A_t\tilde{J}_t + A_{xx}J_{xx} + A_{xy}J_{xy} + A_{yx}J_{yx} + A_{yy}J_{yy} \quad (3.9)$$

with

$$\tilde{J}_t = J_t + \partial_x J_t^x + \partial_y J_t^y,$$
$$J_{xx} = J_x^x, \; J_{yy} = J_y^y, \; J_{xy} = J_x^y, \; J_{yx} = J_y^x. \quad (3.10)$$

Other terms in the dBF action can be rearranged as well, for instance,

$$b_t^a\big(\partial_x A_y^a - \partial_y A_x^a\big) \rightarrow b_t^x(-\partial_x A_{yx} + \partial_y A_{xx})$$
$$+ b_t^y(\partial_y A_{xy} - \partial_x A_{yy}) \quad (3.11)$$

using

$$\partial_x A_y^x - \partial_y A_x^x = -\partial_x A_{yx} + \partial_y A_{xx},$$
$$\partial_x A_y^y - \partial_y A_x^y = -\partial_x A_{yy} + \partial_y A_{xy}. \quad (3.12)$$

The other terms, $b_y^a\partial_t A_x^a - b_x^a\partial_t A_y^a$, can be combined with (3.6) to give

$$b_x^x(\partial_t A_{yx} - \partial_x\partial_y A_t) - b_y^y(\partial_t A_{xy} - \partial_x\partial_y A_t)$$
$$+ b_x^y\big(\partial_t A_{yy} - \partial_y^2 A_t\big) - b_y^x\big(\partial_t A_{xx} - \partial_x^2 A_t\big). \quad (3.13)$$

Finally, adding (3.11) and (3.13) gives

$$\mathcal{L}_{\text{dBF}} = \frac{N}{2\pi}\big[b_t^x(\partial_y A_{xx} - \partial_x A_{yx}) + b_t^y(\partial_y A_{xy} - \partial_x A_{yy})$$
$$+ b_x^x(\partial_t A_{yx} - \partial_x\partial_y A_t) - b_y^y(\partial_t A_{xy} - \partial_x\partial_y A_t)$$
$$+ b_x^y\big(\partial_t A_{yy} - \partial_y^2 A_t\big) - b_y^x\big(\partial_t A_{xx} - \partial_x^2 A_t\big)\big]. \quad (3.14)$$

In this form, b_μ has been completely integrated out and some second derivatives of the tensor gauge fields A_{ab} appear in the action.

So far we have avoided the assumption of symmetry $A_{xy} = A_{yx}$ and the action (3.14) is valid even for $A_{xy} \neq A_{yx}$. To proceed further, we solve the constraint (3.3) by rewriting

$$A_{xy} \rightarrow A_{xy} + \frac{\pi}{N}K_t, \;\; A_{yx} \rightarrow A_{yx} - \frac{\pi}{N}K_t. \quad (3.15)$$

The new tensor is symmetric, $A_{xy} = A_{yx}$. We caution the reader that the symmetric part A_{ab} is a dynamical gauge field but the antisymmetric part K_t is not, rather it is the background gauge field introduced in the dBF action (2.6). We can rewrite the action (3.14) in light of the substitution (3.15). In addition to the second line of (3.14) simplifying to $(b_x^x - b_y^y)(\partial_t A_{xy} - \partial_x\partial_y A_t)$, there are additional terms showing up in the action due to K_t:

$$-\tfrac{1}{2}b_x^x\partial_t K_t - \tfrac{1}{2}b_y^y\partial_t K_t + \tfrac{1}{2}b_t^x\partial_x K_t + \tfrac{1}{2}b_t^y\partial_y K_t. \quad (3.16)$$

We can also rewrite (b_x, b_y) in terms of other fields using (3.5), and transform the bK coupling to

$$b_x K_x + b_y K_y \rightarrow b_y^y\partial_x K_x + b_x^x\partial_y K_y - b_x^y\partial_y K_x - b_y^x\partial_x K_y, \quad (3.17)$$

where some integration by part has been performed. An additional term $\sim J_t^y K_x - J_t^x K_y$ following from the transformation is a coupling between two background matter fields and can be left out of further consideration. Combining (3.16) and (3.17) with $b_\mu^a K_\mu^a$ gives

$$b_t^x\big(K_t^x - \tfrac{1}{2}\partial_x K_t\big) + b_t^y\big(K_t^y - \tfrac{1}{2}\partial_y K_t\big)$$
$$+ b_x^x\big(K_x^x + \partial_y K_y + \tfrac{1}{2}\partial_t K_t\big) + b_x^y\big(K_x^y - \partial_x K_y\big)$$
$$+ b_y^x\big(K_x^y - \partial_y K_x\big) + b_y^y\big(K_y^y + \partial_x K_x + \tfrac{1}{2}\partial_t K_t\big)$$
$$\equiv b_\mu^a\tilde{K}_\mu^a, \quad (3.18)$$

and the action $\mathcal{L}_{bK}$ is modified to

$$\mathcal{L}_{bK} \rightarrow -b_\mu^a\tilde{K}_\mu^a. \quad (3.19)$$

The charge part $-b_\mu K_\mu$ has been integrated out. Meanwhile, the $\mathcal{L}_{AJ}$ action in (3.9) is modified by $K_t \neq 0$,

$$\mathcal{L}_{AJ} \rightarrow -A_t\tilde{J}_t + A_{xx}J_{xx} + A_{xy}J_{xy} + A_{yy}J_{yy} + \frac{\pi}{N}\big(J_x^y - J_y^x\big)K_t, \quad (3.20)$$

with $A_{xy} = A_{yx}$ and $J_{xy} = J_x^y + J_y^x$.

It turns out K_t appears linearly in (3.19) and (3.20) but does not appear elsewhere. However, integrating out K_t is

not justified as it is not a dynamical gauge field. The terms multiplying K_t can be written as

$$\left(\partial_y b_t^y - \partial_t b_y^y\right) - \left(\partial_t b_x^x - \partial_x b_t^x\right) - \frac{2\pi}{N}\left(J_x^y - J_y^x\right). \quad (3.21)$$

This term, fortunately, is zero as the constraints arising from integrating out A_x^a, A_y^a in the full action gives

$$\partial_y b_t^a - \partial_t b_y^a = (2\pi/N)J_x^a,$$
$$\partial_t b_x^a - \partial_x b_t^a = (2\pi/N)J_y^a. \quad (3.22)$$

The equality of (3.21) to zero follows as a special case.

Finally, we can arrange the two actions $\mathcal{L}_{AJ}$ and $\mathcal{L}_{bK}$ as

$$\mathcal{L}_{AJ} = -A_t \tilde{J}_t + A_{xx} J_{xx} + A_{xy} J_{xy} + A_{yy} J_{yy},$$
$$\mathcal{L}_{bK} = -b_\mu^a \tilde{K}_\mu^a, \quad (3.23)$$

with

$$\tilde{J}_t = J_t + \partial_x J_t^x + \partial_y J_t^y,$$
$$J_{xx} = J_x^x, \quad J_{xy} = J_x^y + J_y^x, \quad J_{yy} = J_y^y, \quad (3.24)$$

and

$$\tilde{K}_t^a = K_t^a,$$
$$\tilde{K}_x^x = K_x^x + \partial_y K_y, \quad \tilde{K}_y^x = K_y^x - \partial_x K_y,$$
$$\tilde{K}_x^y = K_x^y - \partial_y K_x, \quad \tilde{K}_y^y = K_y^y + \partial_x K_x. \quad (3.25)$$

We note that $\mathcal{L}_{dBF}$ and $\mathcal{L}_{AJ}$ are by now written in terms of tensorial gauge fields and currents, but $\mathcal{L}_{bK}$ is not.

To remedy this problem, we redefine certain variables

$$\left(b_t^x, b_t^y\right) \rightarrow \left(E_t^y, -E_t^x\right),$$
$$\left(b_x^x - b_y^y, b_x^y, b_y^x\right) \rightarrow \left(-E_{xy}, -E_{yy}, E_{xx}\right), \quad (3.26)$$

and further rewrite

$$(E_{xx}, E_{yy}, E_{xy}) \rightarrow (E_{yy}, E_{xx}, -E_{xy}),$$
$$(A_{xx}, A_{yy}, A_{xy}) \rightarrow (A_{yy}, A_{xx}, -A_{xy}). \quad (3.27)$$

This gives

$$\mathcal{L}_{dBF} \rightarrow A_{xy}\partial_t E_{xy} + A_{yy}\partial_t E_{yy} + A_{xx}\partial_t E_{xx}$$
$$+ E_t^x(\partial_x A_{xx} + \partial_y A_{xy}) + E_t^y(\partial_y A_{yy} + \partial_x A_{xy})$$
$$+ A_t\left(\partial_y^2 E_{xx} + \partial_x^2 E_{yy} - \partial_x \partial_y E_{xy}\right), \quad (3.28)$$

and

$$\mathcal{L}_{AJ} = -A_t \tilde{J}_t + A_{yy} J_{xx} + A_{xx} J_{yy} - A_{xy} J_{xy}. \quad (3.29)$$

Meanwhile, rewriting $\mathcal{L}_{bK}$ with these new variables gives

$$\mathcal{L}_{bK} \rightarrow -E_t^y \tilde{K}_t^x + E_t^x \tilde{K}_t^y + E_{xx} \tilde{K}_x^y - E_{yy} \tilde{K}_y^x$$
$$- \tfrac{1}{2}E_{xy}\left(\tilde{K}_x^x - \tilde{K}_y^y\right) - \tfrac{1}{2}\left(b_x^x + b_y^y\right)\left(\tilde{K}_x^x + \tilde{K}_y^y\right). \quad (3.30)$$

The last line cannot be written in terms of any of the variables in (3.26). Luckily, $b_x^x + b_y^y$ does not appear elsewhere in the action and can be integrated out to yield

$$\tilde{K}_x^x + \tilde{K}_y^y = K_x^x + K_y^y + \partial_\mu K_\mu = 0, \quad (3.31)$$

which is nothing but the K-current conservation already derived in (2.8).

Combining all three remaining actions, the full action $\mathcal{L} = \mathcal{L}_{dBF} + \mathcal{L}_{AJ} + \mathcal{L}_{bK}$ becomes

$$\mathcal{L}_{R2BF} = \frac{N}{2\pi}\Big[A_{xy}\partial_t E_{xy} + A_{yy}\partial_t E_{yy} + A_{xx}\partial_t E_{xx} + E_t^x(\partial_x A_{xx} + \partial_y A_{xy} - \rho^x) + E_t^y(\partial_y A_{yy} + \partial_x A_{xy} - \rho^y)$$
$$+ A_t\left(\partial_y^2 E_{xx} + \partial_x^2 E_{yy} - \partial_x \partial_y E_{xy} - \tilde{J}_t\right)\Big] - A_{xy}J_{xy} + A_{yy}J_{xx} + A_{xx}J_{yy} - E_{xx}K_{xx} - E_{xy}K_{xy} - E_{yy}K_{yy}, \quad (3.32)$$

with

$$(\rho^x, \rho^y) = \left(-\tilde{K}_t^y, \tilde{K}_t^x\right)$$
$$(K_{xx}, K_{xy}, K_{yy}) = \left(-\tilde{K}_x^y, \tfrac{1}{2}\left(\tilde{K}_x^x - \tilde{K}_y^y\right), \tilde{K}_y^x\right). \quad (3.33)$$

The final action $\mathcal{L}_{R2BF}$ is in complete accord with the R2BF action derived in [34] [Eq. (5.2)]. The R2BF action supports a scalar charge $\tilde{J}_t$ and the associated tensor current (J_{xx}, J_{xy}, J_{yy}), as well as vector charges (ρ^x, ρ^y) and the associated tensor current (K_{xx}, K_{xy}, K_{xy}). Although there are two vector charges, only one tensor current K_{ab} exists for both of them.

There are three conservation laws, one for each of the three charges. One of them is given by $(\tilde{J}_t, J_{xx}, J_{xy}, J_{yy})$, satisfying the equation

$$\partial_t \tilde{J}_t + \partial_x^2 J_{xx} + \partial_x \partial_y J_{xy} + \partial_y^2 J_{yy}$$
$$= \partial_t J_t + \partial_x\left(\partial_\mu J_\mu^x\right) + \partial_y\left(\partial_\mu J_\mu^y\right)$$
$$= \partial_t J_t + \partial_x J_x + \partial_y J_y = 0. \quad (3.34)$$

The relation $\partial_\mu J_\mu^a = J_a$ was used to reach the second line. The charge density $\tilde{J}_t$ is the sum of J_t, the bare charge density, and the dipole-induced charge density $\partial_x J_t^x + \partial_y J_t^y$. A charge (J_t) by itself cannot move due to the dipole moment conservation and must be accompanied by dipole motion. This forces charge to form a composite object with a dipole in order to move freely. Although the observation that a free object in dipole-constrained theory is a charge-dipole composite has been made in [34] already, the dBF formulation is helpful in clarifying the nature of this composite charge as the sum of the bare charge (J_t) and the dipole-induced charge $(\partial_x J_t^x + \partial_y J_t^y)$ through the formula (3.24). The tensor currents (J_{xx}, J_{xy}, J_{yy}) and (K_{xx}, K_{xy}, K_{yy}), introduced through heuristic reasoning in [34], are also given concrete definitions in terms of underlying dipole currents.

The second continuity equation follows from defining a three current

$$(\rho^x, K_{xx}, K_{xy}) = \left(-\tilde{K}_t^y, -\tilde{K}_x^y, \left(\tilde{K}_x^x - \tilde{K}_y^y\right)/2\right)$$

and working out

$$\partial_t \rho^x + \partial_x K_{xx} + \partial_y K_{xy}$$
$$= \partial_t \rho^x + \partial_x K_{xx} + \partial_y K_{xy} - \tfrac{1}{2}\partial_y\big(\tilde{K}_x^x + \tilde{K}_y^y\big)$$
$$= -\partial_\mu \tilde{K}_\mu^y = -\partial_\mu K_\mu^y = 0. \tag{3.35}$$

Note that in the second line we added an extra term $\partial_y(\tilde{K}_x^x + \tilde{K}_y^y)/2$ to complete the continuity equation. This is allowed since the added term is zero by the constraint (3.31). It also implies a degree of flexibility in defining K_{xy} as

$$K_{xy} = \tfrac{1}{2}\big(\tilde{K}_x^x - \tilde{K}_y^y\big) + \frac{\theta}{2}\big(\tilde{K}_x^x + \tilde{K}_y^y\big) \tag{3.36}$$

with θ being an arbitrary real number.

A third conservation law exists for the three current

$$(\rho^y, K_{xy}, K_{yy}) = \big(\tilde{K}_t^x, (\tilde{K}_x^x - \tilde{K}_y^y)/2, \tilde{K}_y^x\big),$$

obeying the continuity equation

$$\partial_t \rho^y + \partial_x K_{xy} + \partial_y K_{yy}$$
$$= \partial_t \rho^y + \partial_x K_{xy} + \partial_y K_{yy} + \tfrac{1}{2}\partial_x\big(\tilde{K}_x^x + \tilde{K}_y^y\big)$$
$$= \partial_\mu \tilde{K}_\mu^x = \partial_\mu K_\mu^x = 0. \tag{3.37}$$

In conclusion, we have derived all three conservation laws as well as the exact form of the R2BF given in [34] by starting from the dBF action of [34] and integrating out some degrees of freedom. In the course of the transformation we found explicit representations of the various charge densities and tensor currents in the R2BF in terms of the charge and dipole currents of the original dBF—see (3.24) and (3.33). These expressions will play a vital role in deriving dipolar braiding statistics.

IV. DIPOLAR BRAIDING

A key feature of the dBF (or R2BF) theory is that its quasi-particles show braiding statistics that depend on the location of the particle being braided around [25,34]. This point can be highlighted already in the dCS theory, which we discuss first due to its simplicity.

It is well known that $\partial_\mu J_\mu = 0$ admits a solution

$$J_\mu^c = (1, \dot{x}_c, \dot{y}_c)\delta^2(\mathbf{r} - \mathbf{r}_c(t)), \tag{4.1}$$

where $\mathbf{r}_c(t)$ is the position of the charge at time t, and $(\dot{x}_c, \dot{y}_c)$ is its velocity. By virtue of continuity equation $\partial_\mu[J_\mu^x - (x - x_0)J_\mu] = 0$ and $\partial_\mu[J_\mu^y - (y - y_0)J_\mu] = 0$ [Eq. (2.4)], one has solutions for the dipole currents:

$$J_\mu^x = (x_c - x_0)J_\mu^c + J_\mu^{d,x},$$
$$J_\mu^y = (y_c - y_0)J_\mu^c + J_\mu^{d,y}, \tag{4.2}$$

with J_μ^c given in (4.1) and

$$J_\mu^{d,x} = (1, \dot{x}_x, \dot{y}_x)\delta^2(\mathbf{r} - \mathbf{r}_x(t)),$$
$$J_\mu^{d,y} = (1, \dot{x}_y, \dot{y}_y)\delta^2(\mathbf{r} - \mathbf{r}_y(t)) \tag{4.3}$$

representing currents associated with a single x or y dipole located at $\mathbf{r}_x(t)$ and $\mathbf{r}_y(t)$, respectively. We will refer to solutions given in (4.1) and (4.3) as single-charge and single-dipole solutions, respectively. Current due to a single-charge motion

J_μ^c generates accompanying dipole current $J_\mu^x = (x - x_0)J_\mu^c$ and $J_\mu^y = (y - y_0)J_\mu^c$. To distinguish them from the intrinsic dipole currents $J_\mu^{d,x}$ and $J_\mu^{d,y}$, we refer to them as orbital dipole currents of a orientation. The name has been suggested in [35].

We examine the consequence of inserting various solutions of the charge and the dipole currents into the action $\mathcal{L}_{AJ}$. First, inserting the single-charge solution $J_\mu = J_\mu^c$ into $J_\mu A_\mu$ (and integrating over spacetime) gives

$$\oint [A_x(\mathbf{r}_c)dx_c + A_y(\mathbf{r}_c)dy_c] = \int (\partial_x A_y - \partial_y A_x)dx_c dy_c, \tag{4.4}$$

where the integral is over the area encompassed by the closed trajectory of $\mathbf{r}_c$. Since there is no Chern-Simons term for A_μ, this flux is not quantized. In fact, $\partial_x A_y - \partial_y A_x$ is not even gauge invariant under the dipolar gauge transformation by β in (2.2), and will soon be combined with another term to make it fully gauge invariant.

When we insert the single-dipole solution (4.3) into the action $J_\mu^a A_\mu^a$, we find for $a = x$,

$$\int J_\mu^{d,x} A_\mu^x = \int \big(\partial_x A_y^x - \partial_y A_x^x\big). \tag{4.5}$$

Invoking the flux attachment rule,

$$\varepsilon_{\lambda\mu\nu}\partial_\mu A_\nu^a = \frac{2\pi}{N}J_\lambda^a, \tag{4.6}$$

which can be derived by integrating out A_μ^a from the dCS action, one can further write

$$\int \big(\partial_x A_y^x - \partial_y A_x^x\big) = \frac{2\pi}{N}\int J_t^x, \tag{4.7}$$

showing that the geometric phase is $2\pi/N$ per x dipole enclosed within the path. We conclude that an a-dipole braiding around another dipole of the same orientation results in the fractional phase of $2\pi/N$.

When a charge is being braided, there is an accompanying orbital dipole current $(x - x_0)J_\mu^c$ contributing to J_μ^x. Its contribution to the phase is

$$\oint (x - x_0)J_\mu^c A_\mu^x = \int (x_c - x_0)\big[A_x^x(\mathbf{r}_c)dx_c + A_y^x(\mathbf{r}_c)dy_c\big]$$
$$= \oint (x - x_0)\big[\partial_x A_y^x - \partial_y A_x^x\big]dxdy$$
$$+ \int A_y^x dxdy. \tag{4.8}$$

There is also the orbital y-dipole current $(y - y_0)J_\mu^c$ contributing the phase

$$\oint (y - y_0)J_\mu^c A_\mu^y = \oint (y - y_0)\big[\partial_x A_y^y - \partial_y A_x^y\big]dxdy$$
$$- \int A_x^y dxdy. \tag{4.9}$$

Overall, there are three different contributions to the phase, given in (4.4), (4.8), and (4.9), as a charge is moved

adiabatically around. Combining all three contributions gives

$$\oint (x - x_0)[\partial_x A_y^x - \partial_y A_x^x] dx dy$$

$$+ \oint (y - y_0)[\partial_x A_y^y - \partial_y A_x^y] dx dy$$

$$+ \int (\partial_x A_y - \partial_y A_x + A_y^x - A_x^y) dx dy. \quad (4.10)$$

The second line represents a contribution from the gauge-invariant flux [37] $F_{xy} = \partial_x A_y - \partial_y A_x + A_y^x - A_x^y$, but the value is not tied to the charge density as there is no CS term involving A_μ in the dCS action. The first line, after substituting $\partial_x A_y^a - \partial_y A_x^a = (2\pi/N) J_t^a$, gives

$$-\frac{2\pi}{N} \left[\sum_i (x_i - x_0) + \sum_i (y_i - y_0) \right] \quad (4.11)$$

where x_i (y_i) is the a coordinate of the ith a dipole inside the loop. The minus sign occurs because the action is $\mathcal{L}_{AJ} = -A_\mu J_\mu - A_\mu^a J_\mu^a$. The braiding statistics between a charge and the dipoles are therefore dipolar, in the sense that the phase acquired in the braiding process depends linearly on the coordinates of the particles.

In summary, when an a dipole is adiabatically moved, it picks up the fractional phase of $2\pi/N$ by braiding around other a dipoles. Meanwhile, a charge braiding around dipoles picks up dipolar phase because the charge motion induces orbital dipole current. The orbital dipole current accompanying the charge current is necessitated by the need to satisfy the dipole symmetry.

Next we examine the braiding statistics in the dBF (or R2BF) theory. From the various conservation laws of J currents and K currents derived in (2.8), we can write down the solutions

$$J_\mu = J_\mu^c,$$

$$J_\mu^x = J_\mu^{d,x} + (x - x_0) J_\mu^c, \quad J_\mu^y = J_\mu^{d,y} + (y - y_0) J_\mu^c,$$

$$K_\mu^a = K_\mu^{d,a},$$

$$K_\mu = K_\mu^c - (x - x_0) K_\mu^{d,x} - (y - y_0) K_\mu^{d,y}, \quad (4.12)$$

where $J_\mu^c, K_\mu^c, J_\mu^{d,a}, K_\mu^{d,a}$ are the single-particle solutions of (4.1) and (4.3).

Since our focus is on the dipolar braiding statistics, we can "turn off" the intrinsic dipolar current $J_\mu^{d,a}$ and write the orbital dipole currents as

$$J_\mu^x = (x - x_0) J_\mu^c = (x - x_0)(1, \dot{x}_c, \dot{y}_c) \delta^2(\mathbf{r} - \mathbf{r}_c),$$

$$J_\mu^y = (y - x_0) J_\mu^c = (y - x_0)(1, \dot{x}_c, \dot{y}_c) \delta^2(\mathbf{r} - \mathbf{r}_c). \quad (4.13)$$

This gives

$$\tilde{J}_t = J_t + \partial_x J_t^x + \partial_y J_t^y$$

$$= [1 + (x - x_0)\partial_x + (y - y_0)\partial_y] \delta^2(\mathbf{r} - \mathbf{r}_c),$$

$$J_{xx} = J_x^x = (x - x_0)\dot{x}_c \delta^2(\mathbf{r} - \mathbf{r}_c),$$

$$J_{xy} = J_x^y + J_y^x = [(x - x_0)\dot{y}_c + (y - y_0)\dot{x}_c]\delta^2(\mathbf{r} - \mathbf{r}_c),$$

$$J_{yy} = J_y^y = (y - y_0)\dot{y}_c \delta^2(\mathbf{r} - \mathbf{r}_c). \quad (4.14)$$

Inserting (J_{xx}, J_{xy}, J_{yy}) obtained here into the action $\int (A_{xx} J_{yy} + A_{yy} J_{xx} - A_{xy} J_{xy})$ gives

$$\oint \mathbf{a}(\mathbf{r}_c) \cdot d\mathbf{r}_c, \quad (4.15)$$

where

$$\mathbf{a} = ((x - x_0) A_{yy} - (y - y_0) A_{xy}, (y - y_0) A_{xx} - (x - x_0) A_{xy}).$$

Taking the curl of $\mathbf{a}$ gives

$$\nabla \times \mathbf{a} = (y - y_0)(\partial_x A_{xx} + \partial_y A_{xy}) - (x - x_0)(\partial_x A_{xy} + \partial_y A_{yy})$$

$$= \frac{2\pi}{N}[(y - y_0)\rho^x - (x - x_0)\rho^y], \quad (4.16)$$

where we used the relations $\partial_x A_{xx} + \partial_y A_{xy} = \rho^x$ and $\partial_y A_{yy} + \partial_x A_{xy} = \rho^y$ obtained from integrating out E_t^x, E_t^y in the R2BF action. Since we further have $\rho^x = -K_t^y$ and $\rho^y = K_t^x$, we can rewrite the curl

$$\nabla \times \mathbf{a} = -\frac{2\pi}{N}[(x - x_0)K_t^x + (y - y_0)K_t^y]. \quad (4.17)$$

The Berry phase acquired by moving a J charge along a closed path is therefore

$$-\frac{2\pi}{N} \left[\sum_i (x_i - x_0) + \sum_i (y_i - y_0) \right], \quad (4.18)$$

where x_i (y_i) are the x (y) coordinate of the x dipole (y dipole) enclosed inside the path. Adiabatic motion of J charge is interpreted as that of a bare J charge and J-dipole composite.

We next turn to single-dipole solutions for $\partial_\mu K_\mu^a = 0$:

$$K_\mu^{d,x} = (1, \dot{x}_x, \dot{y}_x)\delta^2(\mathbf{r} - \mathbf{r}_x),$$

$$K_\mu^{d,y} = (1, \dot{x}_y, \dot{y}_y)\delta^2(\mathbf{r} - \mathbf{r}_y). \quad (4.19)$$

The K dipoles, being exempt from dipole constraint, can move freely. The solution for $\partial_\mu[K_\mu + (x - x_0)K_\mu^x + (y - y_0) K_\mu^y] = 0$ is

$$K_\mu = -(x - x_0)K_\mu^{d,x} - (y - y_0)K_\mu^{d,y}. \quad (4.20)$$

It is the sum of contributions from x dipoles and y dipoles. One can keep only one of the terms on the right when there are dipoles of one orientation present but not the other.

The two vector charges $\rho^x = -K_t^y = -\delta^2(\mathbf{r} - \mathbf{r}_y)$ and $\rho^y = K_t^x = \delta^2(\mathbf{r} - \mathbf{r}_x)$ refer to situations where there is only a single y dipole or a single x dipole. When we work out the associated K currents (K_{xx}, K_{xy}, K_{yy}), we can separately consider situations with only the x dipole or the y dipole present. For instance, we choose $\rho^x = -\delta^2(\mathbf{r} - \mathbf{r}_y)$, and the associated K currents are

$$K_{xx} = -\tilde{K}_x^y \rightarrow -K_x^{d,y} - (y - y_0)\partial_y K_x^{d,y},$$

$$K_{xy} = \tilde{K}_x^x \rightarrow -(y - y_0)\partial_y K_y^{d,y},$$

$$K_{yy} = \tilde{K}_y^x \rightarrow (y - y_0)\partial_x K_y^{d,y}. \quad (4.21)$$

That is, from the generic definition of $(\tilde{K}_x^y, \tilde{K}_x^x, \tilde{K}_y^x)$ given in (3.33), we extract those single-dipole solutions associated with the y dipole (terms on the right side of the arrow).

Inserting (4.21) into the R2BF action gives

$$-\int (E_{xx}K_{xx} + E_{xy}K_{xy} + E_{yy}K_{yy}) \rightarrow \oint \mathbf{a}^y \cdot d\mathbf{r}_y, \quad (4.22)$$

where

$$\mathbf{a}^y = (E_{xx} - (y - y_0)\partial_y E_{xx}, (y - y_0)(\partial_x E_{yy} - \partial_y E_{xy})). \quad (4.23)$$

Taking the curl of $\mathbf{a}^y$ gives the dipolar flux attachment rule:

$$\nabla \times \mathbf{a}^y = (y - y_0)\left(\partial_x^2 E_{yy} + \partial_y^2 E_{xx} - \partial_x\partial_y E_{xy}\right)$$

$$= \frac{2\pi}{N}(y - y_0)\tilde{J}_t. \quad (4.24)$$

The second line follows from integrating out A_t in the R2BF action which gives $\partial_x^2 E_{yy} + \partial_y^2 E_{xx} - \partial_x\partial_y E_{xy} = (2\pi/N)\tilde{J}_t$. The phase picked up by a y dipole is

$$\frac{2\pi}{N}\sum_i (y_i - y_0) \quad (4.25)$$

from braiding around the composite J particles enclosed inside, in addition to the fractional phase $2\pi/N$ acquired by braiding around other y dipoles.

A similar conclusion is reached for the braiding of x dipoles. For $\rho^y = \delta^2(\mathbf{r} - \mathbf{r}_x)$, we find

$$K_{xx} = \tilde{K}_x^y \rightarrow -(x - x_0)\partial_y K_x^{d,x},$$

$$K_{xy} = -\tilde{K}_y^y \rightarrow (x - x_0)\partial_x K_x^{d,x},$$

$$K_{yy} = \tilde{K}_y^x \rightarrow K_y^{d,x} + (x - x_0)\partial_x K_y^{d,x} \quad (4.26)$$

as the K current associated with the adiabatic motion of x dipoles. Inserting (4.26) into the R2BF action gives

$$-\int (E_{xx}K_{xx} + E_{xy}K_{xy} + E_{yy}K_{yy}) \rightarrow \oint \mathbf{a}^x \cdot d\mathbf{r}_x, \quad (4.27)$$

where

$$\mathbf{a}^x = ((x - x_0)(\partial_x E_{xy} - \partial_y E_{xx}), (x - x_0)\partial_x E_{yy} - E_{yy}). \quad (4.28)$$

Taking the curl of $\mathbf{a}^x$ results in the dipolar flux attachment rule:

$$\nabla \times \mathbf{a}^x = \frac{2\pi}{N}(x - x_0)\tilde{J}_t. \quad (4.29)$$

Braiding an x dipole around the composite J particles results in the dipolar phase

$$\frac{2\pi}{N}\sum_i (x_i - x_0), \quad (4.30)$$

in addition to $2\pi/N$ from braiding around other x dipoles.

To conclude, dipolar braiding occurs between a composite J charge and K dipoles and are proportional to the x (y) coordinate of the particle being braided around when the K dipole involved is an x dipole (y dipole). The braiding statistics between K dipoles of the same orientation is that of the usual anyons with fractional phases.

V. DISCUSSION

The picture underlying the rank-2 toric code was that of two layers of $\mathbb{Z}_N$ lattice gauge theories being coupled through some constraint [25,38]. The constraint was not related to the dipole symmetry in any way, or at least the connection to dipole symmetry was not obvious. The dipolar braiding statistics among the anyonic excitations in the rank-2 toric code was therefore an unexpected surprise, and its understanding prompted the formulation of BF-like theory employing rank-2 tensor gauge fields [34]. The dipolar braiding among the quasiparticles was derived from the rank-2 BF theory, but the precise relation between the dipolar braiding statistics and the dipole symmetry of the action remained obscure. In this work, the origin of dipolar braiding among the anyons is clarified as a direct consequence of the dipole symmetry built into the dipolar BF theory.

Analysis of the present paper suggests that dipolar braiding is a natural consequence of topological models with dipole symmetry. Extension of the ideas and techniques explored in this paper to the BF theory with quadrupolar symmetry [36] with the resulting quadrupolar braiding statistics should be straightforward, albeit technically challenging.

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