Superfluidity of total angular momentum

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Spontaneous symmetry breaking of a U(1) symmetry in interacting systems leads to superfluidity of a corresponding conserved charge. We generalize the superfluidity to systems with U(1) symmetries acting on both matter fields and 2D spatial coordinates. Such systems can be effectively realized in easy-plane ferromagnetic systems with spin-orbit coupling where the conserved charge is a total angular momentum. We clarify that under a steady injection of spin angular momentum, the superfluid of the total angular momentum shows spacetime oscillations of the spin density and geometry-dependent spin hydrodynamics. We also demonstrate that the steady spin injection destabilizes the superfluid of total angular momentum, causing a dissipation effect in its spin hydrodynamic properties. Although a stability analysis shows that the superfluid under the spin injection is nonideal, the unique spin-transport features persist with weak dissipation of the spin angular momentum. Our study broadens the comprehension of superfluidity and sheds new light on the interplay between symmetries and phases of matter.

I. INTRODUCTION

The discovery of superfluidity [1–3] is a milestone in the history of physics. Exotic macroscopic quantum phenomena in superfluids are explained by the condensation of bosonic atoms [4,5] or neutral Cooper pairs [6]. Spontaneous symmetry breaking (SSB) of a U(1) global gauge symmetry leads to Goldstone modes with gapless and linear dispersions [7–9], which enables dissipationless mass currents. By alternative U(1) symmetries, the superfluidity can be generalized to spin [10–17] and excitonic [18–23] currents.

General relations between Goldstone modes and SSB of continuous symmetries are derived in the literature [24–27], while they mostly considered continuous *internal symmetries* that transform only field operators locally. *Spacetime symmetries* act on both field operators and spacetime coordinates [28], and the symmetries bring about fundamental physical consequences such as the relativistic spin-orbit coupling (SOC). The continuous spacetime symmetries can be spontaneously broken in spinful superfluids in cold-atom systems [29–34]. Nonetheless, it remains largely unexplored how the SSB of the continuous spacetime symmetries affects the hydrodynamic transport of "charges" associated with the broken spacetime symmetries.

In this paper, we generalize the concept of superfluidity to the SSB of continuous spacetime symmetries. As a physical system, we consider the superfluidity of total angular momentum, where a joint U(1) rotational symmetry of an in-plane spin vector and two-dimensional (2D) spatial coordinates is spontaneously broken. The superfluid of total angular momentum is nothing but a spin superfluid [10–17] in the presence of the SOC. It can be effectively realized in a ferromagnet and a spin-triplet exciton condensate [35–37] with easy-plane spin anisotropy.

We derive an effective field theory of a Goldstone mode in the total-angular-momentum superfluid and solve its classical equation of motion in the presence of a steady injection of spin. We find that the total-angular-momentum superfluid shows spacetime oscillations of spin density and current under the spin injection, which contrasts with conventional spin superfluid without SOC [10–17]. We also uncover unique geometry dependence and nonreciprocity in its hydrodynamic spin transport, which are absent in systems only with discrete internal rotational symmetry [13,19,38]. Especially, when the system is in a circular geometry with finite curvature, the spin hydrodynamics depends on the direction of the spin flow as well as the curvature of the system. The proposed spatial and temporal spin textures can be experimentally detected by magnetic force microscopy [39,40] and x-ray pump-probe microscopy [41], respectively.

We also show that unlike in the conventional spin superfluid, the steady spin injection destabilizes the total-angular-momentum superfluid. Landau argued that uniform superfluids moving slower than a critical velocity realize states at local minima of energy, so the superfluidity is protected from any dissipative perturbation [13,42–44]. Based on the same spirit as Landau's argument, we demonstrate that the total-angular-momentum superfluid is *not* an energy-local-minimum state in the presence of the spin injection and decay processes to lower energy states bring about a dissipation effect in the spin hydrodynamic properties of the superfluid. Nonetheless, we show that the qualitative behavior of the hydrodynamic spin transport remains unchanged and distinct from a nonsuperfluid [13].

The structure of the remaining part of the paper is as follows. In Sec. II, we introduce the field theory with the U(1) spacetime symmetry and offer possible realizations of

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the theory in two microscopic models. In Sec. III, we show spin and orbital parts of Noether's current corresponding to the U(1) spacetime symmetry. In Sec. IV, we propose a spin-injection model with different geometries. The spacetime distribution of a Goldstone mode of the theory are proposed from a classical solution of the mesoscopic model under a steady injection of spin current. In Sec. V, we demonstrate the possibility of energy dissipation by a stability analysis against a local deformation. Thereby, we show that the energy of the classical solution can be further lowered by local deformations, indicating a decay process to lower energy states. The effect of the dissipation on the classical motion of the Goldstone mode is also discussed. Section VI is devoted to a summary.

A number of Appendices are offered to help understand the main text of the paper. In Appendix A, we provide detailed derivations of the field theory from the microscopic models proposed in Sec. II. Detailed derivations and solutions for Secs. III and IV are displayed in Appendices B and C, respectively. Appendices D and E provide instrumental details of Sec. V, where we thoroughly discuss the possibility of the dissipation proposed in Sec. V and its effects on the equation of motion (EOM). In Appendix F, we discuss solutions of the spin-injection model at some special parameter points. In Appendix G, we use the same stability analysis as in Sec. V and derive the Landau criterion of a conventional superfluid. We use this classic and simple example to demonstrate the validity of our stability analysis in Sec. V. In Appendix H, we present how to construct the local deformation of the classical solution of the EOM.

II. MODEL

Consider a complex bosonic field $\phi \equiv \phi_x + i\phi_y$ in three dimensions (3D), where the 2D real and time-reversally-odd vector field (ϕ_x, ϕ_y) and two of three spatial coordinates (x, y) transform under a joint U(1) rotation around z direction,

$$\phi \to \phi e^{i\epsilon}, \quad x + iy \to (x + iy)e^{i\epsilon}. \tag{1}$$

The vector field here stands for a spin vector in physical systems. In the presence of the time-reversal symmetry, $\phi \to -\phi^\dagger$, $t \to -t$, $i \to -i$, the SSB of the joint U(1) symmetry is characterized by a real-time field theory of ϕ,

$$\mathcal{L}_\phi = \frac{\eta_1^2}{2}(\partial_t \phi^\dagger)(\partial_t \phi) - \frac{\eta_1^2 c_\perp^2}{2}(\partial_j \phi^\dagger)(\partial_j \phi)$$
$$- \frac{\eta_1^2 c_z^2}{2}(\partial_z \phi^\dagger)(\partial_z \phi) - \frac{\alpha \eta_1^2 c_\perp^2}{4}[(\partial_- \phi)^2 + (\partial_+ \phi^\dagger)^2]$$
$$- \frac{U}{2}(\phi^\dagger \phi - \rho_0)^2, \tag{2}$$

where $\partial_\pm \equiv \partial_x \pm i\partial_y$, $j = x, y$. A global phase of ϕ is properly chosen so that α is real and positive. $\alpha \in \mathbb{R}$ and $\alpha > 0$. We assume $0 < \alpha < 1$ for the stability of the theory. Ground states for $\rho_0 > 0$ break the U(1) symmetry by uniform field configurations $\phi = \sqrt{\rho_0} e^{i\theta_0}$.

The joint nature of the rotational symmetry results from spin-orbit locking in solid-state materials with SOC. An example of the joint rotational symmetry breaking is in the XY ferromagnet phase on a 3D trigonal or hexagonal lattice. A

localized spin model for spin-orbit coupled magnets generally comprises symmetric and antisymmetric exchange interactions,

$$H_{\text{spin}} = \frac{1}{2} \sum_{i,j} \sum_{\mu,\nu=x,y,z} (\mathcal{J}_{ij,\mu\nu} + \mathcal{D}_{ij,\mu\nu}) S_{i,\mu} S_{j,\nu}, \tag{3}$$

with lattice sites $\boldsymbol{i} \equiv (\boldsymbol{i}_\perp, i_z)$, $\boldsymbol{j} \equiv (\boldsymbol{j}_\perp, j_z)$, spin operators $S_{i,\mu}$ ($\mu = x, y, z$), $\mathcal{J}_{ij,\mu\nu} = \mathcal{J}_{ij,\nu\mu}$, and $\mathcal{D}_{ij,\mu\nu} = -\mathcal{D}_{ij,\nu\mu}$. $\boldsymbol{i}_\perp$ and i_z are xy and z coordinates of the lattice site $\boldsymbol{i}$ on the lattices, respectively. For $\boldsymbol{i} = \boldsymbol{j}$, $\mathcal{D}_{ii,\mu\nu} = 0$ while $\mathcal{J}_{ii,\mu\nu}$ gives an single-ion spin anisotropy energy. Suppose that H_{spin} has an easy-plane spin anisotropy in the XY spin plane and undergoes a quantum phase transition of ferromagnetic ordering of the XY spins, $\vec{S}_{i,\perp} \equiv (S_{i,x}, S_{i,y})$. When H_{spin} belongs to a point group of C_{3i}, D_{3d}, C_{3v}, C_{3h}, C_{6h}, C_{6v}, D_{3h}, or D_{6h}, spin hydrodynamics of the XY spin near the ferromagnetic transition point is described by Eq. (2), where $\phi(\boldsymbol{r}_i) \equiv S_{i,x} + iS_{i,y}$ and a 2 by 2 symmetric matrix comprised of $\mathcal{J}_{ij,\mu\nu}$ ($\mu, \nu = x, y$) determines the strength of the α term. Specifically, for each bond $(\boldsymbol{i}, \boldsymbol{j})$, the 2 by 2 matrix $\mathcal{J}_{ij,\mu\nu}$ has real eigenvalues $\lambda_{ij,m}$ and eigenvectors $\boldsymbol{t}_{ij,m}$ ($m = 1, 2$). Defining $\Delta \lambda_{ij} \equiv \lambda_{ij,1} - \lambda_{ij,2}$, $a_{ij,\perp} \equiv |\boldsymbol{i}_\perp - \boldsymbol{j}_\perp|$, Ω as the total volume of the material, and $\varepsilon_{ij,\perp}$ as an angle between $\boldsymbol{t}_{ij,1}$ in the XY spin plane and $\boldsymbol{i}_\perp - \boldsymbol{j}_\perp$ in the xy coordinate plane, α in Eq. (2) is given by a sum of $\Delta \lambda_{ij}$ with a phase $e^{2i\varepsilon_{ij,\perp}}$ over all the bonds (see Appendix A 1),

$$\alpha \eta_1^2 c_\perp^2 = \frac{1}{8\Omega} \sum_{i,j} a_{ij,\perp}^2 \Delta \lambda_{ij} e^{2i\varepsilon_{ij,\perp}}. \tag{4}$$

The antisymmetric exchange interactions $\mathcal{D}_{ij,\mu\nu}$ do not contribute to the spin hydrodynamic equation near the critical point. Note that the joint rotational symmetry in solid-state materials with periodic lattices must be discrete due to the lattices. In fact, $\mathcal{L}_\phi$ for the XY ferromagnet on the hexagonal lattices generally acquires additional hexagonal easy spin axes within the XY spin plane in the form of $\tilde{c}_6 \phi^6 + \text{H.c.}$ The symmetries also allow $\tilde{\alpha}_6 \phi^3 (\partial_+^2 \phi) + \text{H.c.}$ in the action $\mathcal{L}_\phi$. Near the ferromagnetic transition point, however, the additional term becomes effectively negligible compared to the α term in a hydrodynamic regime with an intermediate crossover length scale (see Appendix A 1).

Another example of Eq. (2) can be found in the triplet exciton condensate phase in semiconductors with Rashba spin-orbit interaction. We consider a 2D model for simplicity. It may be regarded as an effective model of 3D systems. Suppose that electron energy bands near the conduction-band bottom and valence-band top in the semiconductors can be approximated by a model with continuous rotational symmetry ($\hbar = 1$),

$$H_{\text{ex}} = \int d^2r a^\dagger \left[\left(-\frac{\partial_i^2}{2m_0} + \epsilon_{g0} \right) \sigma_0 + \xi_{R0}(-i\partial_y \sigma_x + i\partial_x \sigma_y) \right] \boldsymbol{a}$$
$$+ \int d^2r b^\dagger \left[\left(\frac{\partial_i^2}{2m_0'} - \epsilon_{g0} \right) \sigma_0 + \xi_{R0}'(i\partial_y \sigma_x - i\partial_x \sigma_y) \right] \boldsymbol{b}$$
$$+ \int d^2r (\Delta_t \boldsymbol{a}^\dagger \sigma_0 \boldsymbol{b} + \Delta_t^* \boldsymbol{b}^\dagger \sigma_0 \boldsymbol{a})$$
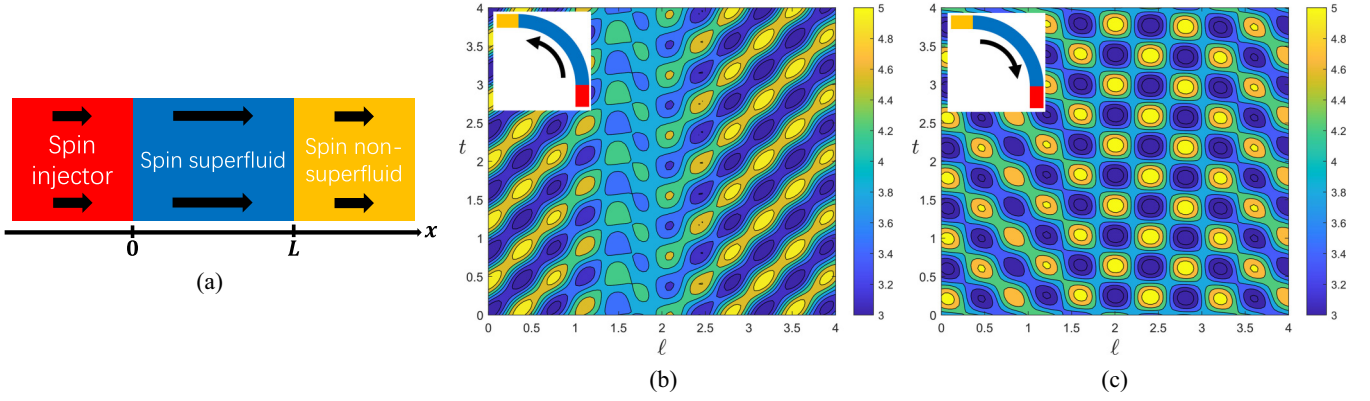
FIG. 1. A spin-injection model. A steady spin current j_0 is injected from a spin injector (red) to the total-angular-momentum superfluid (blue). The spin current passes through the superfluid (blue) and flows into a spin nonsuperfluid (yellow). The direction of the dc component of the current is indicated by black arrows. (a) A straight geometry. (b) A contour plot of $s(\ell, t)$ in a circular geometry with a positive current $j_0 = 4$. (c) A contour plot of $|s(\ell, t)|$ in a circular geometry with a negative current $j_0 = -4$. $\chi = \chi' = D_s = 1$, $T_1' = 4$, $\beta_t = 2$, $r = 1$, $L = 4$, $\alpha = 0.1$ are used in the contour plots.

$$+ \frac{g_{s0}}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int d^2\boldsymbol{r}(a_\sigma^\dagger a_{\sigma'}^\dagger a_{\sigma'} a_\sigma + b_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} b_\sigma$$

$$+ 2\xi_1 a_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} a_\sigma), \tag{5}$$

with $i = x, y, z$, $\boldsymbol{a} \equiv (a_\uparrow, a_\downarrow)$ and $\boldsymbol{b} \equiv (b_\uparrow, b_\downarrow)$ for spin-$\frac{1}{2}$ electrons in conduction and valence bands, respectively. In the presence of the Rashba interactions ($\xi_{R,0}$, $\xi_{R,0}'$) and spinless inter-band coupling (Δ_t, Δ_t^*), an attractive interaction g_{s0} between conduction electrons and valence holes induces a condensation of the XY components of the real part of the s-wave exciton pairing, $O_\mu \equiv \langle \boldsymbol{b}^\dagger \sigma_j \boldsymbol{a} \rangle$ ($j = x, y$) (see Appendix A 2). Thereby, the spin hydrodynamics of the XY-components can be well described by Eq. (2) with $\phi \propto \mathrm{Re}O_x + i\mathrm{Re}O_y$, where α is determined by the Rashba interactions (Appendix A 2).

III. CLASSICAL MOTION AND CONSERVED CURRENT

Motivated by these physical realizations, we study classical motion around the ground states. Taking $\phi = \sqrt{\rho_0 + \delta\rho}e^{i\theta}$, integrating a gapped amplitude mode $\delta\rho$, and neglecting fluctuations along z, we obtain a 2D effective field theory for a Goldstone mode θ in the SSB phase,

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)]$$

$$- \frac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta). \tag{6}$$

We set $\eta_1 = c_\perp = \rho_0 = 1$ without loss of generality. For a given ground state $\phi = \sqrt{\rho_0}e^{i\theta_0}$, the dispersion of a phase fluctuation $\delta\theta = \theta - \theta_0$ is gapless with a linear dispersion, where velocities are anisotropic and depend on θ_0. Note that the joint U(1) symmetry generally allows higher-order terms in derivatives or fields in the effective theory Eq. (2), while they do not affect the hydrodynamic transport of low-energy excitations near the ground states.

According to Noether's theorem [28,45], the U(1) continuous spacetime symmetry endows the classical motion with a conserved current of total angular momentum, which can be divided into a spin part (j_μ^s) and an orbital part (j_μ^l),

$$j_\mu^s = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta,$$

$$j_\mu^l = \left[\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)\right]\Delta x_\nu, \tag{7}$$

with $\mu, \nu \in \{t, x, y\}$, $\Delta x_\nu \in \{\Delta t, \Delta x, \Delta y\}$, $\Delta\theta = 1$, and $(\Delta t, \Delta x, \Delta y) = (0, -y, x)$. The two parts are not conserved by themselves, $\partial_\mu j_\mu^s = -\partial_\mu j_\mu^l = G$, where a spin torque G can be defined by the divergence of the spin current. The spin torque (G), spin currents (j_x^s, j_y^s), and a spin angular momentum along z direction (j_t^s) are given by the following equations (see Appendix B):

$$G = -\alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta),$$

$$j_x^s = -(\partial_x\theta)[1 - \alpha\cos(2\theta)] + \alpha(\partial_y\theta)\sin(2\theta),$$

$$j_y^s = -(\partial_y\theta)[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)\sin(2\theta),$$

$$s \equiv j_t^s = \partial_t\theta. \tag{8}$$

Though the orbital part j_μ^l is nonlocal [see Eqs. (B13)–(B15)], the spin torque G as well as the spin part j_μ^s are local. The locality of the spin torque results from a continuous spacetime translational symmetry of $\mathcal{L}$ (Appendix B).

IV. SPIN INJECTION AND TRANSPORT

To illustrate observables of a total-angular-momentum superfluid, consider a uniform spin current j_0 ($j_0 > 0$) injected into one end ($x = 0$) of the superfluid ($0 < x < L$). The spin current passes through the superfluid and flows into a spin nonsuperfluid at the other end $x = L$ [see Fig. 1(a)] [11,13]. The nonsuperfluid "lead" has diffusive spin transport. Hydrodynamic spin transport in the superfluid is determined by a one-dimensional (1D) EOM of the Goldstone mode $\theta(x, t)$ in Eq. (6) with $\partial_y\theta = 0$,

$$\partial_t^2\theta - (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta) = 0. \tag{9}$$

The EOM (9) will be solved together with proper boundary conditions. To determine the boundary conditions, note that spin transport in the nonsuperfluid ($x > L$) is described by diffusion equations [11,13],

$$\frac{\partial s}{\partial t} + \frac{\partial j_x^s}{\partial x} = -\frac{s}{T_1'}, \quad j_x^s = -D_s \frac{\partial s}{\partial x}, \tag{10}$$

with relaxation time T_1' and a diffusion coefficient D_s. The diffusive spin current is caused by the gradient of the spin density. Due to the relaxation time, the density and current decay exponentially in space for $L > 0$,

$$s(x, t) \equiv \sum_{c \in \mathbb{R}} s_c(x, t) = \sum_{c \in \mathbb{R}} a_c e^{ict} e^{-\sqrt{D_s^{-1} \omega_c} x},$$

$$j_x^s(x, t) \equiv \sum_{c \in \mathbb{R}} j_{x,c}^s(x, t) = \sum_{c \in \mathbb{R}} \sqrt{D_s \omega_c} a_c e^{ict} e^{-\sqrt{D_s^{-1} \omega_c} x}. \tag{11}$$

From Eq. (11) we get $j_{x,c}^s = \sqrt{D_s \omega_c} s_c$, where $j_{x,c}^s$ and s_c are Fourier components of j_x^s and s with a time periodicity $2\pi c^{-1}$. Here $\omega_c = ic + \frac{1}{T_1'}$, a_c are complex coefficients, and the square roots of $D_s^{-1} \omega_c$ take positive real parts. The spin current is assumed to be continuous at the junction between the superfluid and nonsuperfluid, and it is proportional to the gradient of an effective local magnetic field felt by the spin density [11],

$$j_x^s(x = L-, t)$$
$$= j_x^s(x = L+, t)$$
$$= -\beta_t \left[\frac{1}{\chi'} s(x = L+, t) - \frac{1}{\chi} s(x = L-, t) \right]. \tag{12}$$

Here χ, χ' are magnetic susceptibilities at $x = L-$ and $x = L+$, respectively, β_t is a response coefficient of the junction, and they are all positive. Equation (12) imposes a boundary condition (BC) on the spin density and current at $x = L-$ for each frequency c,

$$s_c(x = L-, t) = k_c j_{x,c}^s(x = L-, t), \tag{13}$$

with $k_c \equiv \frac{\chi}{\chi'} [D_s(\frac{1}{T_1'} + ic)]^{-\frac{1}{2}} + \frac{\chi}{\beta_t}$, $k_{-c} = k_c^*$, and $\mathrm{Re}(k_c) > 0$. The steady injection of spin imposes another boundary condition at $x = 0+$, $j_x^s(x = 0+, t) = j_0$ [11]. In the following, the EOM (9) is solved for $\theta(x, t)$ such that $s(x, t)$ and $j_x^s(x, t)$ satisfy the BCs.

An analytical solution of $\theta(x, t)$ can be obtained perturbatively in the SOC. The solution at the first order consists of three parts,

$$\theta(x, t) = \theta_0(x, t) + \theta_1(x, t) + \theta_2(x, t) + \mathcal{O}(\alpha^2). \tag{14}$$

θ_0 is the zeroth-order solution satisfying the EOM and BCs [11,13],

$$\theta_0(x, t) = k_0 j_0 t - j_0 x, \tag{15}$$

with $k_0 = \frac{\chi}{\chi'} \sqrt{\frac{T_1'}{D_s}} + \frac{\chi}{\beta_t}$. An oscillation is absent at the zeroth order due to the BCs with $\mathrm{Re}(k_c) > 0$. θ_1 and θ_2 are at the first order in α. θ_1 is a special solution of an inhomogeneous linear differential equation,

$$\partial_t^2 \theta_1 - \partial_x^2 \theta_1 = -\alpha(\partial_x^2 \theta_0)\cos(2\theta_0) + \alpha(\partial_x \theta_0)^2 \sin(2\theta_0). \tag{16}$$

θ_2 is a solution of a homogeneous linear differential equation such that θ satisfies the BCs at the first order in α,

$$\partial_t^2 \theta_2 - \partial_x^2 \theta_2 = 0. \tag{17}$$

The solution at the first order oscillates with two spatial wave numbers, $2j_0$ and $2k_0 j_0$, and one temporal frequency $2k_0 j_0$ (see Appendix C 1),

$$\theta(x, t) = j_0(k_0 t - x) - \frac{\alpha}{4(k_0^2 - 1)} \sin[2j_0(k_0 t - x)]$$

$$- \frac{\alpha(2k_0^2 - 1)}{4(k_0^2 - 1)} \cos(2k_0 j_0 t)\sin(2k_0 j_0 x)$$

$$+ \alpha \mathrm{Im}(\eta)\cos(2k_0 j_0 t)\cos(2k_0 j_0 x)$$

$$+ \alpha \mathrm{Re}(\eta)\sin(2k_0 j_0 t)\cos(2k_0 j_0 x) + \mathcal{O}(\alpha^2). \tag{18}$$

η is a constant depending on k_0, $k_{c=2k_0 j_0}$, and $2j_0 L$ [see Eqs. (C29) and (C22)]. The two spacetime frequencies $(2j_0, 2k_0 j_0)$ and $(2k_0 j_0, 2k_0 j_0)$ come from θ_1 and θ_2 respectively, and are determined by the BCs. Note that the perturbative solution is divergent and fails near a "resonant" point $k_0 = 1$ [46] (see Appendix F). The divergence can be resolved by adding a finite dissipation term, $-T_1^{-1} \partial_t \theta$, to EOM (9).

Higher-order solutions can be systematically obtained by the perturbative iteration, where the spin density and current have the same periodicity in time as the first-order solution, $\pi(k_0 j_0)^{-1}$. The time periodicity can be detected by a time-resolved measurement of the spin density in the nonsuperfluid "lead," which depends on the injected spin current (j_0) and properties of the junction (k_0). The higher-order solution has no spatial periodicity in general, while its Fourier transform in space has two major peaks at $2j_0$ and $2k_0 j_0$ as in the first-order solution. The two major wave numbers can be observed by a local measurement of the spin density in the superfluid.

The spin hydrodynamics under the spin current has a unique geometric effect in a geometry with a finite curvature [Figs. 1(b) and 1(c)]. To demonstrate this, suppose that the width of the junction in the circular geometry is small enough that the radius of the junction is taken as a constant r and the field depends only on time and a 1D angular coordinate ϑ. With $(x, y) = r(\cos\vartheta, \sin\vartheta)$, Eq. (6) leads to a 1D Lagrangian (see Appendix C 2),

$$\mathcal{L} = \frac{1}{2}(\partial_t \theta)^2 - \frac{1}{2}(\partial_\ell \theta)^2 \left[1 + \alpha \cos\left(2\theta - \frac{2}{r}\ell\right) \right], \tag{19}$$

where $\ell \equiv r\vartheta$. The corresponding EOM under the injected spin current j_0 together with the junction parameter k_0 has a zeroth-order solution, $\theta_0(\ell, t) = k_0 j_0 t - j_0 \ell$, and a first-order solution, $\theta_0(\ell, t) + \theta_1(\ell, t) + \theta_2(\ell, t)$. Here θ_1 is a special solution of an inhomogeneous differential equation,

$$\partial_t^2 \theta_1 - \partial_\ell^2 \theta_1 = -\alpha j_0\left(j_0 + \frac{2}{r}\right)\sin\left[2k_0 j_0 t - 2\left(j_0 + \frac{1}{r}\right)\ell\right]. \tag{20}$$

θ_1 and θ_2 introduce two wave numbers, $2j_0 + \frac{2}{r}$ and $2k_0 j_0$, in the observables, respectively, where the wave number of θ_1 acquires a linear curvature ($\frac{1}{r}$) dependence. Due to the curvature dependence, two injected spin currents

with opposite signs [$j_0 \equiv \mathrm{j}_0$ from Fig. 1(b) and $j_0 = -\mathrm{j}_0$ from Fig. 1(c)] lead to different spatial distributions of the observables (nonreciprocal spin hydrodynamics, see Appendix C 2). The nonreciprocity in the curved geometry contradicts neither the time-reversal symmetry nor an inversion at the origin ($r = 0$), as a uniform circular spin current is even under those symmetries.

When the discreteness of the lattice rotational symmetry becomes relevant, the Lagrangian and EOM acquire $\mathbb{Z}_n$ terms, where the U(1) spacetime symmetry reduces to the $\mathbb{Z}_n$ spacetime symmetry (see Appendix A 1). The $\mathbb{Z}_n$ theory leads to a gapped ground state at equilibrium, whose low-energy spin transport is characterized by the dynamics of domain walls. A $\mathbb{Z}_n$ term $\tilde{c}_n \sin(n\theta)$ also gives rise to similar spacetime oscillations in the observables under the spin injection [11,13,20,38], while there is no geometric dependence. On the contrary, as described above, the spacetime oscillations induced by the SOC (α) have nonreciprocal and curvature-dependent hydrodynamics in the curved geometry.

V. DISSIPATION EFFECT

In the presence of the Galilean covariance, a uniform superfluid moving slower than the velocity of its Goldstone mode achieves a local energy minimum so that it is stable against dissipation by local perturbations, e.g., elastic scattering by disorder [13]. To see the stability of a supercurrent state with the broken U(1) spacetime symmetry, we compare classical energies of the 1D solution $\theta(x, t)$ and its local deformation $\theta(x, t) + \delta\theta(x, t)$. The deformation $\delta\theta(x, t)$ is induced by local perturbations, so the spacetime derivatives of $\delta\theta$ do not contain any uniform component in spacetime. $\theta(x, t) + \delta\theta(x, t)$ as well as $\theta(x, t)$ is a classical solution of Eq. (9), while they do not necessarily share the same boundary conditions. The classical energy in the 1D model can be evaluated from a Hamiltonian,

$$H[\theta] = \int dx \left\{ \frac{1}{2}(\partial_t \theta)^2 + \frac{1}{2}(\partial_x \theta)^2[1 - \alpha \cos(2\theta)] \right\}. \quad (21)$$

As the classical energies are independent of time, for simplicity, we compare time averages of the energies (with $k_0 \neq 1$) over a large period of time T (see Appendix D),

$$\Delta J \equiv \lim_{T \to \infty} \frac{1}{T} \left(\int_0^T dt H[\theta + \delta\theta] - \int_0^T dt H[\theta] \right)$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int dx \{(\partial_t \theta)(\partial_t \delta\theta) + (\partial_x \theta)(\partial_x \delta\theta)$$

$$\times [1 - \alpha \cos(2\theta)] + \alpha(\partial_x \theta)^2 \sin(2\theta)(\delta\theta)\} + \mathcal{O}((\delta\theta)^2)$$

$$= \frac{2}{T} \lim_{T \to \infty} \int_0^T dt \int dx (\partial_x \theta_2)(\partial_x \delta\theta_0) + \mathcal{O}(\alpha^2 \delta\theta, (\delta\theta)^2),$$

$$(22)$$

with $\theta + \delta\theta = \theta_0 + \delta\theta_0 + \mathcal{O}(\alpha)$. Terms oscillating in space or time vanish after the spacetime integrals. $\delta\theta_0$, as well as θ_2, is a solution of Eq. (17), and both are given by linear superpositions of $e^{iq(t-x)}$ and $e^{iq(x+t)}$ over q. Thus the right-hand side of Eq. (22) indicates that for a given $\theta_2 \neq 0$, one can always choose $\delta\theta_0$ with $\Delta J < 0$. This means that the supercurrent state is classically unstable toward other states,

and energy always dissipates by the local perturbations. The instability results from the absence of the Galilean covariance. The superflow state is distinct from the ground state by the spacetime oscillation feature, and the energy of the superflow state can be lowered by excitations $\delta\theta_0$ which match the oscillation periodicity.

The dissipation effect on the spin hydrodynamics can be phenomenologically studied through addition of the simplest time-reversal breaking term, $-T_1^{-1}\partial_t \theta$, into the classical EOM (see Appendix E). With finite T_1^{-1}, the zeroth-order solution of spin current acquires linear spatial decay [11,13], which contrasts with the exponential decay in the nonsuperfluid [13]. Thereby, the hydrodynamic feature of spin transport survives against dissipation. When a phase accumulation γ due to the spatially dependent current is small, a double expansion in α and γ enables a perturbative solution of θ systematically (see Appendix E). The lowest-order solution suggests that the spin density and current remain periodic in time, and they show nonreciprocal and curvature-dependent spin transport in the curved geometry.

VI. SUMMARY

In this paper, we generalize the U(1) internal symmetry in conventional superfluid theories into the U(1) spacetime symmetry. Due to the joint symmetry, the supercurrent state shows geometry-dependent spacetime oscillations, and it is unstable against the dissipation effect. Note that in the spin and excitonic systems, the orbital angular momentum density j_t^l has much smaller coupling with local magnetic probes than the spin angular momentum density j_t^s, because both the spin and exciton are charge neutral, and their motions have no direct coupling with a magnetic field. For this reason, we expect that the local measurements of spin angular momentum density in the proposed physical systems are experimentally feasible. Our study paves the way for further exploration of multiple spacetime symmetries and their coupling with internal symmetries.

APPENDIX A: MICROSCOPIC MODELS

In this Appendix, we discuss physical realizations of the U(1) spacetime symmetry in two physical models; (i) an XY spin model for spin-orbit coupled magnets near a critical point, and (ii) a triplet excitonic model for semiconductors. The U(1) spacetime symmetry is a joint continuous rotational symmetry that acts on both matter field and 2D spatial coordinates. The joint nature of the symmetry results from a locking between the rotation of the matter field and that of the spatial coordinate. Such locking is ubiquitous in solid-state materials with relativistic spin-orbit interaction, where spin or an interband component of spin forms the matter field. In solid-state materials with periodic lattices, the spatial rotation

must be discrete due to the lattices, so the joint rotation symmetry is also discrete; the U(1) spacetime symmetry cannot be an exact symmetry and it is valid only approximately. Nonetheless, for some solid-state systems, the approximation becomes effective where the difference between discrete and continuous joint rotations becomes irrelevant.

1. Easy-plane ferromagnetic spin model

To see the effectiveness of the U(1) theory in magnetic systems, let us consider a XY ferromagnetic spin system in 3D lattices that is symmetric under C_n rotation around a z axis ($n = 3, 4, 6, \ldots$) and time reversal. We will first impose a spatial inversion symmetry; at the end of this section, we will show that the inversion symmetry is not necessary to derive the U(1) theory for some cases. We suppose that the spin system is a spin-orbit coupled magnet with an easy-plane spin anisotropy (an XY plane being the easy plane), and it undergoes a continuous phase transition from a disordered phase to a ferromagnetic ordered phase of XY components of spins, $S_{i,x}$ and $S_{i,y}$. In this section, we will discuss the effective symmetry of spin hydrodynamics near the phase transition point.

The second-order phase transition in the XY ferromagnet with the C_n rotation can be described by a partition function $\mathcal{Z}_n$ ($n = 3, 4, 6, \ldots$) with a Ginzburg-Landau (GL) action for a 2D complex variable $\phi(\boldsymbol{x})$ ($\hbar = 1$),

$$\mathcal{Z}_n = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \exp[-\mathcal{S}_{n,\phi}],$$

$$\mathcal{S}_{n,\phi} = \int d^3\boldsymbol{r} \int_0^{1/k_BT} d\tau \, s_n[\phi(\boldsymbol{r}, \tau)]. \tag{A1}$$

Here the internal field $\phi(\boldsymbol{r})$ is a spatial average of $S_{i,x} + iS_{i,y}$ with respect to a lattice site $\boldsymbol{i}$ over some hydrodynamic volume element. i is the complex unit. $\boldsymbol{r}$ is a spatial coordinate of the hydrodynamic volume element. The transition can happen either at the zero-temperature $T = 0$ critical point (quantum critical point) or at finite-temperature $T \neq 0$ critical point (classical critical point). In this Appendix, we consider the spin hydrodynamics near the $T = 0$ quantum critical point. The time-reversal symmetry means the absence of an external magnetic (Zeeman) field in the spin model, and the ferromagnetic order breaks the symmetry spontaneously. The time-reversal symmetry in the model requires the dynamical exponent z at the quantum critical point to be one, $z = 1$ (see below).

For $n = 4$, the spin system is defined on a 3D tetragonal lattice with a C_4 rotational symmetry around the z axis, such as a layered square lattice. For $n = 3$ or 6, the system is defined on a 3D trigonal or hexagonal lattice with a C_3 or C_6 rotation, e.g., layered honeycomb or triangle lattices. In the following, we first employ a symmetry argument to determine the form of the GL action $s_n[\phi]$ for $n = 4$, 3, and 6 and show that for the $n = 3$ or 6 case, the U(1) joint rotational symmetry is an effective symmetry of the GL action for the xy spins near the critical point, while for the $n = 4$ case, the effective symmetry remains discrete (a $\mathbb{Z}_4$ joint rotational symmetry). To this end, note that under the C_n rotation around the z axis, the complex field $\phi(\boldsymbol{r})$ of the xy spins, and the 3D spatial coordinate, $\boldsymbol{r} \equiv (x, y, z)$, are rotated together due to the spin-orbit

locking,

$$\boldsymbol{r} \to \boldsymbol{r}' = (x', y', z), \quad x' + iy' = e^{\frac{2\pi}{n}}(x + iy),$$

$$\phi(\boldsymbol{r}) \equiv S_x(\boldsymbol{r}) + iS_y(\boldsymbol{r}) \to \phi'(\boldsymbol{r}') = \phi(\boldsymbol{r})e^{i\frac{2\pi}{n}}. \tag{A2}$$

The spatial inversion changes the sign of the coordinate vector, while the time-reversal changes the sign of ϕ_x, ϕ_y, and i. These symmetries constrain forms of the actions for the ϕ field.

a. GL action for XY ferromagnets with C_4 rotation

The symmetries of the joint C_4 rotation around z and time reversal allow the following terms and their complex conjugates in the GL action,

$$|\phi|^2 \equiv \phi^\dagger\phi, \quad i[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi], \quad (\partial_m\phi^\dagger)(\partial_m\phi),$$

$$\phi\partial_-^2\phi, \phi\partial_+^2\phi, (|\phi|^2)^2, \phi^4, \ldots, \tag{A3}$$

with $m = x, y, z$, and $\partial_\pm \equiv \partial_x \pm i\partial_y$. Higher-order terms in ϕ, higher-order spatial gradient terms, and total-derivative terms are omitted as "$\ldots$". The higher-order ϕ terms are irrelevant near the critical point where the amplitude of ϕ becomes smaller. The higher-order spatial gradient terms are irrelevant in the hydrodynamic regime where the volume element over which the spin operator is averaged becomes larger. Here the time reversal forbids odd-order terms of ϕ. The spatial inversion further forbids $i[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi]$ from the action. Accordingly, the GL functional form allowed by the symmetries is given by

$$\mathcal{Z}_{n=4} = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \exp[-\mathcal{S}_{n=4,\phi}],$$

$$\mathcal{S}_{n=4,\phi} \equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_BT} d\tau \left\{ \frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) \right.$$

$$+ \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi)$$

$$+ \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2$$

$$+ \beta(\partial_+\phi)^2 + \beta^*(\partial_-\phi^\dagger)^2] + \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2$$

$$\left. + \frac{1}{2}[\tilde{c}_4\phi^4 + \tilde{c}_4^*(\phi^\dagger)^4] + \cdots \right\}, \tag{A4}$$

with $j = x, y$, imaginary time τ, real numbers ρ_0, η_1, c, U, and complex numbers $\alpha, \beta, \tilde{c}_4$. Here the first-order time-derivative of ϕ, such as $\phi^\dagger\partial_\tau\phi$, is forbidden by the time reversal symmetry. The second-order time-derivative term is induced by an integration of the z-component spin. To be specific, one can start from a path integral of a spin Lagrangian with a Wess-Zumino term that ensures correct commutation relations between spin operators. In the presence of easy-plane anisotropy, the out-of-plane component of spin (S_z) is gapped, and one can integrate out the gapped degree of freedom, leading to an effective field theory of $\phi = S_x + iS_y$. In the effective field theory, the leading-order time-dependent term allowed by time-reversal symmetry is $\frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi)$. Note

that the dynamical exponent z around the $T = 0$ quantum critical point becomes one, namely $z = 1$.

A partition function $\mathscr{Z}_{n=4}$ describes the phase transition from the disordered phase ($\rho_0 < 0$) to the ordered phase of the xy spins ($\rho_0 > 0$). In the ordered phase, a phase of ϕ is locked into four minima determined by the $\tilde{c}_4$ term. At the quantum critical point, the terms with higher-order gradient and/or ϕ terms become irrelevant in the long-wavelength limit, and the effective symmetry of the GL action at the critical point is determined by a gapless free theory part,

$$\overline{\mathscr{S}}_{n=4,\phi} \equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_\mathrm{B}T} d\tau \left\{ \frac{\eta_1^2}{2}(\partial_\tau \phi^\dagger)(\partial_\tau \phi) \right.$$
$$+ \frac{\eta_1^2 c_\perp^2}{2}(\partial_j \phi^\dagger)(\partial_j \phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z \phi^\dagger)(\partial_z \phi)$$
$$+ \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2$$
$$\left. + \beta(\partial_+\phi)^2 + \beta^*(\partial_-\phi^\dagger)^2] \right\}. \tag{A5}$$

Importantly, though the α term in the second line is symmetric under the U(1) spacetime symmetry, $\partial_\pm \to \partial_\pm' = e^{\pm i\epsilon}\partial_\pm$, $\phi \to \phi' = e^{i\epsilon}\phi$ for $\forall\epsilon$, the β term is symmetric only under the joint $\mathbb{Z}_4$ rotational symmetry,

$$\partial_\pm \to \partial_\pm' = e^{\pm i\frac{\pi}{2}}\partial_\pm, \quad \phi \to \phi' = e^{i\frac{\pi}{2}}\phi,$$
$$\phi^\dagger \to (\phi')^\dagger = e^{-i\frac{\pi}{2}}\phi^\dagger. \tag{A6}$$

Due to the β term, the effective symmetry at the critical point remains discrete for the $n = 4$ case.

b. GL action for XY ferromagnets with C_3 or C_6 rotation

For the hexagonal crystal family, on the contrary, the joint C_3 or C_6 rotational symmetry forbids the β term, so the corresponding gapless free theory does have the U(1) spacetime symmetry. To this end, we analyze the terms allowed in the action. The symmetries of the C_6 rotation around z, spatial inversion, and time reversal allow the following terms and their complex conjugates in the action,

$$|\phi|^2, (\partial_m\phi^\dagger)(\partial_m\phi), \phi\partial_-^2\phi, (|\phi|^2)^2, \phi^3\partial_+^2\phi, \phi^6, \dots, \tag{A7}$$

with $m = x, y, z$. When the C_6 rotation is substituted by the C_3 rotation, Eq. (A7) also exhausts all symmetry-allowed terms apart from higher-order ϕ terms, higher-order spatial-gradient terms, and total derivative terms. In fact, the C_3 rotational symmetry alone allows ϕ^3 and $\phi\partial_+\phi$, while the ϕ^3 term is prohibited by the time-reversal symmetry and $\phi\partial_+\phi$ is a total derivative term. Thus the partition function $\mathscr{Z}_{n=3,6}$ near the critical point is given by

$$\mathscr{Z}_{n=3,6} = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \exp[-\mathscr{S}_{n=3,6,\phi}]$$

$$\mathscr{S}_{n=3,6,\phi} \equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_\mathrm{B}T} d\tau \left\{ \frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) \right.$$
$$+ \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi)$$
$$+ \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2 + \tilde{\alpha}_6\phi^3(\partial_+^2\phi)$$

$$+ \tilde{\alpha}_6^*(\phi^\dagger)^3(\partial_-^2\phi^\dagger)] + \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2$$
$$+ \frac{1}{2}[\tilde{c}_6\phi^6 + \tilde{c}_6^*(\phi^\dagger)^6] + \cdots \right\}, \tag{A8}$$

where $\rho_0 > 0$ and $\rho_0 < 0$ correspond to ordered and disordered phases, respectively. Importantly, a gapless free theory part $\overline{\mathscr{S}}_{n=3,6,\phi}$ of the action,

$$\overline{\mathscr{S}}_{n=3,6,\phi} \equiv \int d^3\boldsymbol{r} \int_0^{\hbar/k_\mathrm{B}T} d\tau \left\{ \frac{\eta_1^2}{2}(\partial_\tau\phi^\dagger)(\partial_\tau\phi) \right.$$
$$+ \frac{\eta_1^2 c_\perp^2}{2}(\partial_j\phi^\dagger)(\partial_j\phi) + \frac{\eta_1^2 c_z^2}{2}(\partial_z\phi^\dagger)(\partial_z\phi)$$
$$\left. + \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2] \right\}, \tag{A9}$$

is symmetric under the U(1) spacetime symmetry,

$$\partial_\pm \to \partial_\pm' = e^{\pm i\epsilon}\partial_\pm, \quad \phi \to \phi' = e^{i\epsilon}\phi,$$
$$\phi^\dagger \to (\phi')^\dagger = e^{-i\epsilon}\phi^\dagger, \quad \text{for} \quad \forall\epsilon. \tag{A10}$$

This contrasts with the free theory for the $n = 4$ case which is symmetric only under the joint discrete rotational symmetry.

In the ordered phase ($\rho_0 > 0$) for the C_6 case, a phase of ϕ is locked into six minima by the $\tilde{c}_6$ term. The $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms reduce the symmetry of the whole action into a joint discrete ($\mathbb{Z}_6$) rotational symmetry,

$$\partial_\pm \to \partial_\pm' = e^{\pm i\frac{\pi}{3}}\partial_\pm, \quad \phi \to \phi' = e^{i\frac{\pi}{3}}\phi,$$
$$\phi^\dagger \to (\phi')^\dagger = e^{-i\frac{\pi}{3}}\phi^\dagger. \tag{A11}$$

In the ordered phase for the C_3 case, the phase of ϕ is locked into three minima by the $\tilde{c}_6$ term and other higher-ordered terms omitted as "..." in Eq. (A8). Nonetheless, unlike the β-term in $\mathscr{Z}_{n=4}$, $\tilde{\alpha}_6$ and $\tilde{c}_6$ terms as well as the higher-order terms are *irrelevant* in the long-wavelength limit at the quantum critical point, since their scaling dimensions at the critical point are all negative.

The scaling dimensions of $\tilde{\alpha}_6$ and $\tilde{c}_6$ terms at the critical point, y_{α_6} and y_{c_6}, can be evaluated from a dimensional analysis of the gapless free theory part at $T = 0$; $y_{\alpha_6} = 2 - D = -2$ and $y_{c_6} = 6 - 2D = -2$ with $D = 3 + 1$. Scaling dimensions of the higher-order ϕ terms and higher-order spatial gradient terms are also negative and smaller than y_{α_6} and y_{c_6}. When the hydrodynamic volume element becomes larger, the terms with negative scaling dimensions get smaller at the critical point. Thanks to their irrelevance at the critical point, the GL action respects effectively the U(1) spacetime symmetry in the long-wavelength limit. In other words, the spin hydrodynamics at the critical point becomes U(1) spacetime symmetric more effectively for larger hydrodynamic volume element. The hydrodynamic regime with the effective U(1) symmetry has a lower crossover boundary in its length scale; in order that the hydrodynamics has the effective U(1) spacetime symmetry, the length scale Λ of the volume element should be greater than a certain crossover length $\Lambda_{c,1}$,

$$\Lambda \gg \Lambda_{c,1}. \tag{A12}$$

The crossover length scale is dependent on $\tilde{c}_6$, $\tilde{\alpha}_6$, and other higher-order terms that manifest the joint discrete rotational

symmetries. In the C_6 case, for example, $\Lambda_{c,1}$ is primarily dependent on $\tilde{c}_6$, $\tilde{\alpha}_6$ and their scaling dimensions with the following scalings,

$$\Lambda_{c,1} \propto |\tilde{c}_6|^{\frac{1}{|y_{c_6}|}} \quad \text{or} \quad |\tilde{\alpha}_6|^{\frac{1}{|y_{\alpha_6}|}}. \tag{A13}$$

When the system is in the ordered phase but close to the critical point ($\rho_0 \gtrsim 0$), the hydrodynamics regime with the effective U(1) symmetry has also an upper bound in its length scale,

$$\Lambda_{c,2} \gg \Lambda \gg \Lambda_{c,1}. \tag{A14}$$

The upper bound is because the ground state for $\rho_0 > 0$ breaks the joint rotational symmetry spontaneously, and in this sense, $\tilde{c}_6$, $\tilde{\alpha}_6$, and the other higher-order terms manifesting the discrete symmetry are *dangerously* irrelevant. In typical renormalization group (RG) flow trajectory, they get smaller around a saddle-point fixed point for the critical point upon the increase of the length scale, while in the very long wavelength limit, they become larger again around another fixed point that describes an ordered phase with broken joint U(1) symmetry (a Nambu-Goldstone fixed point). The upper bound $\Lambda_{c,2}$ defines a length scale for this upturn behavior of the dangerously irrelevant scaling variables. Generally, $\Lambda_{c,2}$ has a complicated scaling form of ρ, as it also depends on scaling of the coupling constants around the Nambu-Goldstone fixed point. Nonetheless, $\Lambda_{c,2}$ is always greater than the lower bound, $\Lambda_{c,2} \gg \Lambda_{c,1}$, for smaller ρ_0. In the C_6 case, for example, $\Lambda_{c,2} \gg \Lambda_{c,1}$ is sastified when ρ_0, $\tilde{c}_6$, and $\tilde{\alpha}_6$ are in the following regimes:

$$\rho_0^{\frac{|y_{c_6}|}{y_{\rho_0}}} \ll \frac{1}{|\tilde{c}_6|}, \quad \rho_0^{\frac{|y_{\alpha_6}|}{y_{\rho_0}}} \ll \frac{1}{|\tilde{\alpha}_6|}. \tag{A15}$$

Here y_{ρ_0} is the scaling dimension of ρ_0 around the critical point; $y_{\rho_0} = 2$.

c. Continuum limit of generic XY ferromagnetic spin models in the 3D hexagonal crystal family

The above argument is solely based on the symmetry and scaling arguments, suggesting that *any XY* ferromagnetic spin models with the C_3 or C_6 rotational, spatial inversion, and time reversal symmetries has the effective U(1) spacetime symmetry near the quantum critical point, if the models undergo the continuous phase transition of the ferromagnetic ordering. In the following, we will argue this by deriving explicitly a continuum limit of *generic XY* ferromagnetic spin models with the symmetries.

Exchange interactions in spin-orbit coupled magnets generally comprise of symmetric part $\mathcal{J}_{ij,\mu\nu} = \mathcal{J}_{ij,\nu\mu}$ and

antisymmetric part $\mathcal{D}_{ij,\mu\nu} = -\mathcal{D}_{ij,\nu\mu}$ [Eq. (3)],

$$H_{\text{spin}} = \frac{1}{2} \sum_{i,j} \sum_{\mu,\nu} S_{i,\mu}(\mathcal{J}_{ij,\mu\nu} + \mathcal{D}_{ij,\mu\nu})S_{j,\nu}, \tag{A16}$$

with spin vector $\boldsymbol{S}_i \equiv (S_{i,x}, S_{i,y}, S_{i,z})$. We first consider that the spins live on the 3D hexagonal lattice with C_6 rotational and spatial inversion symmetries, namely the lattice belongs to either C_{6h} or D_{6h} point group. Further discussion of other possibilities of point groups will be provided below. Here, the exchange interactions are not only limited to those between the nearest neighboring sites on the lattice, but they can also be between further neighboring sites.

Near the transition point of the ferromagnetic ordering of the XY spins, the Z component of the spins fluctuates rapidly in space and time, so that one can legitimately integrate out the Z component, yielding effective spin models for the XY spins,

$$H^{\text{eff}} = \frac{1}{2} \sum_{i,j} \sum_{\mu,\nu=x,y} S_{i,\mu}(J_{ij,\mu\nu} + D_{ij,\mu\nu})S_{j,\nu}$$

$$= \frac{1}{2} \sum_{i,j} \left\{ (S_{i,x} \quad S_{i,y}) \begin{pmatrix} J_{ij,xx} & J_{ij,xy} \\ J_{ij,yx} & J_{ij,yy} \end{pmatrix} \begin{pmatrix} S_{j,x} \\ S_{j,y} \end{pmatrix} \right.$$

$$\left. + D_{ij,xy}[S_{i,x}S_{j,y} - S_{i,y}S_{j,x}] \right\}, \tag{A17}$$

with 2 by 2 symmetric and antisymmetric interactions, $J_{ij,\mu\nu} = J_{ij,\nu\mu}$ and $D_{ij,\mu\nu} = -D_{ij,\nu\mu}$ for $\mu, \nu = x, y$. The effective exchange interactions in Eq. (A17) as well as the exchange interactions in Eq. (A16) respect the joint C_6 rotational symmetry and inversion symmetry. In the following, we show that due to the C_6 rotational symmetry, the continuum limit of the symmetric exchange interactions in the effective spin models always take the same form as in Eq. (A8),

$$\frac{1}{2} \sum_{i,j} \sum_{\mu,\nu=x,y} S_{i,\mu}J_{ij,\mu\nu}S_{j,\nu}$$

$$\simeq \int d\boldsymbol{r}^3 \left\{ \frac{r}{2}|\phi|^2 + \frac{\eta_1^2 c_z^2}{2}\partial_z\phi^\dagger\partial_z\phi + \frac{\eta_1^2 c_\perp^2}{2}\sum_{i=x,y}\partial_i\phi^\dagger\partial_i\phi \right.$$

$$\left. + \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2] + \cdots \right\}. \tag{A18}$$

To see this, note first that any bond of two spin sites, $(\boldsymbol{i}, \boldsymbol{j})$, in a sum of Eq. (A17) has 5 other bonds in the sum that are derived from the first bond $(\boldsymbol{i}, \boldsymbol{j})$ by the C_6 rotation, i.e., $(C_6^n(\boldsymbol{i}), C_6^n(\boldsymbol{j}))$ ($n = 1, 2, \ldots, 5$). Due to the joint C_6 rotation symmetry, $J_{C_6^n(\boldsymbol{i})C_6^n(\boldsymbol{j}),\cdots}$ and $J_{\boldsymbol{ij},\cdots}$ are related by the C_6 spin rotation around z;

$$\begin{pmatrix} J_{C_6(i)C_6(j),xx} & J_{C_6(i)C_6(j),xy} \\ J_{C_6(i)C_6(j),yx} & J_{C_6(i)C_6(j),yy} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} J_{ij,xx} & J_{ij,xy} \\ J_{ij,yx} & J_{ij,yy} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}. \tag{A19}$$

Then, by using a gradient expansion, $S_{j,\mu} = S_{i,\mu} + (\boldsymbol{j} - \boldsymbol{i})_\lambda \partial_\lambda S_{i,\mu} + \frac{1}{2}(\boldsymbol{j} - \boldsymbol{i})_\lambda(\boldsymbol{j} - \boldsymbol{i})_\epsilon \partial_\lambda\partial_\epsilon S_{i,\mu} + \cdots$, one can explicitly show that a sum of the symmetric exchange interactions over the six bonds reduce to the same form of the continuum limit as Eq. (A18)

up to the second order in the gradient expansion;

$$\frac{1}{2} \sum_{n=0,1,\ldots,5} \begin{pmatrix} S_{C_6^n(i),x} & S_{C_6^n(i),y} \end{pmatrix} \begin{pmatrix} J_{C_6^n(i)C_6^n(j),xx} & J_{C_6^n(i)C_6^n(j),xy} \\ J_{C_6^n(i)C_6^n(j),yx} & J_{C_6^n(i)C_6^n(j),yy} \end{pmatrix} \begin{pmatrix} S_{C_6^n(j),x} \\ S_{C_6^n(j),y} \end{pmatrix}$$

$$\simeq \frac{3}{2}(\lambda_{ij,1}+\lambda_{ij,2})\phi^\dagger\phi + \frac{3a_{ij,\perp}^2}{16}\left\{ 2(\lambda_{ij,1}+\lambda_{ij,2})\phi^\dagger(\partial_x^2+\partial_y^2)\phi \right.$$

$$\left. + \sum_{m=1,2}\lambda_{ij,m}e^{2i(\varphi_{ij,\perp}-\psi_{ij,\perp,m})}\phi(\partial_x-i\partial_y)^2\phi + \text{c.c.} \right\} + \frac{3a_{ij,z}^2}{4}(\lambda_{ij,1}+\lambda_{ij,2})\phi^\dagger\partial_z^2\phi + \cdots \tag{A20}$$

Here ϕ on the right-hand side is from $\phi(\boldsymbol{r}_i) \equiv S_{i,x} + iS_{i,y}$ in Eq. (A17), and the higher-order derivative and total derivative terms are omitted. We also regard that $\boldsymbol{i}$ and $C_6^n(\boldsymbol{i})$ ($n = 1,\ldots,5$) are the same for the argument of ϕ, because their differences (if exist) can be controlled by the microscopic length. $a_{ij,\perp}$ and $a_{ij,z}$ are the spatial length of the bond $(\boldsymbol{i}, \boldsymbol{j})$ within the xy plane and along z axis, respectively; $a_{ij,\perp} \equiv |\boldsymbol{i}_\perp - \boldsymbol{j}_\perp|$, $a_{ij,z} \equiv |i_z - j_z|$, with $\boldsymbol{i} = (\boldsymbol{i}_\perp, i_z)$ and $\boldsymbol{j} = (\boldsymbol{j}_\perp, j_z)$. $\varphi_{ij,\perp}$ is the angle between $\boldsymbol{j}_\perp - \boldsymbol{i}_\perp$ and the x axis. $\lambda_{ij,m}$ and $\boldsymbol{t}_{ij,m}$ are real-valued eigenvalues and eigenvectors of the 2 by 2 symmetric matrix J_{ij} ($m = 1, 2$). $\psi_{ij,\perp,m}$ is the angle between $\boldsymbol{t}_{ij,m}$ and the x-axis in the xy plane. As $\boldsymbol{t}_{ij,1}$ and $\boldsymbol{t}_{ij,2}$ are orthogonal to each other, $\psi_{ij,2} = \psi_{ij,1} + \pi/2$ and $\sum_{m=1,2}\lambda_{ij,m}e^{2i(\varphi_{ij,\perp}-\psi_{ij,\perp,m})} = (\lambda_{ij,1}-\lambda_{ij,2})e^{2i(\varphi_{ij,\perp}-\psi_{ij,\perp,1})}$. A sum of Eq. (A20) over different types of bonds leads to Eq. (A18), where α is simply given by the sum of $a_{ij,\perp}^2 \sum_{m=1,2}\lambda_{ij,m}e^{2i(\varphi_{ij,\perp}-\psi_{ij,\perp,m})}$. Note that in the absence of the spin-orbit interaction, J_{ij} are always proportional to the unit matrix, where $\lambda_{ij,1} = \lambda_{ij,2}$, $(\lambda_{ij,1}-\lambda_{ij,2})e^{2i(\varphi_{ij,\perp}-\psi_{ij,\perp,1})} = 0$ for any bond $(\boldsymbol{i}, \boldsymbol{j})$, and α vanishes.

The continuum limit of the antisymmetric exchange interaction yields the first-order spatial gradient terms,

$$D_{ij,xy}(S_{i,x}S_{j,y} - S_{i,y}S_{j,x})$$
$$= iD_{ij,xy}(\boldsymbol{i}-\boldsymbol{j})_\mu(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) + \mathcal{O}(\partial^3). \tag{A21}$$

In the presence of the spatial inversion, they are canceled by its inversion symmetric counterpart;

$$D_{I(i)I(j),xy}(S_{I(i),x}S_{I(j),y} - S_{I(i),y}S_{I(j),x})$$
$$= -iD_{ij,xy}(\boldsymbol{i}-\boldsymbol{j})_\mu(\phi^\dagger\partial_\mu\phi - (\partial_\mu\phi^\dagger)\phi) + \mathcal{O}(\partial^3) \tag{A22}$$

with $D_{I(i)I(j),xy} = D_{ij,xy}$. Thus the antisymmetric interaction gives only higher-order gradient terms in the continuum limit for the GL action; they are all irrelevant in the sense that their scaling dimensions around the critical point are negative.

Near the quantum critical point of the ferromagnetic order of the xy spin, the systems effectively have the U(1) spacetime symmetry. Note that apart from the C_{6h} and D_{6h} point groups, the hexagonal crystal family (including the trigonal crystal system and the hexagonal crystal system) has 10 other point groups: C_3, C_{3i}, D_3, C_{3v}, D_{3d} from the trigonal crystal system and C_6, C_{3h}, D_6, C_{6v}, D_{3h} from the hexagonal crystal system. The GL action for C_3, D_3, C_6, and D_6 has an additional term, $i\gamma[\phi^\dagger\partial_z\phi - (\partial_z\phi^\dagger)\phi]$, that comes from the antisymmetric exchange interaction in Eq. (A21). Such a term is prohibited

for C_{3i}, D_{3d}, C_{6h}, and D_{6h} because there is the inversion symmetry. The term is also prohibited for C_{3v}, C_{3h}, C_{6v}, and D_{3h}. Although there is no inversion symmetry, for C_{3v}, C_{6v}, and D_{3h}, there is a vertical mirror symmetry that reflects only one component of the in-plane spin vector; for C_{3h} and D_{3h}, there is a horizontal mirror symmetry which makes the term opposite. In conclusion, a continuum limit of *generic XY* spin models in the hexagonal crystal family with a spatial inversion or rotoinversion symmetry as well as the time-reversal symmetry is described by Eq. (A8). For the C_3, D_3, C_6, and D_6 cases, the first-order z-derivative term can be eliminated by redefinitions of ϕ and ρ_0; $\phi_{\text{new}} = e^{-izA_z}\phi_{\text{old}}$ and $-U(\rho_0)_{\text{new}} = -U(\rho_0)_{\text{old}} - (\eta_1^2c_z^2A_z^2)/2$ with $\gamma = \eta_1^2c_z^2A_z/2$. If A_z is commensurate to the phase locking by the $\tilde{c}_6$ term, $3A_za_z = \mathbb{Z}\pi$ with an integer $\mathbb{Z}$ ($a_{ij,z} = a_z$), the partition function has no magnetic frustration and it describes the continuous phase transition from a disordered phase ($\rho_0 < 0$) to the XY ferromagnetic order phase with a spin-helix along z ($\rho_0 > 0$). As the first-order z-derivative term also respects the U(1) spacetime symmetry, the systems near the transition point have also the effective U(1) spacetime symmetry for these cases.

d. Hydrodynamics in an intermediate length scale near the quantum critical point

An analytic continuation of Eq. (A8) at $T = 0$ ($\tau \to it$) leads to the following real-time complex field theory $\tilde{\mathcal{L}}_\phi$:

$$\tilde{\mathcal{L}}_\phi = \frac{\eta_1^2}{2}(\partial_t\phi^\dagger)(\partial_t\phi) - \frac{\eta_1^2c_\perp^2}{2}(\partial_i\phi^\dagger)(\partial_i\phi)$$
$$- \frac{\eta_1^2c^2}{4}[\alpha(\partial_-\phi)^2 + \alpha^*(\partial_+\phi^\dagger)^2]$$
$$- \frac{\eta_1^2c^2}{4}[\tilde{\alpha}_6(\partial_+\phi)^2\phi^2$$
$$+ \tilde{\alpha}_6^*(\partial_-\phi^\dagger)^2(\phi^\dagger)^2] - \frac{U}{2}(\phi^\dagger\phi - \rho_0)^2$$
$$- \frac{1}{2}[\tilde{c}_6\phi^6 + \tilde{c}_6^*(\phi^\dagger)^6], \tag{A23}$$

where we take classical solutions of ϕ independent of z, so the term of $(\partial_z\phi^\dagger)(\partial_z\phi)$ is negligible. Here, for simplicity, let us take α, $\tilde{\alpha}_6$ and $\tilde{c}_6$ to be real, and assume that a coupling between the phase mode θ and the amplitude mode $\rho = \phi^\dagger\phi$ can be neglected. Then, we obtain an effective theory of the

phase mode θ,

$$
\begin{aligned}
\tilde{\mathcal{L}} =\ & \frac{\eta_1^2 \rho_0}{2}(\partial_t \theta)^2 - \frac{\eta_1^2 c_\perp^2 \rho_0}{2}(\partial_x \theta)^2 \\
& \times \left[1 - \alpha\cos(2\theta) - \tilde{\alpha}_6 \rho_0^2 \cos(4\theta) \right] \\
& - \frac{\eta_1^2 c_\perp^2 \rho_0}{2}(\partial_y \theta)^2 [1 + \alpha\cos(2\theta) + \tilde{\alpha}_6 \rho_0 \cos(4\theta)] \\
& + \eta_1^2 c_\perp^2 \rho_0 (\partial_x \theta)(\partial_y \theta)[\alpha\sin(2\theta) - \tilde{\alpha}_6 \rho_0 \sin(4\theta)] \\
& - \tilde{c}_6 \rho_0^3 \cos(6\theta).
\end{aligned}
\tag{A24}
$$

As in Eq. (A8), terms with higher-order derivatives or higher order in ρ_0 are neglected in Eq. (A24). Equation (A24) is symmetric under the joint $\mathbb{Z}_6$ rotation,

$$
\theta \to \theta + \frac{n\pi}{3},
$$
$$
\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\left(\frac{n\pi}{3}\right) & -\sin\left(\frac{n\pi}{3}\right) \\ \sin\left(\frac{n\pi}{3}\right) & \cos\left(\frac{n\pi}{3}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\tag{A25}
$$

while in the absence of $\tilde{\alpha}_6$ and $\tilde{c}_6$, it is symmetric under the joint U(1) rotation;

$$
\theta \to \theta + \epsilon, \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\tag{A26}
$$

for $\forall \epsilon$.

The U(1) theory becomes a good approximation theory for Eq. (A24), when ρ_0 approaches zero before $\partial_\mu \theta$ ($\mu = t, x, y$) approach zero. This is the case for the 3D spin model Eq. (A14) in the intermediate length scale near the quantum critical point. Thereby, the internal field ϕ is introduced as a spatial average of $S_{i,x} + iS_{i,y}$ over some hydrodynamic volume element. When the length scale of the volume element increases within the intermediate length scale, $\Lambda_{c,1} < \Lambda < \Lambda_{c,2}$, a scaling of ρ_0 and $\partial_\mu \theta$ is controlled by the quantum critical point; ρ_0 gets small faster than $\partial_\mu \theta$, and the approximation becomes better. On the other hand, when the length scale of the element becomes larger than the upper bound $\Lambda_{c,2}$, another scaling law from the Nambu-Goldstone fixed point kicks in, and the $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms become relevant again [47,48]. Besides, for a 2D quantum spin model, although the $\tilde{c}_6$ term is dangerously marginal instead of dangerously irrelevant from simple dimensional counting, as long as a bare value of $\tilde{c}_6$ is small enough, there is still an intermediate length scale where the U(1) theory is applicable. To summarize, the U(1) theory is effective near the quantum critical point only when θ fluctuates over a length in the intermediate length scale. When θ fluctuates more slowly than $\Lambda_{c,2}$, $\partial_\mu \theta$ becomes smaller than a small but finite ρ_0, and the $\tilde{c}_6$ and $\tilde{\alpha}_6$ terms dominate over the others, giving a large contribution to the EOM.

2. Spin-triplet exciton model

As another example of solid-state materials where the U(1) theory of spin dynamics is applicable, we consider semiconductors with electron excitations near a conduction-band bottom and hole excitations near a valence-band top around a high-symmetric k point, e.g., the Γ point. Near the band top and bottom, suppose the kinetic-energy bands can be approximately described by a rotational-symmetric continuous theory. The theory with relativistic spin-orbit interaction is expected to have joint continuous rotational symmetry.

To be specific, we consider a condensate of spin-triplet excitons in a 2D semiconductor model with Rashba-type spin-orbit interactions. The semiconductor model is given by [Eq. (5)]

$$
\begin{aligned}
H_{ex} =\ & \int d^2 r\, \boldsymbol{a}^\dagger \left[\left(-\frac{\partial_i^2}{2m_0} + \epsilon_{g0} \right)\boldsymbol{\sigma}_0 + \xi_{R0}(-i\partial_y \boldsymbol{\sigma}_x + i\partial_x \boldsymbol{\sigma}_y) \right]\boldsymbol{a} \\
& + \int d^2 r\, \boldsymbol{b}^\dagger \left[\left(\frac{\partial_i^2}{2m_0'} - \epsilon_{g0} \right)\boldsymbol{\sigma}_0 + \xi_{R0}'(i\partial_y \boldsymbol{\sigma}_x - i\partial_x \boldsymbol{\sigma}_y) \right]\boldsymbol{b} \\
& + \int d^2 r(\Delta_t \boldsymbol{a}^\dagger \boldsymbol{\sigma}_0 \boldsymbol{b} + \Delta_t^* \boldsymbol{b}^\dagger \boldsymbol{\sigma}_0 \boldsymbol{a}) + \frac{g_{s0}}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int d^2 r \\
& \times (a_\sigma^\dagger a_{\sigma'}^\dagger a_{\sigma'} a_\sigma + b_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} b_\sigma + 2\xi_1 a_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} a_\sigma),
\end{aligned}
\tag{A27}
$$

with $i = x, y$. Here $\boldsymbol{a}$ and $\boldsymbol{b}$ are spin-$\frac{1}{2}$ electron annihilation operators near the Γ point in the conduction and valence bands, respectively. We suppose inter-band interaction is smaller than intraband interaction, namely $0 < \xi_1 < 1$. Due to the attraction between electrons and holes ($\xi_1 g_{s0}$), the quasiparticles form bound states inside a band gap (ϵ_{g0}). The bound states have a spin-singlet component and spin-triplet components. In the presence of Rashba interaction (ξ_{R0}, ξ_{R0}') and interband "spinless" hopping (Δ_t), the in-plane component of the spin-triplet states undergoes Bose-Einstein condensation at $q = 0$. In the following, we will show that this condensation is described by Eq. (2) (without the c_z term).

For simplicity, we take the electron band and the hole band in a symmetric form, $m_0 = m_0'$, $\xi_{R0} = \xi_{R0}'$, while the derivation can be generalized into the case with $m_0 \xi_{R0} = m_0' \xi_{R0}'$. The derivation can be also applicable to a 3D model with a finite effective mass along z. Due to the Rashba interaction (ξ_{R0}) and interband tunneling (Δ_t), the system has only a U(1) rotational symmetry and a time-reversal symmetry,

$$
\boldsymbol{a} \to e^{-i\epsilon\sigma_z/2}\boldsymbol{a}, \quad \boldsymbol{b} \to e^{-i\epsilon\sigma_z/2}\boldsymbol{b},
$$
$$
\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\tag{A28}
$$
$$
\boldsymbol{a} \to i\sigma_y \boldsymbol{a}, \quad \boldsymbol{b} \to i\sigma_y \boldsymbol{b}, \quad t \to -t, \quad i \to -i.
\tag{A29}
$$

The quadratic part of the Hamiltonian Eq. (A27) is diagonalized,

$$
\begin{aligned}
H_{ex} =\ & \sum_{\boldsymbol{k}} \left\{ a_{\boldsymbol{k},\sigma_k}^\dagger \left[\frac{(|\boldsymbol{k}| - k_R)^2}{2m_0} + E_{g0} \right] a_{\boldsymbol{k},\sigma_k} \right. \\
& \left. - b_{\boldsymbol{k},\sigma_k}^\dagger \left[\frac{(|\boldsymbol{k}| - k_R)^2}{2m_0} + E_{g0} \right] b_{\boldsymbol{k},\sigma_k} \right\} \\
& + \sum_{\boldsymbol{k}} \left[\Delta_t a_{\boldsymbol{k},\sigma_k}^\dagger b_{\boldsymbol{k},\sigma_k} + \Delta_t^* b_{\boldsymbol{k},\sigma_k}^\dagger a_{\boldsymbol{k},\sigma_k} \right] + \frac{g_s}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \\
& \times \int d^2 r(a_\sigma^\dagger a_{\sigma'}^\dagger a_{\sigma'} a_\sigma + b_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} b_\sigma + 2 a_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} a_\sigma),
\end{aligned}
\tag{A30}
$$

where

$$k_R = m_0 \xi_{R0}, \quad E_{g0} = \epsilon_{g0} - \frac{k_R^2}{2m_0}, \tag{A31}$$

a_k and b_k are Fourier transforms of $a(r)$ and $b(r)$, σ_k denotes up spin along the direction of $\hat{z} \times \hat{k}$, $\hat{k} \equiv \frac{k}{|k|}$. Here we discard the down-spin bands of the conduction and valence bands, because they are higher in energy and they do not constitute low-energy exciton levels. Since excitons are formed by electrons and holes around the Γ point, we neglect $|k|$ dependence of the hybridization coefficients of the conduction and valence bands,

$$a_\sigma = \alpha_\sigma \cos\Theta - \beta_\sigma e^{i\Phi} \sin\Theta, \quad b_\sigma = \alpha_\sigma e^{-i\Phi} \sin\Theta + \beta_\sigma \cos\Theta, \tag{A32}$$

where

$$E_{g0} = \sqrt{E_{g0}^2 + |\Delta_t|^2} \cos 2\Theta, \quad \Delta_t = \sqrt{E_{g0}^2 + |\Delta_t|^2} e^{i\Phi} \sin 2\Theta. \tag{A33}$$

Taking Eq. (A32) into the interaction term and taking $\Phi = 0$ for simplicity, we get

$$g_{s0}(a_\sigma^\dagger a_{\sigma'}^\dagger a_{\sigma'} a_\sigma + b_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} b_\sigma + \xi_1 a_\sigma^\dagger b_{\sigma'}^\dagger b_{\sigma'} a_\sigma + \xi_1 b_\sigma^\dagger a_{\sigma'}^\dagger a_{\sigma'} b_\sigma)$$
$$= g_s(\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma + \beta_\sigma^\dagger \alpha_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma) + w g_s(\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma + \beta_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma) + w g_s(\alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma + \beta_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \alpha_\sigma) + \cdots, \tag{A34}$$

with

$$g_s = \frac{g_{s0}}{2}[\sin^2(2\Theta) + \xi_1 + \xi_1 \cos^2(2\Theta)], \quad w = \frac{(1-\xi_1)\sin^2(2\Theta)}{\sin^2(2\Theta) + \xi_1[1 + \cos^2(2\Theta)]}, \tag{A35}$$

and $0 < w < 1$. Here we only keep terms in exciton-pairing channels in the basis of α and β, $\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma$, $\alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma$, and $\beta_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \alpha_\sigma$. Neglected terms contain also hybridization between excitons and intraband collective modes, $\alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma$, which in the absence of the time-reversal symmetry leads to an additional cubic term $(\partial_+ \phi)(\phi^\dagger)^2$ in Eq. (2). The hybridization and other neglected terms can be safely omitted as the intraband collective modes are gapped excitations in the semiconductor. The Hamiltonian can be rewritten by the new basis,

$$H_{ex} = \sum_k \left\{ \alpha_{k,\sigma_k}^\dagger \left[\frac{(|k| - k_R)^2}{2m} + E_g \right] \alpha_{k,\sigma_k} - \beta_{k,\sigma_k}^\dagger \left[\frac{(|k| - k_R)^2}{2m} + E_g \right] \beta_{k,\sigma_k} \right\}$$
$$+ g_s \sum_{\sigma,\sigma'=\uparrow,\downarrow} \int d^2r \left(\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma + w \alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma + \frac{w}{2} \alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma + \frac{w}{2} \beta_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \alpha_\sigma \right), \tag{A36}$$

where

$$\sqrt{\left[\frac{(|k| - k_R)^2}{2m_0} + E_{g0} \right]^2 + |\Delta_t|^2} \approx \sqrt{E_{g0}^2 + |\Delta_t|^2} + \frac{E_{g0}}{\sqrt{E_{g0}^2 + |\Delta_t|^2}} \frac{(|k| - k_R)^2}{2m_0} = E_g + \frac{(|k| - k_R)^2}{2m}. \tag{A37}$$

Exciton operators are defined by $O_\mu = b^\dagger \sigma_\mu a$, where $\mu = 0, x, y, z$. In terms of a completeness relation

$$\frac{1}{2} \sum_\mu (\sigma_\mu)_{\alpha\beta} (\sigma_\mu)_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\beta\gamma}, \tag{A38}$$

the interaction terms are decomposed as follows:

$$\sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma = -\sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_{\sigma'} \beta_{\sigma'}^\dagger \alpha_\sigma = -\frac{1}{2} \sum_\mu O_\mu^\dagger O_\mu, \tag{A39}$$

$$\sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma = \sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_\sigma \beta_{\sigma'}^\dagger \alpha_{\sigma'} = O_0^\dagger O_0, \tag{A40}$$

$$\sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma \sim -\sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_{\sigma'} \alpha_{\sigma'}^\dagger \beta_\sigma + \sum_{\sigma,\sigma'} \alpha_\sigma^\dagger \beta_\sigma \alpha_{\sigma'}^\dagger \beta_{\sigma'} = -\frac{1}{2} \sum_\mu O_\mu^\dagger O_\mu^\dagger + O_0^\dagger O_0^\dagger. \tag{A41}$$

In Eq. (A41), we decompose the interaction in two different channels, so we do not divide the result by two. Adding Eqs. (A39)–(A41) together, we get

$$
g_s \sum_{\sigma,\sigma'} \left(\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma + w \alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma + \frac{w}{2} \alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma + \frac{w}{2} \beta_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \alpha_\sigma \right)
$$

$$
= g_s \left[-\frac{1}{2} \sum_\mu O_\mu^\dagger O_\mu + w O_0^\dagger O_0 - \frac{w}{4} \sum_\mu (O_\mu^\dagger O_\mu^\dagger + O_\mu O_\mu) + \frac{w}{2} (O_0^\dagger O_0^\dagger + O_0 O_0) \right]
$$

$$
= -\frac{g_s}{4} \left[\sum_r (2 O_r^\dagger O_r + w O_r^\dagger O_r^\dagger + w O_r O_r) + (2 - 4w) O_0^\dagger O_0 - w O_0^\dagger O_0^\dagger - w O_0 O_0 \right]
$$

$$
= -\frac{g_s}{2} \left[\sum_r (1 + w\cos 2q_r) P_r^2 + (1 - 2w - w\cos 2q_0) P_0^2 \right], \tag{A42}
$$

where $r = x, y, z$, $O_\mu \equiv P_\mu e^{iq_\mu}$. Note that due to w, the U(1) symmetry of the four-component exciton field reduces to a $\mathbb{Z}_2$ symmetry. Since $0 < w < 1$, $q_0 = \pm\frac{\pi}{2}$ and $q_r = 0, \pi$ are preferred by the interaction. Fluctuations of q_μ around the minima are gapped excitations, so they can be safely neglected. This leads to

$$
g_s \sum_{\sigma,\sigma'} \left(\alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \beta_{\sigma'} \alpha_\sigma + w \alpha_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \beta_\sigma - \frac{w}{2} \alpha_\sigma^\dagger \alpha_{\sigma'}^\dagger \beta_{\sigma'} \beta_\sigma - \frac{w}{2} \beta_\sigma^\dagger \beta_{\sigma'}^\dagger \alpha_{\sigma'} \alpha_\sigma \right)
$$

$$
= -\frac{g_s}{2} \left[\sum_r (1 + w) P_r^2 + (1 - w) P_0^2 \right]
$$

$$
= -\frac{g_s}{2} \left[\sum_r (1 + w) \left(\frac{O_r + O_r^\dagger}{2} \right)^2 + (1 - w) \left(\frac{O_0 - O_0^\dagger}{2i} \right)^2 \right]. \tag{A43}
$$

By the Hubbard-Stratonovich transformation, we can introduce real exciton fields ϕ_μ,

$$
\exp\left\{ \int d\tau d^2 \boldsymbol{r} \frac{g_s}{2} \left[\sum_r (1 + w) P_r^2 + (1 - w) P_0^2 \right] \right\}
$$

$$
= \int \mathcal{D}\phi_\mu \exp\left\{ -\int d\tau d^2 \boldsymbol{r} \left[-\sum_r \phi_r (O_r^\dagger + O_r) - i\phi_0 (O_0^\dagger - O_0) + \sum_r \frac{2}{g_s(1+w)} \phi_r^2 + \frac{2}{g_s(1-w)} \phi_0^2 \right] \right\}, \tag{A44}
$$

where ϕ_r and ϕ_0 have the physical meanings of $\frac{g_s(1+w)}{2}\langle P_r \rangle$ and $\frac{g_s(1-w)}{2}\langle P_0 \rangle$, respectively. Since $0 < w < 1$, the interaction term [Eq. (A43)] favors the triplet excitons (ϕ_r) over the singlet excitons (ϕ_0). The quadratic part of Eq. (A36) also lifts the four-fold degeneracy of ϕ_μ, while mass terms for ϕ_x and ϕ_y remain degenerate.

Due to the adjustment of the conduction band and valence band, $m_0 \xi_{R0} = m_0' \xi_{R0}'$, one can expect that momentum-energy dispersions of the exciton bands have minima at $q = 0$, so condensation of the exciton fields happen at the zero momentum. In the following, we keep track of all the four components, ϕ_μ ($\mu = 0, x, y, z$), in the derivation of Eq. (2), to see whether and when ϕ_x and ϕ_y achieve the lowest energy (smallest mass at $q = 0$) among others. Fermion fields can be integrated out,

$$
\int \mathcal{D}[\boldsymbol{a}^\dagger, \boldsymbol{b}^\dagger, \boldsymbol{a}, \boldsymbol{b}] \exp\left[-(\boldsymbol{a}^\dagger \quad \boldsymbol{b}^\dagger) \boldsymbol{G}^{-1} \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} \right] = \det(\boldsymbol{G}^{-1}) = e^{\mathrm{Tr}\ln(\boldsymbol{G}^{-1})}, \tag{A45}
$$

where

$$
\boldsymbol{G}^{-1} = \boldsymbol{G}_0^{-1} + \boldsymbol{G}_\phi, \tag{A46}
$$

$$
\boldsymbol{G}_0 = \begin{pmatrix} g_{0,k}^a \boldsymbol{P}_{\sigma_k} & 0 \\ 0 & g_{0,k}^b \boldsymbol{P}_{\sigma_k} \end{pmatrix} \equiv \begin{pmatrix} \left[-i\omega_n + \frac{(|\boldsymbol{k}| - k_R)^2}{2m} + E_g \right]^{-1} \frac{\sigma_0 + \sigma_{\hat{z} \times \hat{k}}}{2} & 0 \\ 0 & \left[-i\omega_n - \frac{(|\boldsymbol{k}| - k_R)^2}{2m} - E_g \right]^{-1} \frac{\sigma_0 + \sigma_{\hat{z} \times \hat{k}}}{2} \end{pmatrix}, \tag{A47}
$$

$$
\boldsymbol{G}_\phi = \begin{pmatrix} 0 & -\sum_r \phi_r \boldsymbol{\sigma}_r - i\phi_0 \boldsymbol{\sigma}_0 \\ -\sum_r \phi_r \boldsymbol{\sigma}_r + i\phi_0 \boldsymbol{\sigma}_0 & 0 \end{pmatrix}. \tag{A48}
$$

G_0^{-1} and G_ϕ are block-diagonal in the momentum-frequency space and the coordinate space, respectively. G_0 is diagonal in spin along $\hat{z} \times \hat{k}$ and its diagonal element is zero for the down spin,

$$P_{\sigma_k} = \tfrac{1}{2}(\sigma_0 + \sigma_{\widehat{z}\times\widehat{k}}) = \tfrac{1}{2}(\sigma_0 - \sigma_x\sin\theta_{\widehat{k}} + \sigma_y\cos\theta_{\widehat{k}}). \tag{A49}$$

The integration leads to an effective theory of the exciton fields,

$$\mathcal{S}_\phi[\phi_\mu] = -\mathrm{Trln}(\mathbf{1} + G_0 G_\phi) + \frac{2}{g_s}\int d\tau d^2\mathbf{r}\left(\sum_r \frac{1}{1+w}\phi_r^2 + \frac{1}{1-w}\phi_0^2\right), \tag{A50}$$

$$-\mathrm{Trln}(\mathbf{1} + G_0 G_\phi) = \frac{1}{2}\mathrm{Tr}(G_0 G_\phi G_0 G_\phi) + \frac{1}{4}\mathrm{Tr}(G_0 G_\phi G_0 G_\phi G_0 G_\phi G_0 G_\phi) + \cdots \tag{A51}$$

Note that "Tr" stands for traces over both spacetime and spin indices, while "tr" is trace over only spin indices (see below). To determine the form of the effective theory, we use the following relations:

$$P_{\sigma_k}\sigma_z P_{\sigma_k} = 0, \tag{A52}$$

$$P_{\sigma_k}\sigma_x P_{\sigma_k} = P_{\sigma_k}(\sigma_{\widehat{k}}\cos\theta_{\widehat{k}} - \sigma_{\widehat{z}\times\widehat{k}}\sin\theta_{\widehat{k}})P_{\sigma_k} = -\sin\theta_{\widehat{k}}P_{\sigma_k}\sigma_{\widehat{z}\times\widehat{k}} = -\sin\theta_{\widehat{k}}\sigma_{\widehat{z}\times\widehat{k}}P_{\sigma_k}, \tag{A53}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_0 P_{\sigma_k}\sigma_0) = \mathrm{tr}(P_{\sigma_k}^2) = \mathrm{tr}(P_{\sigma_k}) = 1, \tag{A54}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_x P_{\sigma_{k+q}}\sigma_x) = \tfrac{1}{4}\mathrm{tr}(\sigma_x\sigma_x) + \tfrac{1}{4}\mathrm{tr}(\sigma_{\widehat{z}\times\widehat{k}}\sigma_x\sigma_{\widehat{z}\times\widehat{k+q}}\sigma_x) = \tfrac{1}{2} + \tfrac{1}{4}\mathrm{tr}[(-\sin\theta_{\widehat{k}}\sigma_x + \cos\theta_{\widehat{k}}\sigma_y)\sigma_x(-\sin\theta_{\widehat{k+q}}\sigma_x + \cos\theta_{\widehat{k+q}}\sigma_y)\sigma_x]$$

$$= \tfrac{1}{2}(1 + \sin\theta_{\widehat{k}}\sin\theta_{\widehat{k+q}} - \cos\theta_{\widehat{k}}\cos\theta_{\widehat{k+q}}) = \tfrac{1}{2}[1 - \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}})], \tag{A55}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_y P_{\sigma_{k+q}}\sigma_y) = \tfrac{1}{2}[1 + \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}})], \tag{A56}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_x P_{\sigma_{k+q}}\sigma_y) = \tfrac{1}{4}\mathrm{tr}(\sigma_{\widehat{z}\times\widehat{k}}\sigma_x\sigma_{\widehat{z}\times\widehat{k+q}}\sigma_y) = \tfrac{1}{4}\mathrm{tr}[(-\sin\theta_{\widehat{k}}\sigma_x + \cos\theta_{\widehat{k}}\sigma_y)\sigma_x(-\sin\theta_{\widehat{k+q}}\sigma_x + \cos\theta_{\widehat{k+q}}\sigma_y)\sigma_y]$$

$$= -\tfrac{1}{2}(\sin\theta_{\widehat{k}}\cos\theta_{\widehat{k+q}} + \cos\theta_{\widehat{k}}\sin\theta_{\widehat{k+q}}) = -\tfrac{1}{2}\sin(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}), \tag{A57}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_y P_{\sigma_{k+q}}\sigma_x) = -\tfrac{1}{2}\sin(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}), \tag{A58}$$

$$\mathrm{tr}(P_{\sigma_k}\sigma_x P_{\sigma_k}\sigma_x P_{\sigma_k}\sigma_x P_{\sigma_k}\sigma_x) = \mathrm{tr}[(P_{\sigma_k}\sigma_x P_{\sigma_k})^4] = \mathrm{tr}[(P_{\sigma_k}\sigma_{\widehat{z}\times\widehat{k}}\sigma_{\widehat{z}\times\widehat{k}}P_{\sigma_k})^2]\sin^4\theta_{\widehat{k}} = \mathrm{tr}(P_{\sigma_k})\sin^4\theta_{\widehat{k}} = \sin^4\theta_{\widehat{k}}. \tag{A59}$$

As the exciton field $\phi_\mu(\mathbf{q})$ [Fourier transform of $\phi_\mu(\mathbf{r})$] at the zero momentum ($\mathbf{q} = 0$) is expected to have the smallest energy, we expand the effective theory in terms of small $\mathbf{q}$. The zeroth order in $\mathbf{q}$ gives the mass (M_μ) and the quartic term (U) from the first and the second terms in Eq. (A51), respectively. Let us first calculate the masses for the four exciton components ($\mu = 0, x, y, z$),

$$\frac{1}{2}\mathrm{Tr}(G_0 G_\phi G_0 G_\phi) \supset \sum_\mu \int d\tau d^2\mathbf{r}\phi_\mu^2\left[\frac{1}{\beta L^2}\sum_k g_{0,k}^a g_{0,k}^b \mathrm{tr}(P_{\sigma_k}\sigma_\mu P_{\sigma_k}\sigma_\mu)\right]$$

$$= \int d\tau d^2\mathbf{r}\phi_0^2\left(\frac{1}{\beta L^2}\sum_k g_{0,k}^a g_{0,k}^b\right) + \frac{1}{2}\int d\tau d^2\mathbf{r}(\phi_x^2 + \phi_y^2)\left(\frac{1}{\beta L^2}\sum_k g_{0,k}^a g_{0,k}^b\right), \tag{A60}$$

where we used Eqs. (A52) and (A54)–(A56) and $\sum_k \cos(2\theta_{\widehat{k}}) = \sum_k \sin(2\theta_{\widehat{k}}) = 0$. Taking Eqs. (A50) and (A60) together, we get

$$\frac{1}{2}\sum_\mu M_\mu \phi_\mu^2 = \frac{2}{g_s}\left(\sum_r \frac{1}{1+w}\phi_r^2 + \frac{1}{1-w}\phi_0^2\right) - \frac{D_0}{2}(\phi_x^2 + \phi_y^2 + 2\phi_0^2), \tag{A61}$$

where

$$D_0 = -\frac{1}{\beta L^2}\sum_k g_{0,k}^a g_{0,k}^b = -\frac{1}{\beta L^2}\sum_{n,k}\frac{1}{i\omega_n - \xi_k}\frac{1}{i\omega_n + \xi_k} = -\frac{1}{L^2}\sum_k \frac{n_F(\xi_k) - n_F(-\xi_k)}{2\xi_k} = \int \frac{d^2k}{(2\pi)^2}\frac{\tanh\left(\frac{1}{2}\beta\xi_k\right)}{2\xi_k} > 0, \tag{A62}$$

$$\xi_k \equiv \frac{(|\mathbf{k}| - k_R)^2}{2m} + E_g, \quad n_F(\xi_k) \equiv \frac{1}{e^{\beta\xi_k} + 1}, \tag{A63}$$

β is the inverse of temperature. So we have

$$M_z = \frac{4}{g_s(1+w)} > 0, \quad M_0 = \frac{4}{g_s(1-w)} - 2D_0, \quad M_i = \frac{4}{g_s(1+w)} - D_0, \tag{A64}$$

where $i = x, y$. $M_i < 0 < M_0$ is realized by

$$\frac{1}{1+w} < \frac{1}{4}g_s D_0 < \frac{1}{2(1-w)}. \tag{A65}$$

Given that $w > \frac{1}{3}$, the condition can be realized by proper g_s, β, and E_g. Given that the condition is satisfied, we henceforth consider the effective theory only of $\phi = \phi_x + i\phi_y$ and neglect the other exciton components as they are gapped modes.

Equation (2) takes the following form in the imaginary-time representation:

$$\mathcal{L}_{\phi,E} = \frac{\eta_1^2}{2}(\partial_\tau \phi^\dagger)(\partial_\tau \phi) + \frac{\eta_1^2 c_\perp^2}{2}(\partial_i \phi^\dagger)(\partial_i \phi) + \frac{\eta_1^2 c_\perp^2}{4}[\alpha(\partial_- \phi)^2 + \alpha^*(\partial_+ \phi^\dagger)^2] + \frac{U}{2}(\phi^\dagger \phi - \rho_0)^2, \tag{A66}$$

where the mass term was already obtained,

$$2U\rho_0 = M_i = D_0 - \frac{4}{g_s(1+w)}. \tag{A67}$$

To determine U, we calculate the quartic term in ϕ_x from the second term of Eq. (A51), using Eq. (A59) and $\sum_k \sin^4 \theta_{\widehat{k}} = \frac{3}{8}\sum_k$,

$$\frac{1}{4}\text{Tr}[(G_0 G_\phi)^4] \supset \frac{1}{\beta L^2}\sum_{q_1,q_2,q_3}\phi_x(q_1)\phi_x(q_2)\phi_x(q_3)\phi_x(-q_1-q_2-q_3)\left\{\frac{1}{2\beta L^2}\sum_k (g_{0,k}^a g_{0,k}^b)^2 \text{tr}[(P_{\sigma_k}\sigma_x)^4]\right\}$$

$$= \int d\tau d^2 r \phi_x^4\left\{\frac{1}{2\beta L^2}\sum_k (g_{0,k}^a g_{0,k}^b)^2 \text{tr}[(P_{\sigma_k}\sigma_x)^4]\right\} = \frac{3}{8}\int d\tau d^2 r \phi_x^4\left[\frac{1}{2\beta L^2}\sum_k (g_{0,k}^a g_{0,k}^b)^2\right]. \tag{A68}$$

Thus U is given by

$$U = \frac{3}{8\beta L^2}\sum_k (g_{0,k}^a g_{0,k}^b)^2 = \frac{3}{8}\int \frac{d^2 k}{(2\pi)^2}\left\{\frac{d}{dz}\left[\frac{1}{e^{\beta z}+1}\frac{1}{(z-\xi_k)^2}\right]\Big|_{z=-\xi_k} + \frac{d}{dz}\left[\frac{1}{e^{\beta z}+1}\frac{1}{(z+\xi_k)^2}\right]\Big|_{z=\xi_k}\right\}$$

$$= \frac{3}{8}\int \frac{d^2 k}{(2\pi)^2}\frac{1}{(2\xi_k)^2}\left[\frac{\tanh(\frac{1}{2}\beta\xi_k)}{\xi_k} - \frac{\beta}{1+\cosh(\beta\xi_k)}\right] > 0. \tag{A69}$$

To determine the coefficients of the lowest-order gradient terms, we define $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\phi = \phi_x + i\phi_y$, $\phi^\dagger = \phi_x - i\phi_y$, and set $\phi_0 = \phi_z = 0$ in G_ϕ in Eq. (A48). Note also that

$$G_\phi = \begin{pmatrix} 0 & -(\phi\sigma_- + \phi^\dagger\sigma_+) \\ -(\phi\sigma_- + \phi^\dagger\sigma_+) & 0 \end{pmatrix}, \tag{A70}$$

$$\text{tr}(P_{\sigma_k}\sigma_+ P_{\sigma_{k+q}}\sigma_-) = \frac{1}{8}[1 - \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}) + 1 + \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}})] = \frac{1}{4}, \tag{A71}$$

$$\text{tr}(P_{\sigma_k}\sigma_+ P_{\sigma_{k+q}}\sigma_+) = \frac{1}{8}[1 - \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}) - 1 - \cos(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}) - i\sin(\theta_{\widehat{k}} + \theta_{\widehat{k+q}}) - i\sin(\theta_{\widehat{k}} + \theta_{\widehat{k+q}})] = -\frac{1}{4}e^{i(\theta_{\widehat{k}}+\theta_{\widehat{k+q}})}, \tag{A72}$$

$$\text{tr}(P_{\sigma_k}\sigma_- P_{\sigma_{k+q}}\sigma_-) = -\frac{1}{4}e^{-i(\theta_{\widehat{k}}+\theta_{\widehat{k+q}})}. \tag{A73}$$

In terms of Eq. (A70), the first term of Eq. (A51) is given by

$$\frac{1}{2}\text{Tr}(G_0 G_\phi G_0 G_\phi) = \frac{1}{2\beta L^2}\sum_{k,q}(g_{0,k}^a g_{0,k+q}^b + g_{0,k}^b g_{0,k+q}^a)[\phi_q^\dagger \phi_q \text{tr}(P_{\sigma_k}\sigma_+ P_{\sigma_{k+q}}\sigma_-) + \phi_{-q}\phi_q \text{tr}(P_{\sigma_k}\sigma_- P_{\sigma_{k+q}}\sigma_-)] + \text{H.c.} \tag{A74}$$

By an expansion of small $q \equiv (\boldsymbol{q}, i\omega_m)$, we get

$$\eta_1^2 = -\frac{1}{4\beta L^2}\partial_{i\omega_m}^2\Big|_{q=0}\sum_k (g_{0,k}^a g_{0,k+q}^b + g_{0,k+q}^a g_{0,k}^b) = -\frac{1}{2\beta L^2}\partial_{i\omega_m}^2\Big|_{q=0}\sum_k g_{0,k-\frac{q}{2}}^a g_{0,k+\frac{q}{2}}^b, \tag{A75}$$

$$\eta_1^2 c^2 = \frac{1}{4\beta L^2}\partial_{q_x}^2\Big|_{q=0}\sum_k (g_{0,k}^a g_{0,k+q}^b + g_{0,k+q}^a g_{0,k}^b) = \frac{1}{2\beta L^2}\partial_{q_x}^2\Big|_{q=0}\sum_k g_{0,k-\frac{q}{2}}^a g_{0,k+\frac{q}{2}}^b, \tag{A76}$$

$$\eta_1^2 c^2 \alpha = \frac{1}{4\beta L^2}\partial_{q_x}^2\Big|_{q=0}\sum_k (g_{0,k}^a g_{0,k+q}^b + g_{0,k+q}^a g_{0,k}^b)e^{-i(\theta_{\widehat{k}}+\theta_{\widehat{k+q}})} = \frac{1}{2\beta L^2}\partial_{q_x}^2\Big|_{q=0}\sum_k g_{0,k-\frac{q}{2}}^a g_{0,k+\frac{q}{2}}^b e^{-i(\theta_{\widehat{k-\frac{q}{2}}}+\theta_{\widehat{k+\frac{q}{2}}})}, \tag{A77}$$

where we used

$$\sum_k g^a_{0,k} g^b_{0,k+q} = \sum_k g^a_{0,k-q} g^b_{0,k} = \sum_k g^a_{0,k-\frac{q}{2}} g^b_{0,k+\frac{q}{2}}. \tag{A78}$$

Equivalently, we can also use

$$\sum_k \frac{1}{2} \left(g^a_{0,k}\right)'' g^b_{0,k} = \sum_k \frac{1}{2} g^a_{0,k} \left(g^b_{0,k}\right)'' = \sum_k \left[\frac{1}{8} \left(g^a_{0,k}\right)'' g^b_{0,k} + \frac{1}{8} g^a_{0,k} \left(g^b_{0,k}\right)'' - \frac{1}{4} \left(g^a_{0,k}\right)' \left(g^b_{0,k}\right)'\right], \tag{A79}$$

or

$$\sum_k \left(g^a_{0,k}\right)'' g^b_{0,k} = \sum_k g^a_{0,k} \left(g^b_{0,k}\right)'' = -\sum_k \left(g^a_{0,k}\right)' \left(g^b_{0,k}\right)'. \tag{A80}$$

Here primes and double primes denote first-order and second-order derivatives with respect to one of the spacetime components of k. Note that Eqs. (A78) and (A80) are valid given that associated integrals vanish or are sufficiently small in the ultraviolet regime (large k region). Using them, we can determine η_1 and $\eta_1 c_\perp$ as follows:

$$
\begin{aligned}
\eta_1^2 &= \frac{1}{2\beta L^2} \sum_k \left(\partial_{i\omega_m}\big|_{q=0} g^a_{0,k+q}\right) \left(\partial_{i\omega_m}\big|_{q=0} g^b_{0,k+q}\right) \\
&= \frac{1}{2\beta L^2} \sum_k \frac{1}{(i\omega_n - \xi_{\boldsymbol{k}})^2} \frac{1}{(i\omega_n + \xi_{\boldsymbol{k}})^2} = \frac{1}{2\beta L^2} \sum_k \left(g^a_{0,k} g^b_{0,k}\right)^2 = \frac{4U}{3} > 0,
\end{aligned} \tag{A81}
$$

$$
\begin{aligned}
\eta_1^2 c_\perp^2 &= -\frac{1}{2\beta L^2} \sum_k \left(\partial_{q_x}\big|_{q=0} g^a_{0,k+q}\right) \left(\partial_{q_x}\big|_{q=0} g^b_{0,k+q}\right) \\
&= \frac{1}{2\beta L^2} \sum_k \frac{1}{(i\omega_n - \xi_{\boldsymbol{k}})^2} \frac{1}{(i\omega_n + \xi_{\boldsymbol{k}})^2} \frac{(|\boldsymbol{k}| - k_R)^2 k_x^2}{m^2 |\boldsymbol{k}|^2} \\
&= \frac{1}{2} \int \frac{d^2 \boldsymbol{k}}{(2\pi)^2} \frac{1}{(2\xi_{\boldsymbol{k}})^2} \left[\frac{\tanh\left(\frac{1}{2}\beta\xi_{\boldsymbol{k}}\right)}{\xi_{\boldsymbol{k}}} - \frac{\beta}{1 + \cosh(\beta\xi_{\boldsymbol{k}})}\right] \frac{(|\boldsymbol{k}| - k_R)^2}{2m^2} > 0.
\end{aligned} \tag{A82}
$$

To determine the coefficient of the spin-coordinate coupling term ($\eta_1^2 c_\perp^2 \alpha$), We can use a similar trick as Eqs. (A78) and (A80) to simplify Eq. (A77),

$$\eta_1^2 c_\perp^2 \alpha = -\frac{1}{2\beta L^2} \sum_k \partial_{q_x}\big|_{q=0} \left[g^a_{0,k+q} e^{-i\theta_{\widehat{k+q}}}\right] \partial_{q_x}\big|_{q=0} \left[g^b_{0,k+q} e^{-i\theta_{\widehat{k+q}}}\right], \tag{A83}$$

where

$$\partial_{q_x}\big|_{q=0} e^{-i\theta_{\widehat{k+q}}} = -ie^{-i\theta_{\widehat{k}}} \partial_{q_x}\big|_{q=0} \arctan\frac{k_y}{k_x + q_x} = ie^{-i\theta_{\widehat{k}}} \frac{k_y}{k_x^2 + k_y^2} = ie^{-i\theta_{\widehat{k}}} \frac{\sin\theta_{\widehat{k}}}{|\boldsymbol{k}|}. \tag{A84}$$

Then we have

$$
\begin{aligned}
\eta_1^2 c^2 \alpha &= -\frac{1}{2\beta L^2} \sum_k \left[-\left(g^a_{0,k}\right)^2 \frac{(|\boldsymbol{k}| - k_R)\cos\theta_{\widehat{k}}}{m} + ig^a_{0,k} \frac{\sin\theta_{\widehat{k}}}{|\boldsymbol{k}|}\right] \left[\left(g^b_{0,k}\right)^2 \frac{(|\boldsymbol{k}| - k_R)\cos\theta_{\widehat{k}}}{m} + ig^b_{0,k} \frac{\sin\theta_{\widehat{k}}}{|\boldsymbol{k}|}\right] e^{-2i\theta_{\widehat{k}}} \\
&= -\frac{1}{2\beta L^2} \sum_k \left[-\left(g^a_{0,k}\right)^2 \frac{|\boldsymbol{k}| - k_R}{2m} + g^a_{0,k} \frac{1}{2|\boldsymbol{k}|}\right] \left[\left(g^b_{0,k}\right)^2 \frac{|\boldsymbol{k}| - k_R}{2m} + g^b_{0,k} \frac{1}{2|\boldsymbol{k}|}\right] \\
&= -\frac{1}{8\beta L^2} \sum_k \left\{-\left(g^a_{0,k} g^b_{0,k}\right)^2 \frac{(|\boldsymbol{k}| - k_R)^2}{m^2} + g^a_{0,k} g^b_{0,k} \frac{1}{|\boldsymbol{k}|^2} + \left[g^a_{0,k}\left(g^b_{0,k}\right)^2 - \left(g^a_{0,k}\right)^2 g^b_{0,k}\right] \frac{|\boldsymbol{k}| - k_R}{m|\boldsymbol{k}|}\right\},
\end{aligned} \tag{A85}
$$

where

$$
\begin{aligned}
\sum_n \left[g^a_{0,k}\left(g^b_{0,k}\right)^2 - \left(g^a_{0,k}\right)^2 g^b_{0,k}\right] &= \sum_n \left[\frac{1}{-i\omega_n + \xi_{\boldsymbol{k}}}\left(\frac{1}{-i\omega_n - \xi_{\boldsymbol{k}}}\right)^2 - \left(\frac{1}{-i\omega_n + \xi_{\boldsymbol{k}}}\right)^2 \frac{1}{-i\omega_n - \xi_{\boldsymbol{k}}}\right] \\
&= \frac{(i\omega_n + \xi_{\boldsymbol{k}}) - (i\omega_n - \xi_{\boldsymbol{k}})}{(i\omega_n + \xi_{\boldsymbol{k}})^2 (i\omega_n - \xi_{\boldsymbol{k}})^2} = \sum_n 2\xi_{\boldsymbol{k}} \left(g^a_{0,k} g^b_{0,k}\right)^2,
\end{aligned} \tag{A86}
$$

$$\frac{|\boldsymbol{k}| - k_R}{m|\boldsymbol{k}|} \left[2\xi_{\boldsymbol{k}} - \frac{(|\boldsymbol{k}| - k_R)|\boldsymbol{k}|}{m}\right] = \frac{|\boldsymbol{k}| - k_R}{m|\boldsymbol{k}|} \left[2E_g - \frac{(|\boldsymbol{k}| - k_R)k_R}{m}\right]. \tag{A87}$$

In terms of Eqs. (A62), (A69), (A86), and (A87), we finally determine $\eta_1^2 c^2 \alpha$ as follows:

$$\eta_1^2 c_\perp^2 \alpha = \frac{1}{2} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \left\{ \frac{\tanh\left(\frac{1}{2}\beta\xi_{\mathbf{k}}\right)}{2\xi_{\mathbf{k}}} \frac{1}{4|\mathbf{k}|^2} + \frac{1}{(2\xi_{\mathbf{k}})^2} \left[\frac{\tanh\left(\frac{1}{2}\beta\xi_{\mathbf{k}}\right)}{\xi_{\mathbf{k}}} - \frac{\beta}{1 + \cosh(\beta\xi_{\mathbf{k}})} \right] \left[\frac{(|\mathbf{k}| - k_R)k_R}{m} - 2E_g \right] \frac{|\mathbf{k}| - k_R}{4m|\mathbf{k}|} \right\}. \quad \text{(A88)}$$

To evaluate α and $c_\perp$, note first that the omission of the down-spin bands in Eq. (A30) is justified when

$$1 \ll \beta E_g \ll \beta \frac{k_R^2}{2m}. \quad \text{(A89)}$$

The condition also implies that when the temperature is low enough, electrons and holes are excited only around k_R. This naturally lets us introduce an "ultraviolet" cutoff k_g in the integral over $|\mathbf{k}|$ in Eqs. (A82) and (A88),

$$\eta_1^2 c_\perp^2 = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{16\xi_{\mathbf{k}}^3} \frac{(|\mathbf{k}| - k_R)^2}{m^2} = \int_{k_R - k_g}^{k_R + k_g} \frac{dk}{2\pi} \frac{k}{16\xi_k^3} \frac{(k - k_R)^2}{m^2}, \quad \text{(A90)}$$

$$\eta_1^2 c_\perp^2 \alpha = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{16\xi_{\mathbf{k}}^3 |\mathbf{k}|^2} \left\{ \frac{k_R |\mathbf{k}|(|\mathbf{k}| - k_R)^2}{2m^2} - \frac{E_g |\mathbf{k}|(|\mathbf{k}| - k_R)}{m} + \left[\frac{(|\mathbf{k}| - k_R)^2}{2m} + E_g \right]^2 \right\}$$

$$= \int_{k_R - k_g}^{k_R + k_g} \frac{dk}{2\pi} \frac{1}{16\xi_k^3 k} \left\{ \frac{(k - k_R)^2 (k^2 + k_R^2)}{4m^2} - \frac{E_g k_R (k - k_R)}{m} + E_g^2 \right\}. \quad \text{(A91)}$$

The cutoff k_g satifies that $k_g = \mathcal{O}(\sqrt{2mE_g}) \ll k_R$. Note also that without the cutoff the integral in Eq. (A88) has the logarithmic divergence at $k = 0$. With the cutoff, We finally obtain

$$\eta_1^2 c_\perp^2 E_g = \int_{-k_g}^{k_g} \frac{dk}{2\pi} \frac{(k_R + k)E_g}{16} \left(\frac{k^2}{2m} + E_g \right)^{-3} \frac{k^2}{m^2} = \int_0^{k_g} \frac{dk}{\pi} \frac{k_R E_g}{16} \left(\frac{k^2}{2m} + E_g \right)^{-3} \frac{k^2}{m^2} = \mathcal{O}\left(\frac{k_R}{k_g} \right), \quad \text{(A92)}$$

$$\alpha \eta_1^2 c_\perp^2 E_g = \int_{-k_g}^{k_g} \frac{dk}{2\pi} \frac{(k_R + k)^{-1} E_g}{16} \left(\frac{k^2}{2m} + E_g \right)^{-3} \left\{ \frac{k\left[(k_R + k)^2 + k_R^2\right]}{4m^2} - \frac{E_g k_R k}{m} + E_g^2 \right\}$$

$$= \frac{1}{2} \eta_1^2 c_\perp^2 E_g + \mathcal{O}(1) < \frac{1}{2} \eta_1^2 c_\perp^2 E_g + \left| \mathcal{O}\left(\frac{k_R}{k_g} \right) \right|. \quad \text{(A93)}$$

A comparison between Eqs. (A92) and (A93) suggests that $|\alpha| = \frac{1}{2} < 1$ in the limit of Eq. (A89). $\alpha = \mathcal{O}(1)$ is due to the large spin-orbit-coupling limit, while $\alpha \ll \mathcal{O}(1)$ for smaller spin-orbit coupling. Nonetheless, the competition between different components of excitons will be more complicated in the smaller spin-orbit coupling case. $\eta_1^2 c_\perp^2 E_g \gg 1$ is consistent with the physical picture of Eq. (A89). In the large spin-orbit coupling limit, We can also simplify Eqs. (A62) and (A81) and all the coefficients in Eq. (2),

$$D_0 = \frac{4}{g_s(1 + w)} + 2U\rho_0 = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{2\xi_{\mathbf{k}}} = \int_0^{k_g} \frac{dk}{\pi} \frac{1}{2} \left(\frac{k^2}{2m} + E_g \right)^{-1}, \quad \text{(A94)}$$

$$\eta_1^2 = \frac{4U}{3} = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{8\xi_{\mathbf{k}}^3} = \int_0^{k_g} \frac{dk}{\pi} \frac{1}{8} \left(\frac{k^2}{2m} + E_g \right)^{-3}. \quad \text{(A95)}$$

To summarize, the U(1) theory of Eq. (2) can be derived as an effective theory for the spin-triplet exciton condensate phase in semiconductors with Rashba spin-orbit interaction. Thereby, the spinless inter-band mixing (Δ_t, Δ_t^*) and an attractive interaction g_s between electrons in the conduction band and holes in the valence band induces a condensation of the XY component of

the real part of the excitonic pairing, $\phi \propto \mathrm{Re}O_x + i\mathrm{Re}O_y$ with $O_j \equiv \langle \boldsymbol{b}^\dagger \sigma_j \boldsymbol{a} \rangle$ $(j = x, y)$. Physically reasonable values of the coefficients in the U(1) theory are obtained within certain limits.

APPENDIX B: DERIVATION AND CONSERVATION OF NOETHER'S CURRENT

In this Appendix, we derive the spin (j_μ^s) and orbital (j_μ^l) parts of Noether's current in Eqs. (7) and (8) and verify that the total angular momentum is conserved. We start with the classical effective theory Eq. (6),

$$\mathcal{L} = \tfrac{1}{2}(\partial_t\theta)^2 - \tfrac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)] - \tfrac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)(\partial_y\theta)\sin(2\theta). \tag{B1}$$

The theory has a U(1) spacetime symmetry,

$$\theta \to \theta + \epsilon\Delta\theta = \theta + \epsilon, \quad x \to x + \epsilon\Delta x = x - \epsilon y, \quad y \to y + \epsilon\Delta y = y + \epsilon x, \quad t \to t + \epsilon\Delta t = t. \tag{B2}$$

With the continuous symmetry Eq. (B2), Noether's theorem gives a conserved current,

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta + \left[\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)\right]\Delta x_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta + T_{\mu\nu}\Delta x_\nu, \tag{B3}$$

where $\mu, \nu \in \{t, x, y\}$, $\Delta x_\nu \in \{\Delta t, \Delta x, \Delta y\}$. $T_{\mu\nu} \equiv \delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)$ is a stress-energy tensor. The conserved current obeys $\partial_\mu j_\mu = 0$ as long as an EOM is satisfied,

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\right] - \frac{\partial\mathcal{L}}{\partial\theta} = 0. \tag{B4}$$

The EOM is given by

$$\begin{aligned}
&\partial_t^2\theta - \left(\partial_x^2\theta\right)[1 - \alpha\cos(2\theta)] - 2\alpha(\partial_x\theta)^2\sin(2\theta) - \left(\partial_y^2\theta\right)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_y\theta)^2\sin(2\theta) \\
&\quad + 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) + 4\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) + \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) \\
&= \partial_t^2\theta - \left(\partial_x^2\theta\right)[1 - \alpha\cos(2\theta)] - \left(\partial_y^2\theta\right)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) \\
&\quad - \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) \\
&= 0. \tag{B5}
\end{aligned}$$

The theory has spatial and temporal translational symmetries, which imposes a conservation rule on the stress-energy tensor,

$$\partial_\mu T_{\mu\nu} = \partial_\mu\left[\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)\right] = 0. \tag{B6}$$

The total angular momentum current of Eq. (B3) can be divided into spin angular momentum current j_s that does not depend on $T_{\mu\nu}$, and orbital angular momentum current j_l that depends on $T_{\mu\nu}$.

$$j_\mu^s = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\Delta\theta, \quad j_\mu^l = \left[\delta_{\mu\nu}\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}(\partial_\nu\theta)\right]\Delta x_\nu. \tag{B7}$$

Let us focus on the spin part. Spin angular momentum density is

$$s = j_t^s = \partial_t\theta, \tag{B8}$$

and corresponding spin currents are

$$j_x^s = -(\partial_x\theta)[1 - \alpha\cos(2\theta)] + \alpha(\partial_y\theta)\sin(2\theta), \tag{B9}$$

$$j_y^s = -(\partial_y\theta)[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)\sin(2\theta). \tag{B10}$$

The spin density and current are zero at equilibrium, $j_\mu^s = 0$. The spin angular momentum is not conserved,

$$\partial_\mu j_\mu^s = G = -\partial_\mu j_\mu^l. \tag{B11}$$

Local sources of the spin angular momentum are given by spin torque G. The torque represents the mutual conversion between orbital and spin angular momenta. Taking EOM (B5) into Eqs. (B8)–(B10), we get the spin torque as follows:

$$\begin{aligned}
G = \partial_\mu j_\mu^s &= \partial_t^2\theta - \left(\partial_x^2\theta\right)[1 - \alpha\cos(2\theta)] - \left(\partial_y^2\theta\right)[1 + \alpha\cos(2\theta)] - 2(\partial_x\theta)^2\alpha\sin(2\theta) + 2(\partial_y\theta)^2\alpha\sin(2\theta) \\
&\quad + 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) + 4\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) \\
&= -\alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta). \tag{B12}
\end{aligned}$$

The orbital-angular-momentum density and current are given by

$$l = j_t^l = y(\partial_t\theta)(\partial_x\theta) - x(\partial_t\theta)(\partial_y\theta), \tag{B13}$$

$$j_x^l = -y\left\{\tfrac{1}{2}(\partial_t\theta)^2 - \tfrac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)] + \tfrac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)]\right\} + x\{(\partial_x\theta)(\partial_y\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2\sin(2\theta)\}, \tag{B14}$$

$$j_y^l = x\left\{\tfrac{1}{2}(\partial_t\theta)^2 - \tfrac{1}{2}(\partial_x\theta)^2[1 - \alpha\cos(2\theta)] + \tfrac{1}{2}(\partial_y\theta)^2[1 + \alpha\cos(2\theta)]\right\} - y\{(\partial_x\theta)(\partial_y\theta)[1 + \alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta)\}. \tag{B15}$$

The orbital angular momentum density and current depends explicitly on spatial coordinates, and they depend on a choice of the origin for the spatial coordinates. Besides, EOM (B5) gives $\partial_t^2\theta$ instead of $\partial_t\theta$, while j_t^s as well as $\partial_t j_t^s$ contains $\partial_t\theta$. Nonetheless, we can verify the continuity equation Eq. (B11) directly, using Eqs. (B5) and (B13)–(B15). $\partial_\mu j_\mu^l$ is formally given by a term that has no explicit dependence on x and y, and terms that depend explicitly and linearly on the spatial coordinates. The latter terms vanish thanks to Eq. (B5);

$$\begin{aligned}
\frac{\partial(\partial_\mu j_\mu^l)}{\partial y}\bigg|_{x,\theta,\partial_\mu\theta} &= (\partial_t^2\theta)(\partial_x\theta) + (\partial_t\theta)(\partial_x\partial_t\theta) - (\partial_t\theta)(\partial_x\partial_t\theta) + (\partial_y\theta)(\partial_x\partial_y\theta)[1 + \alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2(\partial_x\theta)\sin(2\theta) \\
&\quad - (\partial_x\theta)(\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2(\partial_x\theta)\sin(2\theta) - (\partial_x\theta)(\partial_y^2\theta)[1 + \alpha\cos(2\theta)] \\
&\quad - (\partial_x\partial_y\theta)(\partial_y\theta)[1 + \alpha\cos(2\theta)] + 2\alpha(\partial_x\theta)(\partial_y\theta)^2\sin(2\theta) + 2\alpha(\partial_x\theta)(\partial_x\partial_y\theta)\sin(2\theta) + 2\alpha(\partial_x\theta)^2(\partial_y\theta)\cos(2\theta) \\
&= (\partial_t^2\theta)(\partial_x\theta) - (\partial_x\theta)(\partial_x^2\theta + \partial_y^2\theta) + \alpha(\partial_x\theta)\sin(2\theta)[(\partial_y\theta)^2 - (\partial_x\theta)^2 + 2(\partial_x\partial_y\theta)] + \alpha(\partial_x\theta)\cos(2\theta)[(\partial_x\theta)^2 \\
&\quad - (\partial_y\theta)^2 + 2(\partial_x\theta)(\partial_y\theta)] \\
&= 0, \tag{B16}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(\partial_\mu j_\mu^l)}{\partial x}\bigg|_{y,\theta,\partial_\mu\theta} &= -(\partial_t^2\theta)(\partial_y\theta) - (\partial_t\theta)(\partial_y\partial_t\theta) + (\partial_t\theta)(\partial_y\partial_t\theta) - (\partial_x\theta)(\partial_x\partial_y\theta)[1 - \alpha\cos(2\theta)] \\
&\quad - \alpha(\partial_x\theta)^2(\partial_y\theta)\sin(2\theta) + (\partial_y\theta)(\partial_y^2\theta)[1 + \alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2(\partial_y\theta)\sin(2\theta) + (\partial_y\theta)(\partial_x^2\theta)[1 - \alpha\cos(2\theta)] \\
&\quad + (\partial_x\partial_y\theta)(\partial_x\theta)[1 - \alpha\cos(2\theta)] + 2\alpha(\partial_y\theta)(\partial_x\theta)^2\sin(2\theta) - 2\alpha(\partial_y\theta)(\partial_x\partial_y\theta)\sin(2\theta) - 2\alpha(\partial_y\theta)^2(\partial_x\theta)\cos(2\theta) \\
&= -(\partial_t^2\theta)(\partial_y\theta) + (\partial_y\theta)(\partial_x^2\theta + \partial_y^2\theta) + \alpha(\partial_y\theta)\sin(2\theta)[(\partial_x\theta)^2 - (\partial_y\theta)^2 - 2(\partial_x\partial_y\theta)] \\
&\quad + \alpha(\partial_y\theta)\cos(2\theta)[(\partial_y\theta)^2 - (\partial_x\theta)^2 - 2(\partial_x\theta)(\partial_y\theta)] = 0. \tag{B17}
\end{aligned}$$

The former term is nothing but $-G$,

$$\begin{aligned}
\partial_\mu j_\mu^l - \frac{\partial(\partial_\mu j_\mu^l)}{\partial x}\bigg|_{y,\theta,\partial_\mu\theta} - \frac{\partial(\partial_\mu j_\mu^l)}{\partial y}\bigg|_{x,\theta,\partial_\mu\theta} &= (\partial_x\theta)(\partial_y\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_y\theta)^2\sin(2\theta) \\
&\quad - (\partial_x\theta)(\partial_y\theta)[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta) \\
&= \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta). \tag{B18}
\end{aligned}$$

Thus the total angular momentum is indeed conserved,

$$\partial_\mu j_\mu^l = \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta) - 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta) = -G. \tag{B19}$$

APPENDIX C: SOLUTIONS FOR THE SPIN-INJECTION MODEL

In this Appendix, we solve $\theta(x, t)$ in the spin-injection model, Eq. (9), together with the boundary condition, Eq. (12), and $j_x^s(x = 0, t) = j_0$. We consider a general junction parameter k_0 except $k_0 = 1$ (straight geometry), $k_0^2 = (1 + \frac{1}{j_0 r})^2$ (circular geometry), and $k_0 = 0$, while leaving discussions about solutions at these special parameter points for Appendix F.

1. Straight geometry without curvature

For the straight geometry without the curvature [Fig. 1(a)], let us consider the EOM in the 1D system,

$$\partial_t^2\theta = (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] + \alpha(\partial_x\theta)^2\sin(2\theta). \tag{C1}$$

The boundary conditions are given by

$$j_x^s(0, t) = j_0, \tag{C2}$$

$$s_c(L, t) = k_c j_{x,c}^s(L, t), \tag{C3}$$

with $k_c \equiv \frac{\chi}{\chi'}[D_s(\frac{1}{T_1'} + ic)]^{-1/2} + \frac{\chi}{\beta_t}$. Here "$c$" stands for the frequency of spin density and current at $x = L$, and Eq. (C3) is imposed for each frequency component of the density and current. The density and current are given by θ (Eq. (8)),

$$s = \partial_t \theta, \quad j_x^s = -(\partial_x \theta)[1 - \alpha \cos(2\theta)],$$
$$j_y^s = \alpha \partial_x \theta \sin(2\theta), \quad G = -\alpha(\partial_x \theta)^2 \sin(2\theta). \tag{C4}$$

We solve the EOM by a perturbative expansion of α. With $\theta(x, t) = \theta_0(x, t) + \mathcal{O}(\alpha)$, the zeroth order is

$$\partial_t^2 \theta_0 = \partial_x^2 \theta_0. \tag{C5}$$

The general solution of Eq. (C5) is

$$\theta_0(x, t) = At + Bx + F_0 + \sum_{c \in \mathbb{R}, c \neq 0} [F_c e^{ic(t-x)} + F_c' e^{ic(t+x)}], \tag{C6}$$

where A, B, F_c, F_c' are constants. Eq. (C2) leads to

$$\theta_0(x, t) = At - j_0 x + F_0 + \sum_{c \in \mathbb{R}, c \neq 0} 2F_c \cos(cx) e^{ict}. \tag{C7}$$

Since $\mathrm{Re}(k_c) > 0$, Eq. (C3) requires $F_c = 0$ for $c \neq 0$, and the zeroth order takes the following form:

$$\theta_0(x, t) = s(x, t)t - j_x^s(x, t)x + F(0)$$
$$= k_0 j_0 t - j_0 x + F_0. \tag{C8}$$

Here F_0 can be absorbed by a time translation so we take $F_0 = 0$. Note that without the spin-orbit coupling ($\alpha = 0$), the spin density and current are static,

$$j_x^s(0 < x < L) = j_0 + \mathcal{O}(\alpha),$$
$$s(0 < x < L) = k_0 j_0 + \mathcal{O}(\alpha). \tag{C9}$$

Upon a substitution of Eq. (C9) into Eq. (C1) and an expansion of Eq. (C1) in α, the first-order correction to the solution is given by an inhomogeneous linear differential equation. Thereby, the first-order solution has two parts, θ_1 and θ_2,

$$\theta(x, t) = \theta_0(x, t) + \theta_1(x, t) + \theta_2(x, t) + \mathcal{O}(\alpha^2), \tag{C10}$$

and θ_1 is a special solution of the inhomogenous equation,

$$\partial_t^2 \theta_1 - \partial_x^2 \theta_1 = -\alpha(\partial_x^2 \theta_0)\cos(2\theta_0) + \alpha(\partial_x \theta_0)^2 \sin(2\theta_0)$$
$$= \alpha j_0^2 \sin(2k_0 j_0 t - 2j_0 x). \tag{C11}$$

$\theta_2(x, t)$ is a solution of the homogeneous differential equation,

$$\partial_t^2 \theta_2 - \partial_x^2 \theta_2 = 0. \tag{C12}$$

With $\theta_1(x, t)$ and $\theta_2(x, t)$, the spin density and current should satisfy the BCs up to the first order in α.

Thanks to the linear x and t dependence of $\theta_0(x, t)$ and $k_0 \neq 1$, we can find a special solution,

$$\theta_1(x, t) = -\frac{\alpha}{4(k_0^2 - 1)} \sin(2k_0 j_0 t - 2j_0 x). \tag{C13}$$

$\theta_2(x, t)$ takes the same form as Eq. (C6). With these solutions, the spin density and current are given by the following up to

the first order in α,

$$s = k_0 j_0 - \frac{k_0 j_0 \alpha}{2(k_0^2 - 1)} \cos(2k_0 j_0 t - 2j_0 x) + \partial_t \theta_2 + \mathcal{O}(\alpha^2), \tag{C14}$$

$$j_x^s = j_0 - \alpha j_0 \cos(2k_0 j_0 t - 2j_0 x)$$
$$- \frac{j_0 \alpha}{2(k_0^2 - 1)} \cos(2k_0 j_0 t - 2j_0 x) - \partial_x \theta_2 + \mathcal{O}(\alpha^2)$$
$$= j_0 - \frac{j_0 \alpha(2k_0^2 - 1)}{2(k_0^2 - 1)} \cos(2k_0 j_0 t - 2j_0 x) - \partial_x \theta_2 + \mathcal{O}(\alpha^2). \tag{C15}$$

In order that Eqs. (C14) and (C15) satisfy the BCs, $\theta_2(x, t)$ must have the same frequency as $\theta_1(x, t)$,

$$\theta_2(x, t) = \alpha g e^{2ik_0 j_0(t-x)} + \alpha g' e^{2ik_0 j_0(t+x)} + \text{c.c.} \tag{C16}$$

Here g and g' are complex constants. By the same reasoning as in the text below Eq. (C7), other frequency components in $\theta_2(x, t)$ vanish. This leads to

$$s = \frac{1}{2} k_0 j_0 + \alpha k_0 j_0 e^{2ik_0 j_0 t}\left[2ige^{-2ik_0 j_0 x} - \frac{e^{-2ij_0 x}}{4(k_0^2 - 1)}\right]$$
$$+ 2i\alpha k_0 j_0 g' e^{2ik_0 j_0(t+x)} + \mathcal{O}(\alpha^2) + \text{c.c.}, \tag{C17}$$

$$j_x^s = \frac{1}{2} j_0 + \alpha j_0 e^{2ik_0 j_0 t}\left[2ik_0 g e^{-2ik_0 j_0 x} - \frac{(2k_0^2 - 1)e^{-2ij_0 x}}{4(k_0^2 - 1)}\right]$$
$$- 2i\alpha k_0 j_0 g' e^{2ik_0 j_0(t+x)} + \mathcal{O}(\alpha^2) + \text{c.c.} \tag{C18}$$

The boundary conditions Eqs. (C2) and (C3) require

$$\alpha j_0\left[2ik_0 g - \frac{2k_0^2 - 1}{4(k_0^2 - 1)}\right] - 2i\alpha k_0 j_0 g' = 0, \tag{C19}$$

$$\alpha j_0\left[2ik_0 g e^{-2ik_0 j_0 L} - \frac{e^{-2ij_0 L}}{4(k_0^2 - 1)}\right] + 2i\alpha j_0 k_0 g' e^{2ik_0 j_0 L}$$

$$= k\alpha j_0\left[2ik_0 g e^{-2ik_0 j_0 L} - \frac{(2k_0^2 - 1)e^{-2ij_0 L}}{4(k_0^2 - 1)}\right]$$
$$- 2ik\alpha k_0 j_0 g' e^{2ik_0 j_0 L}, \tag{C20}$$

with

$$k_0 = \frac{\chi}{\chi'}\left(\frac{D_s}{T_1'}\right)^{-\frac{1}{2}} + \frac{\chi}{\beta_t}, \tag{C21}$$

$$k \equiv k_{2k_0 j_0} = \frac{\chi}{\chi'}\left[D_s\left(\frac{1}{T_1'} + 2ik_0 j_0\right)\right]^{-\frac{1}{2}} + \frac{\chi}{\beta_t}, \tag{C22}$$

and $k_{-c} = (k_c)^*$ for a real number c. Equations (C19) and (C20) can be simplified,

$$2ik_0(g - g') = \frac{2k_0^2 - 1}{4(k_0^2 - 1)}, \tag{C23}$$

$$2ik_0(ge^{-2ik_0 j_0 L} + g' e^{2ik_0 j_0 L}) - 2ikk_0(ge^{-2ik_0 j_0 L} - g' e^{2ik_0 j_0 L})$$
$$= \frac{1 - (2k_0^2 - 1)k}{4(k_0^2 - 1)} e^{-2ij_0 L}. \tag{C24}$$

The two equations Eqs. (C23) and (C24) determine the two coefficients, g and g',

$$\begin{pmatrix} 1 & -1 \\ (1-k)e^{-ik_0\beta_L} & (1+k)e^{ik_0\beta_L} \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} = \frac{1}{8ik_0(k_0^2-1)} \begin{pmatrix} 2k_0^2-1 \\ (1-(2k_0^2-1)k)e^{-i\beta_L} \end{pmatrix},$$

(C25)

where $\beta_L \equiv 2j_0 L$. The solution of the equations is

$$g = \frac{(2k_0^2-1)(1+k)e^{ik_0\beta_L} + [1-(2k_0^2-1)k]e^{-i\beta_L}}{(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}} \frac{1}{8ik_0(k_0^2-1)},$$

(C26)

$$g' = \frac{[1-(2k_0^2-1)k]e^{-i\beta_L} - (2k_0^2-1)(1-k)e^{-ik_0\beta_L}}{(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}} \frac{1}{8ik_0(k_0^2-1)}.$$

(C27)

From Eqs. (C8), (C10), (C13), and (C16), we obtain

$$\theta(x,t) = j_0(k_0 t - x) - \frac{\alpha}{4(k_0^2-1)}\sin[2j_0(k_0 t - x)] + 2\alpha\mathrm{Re}(g+g')\cos(2k_0 j_0 t)\cos(2k_0 j_0 x)$$

$$+ 2\alpha\mathrm{Re}(g-g')\sin(2k_0 j_0 t)\sin(2k_0 j_0 x) - 2\alpha\mathrm{Im}(g+g')\sin(2k_0 j_0 t)\cos(2k_0 j_0 x)$$

$$+ 2\alpha\mathrm{Im}(g-g')\cos(2k_0 j_0 t)\sin(2k_0 j_0 x) + \mathcal{O}(\alpha^2).$$

(C28)

The solution has one frequency $(2k_0 j_0)$ in time and two wave numbers $(2j_0, 2k_0 j_0)$ in space. According to Eqs. (C23), (C26), and (C27), Eq. (C28) is nothing but Eq. (18) with

$$\eta \equiv 2i(g+g') = \frac{(2k_0^2-1)[(1+k)e^{ik_0\beta_L} - (1-k)e^{-ik_0\beta_L}] + 2[1-(2k_0^2-1)k]e^{-i\beta_L}}{4k_0(k_0^2-1)[(1+k)e^{ik_0\beta_L} + (1-k)e^{-ik_0\beta_L}]}.$$

(C29)

Higher-order solutions can be obtained by the same perturbative iteration method. In the solution, the spin density and current have the same periodicity in time as the first-order solution, $\pi(k_0 j_0)^{-1}$. This is because the inhomogeneous terms at every order keep the same discrete time translational symmetry as that for the first order. For irrational k_0, the solution is not periodic in the space coordinate x because of the superpositions of the two wave numbers. Nonetheless, the Fourier-transform in the space has two major peaks at $2j_0$ and $2k_0 j_0$.

2. Circular geometry with curvature

For the 1D circular geometry with a finite radius r of the curvature [insets of Figs. 1(b) and 1(c)], the Lagrangian is generalized as follows:

$$\mathcal{L} = \frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2r^2}(\partial_\vartheta\theta)^2 + \frac{\alpha}{2r^2}(\partial_\vartheta\theta)^2[\cos(2\theta)(\sin^2\vartheta - \cos^2\vartheta) - 2\sin(2\theta)\sin\theta\cos\theta]$$

$$= \frac{1}{2}(\partial_t\theta)^2 - \frac{1}{2}(\partial_\ell\theta)^2\left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right)\right],$$

(C30)

with a 1D coordinate $\ell \equiv r\vartheta$, and

$$\partial_r\theta(x,y) \equiv \partial_r(r\cos\vartheta, r\sin\vartheta) = 0, \quad \partial_x = -\frac{1}{r}(\sin\vartheta)\partial_\vartheta, \quad \partial_y = \frac{1}{r}(\cos\vartheta)\partial_\vartheta.$$

(C31)

The Lagrangian gives the classical EOM in the 1D system,

$$\partial_t^2\theta - (\partial_\ell^2\theta)\left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right)\right] + 2\alpha(\partial_\ell\theta)\sin\left(2\theta - \frac{2}{r}\ell\right)\left[(\partial_\ell\theta) - \frac{1}{r}\right] - \alpha(\partial_\ell\theta)^2\sin\left(2\theta - \frac{2}{r}\ell\right)$$

$$= \partial_t^2\theta - (\partial_\ell^2\theta)\left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right)\right] + \alpha(\partial_\ell\theta)\left[(\partial_\ell\theta) - \frac{2}{r}\right]\sin\left(2\theta - \frac{2}{r}\ell\right) = 0,$$

(C32)

and the spin density and current,

$$s = \partial_t\theta, \quad j_\ell^s = -(\partial_\ell\theta)\left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right)\right].$$

(C33)

The boundary conditions are imposed on the spin density and current,

$$j_\ell^s(0,t) = j_0, \quad s_c(L,t) = k_c j_{\ell,c}^s(L,t).$$

(C34)

The boundary condition at $\ell = L$ is imposed on every frequency (c) component of the density and current, and $k_c = \frac{\chi}{\chi'}[D_s(\frac{1}{T_1'} + ic)]^{-1/2} + \frac{\chi}{\beta_t}$.

The zeroth-order solution of the EOM that satisfies the BCs is given by

$$\theta_0(\ell, t) = k_0 j_0 t - j_0 \ell. \tag{C35}$$

In the perturbative iteration method, the first-order solution comprises of $\theta_1(\ell, t)$ and $\theta_2(\ell, t)$. $\theta_1(\ell, t)$ is a special solution of the inhomogeneous linear differential equation, Eq. (20), while $\theta_2(\ell, t)$ is a solution of the homogeneous linear differential equation. For $k_0^2 \neq (1 + \frac{1}{j_0 r})^2$, we find the special solution,

$$\theta_1(\ell, t) = \frac{\alpha(1 + \frac{2}{j_0 r})}{4[k_0^2 - (1 + \frac{1}{j_0 r})^2]} \sin[2k_0 j_0 t - 2(j_0 + \frac{1}{r})\ell], \tag{C36}$$

together with

$$\theta_2(\ell, t) = At + B\ell + F_0 + \sum_{c \in \mathbb{R}, c \neq 0} [F_c e^{ic(t-\ell)} + F'_c e^{ic(t+\ell)}]. \tag{C37}$$

A substitution of $\theta = \theta_0 + \theta_1 + \theta_2 + \mathcal{O}(\alpha^2)$ into Eq. (C33) leads to

$$s = k_0 j_0 + \frac{k_0 j_0 \alpha(1 + \frac{2}{j_0 r})}{2[k_0^2 - (1 + \frac{1}{j_0 r})^2]} \cos\left[2k_0 j_0 t - 2\left(j_0 + \frac{1}{r}\right)\ell\right] + \partial_t \theta_2 + \mathcal{O}(\alpha^2), \tag{C38}$$

$$j_\ell^s = j_0 + \frac{\alpha j_0}{2} \frac{2k_0^2 - 2(1 + \frac{1}{j_0 r})^2 + (1 + \frac{1}{j_0 r})(1 + \frac{2}{j_0 r})}{k_0^2 - (1 + \frac{1}{j_0 r})^2} \cos\left[2k_0 j_0 t - 2\left(j_0 + \frac{1}{r}\right)\ell\right] - \partial_x \theta_2 + \mathcal{O}(\alpha^2)$$

$$= j_0 + \frac{j_0 \alpha[2k_0^2 - (1 + \frac{1}{j_0 r})]}{2[k_0^2 - (1 + \frac{1}{j_0 r})^2]} \cos\left[2k_0 j_0 t - 2\left(j_0 + \frac{1}{r}\right)\ell\right] - \partial_x \theta_2 + \mathcal{O}(\alpha^2). \tag{C39}$$

In order that Eqs. (C38) and (C39) satisfy the BCs, $\theta_2(\ell, t)$ must have the same frequency as $\theta_1(\ell, t)$;

$$\theta_2(\ell, t) = \alpha g e^{2ik_0 j_0(t-\ell)} + \alpha g' e^{2ik_0 j_0(t+\ell)} + \text{c.c.} \tag{C40}$$

The complex constants, g and g', are determined by

$$\begin{pmatrix} 1 & -1 \\ (1-k)e^{-ik_0 \beta_L} & (1+k)e^{ik_0 \beta_L} \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} = \frac{1}{8ik_0[k_0^2 - (1 + \frac{1}{j_0 r})^2]} \begin{pmatrix} 2k_0^2 - (1 + \frac{1}{j_0 r}) \\ \{1 + \frac{2}{j_0 r} - [2k_0^2 - (1 + \frac{1}{j_0 r})]k\}e^{-i(1 + \frac{1}{j_0 r})\beta_L} \end{pmatrix}, \tag{C41}$$

where k is given by Eq. (C22).

$\theta_1(\ell, t)$ vanishes when $j_0 = -\frac{2}{r}$. Even when $\theta_1(\ell, t) = 0$, $\theta_2(\ell, t) \neq 0$ in general. The nonzero $\theta_2(\ell, t)$ comes from an $\mathcal{O}(\alpha)$ contribution of j_ℓ^s in Eq. (C33). For $j_0 = -\frac{2}{r}$ and $k_0 = \frac{1}{2}$, both $\theta_1(\ell, t)$ and $\theta_2(\ell, t)$ reduce to zero, and $\theta_0(\ell, t)$ becomes an "exact" solution satisfying the BCs. However, the exactness is not protected by the symmetry of the theory, and there will be a finite $\theta_1(\ell, t)$ when higher-order expansion terms are considered in Eq. (6).

Besides, Eqs. (C25) and (C41) always have unique solutions for g and g', because

$$(1+k)e^{ik_0 \beta_L} + (1-k)e^{-ik_0 \beta_L} = 0 \tag{C42}$$

or

$$k = \frac{e^{-ik_0 \beta_L} + e^{ik_0 \beta_L}}{e^{-ik_0 \beta_L} - e^{ik_0 \beta_L}} = \frac{i}{\tan(2k_0 \beta_L)} \tag{C43}$$

contradicts with $\text{Re}(k) > 0$. The solutions of Eqs. (C25) and (C41) are divergent at $k_0 = 1$ (straight) and $k_0^2 = (1 + \frac{1}{j_0 r})^2$ (circular), respectively. Physically, the divergence could be avoided by finite dissipation time T_1. A more detailed discussion on the divergence is given in Appendix F.

The radius (r) dependence of $\theta(\ell, t)$ leads to the nonreciprocity of the hydrodynamic spin transport. We show the

nonreciprocity in insets of Figs. 1(b) and 1(c). The nonreciprocity is essentially from $\theta_1(\ell, t)$, as the spatial wavelength of $\theta_1(\ell, t)$ depends on the radius r; it also comes from $\theta_2(\ell, t)$ as $\theta_2(\ell, t)$ is different for two opposite currents to satisfy the boundary conditions. From the figures, we can see that $\theta(\ell, t)$ is periodic along t because $\theta_1(\ell, t)$ and $\theta_2(\ell, t)$ share the same temporal frequency; the structure of $\theta(\ell, t)$ along ℓ is more complicated because $\theta_1(\ell, t)$ and $\theta_2(\ell, t)$ give two different spatial wave numbers. To show the spatial structure of $\theta(\ell, t)$ more clearly, we plot them in a larger range of ℓ (see Fig. 2), although physically ℓ should not be greater than L, and L should not be greater than $2\pi r$.

APPENDIX D: POSSIBILITY OF DISSIPATION

In this Appendix, we study the stability of the superfluid state with a finite supercurrent in the U(1) spacetime theory. The Lagrangian Eq. (6) leads to a classical EOM, Eq. (B5). A solution of the EOM was obtained under boundary conditions of a finite current (Sec. III). The solution characterizes the supercurrent state. To study the stability of the supercurrent state, we compare the classical energy of the solution $\theta(x, y, t)$ with the energy of other solutions of the EOM with different
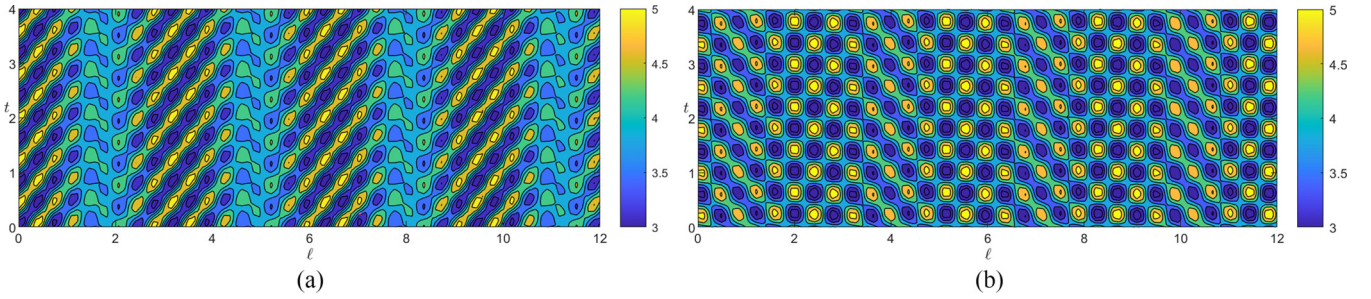
FIG. 2. Contour plots of $\theta(\ell, t)$ in a larger spatial range of ℓ with the same parameters in insets of Figs. 1(b) and 1(c). (a) and (b) above correspond to Figs. 1(b) and 1(c), respectively.

BCs, say $\theta(x, y, t) + \delta\theta(x, y, t)$. Here, we consider that $\delta\theta$ is a deformation induced by spatially local perturbations. Thus the spacetime derivatives of $\delta\theta(x, y, t)$ do not contain any uniform component in spacetime. The classical energy is evaluated by a Hamiltonian that corresponds to the Lagrangian Eq. (6),

$$
H[\theta] = \int d^2 \boldsymbol{r} \left\{ \frac{1}{2}(\partial_t \theta)^2 + \frac{1}{2}(\partial_x \theta)^2 [1 - \alpha \cos(2\theta)] \right.
$$
$$
\left. + \frac{1}{2}(\partial_y \theta)^2 [1 + \alpha \cos(2\theta)] - \alpha(\partial_x \theta)(\partial_y \theta) \sin(2\theta) \right\}.
$$
(D1)

The solution for the supercurrent state with broken U(1) spacetime symmetry depends on time, e.g., Eq. (18), while the Hamiltonian of θ and $\theta + \delta\theta$ are conserved, i.e., time-independent. Thus, for clarity of calculation, we compare the "time averages" of the classical energies over a large period of time T,

$$
\Delta J = \lim_{T \to \infty} \frac{1}{T} \left(\int_0^T H[\theta + \delta\theta] dt - \int_0^T H[\theta] dt \right). \quad \text{(D2)}
$$

When the classical energy of θ is lower than $\theta + \delta\theta$ for arbitrary small $\delta\theta$, the supercurrent state of $\theta(x, t)$ is stable against the local perturbation. If it is not for some $\delta\theta$, the supercurrent state is no longer stable, and it must experience energy dissipation. Effects of the energy dissipation can be included as finite relaxation time into the classical EOM (see Appendix E). To demonstrate the validity of our method used in this Appendix, we also apply the same method to a conventional superfluid moving at a finite velocity and derive its Landau criterion (see Appendix G).

As explained above, $\delta\theta$ is a deformation induced by the local perturbations and the spacetime derivatives of $\delta\theta$ are considered to be always zero on average. The locality of $\delta\theta$ is crucial in the following argument. For example, a finite average of the space derivative of $\delta\theta$ changes a uniform current, and such $\delta\theta$ should be excluded from the local deformation. This is because even for the conventional superfluid, the classical energy with a smaller velocity of the supercurrent will always decrease. In the derivation of the Landau criterion in Appendix G, only single excitations with (k, ω) are considered; perturbation that lowers the average velocity is excluded implicitly.

In this Appendix, we apply the stability analysis to the total-angular-momentum superfluid in the spin-injection

model. We consider a general value of the junction parameter k_0, except for $k_0 = 1$ (straight geometry), $k_0^2 = (1 + \frac{1}{j_0 r})^2$ (circular geometry), and $k_0 = 0$, while leaving discussions about some of these points for Appendix F. In the following, let us study the straight geometry case.

In the 1D spin injection model, θ depends only on x and t; $\theta(x, t)$ and $\theta(x, t) + \delta\theta(x, t)$. The energy difference between θ and $\theta + \delta\theta$ will be evaluated order by order in powers of the SOC (α). To this end, we expand θ and $\delta\theta$ in powers of α,

$$
\theta = \theta_0(x, t) + \theta_1'(x, t) + \mathcal{O}(\alpha^2), \quad \text{(D3)}
$$

$$
\delta\theta(x, t) = \delta\theta_0(x, t) + \delta\theta_1(x, t) + \mathcal{O}(\alpha^2), \quad \text{(D4)}
$$

where

$$
\theta_1'(x, t) = \theta_1(x, t) + \theta_2(x, t), \quad \text{(D5)}
$$

$\delta\theta_1(x, t) = \mathcal{O}(\alpha)$. $\delta\theta_0(x, t)$, $\theta_0(x, t)$, and $\theta_2(x, t)$ are solutions of Eq. (C5). Since the spacetime derivatives of $\delta\theta(x, t)$ is not uniform, $\delta\theta_0(x, t)$, as well as $\theta_2(x, t)$, is given by linear superpositions of $e^{iq(x-t)}$ and $e^{iq(x+t)}$ over q, e.g.

$$
\delta\theta_0(x, t) = \frac{1}{\sqrt{L}} \sum_q [\delta d_q e^{iq(x-t)} + \delta d_q' e^{iq(x+t)}], \quad \text{(D6)}
$$

with the system length L. In Appendix H, we give a perturbation theory in the SOC (α) that determines the higher order corrections (e.g., $\delta\theta_1$) for a given $\delta\theta_0$ in the form of Eq. (D6).

Given the α-expansions of θ and $\theta + \delta\theta$, we now evaluate their energy difference order by order in the power of α. We expand ΔJ in Eq. (D2),

$$
\Delta J = \delta J + \frac{1}{2}\delta^2 J + \mathcal{O}((\delta\theta)^2). \quad \text{(D7)}
$$

Here δJ and $\delta^2 J$ are at the first and second order in $\delta\theta$, respectively. The first-order variation is, i.e., Eq. (22)

$$
\delta J = \frac{1}{T} \int_0^T dt \int dx \{ (\partial_t \theta)(\partial_t \delta\theta) + (\partial_x \theta)(\partial_x \delta\theta)
$$
$$
\times [1 - \alpha \cos(2\theta)] + \alpha(\partial_x \theta)^2 \sin(2\theta)(\delta\theta) \}
$$
$$
= \frac{1}{T} \int_0^T dt \int dx [(\partial_t \theta_0)(\partial_t \delta\theta_0) + (\partial_t \theta_1')(\partial_t \delta\theta_0)
$$
$$
+ (\partial_t \theta_0)(\partial_t \delta\theta_1) + (\partial_x \theta_0)(\partial_x \delta\theta_0) + (\partial_x \theta_1')(\partial_x \delta\theta_0)
$$
$$
+ (\partial_x \theta_0)(\partial_x \delta\theta_1) - \alpha(\partial_x \theta_0)(\partial_x \delta\theta_0)\cos(2\theta_0)
$$
$$
+ \alpha(\partial_x \theta_0)^2 \sin(2\theta_0)(\delta\theta_0)] + \mathcal{O}(\alpha^2)
$$

$$= \frac{1}{T} \int_0^T dt \int dx \{ (\partial_t \theta_1')(\partial_t \delta\theta_0) + (\partial_x \theta_1')(\partial_x \delta\theta_0)$$
$$+ \alpha[(\partial_x^2 \theta_0)\cos(2\theta_0) - (\partial_x \theta_0)^2 \sin(2\theta_0)]\delta\theta_0 \} + \mathcal{O}(\alpha^2)$$

$$= \frac{1}{T} \int_0^T dt \int dx [(\partial_t \theta_1')(\partial_t \delta\theta_0) + (\partial_x \theta_1')(\partial_x \delta\theta_0)$$
$$+ (-\partial_t^2 \theta_1 + \partial_x^2 \theta_1)\delta\theta_0] + \mathcal{O}(\alpha^2)$$

$$= \frac{2}{T} \int_0^T dt \int dx (\partial_x \theta_2)(\partial_x \delta\theta_0) + \mathcal{O}(\alpha^2), \tag{D8}$$

where we neglect boundary contributions in the right-hand side, e.g.,

$$\frac{1}{T} \int_0^T dt (\partial_t \theta_0)(\partial_t \delta\theta_1) = \frac{k_0 j_0}{T} \int_0^T dt (\partial_t \delta\theta_1) = \mathcal{O}(T^{-1}),$$
$$\int dx (\partial_x \theta_0)(\partial_x \delta\theta_1) = -j_0 \int dx (\partial_x \delta\theta_1)$$
$$= \mathcal{O}(1) \ll \mathcal{O}(L). \tag{D9}$$

From the third to the fourth line, we use Eq. (C11). From the fourth line to the last line, we neglect terms that contain θ_1 and $\delta\theta_0$, because for $k_0 \neq 1$, θ_1 and $\delta\theta_0$ have different velocities (ratios between the frequency and wave number), and their product must vanish under the spacetime integral. The second-order variation is

$$\frac{1}{2}\delta^2 J = \frac{1}{T} \int dt dx \left\{ \frac{1}{2}(\partial_t \delta\theta)^2 + \frac{1}{2}(\partial_x \delta\theta)^2[1 - \alpha\cos(2\theta)] \right.$$
$$\left. - \alpha(\delta\theta)^2[(\partial_x \theta)^2\cos(2\theta) + (\partial_x^2 \theta)\sin(2\theta)] \right\}$$

$$= \frac{1}{T} \int dt dx \left\{ \frac{1}{2}(\partial_t \delta\theta)^2 + \frac{1}{2}(\partial_x \delta\theta)^2 - \frac{\alpha}{2}(\partial_x \delta\theta_0)^2\cos(2\theta_0) \right.$$
$$\left. - \alpha(\delta\theta_0)^2[(\partial_x \theta_0)^2\cos(2\theta_0) + (\partial_x^2 \theta_0)\sin(2\theta_0)] \right\}$$
$$+ \mathcal{O}(\alpha^2)$$

$$= \frac{1}{T} \int dt dx \left\{ \frac{1}{2}(\partial_t \delta\theta)^2 + \frac{1}{2}(\partial_x \delta\theta)^2 - \frac{\alpha}{2}(\partial_x \delta\theta_0)^2\cos(2\theta_0) \right.$$
$$\left. - \alpha(\delta\theta_0)^2(\partial_x \theta_0)^2\cos(2\theta_0) \right\} + \mathcal{O}(\alpha^2). \tag{D10}$$

Note that $\frac{1}{2}\delta^2 J \geqslant 0$ at $\mathcal{O}(\alpha)$. This is because the leading-order term [$\mathcal{O}(1)$ term] is positive semidefinite, and negative contributions come from $\mathcal{O}(\alpha)$ terms. Besides, under the spacetime integral, the $\mathcal{O}(\alpha)$ terms can be nonzero only if $\delta\theta_0$ in Eq. (D6) comprises of (more than) two Fourier components, q_1, q_2, ..., and an oscillation function from $(\delta\theta_0)^2$ and that from $\cos(2\theta_0) = \cos(2j_0 k_0 t - 2j_0 x)$ cancel each other, e.g.,

$$q_1 - q_2 = \pm j_0 k_0, \quad q_1 + q_2 = \pm j_0. \tag{D11}$$

In the presence of such components in $\delta\theta_0$, however, the leading-order term is positive definite.

$\delta\theta_0$, as well as θ_2, is a solution of Eq. (C5); both are given by linear superpositions of $e^{iq(t+x)}$ and $e^{iq(t-x)}$ over q. Thus, for given $\theta_2 \neq 0$, one can always choose $\delta\theta_0$ such that the spacetime integral of $(\partial_x \theta_2)(\partial_x \delta\theta_0)$ remains nonzero and negative,

$\delta J < 0$. This suggests that the superflow state is stable only when $\theta_2(x, t) = 0$, while it is not stable when $\theta_2(x, t) \neq 0$.

The same conclusion holds true in the spin-injection model with the circular geometry. In the 1D spin-injection model with finite curvature, the Hamiltonian is given by

$$\mathcal{H} = \int d\ell \left\{ \frac{1}{2}(\partial_t \theta)^2 + \frac{1}{2}(\partial_\ell \theta)^2 \left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right) \right] \right\}, \tag{D12}$$

where θ and $\theta + \delta\theta$ depend only on ℓ and t. Their energy difference ΔJ can be expanded in the powers of small local deformation $\delta\theta$. The first- and second-order variations of the energy in $\delta\theta$ are

$$\delta J = \frac{1}{T} \int_0^T dt \int d\ell \left\{ (\partial_t \theta)(\partial_t \delta\theta) + (\partial_\ell \theta)(\partial_\ell \delta\theta) \right.$$
$$\times \left[1 + \alpha\cos\left(2\theta - \frac{2}{r}\ell\right) \right]$$
$$\left. - \alpha(\partial_\ell \theta)^2\sin\left(2\theta - \frac{2}{r}\ell\right)(\delta\theta) \right\}, \tag{D13}$$

$$\frac{1}{2}\delta^2 J = \frac{1}{T} \int_0^T dt \int d\ell \left\{ \frac{1}{2}(\partial_t \delta\theta)^2 + \frac{1}{2}(\partial_\ell \delta\theta)^2 \right.$$
$$+ \alpha(\partial_\ell \delta\theta)^2\cos\left(2\theta - \frac{2}{r}\ell\right) + \alpha(\delta\theta)^2 \left\{ \left[(\partial_\ell \theta)^2 \right. \right.$$
$$\left. - \frac{2}{r}(\partial_\ell \theta) \right] \cos\left(2\theta - \frac{2}{r}\ell\right) + (\partial_\ell^2 \theta)\sin\left(2\theta - \frac{2}{r}\ell\right) \right\} \right\}, \tag{D14}$$

respectively. Equations (D13) and (D14) have a similar structure as Eqs. (D8) and (D10), respectively. For $k_0^2 \neq (1 + \frac{1}{j_0 r})^2$ (resonance point), one can use the α expansion of θ and $\delta\theta$, and the expressions support the same conclusion in the circular geometry case; $\delta J < 0$ for some $\delta\theta_0$ and $\frac{1}{2}\delta^2 J \geqslant 0$.

In summary, contrary to the conventional superfluid with $\theta_1 = \theta_2 = 0$, the supercurrent state with the broken U(1) spacetime symmetry is classically unstable toward other states, and it must experience the energy dissipation by local perturbation. Physically speaking, the difference in the stabilities between these two types of superfluids comes from the fact that the spin-injection boundary condition does not break the U(1) symmetry of the conventional superfluid, but it breaks the U(1) spacetime symmetry of the total-angular-momentum superfluid; under the U(1) spacetime rotation, the whole junction should also be rotated. Effects of the energy dissipation can be included as finite relaxation time T_1^{-1}, while the motion of θ with vanishing or small T_1^{-1} can be realized only in a superclean limit.

A conventional superfluid described by a nonrelativistic complex field has a critical velocity given by the Landau criterion. Below the critical velocity, a supercurrent is stable. The Landau criterion should be derived from a theory of the complex field instead of an effective theory of a Goldstone mode. This is because when the velocity approaches the critical value, a low-energy condition is already violated. Our analyses only study the stability in the low-energy limit where

a nonrelativistic complex field and a relativistic complex field both lead to a Goldstone mode with linear dispersion.

APPENDIX E: EFFECTS OF DISSIPATION IN THE CLASSICAL EOM

Note first that the classical equation, Eq. (B5), as well as its 1D descendants, Eqs. (C1) and (C32), are all invariant under the time-reversal operation; $t \to -t$, and $\theta \to \theta + \pi$. In the previous Appendix, we demonstrate that the classical energy of the spin supercurrent state is higher than other states due to the finite α. This suggests that the supercurrent state decays into other states with lower energy. Such an energy-nonconserving decay process generally breaks the time-reversal symmetry of the classical equation. To study the effect of the decay process into the spin hydrodynamics predicted in Sec. V, we include the simplest time-reversal-breaking term, $\partial_t \theta$, into the classical equation;

$$\partial_t^2 \theta - (\partial_x^2)\theta[1 - \alpha\cos(2\theta)] - (\partial_y^2\theta)[1 + \alpha\cos(2\theta)]$$
$$+ 2\alpha(\partial_x\partial_y\theta)\sin(2\theta) - \alpha[(\partial_x\theta)^2 - (\partial_y\theta)^2]\sin(2\theta)$$
$$+ 2\alpha(\partial_x\theta)(\partial_y\theta)\cos(2\theta)$$
$$= -\frac{1}{T_1}\partial_t\theta. \tag{E1}$$

From the symmetry point of view, one could also add other time-reversal breaking terms that respect the U(1) spacetime symmetry but breaks the time-reversal symmetry, e.g.,

$$\cdots = -\frac{1}{T_1}\partial_t\theta - \frac{1}{T_2}\partial_t\theta \times (\partial_x^2\theta + \partial_y^2\theta) - \frac{1}{T_3}\partial_t\theta$$
$$\times \{(\partial_x^2\theta - \partial_y^2\theta)\cos(2\theta) + 2\partial_x\partial_y\theta\sin(2\theta)\} + \cdots \tag{E2}$$

Nonetheless, the first term on the right-hand side always dominates the others in the hydrodynamic regime, since the physical variable θ changes much more slowly than any microscopic length scales in the hydrodynamic regime, and in this sense, the other terms in Eq. (E2) are higher-order spatial gradient terms than the first term in Eq. (E2). In this Appendix, we will solve Eq. (E1) or its 1D descendant in the spin-injection model with the straight geometry,

$$\partial_t^2\theta - (\partial_x^2\theta)[1 - \alpha\cos(2\theta)] - \alpha(\partial_x\theta)^2\sin(2\theta) = -\frac{1}{T_1}\partial_t\theta. \tag{E3}$$

$\theta_0(x, t)$ with the dissipation term was previously solved by Refs. [11,13]. It satisfies

$$\partial_t^2\theta_0 + \frac{1}{T_1}\partial_t\theta_0 = \partial_x^2\theta_0. \tag{E4}$$

The general solution (up to a constant F_0) of Eq. (E4) is

$$\theta_0(x, t) = At + \frac{A}{2T_1}x^2 + Bx$$
$$+ \sum_{c\in\mathbb{R}, c\neq 0}[F_c e^{ict-i\kappa_c x} + F_c' e^{ict+i\kappa_c x}], \tag{E5}$$

where A, B, F_c, F_c' are constants and

$$\kappa_c = \sqrt{c^2 + i\frac{c}{T_1}}. \tag{E6}$$

The boundary conditions Eqs. (C2) and (C3) are satisfied by

$$B = -j_0, \quad A = -k_0\left(\frac{A}{T_1}L + B\right), \quad F_c = F_c' = 0, \tag{E7}$$

which leads to

$$A = \left(1 + \frac{k_0 L}{T_1}\right)^{-1} k_0 j_0. \tag{E8}$$

This gives the zeroth-order solution of the EOM with the BCs,

$$\theta_0(x, t) = \frac{T_1 k_0 j_0}{T_1 + k_0 L}t - j_0\left[1 - \frac{k_0 x}{2(T_1 + k_0 L)}\right]x$$
$$\equiv \tilde{k}_0 j_0 t - j_0 h(x)x, \tag{E9}$$

$$j_x^s(0 < x < L) = \frac{T_1 + k_0(L - x)}{T_1 + k_0 L}j_0 + \mathcal{O}(\alpha),$$

$$s(0 < x < L) = \frac{T_1 k_0 j_0}{T_1 + k_0 L} + \mathcal{O}(\alpha). \tag{E10}$$

In the conventional spin superfluid with $T_1^{-1} \neq 0$, the spin density and the spin current are static. Different from the dissipationless case ($T_1^{-1} = 0$), the spin current decreases linearly in the 1D coordinate x, while the spin density is uniform in x.

Due to nonlinear x dependence of $\theta_0(x, t)$, the perturbative analyses in the SOC (α) becomes harder. To obtain the solution of the EOM analytically, we consider a limit that a phase accumulation γ is small when the spatial dependence of the current is small,

$$\gamma \equiv \left[-\frac{dh(x)}{dx}L\right]j_0 L = \frac{k_0 j_0 L^2}{2(T_1 + k_0 L)} \ll 1. \tag{E11}$$

The small γ limit can be achieved by a small dissipation term or a short propagation distance. The zeroth-order solution in the small γ and α limit is

$$\theta_0(x, t) = \tilde{k}_0 j_0\left(1 - \frac{k_0 L}{T_1}\right)t - j_0 h(x)x$$
$$= \tilde{k}_0 j_0 t - j_0 x + \mathcal{O}(\gamma). \tag{E12}$$

We will solve $\theta(x, t)$ up to the first order in α or in γ, namely $\mathcal{O}(\alpha, \gamma)$. Thereby, we keep the zeroth order of $\theta_0(x, t)$ when solving $\theta_1(x, t)$ and $\theta_2(x, t)$. Equation (C11) is slightly modified and becomes

$$\partial_t^2\theta_1 + \frac{1}{T_1}\partial_t\theta_1 - \partial_x^2\theta_1 = \alpha j_0^2\sin(2\tilde{k}_0 j_0 t - 2j_0 x)$$
$$= \frac{\alpha j_0^2}{2i}e^{2i\tilde{k}_0 j_0 t - 2ij_0 x} + \text{c.c.} \tag{E13}$$

Equation (E13) has a special solution,

$$\theta_1(x, t) = \frac{\alpha j_0^2 e^{2i\tilde{k}_0 j_0 t - 2ij_0 x}}{2i\left(-4\tilde{k}_0^2 j_0^2 - \frac{2i}{T_1}\tilde{k}_0 j_0 + 4j_0^2\right)} + \text{c.c.}$$
$$= -\frac{\alpha j_0 T_1 e^{2i\tilde{k}_0 j_0 t - 2ij_0 x}}{8ij_0 T_1(\tilde{k}_0^2 - 1) + 4\tilde{k}_0} + \text{c.c.}$$
$$\equiv \alpha g_0 e^{2ij_0(\tilde{k}_0 t - x)} + \text{c.c.}, \tag{E14}$$

where

$$g_0 = -\frac{j_0 T_1}{8i j_0 T_1 (\tilde{k}_0^2 - 1) + 4\tilde{k}_0}. \tag{E15}$$

Equation (C16) holds true, while Eqs. (C17) and (C18) are modified,

$$s = \frac{1}{2}\tilde{k}_0 j_0 + \alpha \tilde{k}_0 j_0 e^{2i\tilde{k}_0 j_0 t}\left[2ige^{-2i\tilde{k}_0 j_0 x} - \frac{j_0 T_1 e^{-2i j_0 x}}{4 j_0 T_1 (\tilde{k}_0^2 - 1) - 2i\tilde{k}_0}\right] + 2i\alpha \tilde{k}_0 j_0 g' e^{2i\tilde{k}_0 j_0 (t+x)} + \mathcal{O}(\alpha^2, \gamma^2, \alpha\gamma) + \text{c.c.}, \tag{E16}$$

$$j_x^s = \frac{1}{2}j_0 + \alpha j_0 e^{2i\tilde{k}_0 j_0 t}\left[2i\tilde{k}_0 g e^{-2i\tilde{k}_0 j_0 x} - \frac{j_0 T_1 e^{-2i j_0 x}}{4 j_0 T_1 (\tilde{k}_0^2 - 1) - 2i\tilde{k}_0} - \frac{e^{-2i j_0 x}}{2}\right] - 2i\alpha \tilde{k}_0 j_0 g' e^{2i\tilde{k}_0 j_0 (t+x)} + \mathcal{O}(\alpha, \gamma, \gamma j_0 L) + \text{c.c.}$$

$$= \frac{1}{2}j_0 + \alpha j_0 e^{2i\tilde{k}_0 j_0 t}\left\{2i\tilde{k}_0 g e^{-2i\tilde{k}_0 j_0 x} - \frac{[j_0 T_1 (2\tilde{k}_0^2 - 1) - i\tilde{k}_0]e^{-2i j_0 x}}{4 j_0 T_1 (\tilde{k}_0^2 - 1) - 2i\tilde{k}_0}\right\} - 2i\alpha \tilde{k}_0 j_0 g' e^{2i\tilde{k}_0 j_0 (t+x)} + \mathcal{O}(\alpha^2, \gamma^2, \alpha\gamma) + \text{c.c.} \tag{E17}$$

The boundary conditions Eqs. (C2) and (C3) leads to the following secular equation [cf. Eq. (C25)]:

$$\begin{pmatrix} 1 & -1 \\ (1-k)e^{-i\tilde{k}_0 \beta_L} & (1+k)e^{i\tilde{k}_0 \beta_L} \end{pmatrix}\begin{pmatrix} g \\ g' \end{pmatrix} = \frac{1}{8i\tilde{k}_0 j_0 T_1 (\tilde{k}_0^2 - 1) + 4\tilde{k}_0^2}\begin{pmatrix} j_0 T_1 (2\tilde{k}_0^2 - 1) - i\tilde{k}_0 \\ (j_0 T_1 - j_0 T_1 (2\tilde{k}_0^2 - 1)k + i\tilde{k}_0 k)e^{-i\beta_L} \end{pmatrix}. \tag{E18}$$

The solution of Eq. (E18) is

$$g = \frac{[j_0 T_1 (2\tilde{k}_0^2 - 1) - i\tilde{k}_0](1+k)e^{i\tilde{k}_0 \beta_L} + [j_0 T_1 - j_0 T_1 (2\tilde{k}_0^2 - 1)k + i\tilde{k}_0 k]e^{-i\beta_L}}{[(1+k)e^{i\tilde{k}_0 \beta_L} + (1-k)e^{-i\tilde{k}_0 \beta_L}][8i\tilde{k}_0 j_0 T_1 (\tilde{k}_0^2 - 1) + 4\tilde{k}_0^2]}, \tag{E19}$$

$$g' = \frac{[j_0 T_1 - j_0 T_1 (2\tilde{k}_0^2 - 1)k + i\tilde{k}_0 k]e^{-i\beta_L} - [j_0 T_1 (2\tilde{k}_0^2 - 1) - i\tilde{k}_0](1-k)e^{-i\tilde{k}_0 \beta_L}}{[(1+k)e^{i\tilde{k}_0 \beta_L} + (1-k)e^{-i\tilde{k}_0 \beta_L}][8i\tilde{k}_0 j_0 T_1 (\tilde{k}_0^2 - 1) + 4\tilde{k}_0^2]}. \tag{E20}$$

Similar to Eq. (C28), the solution of $\theta(x, t)$ is

$$\theta(x, t) = \tilde{k}_0 j_0 t - j_0\left[1 - \frac{k_0 x}{2(T_1 + k_0 L)}\right]x + 2\alpha \text{Re}(g_0)\cos[2 j_0 (\tilde{k}_0 t - x)] - 2\alpha \text{Im}(g_0)\sin[2 j_0 (\tilde{k}_0 t - x)]$$

$$+ 2\alpha \text{Re}(g)\cos[2\tilde{k}_0 j_0 (t-x)] + 2\alpha \text{Re}(g')\cos[2\tilde{k}_0 j_0 (t+x)] - 2\alpha \text{Im}(g)\sin[2\tilde{k}_0 j_0 (t-x)]$$

$$+ 2\alpha \text{Im}(g')\sin[2\tilde{k}_0 j_0 (t+x)] + \mathcal{O}(\alpha^2, \gamma^2, \alpha\gamma), \tag{E21}$$

where g_0, g, g' are given by Eqs. (E15), (E19), and (E20). k_0, k, and $\tilde{k}_0$ are given by Eqs. (C21), (C22), and (E9). Compared to Eqs. (C25) and (C41), Eqs. (E19)–(E21) have no divergence due to finite T_1. The situation is analogous to periodically driven harmonic oscillators, where the dissipation removes divergence due to resonance [46]. Note also that the solution Eq. (E21) has a periodicity in time, $\pi(\tilde{k}_0 j_0)^{-1}$, and two characteristic wavelengths, $2j_0$, and $2\tilde{k}_0 j_0$. A solution at higher order in γ has all spatial Fourier components, while it is still periodic in time with the same periodicity.

APPENDIX F: SPECIAL PARAMETER POINTS IN THE SPIN-INJECTION MODEL

In this Appendix, we study some special parameter points in the spin-injection model without the spin relaxation term, where solutions in Appendix C do not apply directly and need careful investigations.

1. $k_0 = 1$ (straight geometry) and $k_0^2 = (1 + \frac{1}{j_0 r})^2$ (circular geometry)

Consider $k_0 = 1$ in the straight geometry and $k_0^2 = (1 + \frac{1}{j_0 r})^2$ in the circular geometry. Naive substitutions of $k_0 = 1$ into Eqs. (C13) and (C25) and of $k_0^2 = (1 + \frac{1}{j_0 r})^2$ into

Eqs. (C36) and (C41) lead to divergences in θ_1 and θ_2, respectively. It seems that the divergences in θ_1 and θ_2 cancel each other. For example, Eqs. (C29) and (18) at $k_0 = 1$ become

$$\eta|_{k_0=1} = \frac{[(1+k)e^{i\beta_L} - (1-k)e^{-i\beta_L}] + 2(1-k)e^{-i\beta_L}}{4(k_0^2 - 1)[(1+k)e^{i\beta_L} + (1-k)e^{-i\beta_L}]}$$

$$= \frac{1}{4(k_0^2 - 1)}, \tag{F1}$$

$$\theta(x, t)|_{k_0=1} = j_0(t - x) - \frac{\alpha}{4(k_0^2 - 1)}\sin[2 j_0 (t - x)]$$

$$- \frac{\alpha}{4(k_0^2 - 1)}\cos(2 j_0 t)\sin(2 j_0 x)$$

$$+ \frac{\alpha}{4(k_0^2 - 1)}\sin(2 j_0 t)\cos(2 j_0 x) + \mathcal{O}(\alpha^2)$$

$$= j_0(t - x) + \mathcal{O}(\alpha^2), \tag{F2}$$

respectively, where the final result of $\theta(x, t)$ is apparently finite. However, Eq. (F2) is not a solution to EOM (C1). In fact, with light-cone coordinates

$$\xi = t - x, \quad \zeta = t + x, \tag{F3}$$

Eq. (C11) at $k_0 = 1$ becomes

$$[(\partial_\xi + \partial_\zeta)^2 - (\partial_\xi - \partial_\zeta)^2]\theta_1 = 4\partial_\xi \partial_\zeta \theta_1 = \alpha j_0^2 \sin(2j_0\xi).$$
(F4)

Equation (F4) has a special solution which is not consistent to Eq. (F2),

$$\theta_1 = -\frac{\alpha\zeta}{16}\cos(2j_0\xi) = -\frac{\alpha(t+x)}{16}\cos[2j_0(t-x)].$$
(F5)

Note that $|\theta_1|$ in Eq. (F5) is not bounded for large $\zeta = t + x$. This indicates that the perturbation with respect to α becomes invalid at $k_0 = 1$, leading to the discrepancy. The divergences at $k_0 = 1$ can be regarded as the resonance of the inhomogeneous linearized differential equation [46], and one can expect that the SOC has nonperturbative effects around $k_0 = 1$.

To understand the origin of the nonperturbative effect of α, let us consider a set of solutions of Eq. (C1) that depends on x and t only through $x - vt$. For the later comparison to a special solution developed in Appendix C, $\theta_0 + \theta_1 + \mathcal{O}(\alpha^2)$, let v to be k_0,

$$\theta(x, t) = \theta(x - k_0 t), \quad \theta' \equiv \partial_x \theta = -\frac{1}{k_0}\partial_t \theta.$$
(F6)

Here, the prime denotes an x derivative. Eq. (C1) effectively becomes an ordinary differential equation,

$$(k_0^2 - 1)\theta'' = -\alpha\theta''\cos(2\theta) + \alpha\theta'^2\sin(2\theta).$$
(F7)

To solve this equation, use its analogy to 1D classical mechanics, where the phase $\theta(x)$ as a function of x corresponds to a 1D coordinate as a function of time. The classical mechanics for the 1D coordinate has a Lagrangian whose variation gives Eq. (F7) as a classical EOM,

$$L_{1D} = \frac{1}{2}(k_0^2 - 1)\theta'^2 + \frac{\alpha}{2}\theta'^2\cos(2\theta).$$
(F8)

The classical mechanics has a canonical momentum conjugate to the coordinate,

$$\pi = \frac{\partial L_{1D}}{\partial \theta'} = (k_0^2 - 1)\theta' + \alpha\theta'\cos(2\theta),$$
(F9)

as well as a conserved Hamiltonian,

$$H_{1D} = \pi\theta' - L_{1D} = \frac{1}{2}(k_0^2 - 1)\theta'^2 + \frac{\alpha}{2}\theta'^2\cos(2\theta).$$
(F10)

Utilizing the x independence ("time"-independence) of H_{1D}, we can solve the EOM from Eq. (F10),

$$\frac{d\theta}{dx} = \pm\sqrt{\frac{2H_{1D}}{k_0^2 - 1 + \alpha\cos(2\theta)}}.$$
(F11)

Its formal solution is given by

$$\pm(x - x_0) = \int_{\theta(x_0)}^{\theta(x)}\sqrt{\frac{k_0^2 - 1 + \alpha\cos(2\theta)}{2H_{1D}}}d\theta.$$
(F12)

With Eq. (F6), we get a set of $(1+1)$-dimensional solutions,

$$\pm(x - k_0 t - C_0) = \int_{\theta(x_0,t_0)}^{\theta(x,t)}\sqrt{\frac{k_0^2 - 1 + \alpha\cos(2\theta)}{2H_{1D}}}d\theta,$$
(F13)

where $C_0 = x_0 - k_0 t_0$.

Equation (F13) is inclusive of those perturbative solutions in Sec. III that depend on x and t only through $x - k_0 t$, i.e., $\theta_0 + \theta_1$ given by Eqs. (C8) and (C13). Namely, when $\alpha \ll |k_0^2 - 1|$, we can apply an expansion in α,

$$\pm(x - k_0 t - C_0)$$
$$= \int_{\theta(x_0,t_0)}^{\theta(x,t)}\sqrt{\frac{k_0^2 - 1}{2H_{1D}}}\left[1 + \frac{\alpha}{2(k_0^2 - 1)}\cos(2\theta)\right]d\theta + \mathcal{O}(\alpha^2).$$
(F14)

From this, we obtain

$$\theta(x, t) = \pm\sqrt{\frac{2H_{1D}}{k_0^2 - 1}}(x - k_0 t - C_0')$$
$$\quad - \frac{\alpha}{4(k_0^2 - 1)}\sin(2\theta) + \mathcal{O}(\alpha^2)$$
$$= \theta_0 - \frac{\alpha}{4(k_0^2 - 1)}\sin(2\theta_0) + \mathcal{O}(\alpha^2),$$
(F15)

where

$$\theta_0 = \pm\sqrt{\frac{2H_{1D}}{k_0^2 - 1}}(x - k_0 t - C_0'),$$
(F16)

$$C_0' = C_0 \mp \sqrt{\frac{k_0^2 - 1}{2H_{1D}}}\left\{\theta(x_0, t_0) + \frac{\alpha}{4(k_0^2 - 1)}\sin[2\theta(x_0, t_0)]\right\}.$$
(F17)

Note that due to the absence of $\theta_2(x, t)$ in its α expansion, Eq. (F15) does not satisfy the boundary conditions in the spin-injection model in general.

Nonetheless, Eq. (F13) is still useful to see that the expansion in α is invalid when $|k_0^2 - 1| < \mathcal{O}(\alpha)$. Take $k_0 = 1$ in Eq. (F13) as an example. Thereby, the sign of $\cos(2\theta)$ is conserved from Eq. (F10). Then, the integral of Eq. (F13) gives

$$\pm\sqrt{\frac{2H_{1D}}{\alpha}}(x - k_0 t - C_0) = E(\theta(x, t), 2) - E(\theta(x_0, t_0), 2),$$
(F18)

where $E(\theta, m)$ is the elliptic integral of the second kind. This special solution can be made independent from α because α can be absorbed into another parameter H_{1D}. The α independence as well as the conserved sign of $\cos(2\theta)$ are not consistent with $\theta_0(x, t) + \theta_1(x, t)$ in Eqs. (C8) and (C13). This suggests that the expansion in α is invalid, particularly at $k_0 = 1$.

2. $k_0 = 0$ (straight geometry)

Another special parameter point is $k_0 = 0$. $k_0 = 0$ is a limit where χ in Eq. (C21) is much smaller than $\chi'\sqrt{\frac{D_s}{T_1'}}$ and β_t, and k_c goes to zero for any c;

$$k_c \equiv \frac{\chi}{\chi'}\sqrt{\frac{1}{D_s\left(\frac{1}{T_1'} + ic\right)}} + \frac{\chi}{\beta_t} \rightarrow 0.$$
(F19)

We first consider $k_0 = 0$ in the straight geometry ($r^{-1} = 0$). θ_0 at $k_0 = 0$ has no time dependence in Eq. (C8), so that the phase F_0 in Eq. (C8) cannot be absorbed into the time.

Meanwhile, $s(x = L-) = 0$ because $k_c = 0$ for any c, and $s(x) \equiv \partial_t \theta = 0$ can be always satisfied by a time-independent θ. Thus we have only to make Eqs. (C8) and (C10) with an additional F_0 to satisfy the other boundary condition, $j_x^s(x = 0+) = j_0$. Firstly, let us choose $\theta_0 = -j_0 x + F_0$ that satisfies the boundary condition. The substitution into Eq. (C11) gives $\theta_1 = \frac{\alpha}{4} \sin(-2j_0 x + F_0)$. To satisfy the boundary condition up to the 1st order in α [see also Eq. (C4)],

$$j_x^s(x = 0+) = j_0 - \frac{j_0 \alpha}{2} \cos(2F_0) - \partial_x \theta_2 + \mathcal{O}(\alpha^2) = j_0, \tag{F20}$$

we have

$$\theta_2(x) = -\frac{j_0 \alpha}{2} \cos(2F_0) x. \tag{F21}$$

Here, $|\theta_2(x)|$ for large x is not bounded, where the perturbation in α breaks down. An alternative way to get a consistent perturbative solution is to require $\theta_2(x) = 0$ and take $\theta_0(x) = -\tilde{j}_0 x + F_0$ where $\tilde{j}_0 \neq j_0$. Then we have

$$j_x^s(x = 0+) = \tilde{j}_0 - \frac{\tilde{j}_0 \alpha}{2} \cos(2F_0) - \partial_x \theta_2 + \mathcal{O}(\alpha^2) = j_0. \tag{F22}$$

From this, we get

$$\tilde{j}_0 = j_0 \left[1 - \frac{\alpha}{2} \cos(2F_0) \right]^{-1} = j_0 \left[1 + \frac{\alpha}{2} \cos(2F_0) \right] + \mathcal{O}(\alpha^2). \tag{F23}$$

At $\mathcal{O}(\alpha)$, this solution is equivalent to absorbing $\theta_2(x)$ in Eq. (F21) into $\theta_0(x)$. Note also that from Eq. (D8), $\theta_2(x) = 0$ implies that the steady (but not uniform) current without spin accumulation has no energy dissipation. However, a higher-order calculation in α suggests that δJ can be nonzero and negative. This is because $\delta\theta_1$ and θ_1 can share identical Fourier components [see Eq. (H35) in Appendix H].

3. $k_0 = 0$ (circular geometry)

Let us next consider the circular geometry with $k_0 = 0$. We take $\theta_2(\ell) = 0$ and $\theta_0(\ell) = -\tilde{j}_0 \ell + F_0$ with $\tilde{j}_0 \neq j_0$. Then, similar to Eq. (20), we obtain

$$-\partial_\ell^2 \theta_1 = -\alpha \tilde{j}_0 \left(\tilde{j}_0 + \frac{2}{r} \right) \sin\left(-2\tilde{j}_0 \ell - \frac{2}{r}\ell + 2F_0 \right). \tag{F24}$$

When $\tilde{j}_0 \neq -\frac{1}{r}$, Eq. (F24) leads to

$$\theta_1(\ell) = \frac{\alpha \tilde{j}_0 \left(\tilde{j}_0 + \frac{2}{r} \right)}{4 \left(\tilde{j}_0 + \frac{1}{r} \right)^2} \sin\left(2\tilde{j}_0 \ell + \frac{2}{r}\ell - 2F_0 \right). \tag{F25}$$

Note that when $\tilde{j}_0 = -\frac{2}{r}$, $\theta_1(\ell) = 0$, and $\theta(\ell) = \frac{2\ell}{r} + F_0$ ($F_0 \in \mathbb{R}$) becomes an "exact" solution of Eq. (C32). The solution is not exact in the presence of higher-order derivatives in the model Eq. (6).

From Eq. (F25) and $\theta_2 = 0$, the boundary condition at $\ell = 0+$ reads

$$j_\ell^s(\ell = 0+) = j_0 = \tilde{j}_0 + \alpha \tilde{j}_0 \cos(2F_0)$$
$$- \frac{\alpha \tilde{j}_0 \left(\tilde{j}_0 + \frac{2}{r} \right)}{2 \left(\tilde{j}_0 + \frac{1}{r} \right)} \cos(2F_0) + \mathcal{O}(\alpha^2)$$
$$= \tilde{j}_0 + \frac{\alpha \tilde{j}_0^2}{2 \left(\tilde{j}_0 + \frac{1}{r} \right)} \cos(2F_0) + \mathcal{O}(\alpha^2). \tag{F26}$$

So we have

$$\tilde{j}_0 = j_0 - \frac{\alpha \tilde{j}_0^2}{2 \left(\tilde{j}_0 + \frac{1}{r} \right)} \cos(2F_0) + \mathcal{O}(\alpha^2). \tag{F27}$$

As in the previous case, the steady current without spin accumulation has an energy dissipation.

a. $k_0 = 0$, $\tilde{j}_0 = -\frac{1}{r}$ (circular geometry)

When $\tilde{j}_0 = -\frac{1}{r}$, Eq. (F24) leads to

$$\theta_1 = -\frac{\ell^2}{2} \alpha \tilde{j}_0 \left(\tilde{j}_0 + \frac{2}{r} \right) \sin(2F_0) = \frac{\ell^2 \alpha}{2r^2} \sin(2F_0). \tag{F28}$$

When $F_0 \neq \frac{N}{2}\pi$, the perturbation of α becomes invalid for larger ℓ, where $|\theta_1(\ell)|$ is not bounded. When $F_0 = \frac{N}{2}\pi$, $\theta_1(\ell) = 0$, and the boundary condition requires

$$j_0 = \tilde{j}_0[1 + \alpha(-1)^N] = -\frac{1}{r}[1 + \alpha(-1)^N]. \tag{F29}$$

This solution $\left(\theta(\ell, t) = \frac{\ell}{r} + \frac{N\pi}{2} \right)$ can exist only at $k_0 = 0$, while it is not continuously connected to a solution at finite small k_0 (see below).

b. $k_0 \to 0$, $j_0 = -\frac{1}{r} + \mathcal{O}(\alpha)$ (circular geometry)

To see that the solution Eq. (F29) is not continuously connected to a perturbative solution at finite small k_0, let us keep a finite k_0 and choose $j_0 = -\frac{1}{r} + \mathcal{O}(\alpha)$; $\theta_0(\ell) = \frac{1}{r}(k_0 t - \ell) + \mathcal{O}(\alpha)$. In the perturbation theory, we should neglect $\mathcal{O}(\alpha^2)$ contributions to θ_1, so θ_1 is not affected by the $\mathcal{O}(\alpha)$ component in j_0. Then Eq. (20) becomes

$$\partial_t^2 \theta_1 - \partial_\ell^2 \theta_1 = -\frac{\alpha}{r^2} \sin\left(-\frac{2k_0}{r} t \right), \tag{F30}$$

which leads to a solution,

$$\theta_1 = -\frac{\alpha}{4k_0^2} \sin\left(\frac{2k_0}{r} t \right). \tag{F31}$$

When $k_0 \to 0$, the solution has the divergence. Physically speaking, when $k_0 \to 0$, F_0 changes slowly with respect to time, so we cannot fix the phase F_0 in Eq. (F28).

APPENDIX G: DERIVATION OF THE LANDAU CRITERION

In this Appendix, we use the same framework as in Sec. V and Appendix D and derive the Landau criterion. A similar argument can be found in Ref. [13], while the derivation here is more formal than Ref. [13]. We begin with a 1D superfluid model,

$$\tilde{\mathcal{L}}_\phi = i\hbar \phi^\dagger \partial_t \phi - \frac{\hbar^2}{2m} (\partial_x \phi^\dagger)(\partial_x \phi) - \frac{U}{2} (\phi^\dagger \phi)^2 + \mu \phi^\dagger \phi, \tag{G1}$$

with its classical EOM,

$$i\hbar \partial_t \phi = \left(-\frac{\hbar^2}{2m} \partial_x^2 + U\phi^\dagger \phi - \mu \right) \phi. \tag{G2}$$

Thanks to the Galilean covariance of the Lagrangian Eq. (G1), our discussion will be easier than Sec. V, and we do not need to expand the variation of the motion as in Sec. V. Namely, using the Galilean covariance, we can directly obtain two motions that are close to each other, and compare their energies from the corresponding Hamiltonian,

$$\tilde{H}_\phi[\phi] = \int dx \left[\frac{\hbar^2}{2m} (\partial_x \phi^\dagger)(\partial_x \phi) + \frac{U}{2} (\phi^\dagger \phi)^2 - \mu \phi^\dagger \phi \right]. \tag{G3}$$

Consider a steady flow $\phi_0(x, t)$,

$$\phi_0(x, t) = \sqrt{\rho_0} \exp \left[\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right) \right]. \tag{G4}$$

Let us assume that the following motion $\phi(x, t)$, as well as $\phi_0(x, t)$, satisfies the EOM, Eq. (G2),

$$\phi(x, t) = \phi'(x', t') \exp \left[\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right) \right]$$

$$= \phi'(x - vt, t) \exp \left[\frac{i}{\hbar} \left(mvx - \frac{mv^2}{2} t \right) \right], \tag{G5}$$

with $x' = x - vt$ and $t' = t$. Here $|\phi'(x', t') - \sqrt{\rho_0}| \ll \sqrt{\rho_0}$ and $\phi(x, t)$ is close to $\phi_0(x, t)$. Using a Galilean transformation,

$$x' = x - vt, \quad t' = t, \tag{G6}$$

$$\partial_x = \partial_{x'}, \quad \partial_t = \partial_{t'} - v\partial_x, \tag{G7}$$

we can see that $\phi'(x', t')$ must satisfy a similar equation as Eq. (G2),

$$\left(i\hbar \partial_t + \frac{1}{2} mv^2 \right) \phi'(x', t')$$

$$= \left[\frac{1}{2m} (-i\hbar \partial_x + mv)^2 + U\phi'^\dagger \phi' - \mu \right] \phi'(x', t'), \tag{G8}$$

or equivalently,

$$i\hbar \partial_{t'} \phi'(x', t') = \left(-\frac{\hbar^2}{2m} \partial_{x'}^2 + U\phi'^\dagger \phi' - \mu \right) \phi'(x', t'). \tag{G9}$$

Now we compare the energies of $\phi(x, t)$ and $\phi_0(x, t)$. The energy of $\phi_0(x, t)$ is,

$$\tilde{H}_\phi[\phi_0] = \int dx \left[\frac{\hbar^2}{2m} (\partial_x \phi_0^\dagger)(\partial_x \phi_0) + \frac{U}{2} (\phi_0^\dagger \phi_0)^2 - \mu \phi_0^\dagger \phi_0 \right]$$

$$= E_0 + \frac{1}{2} mv^2 Q_0, \tag{G10}$$

where

$$E_0 = \int dx \left(\frac{U}{2} \rho_0^2 - \mu \rho_0 \right) = -\frac{\mu}{2} \int dx \rho_0,$$

$$Q_0 = \int dx \rho_0. \tag{G11}$$

The energy of $\phi(x, t)$ is

$$\tilde{H}_\phi[\phi] = \int dx \left[\frac{\hbar^2}{2m} (\partial_x \phi^\dagger)(\partial_x \phi) + \frac{U}{2} (\phi^\dagger \phi)^2 - \mu \phi^\dagger \phi \right]$$

$$= \int dx \phi'^\dagger (x - vt, t) \left[\frac{1}{2m} (-i\hbar \partial_x + mv)^2 \right.$$

$$\left. + \frac{U}{2} \phi'^\dagger \phi' - \mu \right] \phi'(x - vt, t)$$

$$= \int dx \phi'^\dagger (x, t) \left[\frac{1}{2m} (-i\hbar \partial_x + mv)^2 \right.$$

$$\left. + \frac{U}{2} \phi'^\dagger \phi' - \mu \right] \phi'(x, t)$$

$$= \int dx \phi'^\dagger (x, t) \left[-\frac{\hbar^2}{2m} \partial_x^2 - iv\hbar \partial_x + \frac{1}{2} mv^2 \right.$$

$$\left. + \frac{U}{2} \phi'^\dagger \phi' - \mu \right] \phi'(x, t)$$

$$= \tilde{H}_\phi[\phi'] + vP_\phi[\phi'] + \frac{1}{2} mv^2 Q_\phi[\phi'], \tag{G12}$$

where

$$P_\phi[\phi'] = \int dx \phi'^\dagger (x, t)(-i\hbar \partial_x) \phi'(x, t),$$

$$Q_\phi[\phi'] = \int dx \phi'^\dagger (x, t) \phi'(x, t). \tag{G13}$$

Thus the energy difference between $\phi(x, t)$ and $\phi_0(x, t)$ is

$$\Delta E_\phi[\phi, \phi_0] = \tilde{H}_\phi[\phi] - \tilde{H}_\phi[\phi_0]$$

$$= (\tilde{H}_\phi[\phi'] - E_0) + vP_\phi[\phi']$$

$$+ \frac{1}{2} mv^2 (Q_\phi[\phi'] - Q_0). \tag{G14}$$

For ϕ' that satisfies EOM (G9), energy $H_\phi[\phi']$, momentum $P_\phi[\phi']$, and U(1) charge $Q_\phi[\phi']$ must be all conserved. This is because from Eq. (G9), $\phi'(x, t)$ is a solution of Eq. (G2).

The average velocity of $\phi(x, t)$ is not small, while $\phi'(x, t)$ can be assumed at the near-equilibrium limit [44]. In this limit, the EOM of $\phi' = \sqrt{\rho_0 + \delta\rho'} e^{i\theta'}$ can be described by a wave equation of $\delta\rho'$ and θ',

$$\partial_t^2 \theta'(x, t) - \frac{\rho_0 U}{m} \partial_x^2 \theta'(x, t) = 0, \tag{G15}$$

$$\delta\rho'(x, t) = -\frac{\hbar}{U} \partial_t \theta'(x, t). \tag{G16}$$

Thereby, $\theta'(x, t)$ is given by a superposition of oscillations,

$$\theta'(x, t) = \frac{1}{\sqrt{L}} \sum_q [f_q e^{iq(x-v_c t)} + f_q' e^{iq(x+v_c t)}],$$

$$v_c = \sqrt{\frac{\rho_0 U}{m}}. \tag{G17}$$

In the near-equilibrium limit of ϕ', we evaluate the energy difference in the leading order in small f_q and f_q',

$$\tilde{H}_\phi[\phi'] - E_0 = \int dx \left[\frac{\hbar^2 \rho_0}{2m} (\partial_x \theta')^2 + \frac{U}{2} (\delta\rho')^2 \right]$$

$$= \int dx \left[\frac{\hbar^2 \rho_0}{2m} (\partial_x \theta')^2 + \frac{\hbar^2}{2U} (\partial_t \theta')^2 \right]$$

$$= \frac{\hbar^2 \rho_0}{m} \sum_q q^2 (|f_q|^2 + |f_q'|^2), \tag{G18}$$

$$P_\phi[\phi'] = \int dx \hbar \delta \rho'(\partial_x \theta') = -\frac{\hbar^2}{U} \int dx (\partial_t \theta')(\partial_x \theta')$$

$$= \frac{\hbar^2 \rho_0}{m v_c} \sum_q q^2 (|f_q|^2 - |f_q'|^2), \tag{G19}$$

$$Q_\phi[\phi'] - Q_0 = \int dx \delta \rho' = 0. \tag{G20}$$

Taking Eqs. (G18)–(G20) into Eq. (G14), we have

$$\Delta E_\phi = \frac{\hbar^2 \rho_0}{m} \sum_q q^2 \left[\left(1 + \frac{v}{v_c}\right) |f_q|^2 + \left(1 - \frac{v}{v_c}\right) |f_q'|^2 \right]. \tag{G21}$$

To make $\Delta E_\phi \geqslant 0$ for any (small) f_q and f_q', we obtain the Landau criterion,

$$|v| \leqslant v_c = \sqrt{\frac{\rho_0 U}{m}}. \tag{G22}$$

APPENDIX H: LOCAL DEFORMATIONS OF CLASSICAL SOLUTIONS OF THE EOM

The superfluid state with a finite supercurrent is characterized by the solution $\theta(x,t)$ of the classical EOM in the 1D spin-injection model (e.g., with the straight geometry). In Sec. V, we introduced its local deformation $\theta(x,t) + \delta\theta(x,t)$ as another solution of the EOM with different boundary conditions. We regarded that $\delta\theta(x,t)$, as well as $\theta(x,t)$, can be determined perturbatively in the SOC (α). At the zeroth order in SOC, $\theta + \delta\theta$, as well as θ, is a solution of $\partial_t^2 \theta - \partial_x^2 \theta = 0$, and so is $\delta\theta$. Since $\delta\theta(x,t)$ is a local deformation and its spacetime derivatives should not contain any uniform components in space, the zeroth order of $\delta\theta(x,t)$ must be given by Eq. (D6);

$$\delta\theta_0(x,t) = \frac{1}{\sqrt{L}} \sum_q [\delta d_q e^{iq(x-t)} + \delta d_q' e^{iq(x+t)}]. \tag{H1}$$

In this Appendix, for a given form of Eq. (D6) as the zeroth order, we will show how to determine the first order of $\delta\theta(x,t)$;

$$\delta\theta(x,t) = \delta\theta_0(x,t) + \delta\theta_1(x,t) + \mathcal{O}(\alpha^2). \tag{H2}$$

We first give a general framework to determine $\delta\theta$. $\theta + \delta\theta$, as well as θ, is a local minimum of the action, $S = \int d^3r \mathcal{L} \equiv \int dt d^2r \mathcal{L}$, and $\delta\theta$ is infinitesimally small. Thus we take a $\delta\theta$ variation of S;

$$S[\theta] \equiv S_{xx}[\theta] + S_{yy}[\theta] + S_{xy}[\theta] + \frac{1}{2} \int d^3r (\partial_t \theta)^2, \tag{H3}$$

$$S_{xx} \equiv -\frac{1}{2} \int d^3r (\partial_x \theta)^2 [1 - \alpha \cos(2\theta)],$$

$$S_{yy} \equiv \frac{1}{2} \int d^3r (\partial_y \theta)^2 [1 + \alpha \cos(2\theta)], \tag{H4}$$

$$S_{xy} \equiv \alpha \int d^3r (\partial_x \theta)(\partial_y \theta) \sin(2\theta).$$

The first-order variation just gives EOM (B5),

$$\delta S = \delta S_{xx} + \delta S_{yy} + \delta S_{xy} - \int d^3r (\delta\theta)(\partial_t^2 \theta), \tag{H5}$$

$$\delta S_{xx} \equiv \int d^3r (\delta\theta) \{ (\partial_x^2 \theta)[1 - \alpha \cos(2\theta)]$$

$$+ \alpha (\partial_x \theta)^2 \sin(2\theta) \}, \tag{H6}$$

$$\delta S_{yy} \equiv \int d^3r (\delta\theta) \{ (\partial_y^2 \theta)[1 + \alpha \cos(2\theta)]$$

$$- \alpha (\partial_y \theta)^2 \sin(2\theta) \}, \tag{H7}$$

$$\delta S_{xy} \equiv -2\alpha \int d^3r (\delta\theta)[(\partial_x \partial_y \theta) \sin(2\theta)$$

$$+ (\partial_x \theta)(\partial_y \theta) \cos(2\theta)]. \tag{H8}$$

δS vanishes since θ is an extremum or a saddle point of S. The second-order variation $\delta^2 S$ determines small deformation $\delta\theta$ in such a way that $\theta + \delta\theta$ is an extremum or a saddle point of S. S_{xx} gives

$$\delta^2 S_{xx} = \int d^3r (\delta\theta) \{ (\partial_x^2 \delta\theta)[1 - \alpha \cos(2\theta)]$$

$$+ 2\alpha (\partial_x^2 \theta) \sin(2\theta)(\delta\theta) + 2\alpha (\partial_x \theta)(\partial_x \delta\theta) \sin(2\theta)$$

$$+ 2\alpha (\partial_x \theta)^2 \cos(2\theta)(\delta\theta) \}$$

$$= -\int d^3r [(\partial_x \delta\theta)^2 + \alpha \cos(2\theta)(\delta\theta) \partial_x^2 (\delta\theta)$$

$$- 2\alpha \sin(2\theta)(\partial_x^2 \theta)(\delta\theta)^2 - 2\alpha (\partial_x \theta)(\delta\theta)(\partial_x \delta\theta) \sin(2\theta)$$

$$- 2\alpha (\partial_x \theta)^2 \cos(2\theta)(\delta\theta)^2], \tag{H9}$$

where

$$\int d^3r \alpha \cos(2\theta)(\delta\theta) \partial_x^2 (\delta\theta)$$

$$= -\int d^3r \alpha \partial_x [\cos(2\theta)(\delta\theta)](\partial_x \delta\theta)$$

$$= \int d^3r [2\alpha \sin(2\theta)(\partial_x \theta)(\delta\theta)(\partial_x \delta\theta) - \alpha \cos(2\theta)(\partial_x \delta\theta)^2]. \tag{H10}$$

Taking Eq. (H10) into Eq. (H9), we get

$$\delta^2 S_{xx} = -\int d^3r \{ (\partial_x \delta\theta)^2 [1 - \alpha \cos(2\theta)] - 2\alpha (\delta\theta)^2$$

$$\times [\cos(2\theta)(\partial_x \theta)^2 + \sin(2\theta)(\partial_x^2 \theta)] \}. \tag{H11}$$

Similarly, we get

$$\delta^2 S_{yy} = -\int d^3r \{ (\partial_y \delta\theta)^2 [1 + \alpha \cos(2\theta)]$$

$$+ 2\alpha (\delta\theta)^2 [\cos(2\theta)(\partial_y \theta)^2 + \sin(2\theta)(\partial_y^2 \theta)] \}. \tag{H12}$$

S_{xy} gives

$$\delta^2 S_{xy} = -2\alpha \int d^3r (\delta\theta)[(\partial_x \partial_y \delta\theta) \sin(2\theta)$$

$$+ 2(\delta\theta) \cos(2\theta)(\partial_x \partial_y \theta) + (\partial_x \delta\theta)(\partial_y \theta) \cos(2\theta)$$

$$+ (\partial_y \delta\theta)(\partial_x \theta) \cos(2\theta) - 2(\delta\theta) \sin(2\theta)(\partial_x \theta)(\partial_y \theta)], \tag{H13}$$

where

$$-\alpha \int d^3r (\delta\theta)(\partial_x\partial_y\delta\theta)\sin(2\theta) = \alpha \int d^3r (\partial_y\delta\theta)[(\partial_x\delta\theta)\sin(2\theta) + 2(\delta\theta)\cos(2\theta)(\partial_x\theta)]$$

$$= \alpha \int d^3r (\partial_x\delta\theta)[(\partial_y\delta\theta)\sin(2\theta) + 2(\delta\theta)\cos(2\theta)(\partial_y\theta)]. \tag{H14}$$

Taking Eq. (H14) into Eq. (H13), we get

$$\delta^2 S_{xy} = 2\alpha \int d^3r \{(\partial_x\delta\theta)(\partial_y\delta\theta)\sin(2\theta) + 2(\delta\theta)^2[\sin(2\theta)(\partial_x\theta)(\partial_y\theta) - \cos(2\theta)(\partial_x\partial_y\theta)]\}. \tag{H15}$$

Besides, we have

$$-\delta \int d^3r (\delta\theta)(\partial_t^2\theta) = \int d^3r (\delta\partial_t\theta)^2. \tag{H16}$$

Combining Eqs. (H11), (H12), (H15), and (H16) together, we obtain the second-order variation of S with respect to small $\delta\theta$,

$$S_{\delta\theta} = \frac{1}{2}\delta^2 S = \int d^3r \left\{ \frac{1}{2}(\partial_t\delta\theta)^2 - \frac{1}{2}(\partial_x\delta\theta)^2[1 - \alpha\cos(2\theta)] - \frac{1}{2}(\partial_y\delta\theta)^2[1 + \alpha\cos(2\theta)] + \alpha(\partial_x\delta\theta)(\partial_y\delta\theta)\sin(2\theta) \right.$$

$$\left. + \alpha(\delta\theta)^2\{2\sin(2\theta)(\partial_x\theta)(\partial_y\theta) + \cos(2\theta)[(\partial_x\theta)^2 - (\partial_y\theta)^2] + \sin(2\theta)(\partial_x^2\theta - \partial_y^2\theta) - 2\cos(2\theta)(\partial_x\partial_y\theta)\} \right\}. \tag{H17}$$

Here θ is a solution of the classical EOM. Given such θ, we have only to find those $\delta\theta$ that makes $\delta S_{\delta\theta}[\delta\theta] = 0$.

In the following, we focus on the 1D solution ($\partial_y\theta = \partial_y\delta\theta = 0$) for simplicity, and neglect the integral over y, while the following derivation can be generalized to $\partial_y\delta\theta \neq 0$. The action becomes in the 1D model

$$K \equiv S_{\delta\theta}|_{\partial_y\theta = \partial_y\delta\theta = 0}$$

$$= \int_{-\infty}^{\infty} dt \int_L dx \left\{ \frac{1}{2}(\partial_t\delta\theta)^2 - \frac{1}{2}(\partial_x\delta\theta)^2[1 - \alpha\cos(2\theta)] + \alpha(\delta\theta)^2[(\partial_x\theta)^2\cos(2\theta) + (\partial_x^2\theta)\sin(2\theta)] \right\}. \tag{H18}$$

We substitute into K a perturbative solution of θ in α, e.g., Eq. (18), and expand K in powers of α. This gives

$$K = K_0 + K_1 + K_2 + \mathcal{O}(\alpha^3), \tag{H19}$$

where $K_n = \mathcal{O}(\alpha^n)$. K is a quadratic function of $\delta\theta$. In terms of a Fourier transform of $\delta\theta_{q,\omega}$,

$$\delta\theta_{q,\omega} = \frac{1}{\sqrt{L}} \int_L dx \int_{-\infty}^{\infty} dt\, \delta\theta(x,t) e^{-iqx+i\omega t}, \tag{H20}$$

the quadratic function can be characterized by matrix elements among wave number q and frequency ω;

$$K = \frac{1}{L}\sum_{q,q'} \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \delta\theta_{q,\omega}^\dagger K_{q,\omega;q',\omega'}[\theta]\delta\theta_{q',\omega'}. \tag{H21}$$

Let us take a solution $\theta(x,t)$ for the spin-injection model with the straight geometry as an example. We first take a part of $\theta(x,t)$ with only one spatial wavelength from Eqs. (C8) and (C13),

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \mathcal{O}(\alpha^2), \tag{H22}$$

$$\theta_0(x,t) = -j_0 x + j_0 k_0 t,$$

$$\theta_1(x,t) = \frac{\alpha}{4(k_0^2 - 1)}\sin(2j_0 x - 2j_0 k_0 t). \tag{H23}$$

An inclusion of $\theta_2(x,t)$ shall be given later. The zeroth order of K is given by

$$K_0 = \frac{1}{2} \int dt \int_L dx [(\partial_t\delta\theta)^2 - (\partial_x\delta\theta)^2]$$

$$= \frac{1}{2}\sum_q \int \frac{d\omega}{2\pi} \delta\theta_{q,\omega}^\dagger (\omega^2 - q^2)\delta\theta_{q,\omega}. \tag{H24}$$

A substitution of Eq. (H23) into $\mathcal{O}(\alpha)$ terms in Eq. (H18) gives K_1 and K_2,

$$\frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2\theta_0 + 2\theta_1) + \alpha(\delta\theta)^2(\partial_x\theta_0 + \partial_x\theta_1)^2\cos(2\theta_0 + 2\theta_1) + \alpha(\delta\theta)^2(\partial_x^2\theta_0 + \partial_x^2\theta_1)\sin(2\theta_0 + 2\theta_1)$$

$$= \frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2\theta_0) + \alpha(\delta\theta)^2(\partial_x\theta_0)^2\cos(2\theta_0) - \alpha(\partial_x\delta\theta)^2\sin(2\theta_0)\theta_1 - 2\alpha(\delta\theta)^2(\partial_x\theta_0)(\partial_x\theta_1)\cos(2\theta_0)$$

$$- 2\alpha(\delta\theta)^2(\partial_x\theta_0)^2\sin(2\theta_0)\theta_1 - \alpha(\delta\theta)^2(\partial_x^2\theta_1)\sin(2\theta_0) + \mathcal{O}(\alpha^3)$$

$$= \frac{\alpha}{2}(\partial_x\delta\theta)^2\cos(2j_0x - 2k_0j_0t) + \alpha j_0^2(\delta\theta)^2\cos(2j_0x - 2j_0k_0t) + \frac{\alpha^2(\partial_x\delta\theta)^2}{4(k_0^2 - 1)}\sin^2(2j_0x - 2j_0k_0t)$$

$$- \frac{\alpha^2 j_0^2(\delta\theta)^2}{k_0^2 - 1}\cos^2(2j_0x - 2j_0k_0t) + \frac{\alpha^2 j_0^2(\delta\theta)^2}{2(k_0^2 - 1)}\sin^2(2j_0x - 2j_0k_0t) + \frac{\alpha^2 j_0^2(\delta\theta)^2}{k_0^2 - 1}\sin^2(2j_0x - 2j_0k_0t) + \mathrm{O}(\alpha^3). \quad \text{(H25)}$$

Equivalently, we have

$$K_1 = \alpha\int dt\int_L dx\left[\frac{1}{2}(\partial_x\delta\theta)^2\cos(2j_0x - 2j_0k_0t) + j_0^2(\delta\theta)^2\cos(2j_0x - 2j_0k_0t)\right]$$

$$= \alpha\int dt\int_L dx\left[\frac{1}{4L}\sum_{q,q'}\int_{\omega,\omega'} qq'\delta\theta_{q,\omega}^\dagger\delta\theta_{q',\omega'} + \frac{j_0^2}{2L}\sum_{q,q'}\int_{\omega,\omega'}\delta\theta_{q,\omega}^\dagger\delta\theta_{q',\omega'}\right]e^{-iqx+i\omega t}e^{iq'x - i\omega' t}e^{2ij_0x - 2ij_0k_0t} + \text{H.c.}$$

$$= \frac{\alpha}{4}\sum_q\int_\omega \delta\theta_{q+2j_0,\omega+2j_0k_0}^\dagger\delta\theta_{q,\omega}[q(q+2j_0) + 2j_0^2] + \text{H.c.} = \frac{\alpha}{4}\sum_q\int_\omega \delta\theta_{q+j_0,\omega+j_0k_0}^\dagger\delta\theta_{q-j_0,\omega-j_0k_0}(q^2 + j_0^2)e^{-2ij_0k_0t} + \text{H.c.},$$
$$\text{(H26)}$$

$$K_2 = \alpha^2\int dt\int_L dx\left\{\frac{(\partial_x\delta\theta)^2}{8(k_0^2-1)}[1 - \cos(4j_0x - 4j_0k_0t)] - \frac{j_0^2(\delta\theta)^2}{k_0^2-1}\cos(4j_0x - 4j_0k_0t) + \frac{j_0^2(\delta\theta)^2}{4(k_0^2-1)}[1 - \cos(4j_0x - 4j_0k_0t)]\right\}$$

$$= \frac{\alpha^2}{8(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\delta\theta_{q,\omega}^\dagger\delta\theta_{q,\omega}q^2 + \frac{\alpha^2 j_0^2}{4(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\delta\theta_{q,\omega}^\dagger\delta\theta_{q,\omega}$$

$$- \frac{\alpha^2}{16(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\{\delta\theta_{q+2j_0,\omega+2j_0k_0}^\dagger\delta\theta_{q-2j_0,\omega-2j_0k_0}[(q^2 - 4j_0^2) + 8j_0^2 + 2j_0^2]e^{-4ij_0k_0t} + \text{H.c.}\}$$

$$= \frac{\alpha^2}{8(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}\delta\theta_{q,\omega}^\dagger\delta\theta_{q,\omega}(2j_0^2 + q^2) - \frac{\alpha^2}{16(k_0^2-1)}\sum_q\int\frac{d\omega}{2\pi}[\delta\theta_{q+2j_0,\omega+2j_0k_0}^\dagger\delta\theta_{q-2j_0,\omega-2j_0k_0}(q^2 + 6j_0^2) + \text{H.c.}].$$
$$\text{(H27)}$$

Taking Eqs. (H24), (H26), and (H27) into Eq. (H20), we obtain the matrix elements,

$$\frac{1}{2\pi}K_{q+p,\omega+\nu;q-p,\omega-\nu} = (\omega^2 - q^2)\delta_{p,0}\delta(\nu) + \frac{\alpha(q^2 + j_0^2)}{2}\left(\delta_{p,j_0}\delta(\nu - j_0k_0) + \delta_{p,-j_0}\delta(\nu + j_0k_0)\right)$$

$$- \frac{\alpha^2}{8(k_0^2-1)}\left[-2(2j_0^2 + q^2)\delta_{p,0}\delta(\nu) + (q^2 + 6j_0^2)\left(\delta_{p,2j_0}\delta(\nu - 2j_0k_0) + \delta_{p,-2j_0}\delta(\nu + 2j_0k_0)\right)\right] + \mathcal{O}(\alpha^3).$$
$$\text{(H28)}$$

To find $\delta\theta$ that satisfies $\delta S_{\delta\theta}[\delta\theta] = 0$, we only have to find an eigenmode of K in Eq. (H21) that belongs to zero eigenvalue ("eigenenergy"). At the zeroth order in α, eigenmodes of K are characterized by q and ω, and the "zero-energy" eigenmodes are obtained by setting ω to be q (on-shell condition). When α is included perturbatively, eigenmodes at q and ω hybridize with eigenmodes at $q \pm 2j_0$ and $\omega \pm 2j_0k_0$ as well as eigenmodes at $q \pm 4j_0$ and $\omega \pm 4j_0k_0$ in terms of off-diagonal mixing terms. Due to the off-diagonal mixing terms, eigenmodes of K are characterized by q and ω modulo $2j_0$ and $2j_0k_0$ respectively, and $(q, \omega) \in [-j_0, j_0] \times [-k_0j_0, k_0j_0]$ plays a role of a first Brillouin zone. In the Brillouin zone, eigenmodes at the same (q, ω) are distinguished by a band index n,

$$K = \frac{1}{2}\sum_{n\in\mathbb{N}}\sum_{-j_0\leqslant q < j_0}\int_{-k_0j_0}^{k_0j_0}\frac{d\omega}{2\pi}\delta\varphi_{q,\omega,n}^\dagger\Lambda_{q,\omega,n}\delta\varphi_{q,\omega,n} + \mathcal{O}(\alpha^3), \quad \text{(H29)}$$

with

$$\delta\theta_{q+2j_0m_1,\omega+2k_0j_0m_2} = \sum_{n\in\mathbb{N}} c_{q,\omega,n;m_1,m_2}\delta\varphi_{q,\omega,n},\tag{H30}$$

$m_1 \in \mathbb{Z}$, $m_2 \in \mathbb{Z}$. Here, $c_{q,\omega,n;m_1,m_2}$ is analogous to the periodic part of a Bloch wave function in the band theory. From Eq. (H28) $K_{q,\omega;q,\omega'}$ is real symmetric, so that $c_{q,\omega,n;m_1,m_2}$ are real. An eigenstate of the lowest energy, say $\delta\varphi_{q,\omega,n=0}$, must approach $\delta\theta_{q,\omega}$ in the limit of $\alpha \to 0$. Such lowest eigenmode ($n=0$) is calculated up to the first order in α as follows:

$$c_{q,\omega,0;m_1,m_2} = \delta_{m_1,0}\delta_{m_2,0} - \frac{\alpha\left[\frac{(q+j_0)^2+j_0^2}{2}\right]}{(\omega+2j_0k_0)^2-(q+2j_0)^2-\omega^2+q^2}\delta_{m_1,1}\delta_{m_2,1}$$

$$- \frac{\alpha\left[\frac{(q-j_0)^2+j_0^2}{2}\right]}{(\omega-2j_0k_0)^2-(q-2j_0)^2-\omega^2+q^2}\delta_{m_1,-1}\delta_{m_2,-1} + \mathcal{O}(\alpha^2)$$

$$= \delta_{m_1,0}\delta_{m_2,0} - \alpha\frac{1+\mathcal{O}(q,\omega)}{4(k_0^2-1)}\delta_{m_1,1}\delta_{m_2,1} - \alpha\frac{1+\mathcal{O}(q,\omega)}{4(k_0^2-1)}\delta_{m_1,-1}\delta_{m_2,-1} + \mathcal{O}(\alpha^2).\tag{H31}$$

A corresponding "eigenenergy" is calculated up to the second order,

$$\Lambda_{q,\omega,0} = \omega^2 - q^2 + \frac{\alpha^2(2j_0^2+q^2)}{4(k_0^2-1)} - \frac{\alpha^2\left[\frac{(q+j_0)^2+j_0^2}{2}\right]^2}{(\omega+2j_0k_0)^2-(q+2j_0)^2-\omega^2+q^2} - \frac{\alpha^2\left[\frac{(q-j_0)^2+j_0^2}{2}\right]^2}{(\omega-2j_0k_0)^2-(q-2j_0)^2-\omega^2+q^2} + \mathcal{O}(\alpha^3)$$

$$= \omega^2 - q^2 + \frac{\alpha^2(2j_0^2+q^2)}{4(k_0^2-1)} - \frac{\alpha^2(2j_0^2+2qj_0+q^2)^2}{4[4j_0^2(k_0^2-1)+4qj_0-4\omega j_0k_0]} - \frac{\alpha^2(2j_0^2-2qj_0+q^2)^2}{4[4j_0^2(k_0^2-1)-4qj_0+4\omega j_0k_0]} + \mathcal{O}(\alpha^3),\tag{H32}$$

namely,

$$\Lambda_{q,\omega,0} = \omega^2 + \frac{\alpha^2 j_0^2}{2}\frac{1}{k_0^2-1} - q^2\left[1-\frac{\alpha^2}{4(k_0^2-1)}\right] - \frac{\alpha^2}{16j_0^2(k_0^2-1)}\left(4j_0^4+8qj_0^3+8q^2j_0^2\right)\left[1-\frac{q-\omega k_0}{j_0(k_0^2-1)}+\frac{(q-\omega k_0)^2}{j_0^2(k_0^2-1)^2}\right]$$

$$+ \mathcal{O}((q+\omega k_0)^3)\Big] - \frac{\alpha^2}{16j_0^2(k_0^2-1)}\left(4j_0^4-8qj_0^3+8q^2j_0^2\right)\left[1+\frac{q-\omega k_0}{j_0(k_0^2-1)}+\frac{(q-\omega k_0)^2}{j_0^2(k_0^2-1)^2}+\mathcal{O}((q+\omega k_0)^3)\right] + \mathcal{O}(\alpha^3)$$

$$= \omega^2 + \frac{\alpha^2 j_0^2}{2}\frac{1}{k_0^2-1} - q^2\left[1-\frac{\alpha^2}{4(k_0^2-1)}\right] - \frac{\alpha^2 j_0^2}{2(k_0^2-1)} + \mathcal{O}(\alpha^2\omega^2,\alpha^2q^2,\alpha^2\omega q,\alpha^3)$$

$$= \omega^2 - q^2\left[1-\frac{\alpha^2}{4(k_0^2-1)}\right] + \mathcal{O}(\alpha^2\omega^2,\alpha^2q^2,\alpha^2\omega q,\alpha^3).\tag{H33}$$

The lowest energy band indicates that $\delta\theta$ evaluated on shell, $\Lambda_{q,\omega,0} = 0$, behaves like a gapless classical wave. This is because the original theory, Eq. (6), has a spacetime translational symmetry. Thus, for any θ, one can choose $\delta\theta$ as a translation of θ, and such $\delta\theta$ does not change the Lagrangian. For a general k_0 (off the resonance point; $k_0 \neq 1$), α can be treated perturbatively, and the classical wave up to the second order in α has a well-defined (i.e., real-valued) velocity v,

$$1 - \frac{\alpha^2}{4(k_0^2-1)} = 1 + \mathcal{O}(\alpha^2) \equiv v^2 > 0.\tag{H34}$$

By evaluating the eigenmode on shell ($|\omega| = |q| + \mathcal{O}(\alpha^2)|q|$), we finally determine the first-order $\delta\theta_1$ for an arbitrary form of $\delta\theta_0$ given by Eq. (H1),

$$\delta\theta_1(x,t) = \frac{1}{\sqrt{L}}\sum_q\int_{-k_0j_0}^{k_0j_0}d\omega[\delta(q-\omega)d_q+\delta(q+\omega)d'_q]\left[-\alpha\frac{1+\mathcal{O}(q)}{4(k_0^2-1)}e^{i(q+2j_0)x-i(\omega+2j_0k_0)t}\right.$$

$$\left. - \alpha\frac{1+\mathcal{O}(q)}{4(k_0^2-1)}e^{i(q-2j_0)x-i(\omega-2j_0k_0)t}\right].\tag{H35}$$

Finally, let us include $\theta_2(x,t)$ into Eq. (H22),

$$\theta(x,t) = \theta_0(x,t) + \theta_1(x,t) + \theta_2(x,t) + \mathcal{O}(\alpha^2),\tag{H36}$$

$$\theta_2(x,t) = 2\alpha\text{Re}(g)\cos[2k_0j_0(t-x)] + 2\alpha\text{Re}(g')\cos[2k_0j_0(t+x)] - 2\alpha\text{Im}(g)\sin[2k_0j_0(t-x)] - 2\alpha\text{Im}(g')\sin[2k_0j_0(t+x)].\tag{H37}$$

Equation (H27) has an additional $\mathcal{O}(\alpha^2)$ contribution,

$$\Delta K_2 = \int dt\,dx \big[-\alpha(\partial_x \delta\theta)^2 \sin(2\theta_0)\theta_2 + 2\alpha(\delta\theta)^2(\partial_x\theta_0)(\partial_x\theta_2)\cos(2\theta_0) - 2\alpha(\delta\theta)^2(\partial_x\theta_0)^2\sin(2\theta_0)\theta_2 + \alpha(\delta\theta)^2\big(\partial_x^2\theta_2\big)\sin(2\theta_0)\big].$$

(H38)

For $k_0 \neq 1$, ΔK_2 contributes only to off-diagonal matrix elements of $K_{q,\omega;q,\omega}$, so that it changes neither Eq. (H33) nor Eq. (H35) at their respective subleading order.

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