Controllable odd-frequency Cooper pairs in multisuperconductor Josephson junctions

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We consider Josephson junctions formed by multiple superconductors with distinct phases and explore the formation of nonlocal or intersuperconductor pair correlations. We find that the multiple superconductor nature offers an additional degree of freedom that broadens the classification of pair symmetries, enabling nonlocal even- and odd-frequency pairings that can be highly controlled by the superconducting phases and the energy of the superconductors. Specially, when the phase difference between two superconductors is π, their associated nonlocal odd-frequency pairing is the only type of intersuperconductor pair correlations. Finally, we show that these nonlocal odd-frequency Cooper pairs dominate the nonlocal conductance via crossed Andreev reflections, which constitutes a direct evidence of odd-frequency pairing.

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I. INTRODUCTION

Superconductivity is caused by electrons binding together into Cooper pairs below a critical temperature and has attracted great interest due to its properties for quantum technologies [1]. The applications of superconductors are thus intimately linked to the Cooper pairs, specially to the symmetries or their wave function or pair amplitude. Due to the fermionic nature of electrons, the pair amplitude is antisymmetric under the exchange of all the quantum numbers describing the paired electron states plus the exchange of their relative time coordinates. Of particular interest is that the antisymmetry enables the formation of odd-frequency Cooper pairs, where the pair amplitude is odd in the relative time, or frequency ω, of the paired electrons [2–7]. As a result, the odd-ω Cooper pairs characterize a unique type of superconducting pairing that is intrinsically dynamic [8–13].

Odd-ω Cooper pairs have been studied as bulk and induced effects in several systems [8–13], such as in superconducting heterostructures [14–69], multiband superconductors [70–82], time-periodic superconductors [83–85], and non-Hermitian superconductors [86]. There also exist experiments supporting the realization of induced triplet odd-ω pairs in hybrid systems between superconductors with magnetic materials [87–101]. All these studies show that, to induce odd-ω pairs, the symmetries linked to the quantum numbers of the paired electrons must break [12]. While this condition guarantees the formation of odd-ω pairs, it does not restrict the appearance of even-ω pairs [81,102,103] which then mask the odd-ω signatures. Yet another issue is that controlling odd-ω pairs, despite the efforts [104–109], is still challenging without magnetic materials.

In this work we demonstrate the generation, control, and direct detection of spin-singlet odd-ω Cooper pairs in Josephson junctions (JJs) formed by multiple superconductors (Fig. 1). In particular, we exploit the degree of freedom offered by the multisuperconductor nature of the setup and find that intersuperconductor even- and odd-ω Cooper pairs naturally arise and can be controlled by the superconducting phases and on-site energies of the superconductors. Interestingly, for a JJ with two superconductors, the even-ω amplitude vanishes either when the superconducting phase difference is π or at zero on-site energy, leaving only odd-ω pairing. This behavior remains when the number of superconductors increases but only at weak couplings between superconductors. Furthermore, we discover that crossed Andreev reflections (CARs) directly probe odd-ω Cooper pairs and can be controlled by the superconducting phases. Our work thus puts forward multisuperconductor JJs as a powerful and entirely different route for odd-ω Cooper pairs.

The remainder of this article is organized as follows. In Sec. II, we introduce the multisuperconductor JJs studied in this work, while in Sec. III we show how to obtain the emerging pair amplitudes. In Sec. IV we present the obtained even- and odd-ω pair amplitudes and discuss their tunability by the superconducting phases. In Sec. V we demonstrate how the nonlocal odd-ω pair amplitude is detected via CAR processes. Finally, in Sec. VI we present our conclusions.

II. MULTISUPERCONDUCTOR JJs

We consider JJs as shown in Fig. 1, where n conventional spin-singlet s-wave superconductors are coupled directly. For

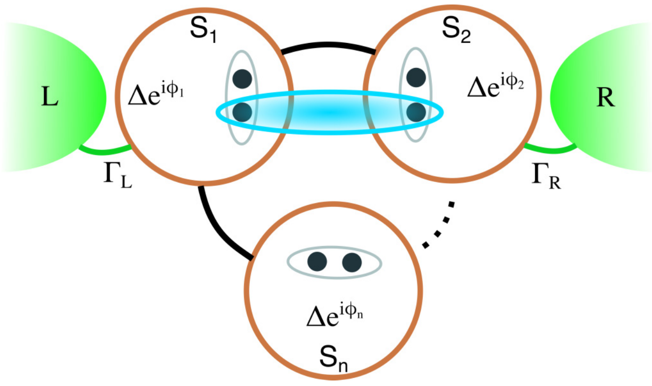

FIG. 1. JJs formed by coupling superconductors S_i with distinct phases ϕ_i, and same induced pair potential Δ. In each S_i local pairs are depicted in gray ellipses containing two electrons (black filled circles), referred to as intrasuperconductor (local) pairs. Due to the tunneling between superconductors, intersuperconductor (nonlocal) pair correlations emerge (cyan) which can be controlled by ϕ_i. Normal leads (green) are attached to two S_i for exploring nonlocal transport and detecting intersuperconductor Cooper pairs.

the sake of simplicity, we model these JJs by only considering the contact regions, with a Hamiltonian given by

$$H_{\rm nJJ} = \sum_{j=1}^{n}[\epsilon_j c^{\dagger}_{j\sigma}c_{j\sigma} + \Delta e^{i\phi_j}c^{\dagger}_{j\sigma}c^{\dagger}_{j\bar{\sigma}} + {\rm H.c.}] + H_{\rm T}, \qquad (1)$$

where the first two terms describe the superconductor S_j, where $c_{j\sigma}$ ($c^{\dagger}_{j\sigma}$) annihilates (creates) an electronic state with spin σ at site j with on-site energy ϵ_j, phase ϕ_j, and induced pair potential Δ from a parent spin-singlet s-wave superconductor with order parameter $\Delta_{\rm sc}$. Moreover, $H_{\rm T} = t_0\sum_{j=1}^{n} c^{\dagger}_{j\sigma}c_{j+1\sigma} + {\rm H.c.}$ represents the coupling between superconductors with equal strength t_0 and $c_{n+1} = c_1$. Away from the bulk gap edges, Δ is determined as $\Delta = \tau^2/\Delta_{\rm sc}$ [110–113], where τ is the coupling between S_j and the bulk superconductor. Below we choose $\tau = 0.7$ and $\Delta = 0.5$ such that $\Delta_{\rm sc} = 1$ is larger than the induced gap and fix it as our energy unit. We also drop the spin index for simplicity but keep in mind that the superconductors in Eq. (1) are spin singlet. Despite the simplicity of our model, it captures the main effects we aim to explore in this work, namely, the multisuperconductor nature and the distinct superconducting phases. Systems involving multiple JJs have been studied before but in the context of topological phases [114–125]. Here, we expand the playground of these multisuperconductor JJs for realizing controllable odd-ω Cooper pairs.

III. SUPERCONDUCTING PAIR AMPLITUDES

We are interested in intersuperconductor pair correlations which we also refer to as nonlocal pair correlations as they reside between superconductors. Pair correlations are described by the anomalous Green's function $\mathcal{F}_{nm}(1,1') = \langle \mathcal{T}c_n(1)c_m(1')\rangle$, where $\mathcal{T}$ is the time ordering operator, c_n annihilates an electronic state with quantum numbers n at time and position $1 = (x_1, t_1)$ [126,127]. The fermionic nature of electrons dictates the antisymmetry condition $\mathcal{F}_{nm}(1,1') = -\mathcal{F}_{mn}(1',1)$, which enables the classification of superconducting pair correlations based on all the quantum numbers, including time and space coordinates [8–13]. Thus this condition enables even- and odd-ω pair correlations when $F_{nm}(\omega) = \pm F_{nm}(-\omega)$, with $F_{nm}(\omega)$ being the Fourier transform of $\mathcal{F}_{nm}(1,1')$ into frequency domain. In the case of multisuperconductor junctions, the multiple superconductor nature introduces an additional quantum number n, the superconductor index, that broadens the classification of pair symmetries in a similar way as the band index in multiband superconductors [12]. In Table I we present all the allowed pair symmetry classes that respect the antisymmetry condition in JJs with spin-singlet and spin-triplet superconductors: four classes correspond to odd-ω pair correlations which are the four bottom classes in Table I; see Supplemental Material [128] for details. It is evident that the superconductor index (sup. index) plays a crucial role for broadening the allowed pair symmetries.

TABLE I. Allowed superconducting pair symmetries in multisuperconductor JJs under the presence of spin-mixing fields. The classes ESEE and OSOE correspond to the pair correlations reported in this work.

Pair symmetries in multisuperconductor JJs				
Frequency ($\omega \leftrightarrow -\omega$)	Spin ($\uparrow\leftrightarrow\downarrow$)	Sup. index ($n \leftrightarrow m$)	Parity ($x \leftrightarrow x'$)	Pair symmetry class (total exchange)
Even	Singlet	Even	Even	ESEE
Even	Singlet	Odd	Odd	ESOO
Even	Triplet	Even	Odd	ETEO
Even	Triplet	Odd	Even	ETOE
Odd	Singlet	Even	Odd	OSEO
Odd	Singlet	Odd	Even	OSOE
Odd	Triplet	Even	Even	OTEE
Odd	Triplet	Odd	Odd	OTOO

In the JJs with spin-singlet s-wave superconductors considered here, the symmetric and antisymmetric combination $F^{+(-)}_{nm} = (F_{nm} \pm F_{mn})/2$ become even- and odd-ω pair symmetry classes, respectively [128]. These two pair symmetry classes correspond to the ESEE and OSOE classes in Table I. In practice, the pair correlations F_{nm} are obtained from the electron-hole component of the Nambu Green's function, whose equation of motion in frequency space reads $[\omega - \mathcal{H}_{\rm nJJ}]G(\omega) = \mathbf{I}$, where $\mathcal{H}_{\rm nJJ}$ is the Nambu Hamiltonian of the JJ with n superconductors described by Eqs. (1) in the basis $\Psi = (c_1, c^{\dagger}_1, c_2, c^{\dagger}_2, \ldots, c_n, c^{\dagger}_n)^{\rm T}$.

IV. INTERSUPERCONDUCTOR PAIR AMPLITUDES IN JJs

To begin, we focus on the pair correlations in a JJ with two superconductors coupled directly. This system is modeled by $H_{\rm 2JJ}$ with $n = 2$ in Eq. (1). As described in the previous section, the pair correlations are obtained from electron-hole components of the Green's function associated to the Nambu Hamiltonian in the basis $\Psi = (c_1, c^{\dagger}_1, c_2, c^{\dagger}_2)^{\rm T}$. Without loss of generality, we assume a phase difference $\phi_2 - \phi_1 = \phi$. Then, considering $\epsilon_{1,2} \equiv \epsilon$, the symmetric and antisymmetric pair

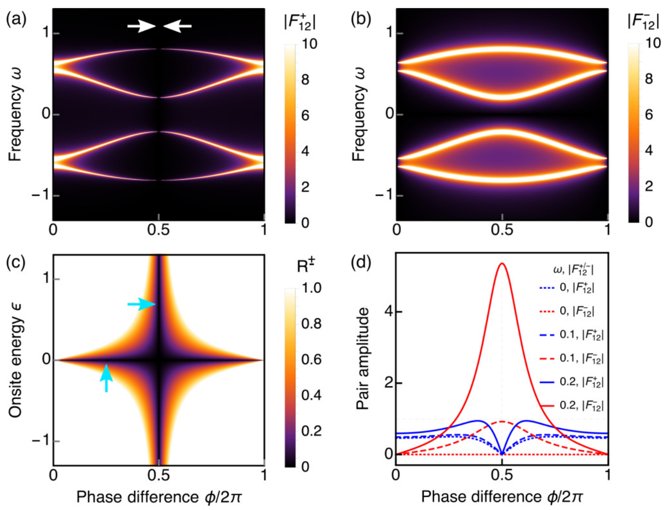

FIG. 2. (a), (b) Symmetric even-ω and antisymmetric odd-ω nonlocal pair amplitudes ($F_{12}^{\pm}$) in a JJ with two superconductors coupled directly as a function of ω and ϕ at $\epsilon = 0.1$, with the color scale cut off at 10 for visualization. White arrows in (a) indicate that $|F_{12}^{+}|$ vanishes at $\phi = \pi$. (c) Ratio $R^{\pm} = |F_{12}^{+}|/|F_{12}^{-}|$ as a function of ϵ and ϕ at $\omega = 0.2$. Cyan arrows indicate that R vanishes either at $\epsilon = 0$ or $\phi = \pi$. (d) Line cuts of (a), (b) at fixed ω. Parameters: $\Delta = 0.5$ and $t_0 = 0.3$.

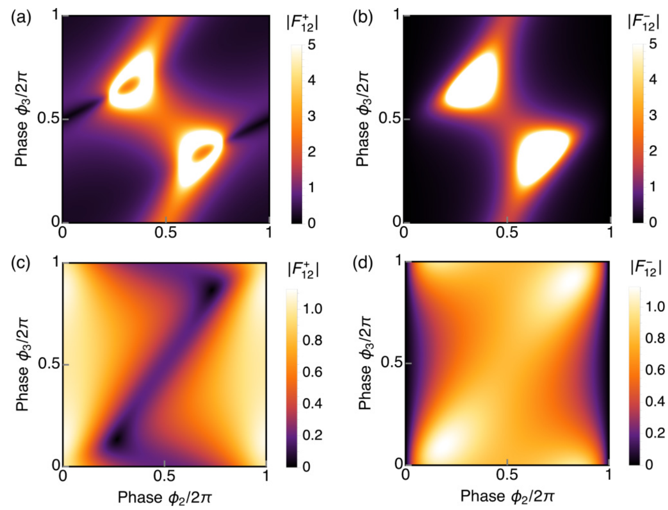

FIG. 3. (a), (b) Symmetric even-ω (F_{12}^{+}) and antisymmetric odd-ω (F_{12}^{-}) nonlocal pair amplitudes in a JJ with three superconductors coupled directly as a function of ϕ_2 and ϕ_3 at $\omega = 0.1$ and $t_0 = 0.3$, with the color scale cut off at 5 for visualization. (c), (d) Same as in (a),(b) but at $\omega = 1$ and $t_0 = 0.5$. Parameters: $\Delta = 0.5$, $\epsilon = 0$, and $\phi_1 = 0$.

amplitudes in superconductor index are given by [128]

$$F_{12}^{+}(\omega) = \frac{2\epsilon\Delta t_0 \cos(\phi/2)}{P + 2\Delta^2 t_0^2 \cos(\phi)},$$
$$F_{12}^{-}(\omega) = \frac{2i\omega\Delta t_0 \sin(\phi/2)}{P + 2\Delta^2 t_0^2 \cos(\phi)}, \qquad (2)$$

where ω represents complex frequencies unless otherwise stated and $P = (\Delta^2 - \omega^2 + \epsilon^2)^2 - 2t_0^2(\omega^2 + \epsilon^2) + t_0^4$. First, both pair amplitudes in Eqs. (2) have the same denominator which is an even function of ω and reveals the formation of Andreev bound states (ABSs) when $P + 2\Delta^2 t_0^2 \cos(\phi) = 0$. This is seen in the bright regions of Fig. 2, where we plot the absolute value of the symmetric and antisymmetric amplitudes as a function of the phase difference ϕ. Second, the numerators of both F_{12}^{+} and F_{12}^{-} have different functional dependences, oscillating with the phase difference ϕ in an alternate fashion as $\cos(\phi/2)$ and $\sin(\phi/2)$, respectively [129]. While the numerator of the symmetric term is an even function of ω with a linear dependence on ϵ, the antisymmetric component is interestingly linear in ω and, therefore, an odd function of frequency. The symmetric even-ω part vanishes either when $\epsilon = 0$ or $\phi = \pi$, while the antisymmetric odd-ω pair amplitude remains remarkably finite at these points and even acquires large values. The surprising features of the nonlocal pair amplitudes can be seen by comparing the panels of Figs. 2(a), 2(b) and 2(d), where the vanishing values of the even-ω part are indicated by white arrows in Fig. 2(a). The vanishing values of the even-ω pairing can be better seen in Fig. 2(c), where we plot the ratio between the two pair amplitudes, $R^{\pm} = |F_{12}^{+}|/|F_{12}^{-}| = |(\epsilon/i\omega)\cot(\phi/2)|$: $R^{\pm}$ vanishes either at $\epsilon = 0$ or $\phi = \pi$. Note, however, that since F_{12}^{-} is an odd function of ω and thus vanishes at $\omega = 0$, $R^{\pm}$ has a clear interpretation only for $\omega \neq 0$. In sum, JJs with two superconductors exhibit highly tunable odd-ω pairing that is the only type of intersuperconductor pair correlations.

For JJs with more superconductors $n > 2$, the expressions for the nonlocal pair amplitudes become lengthy, but still capturing the formation of ABSs in the denominator and with numerators that strongly depend on all ϕ_i [128]. We find that the symmetric and antisymmetric pair amplitudes between nearest neighbor superconductors develop even- and odd-ω symmetries, respectively. While the odd-ω part is proportional to $\sim(e^{i\phi_{j+1}} - e^{i\phi_j})$, the even-$\omega$ term is to $\sim(e^{i\phi_{j+1}} + e^{i\phi_j}) + P(\phi_{1,\dots,n})$, where P is a function of all the system parameters [128]. Thus the odd-ω term depends on the sine of the phase difference of the involved superconductors as in JJs with two superconductors discussed above. However, the even-ω part has a cosine part as for JJs with two superconductors, but also an additional contribution due to the rest of the system. Nevertheless, both pair amplitudes exhibit a high degree of tunability by means of the superconducting phases. To visualize this fact, in Fig. 3 we plot the even-ω and odd-ω pair amplitudes for a JJ with three superconductors as a function of ϕ_2 and ϕ_3 at $\phi_1 = 0$. The main feature of this figure is that the behavior of both pair amplitudes is highly controllable by the superconducting phases. Interestingly, there are regions where the even-ω component acquires vanishing small values while the odd-ω remains sizably large; see dark and bright regions in Figs. 3(a) and 3(c) and Figs. 3(b) and 3(d), respectively.

The vanishing and finite values of the even- and odd-ω pair amplitudes can be further visualized in a simpler regime. Specially, for very weak couplings between superconductors t_0 and for superconductors with the same on-site energy ϵ, the nearest neighbor nonlocal pair amplitudes up to linear order in t_0 are given by [128]

$$F_{j,j+1}^{+}(\omega) \approx \frac{\epsilon\Delta t_0(e^{i\phi_{j+1}} + e^{i\phi_j})}{(\Delta^2 - \omega^2 + \epsilon^2)^2},$$
$$F_{j,j+1}^{-}(\omega) \approx \frac{\omega\Delta t_0(e^{i\phi_{j+1}} - e^{i\phi_j})}{(\Delta^2 - \omega^2 + \epsilon^2)^2}, \qquad (3)$$

where $j = 1, \ldots, n$ and $\phi_{n+1} = \phi_1$. Strikingly, only the pair amplitudes between nearest neighbor superconductors remain finite at leading order in t_0 [130]. As expected, $F^{+}_{j,j+1}$ and $F^{-}_{j,j+1}$ in Eqs. (3) exhibit even- and odd-ω spin-singlet symmetries, respectively. Interestingly, both pair amplitudes acquire the same form as their counterparts in JJs with two superconductors; see Eqs. (2). In this regime, the even-ω pairing thus vanishes either at $\epsilon = 0$ or when $e^{i\phi_{j+1}} + e^{i\phi_j} = 0$, which needs a phase difference of $\phi_{j+1} - \phi_j = \pi$ between superconductors. However, the odd-ω component remains always finite in this regime, exhibiting high tunability by ϕ_j. We have verified that this behavior remains even in JJs with finite superconductors and also in JJs with superconductors coupled via a normal region [128]. Hence multisuperconductor JJs represent a rich platform for the generation and control of nonlocal odd-ω pair correlations that do not require magnetic elements. Before closing this part, we highlight that the odd-ω pair amplitudes presented here are a proximity-induced superconducting effect bound to the device, exhibiting wide controllability by the superconducting phases and with important impact on physical observables as we discuss next.

V. CAR DETECTION OF ODD-ω PAIRING

Having established the emergence of intersuperconductor odd-ω pairs in multisuperconductor JJs, now we inspect a direct detection protocol. Due to the nonlocal character of the pair correlations found here, it is natural to explore nonlocal transport of Cooper pairs [28,36,75,131]. Without loss of generality, we focus on JJs formed by two superconductors and aim at detecting the odd-ω pairs obtained in Eqs. (2). Hence we attach two normal leads at the left and the right of the system as in Fig. 1 and include them in our model via retarded self-energies $\Sigma^{r}_{\rm L(R)}$, such that the system's retarded Green's function is $G^r(\omega) = (\omega + i0^+ - H_{\rm 2JJ} - \Sigma^r_{\rm L} - \Sigma^r_{\rm R})^{-1}$ [132]. Here, $H_{\rm 2JJ}$ describes the JJ described by Eq. (1) with $n = 2$ and ω now represents real frequencies. In the wideband limit, $\Sigma^r_j = -i\Gamma_j/2$, where $\Gamma_j = \pi|\tau|^2\rho_j$ characterizes the coupling to lead j with surface density of states ρ_j and τ is the hopping between leads and superconductors.

At weak Γ_j, the JJ can be probed by nonlocal transport. Specially, the transport of Cooper pairs is characterized by nonlocal Andreev reflection or crossed Andreev reflection ($T_{\rm CAR}$), which competes with electron tunneling ($T_{\rm ET}$) to determine the nonlocal conductance $\sim(T_{\rm CAR} - T_{\rm ET})$ [128]. These CAR and ET processes involve electron-hole (hole-electron) and electron-electron (hole-hole) transfers, $T_{\rm CAR} = T_{eh} + T_{he}$ and $T_{\rm ET} = T_{ee} + T_{hh}$, which can be obtained from G^r as [75]

$$
\begin{aligned}
T_{ee} &= \Gamma^e_{\rm L}\Gamma^e_{\rm R}\left|g^r_{12}\right|^2, \quad T_{hh} = \Gamma^h_{\rm L}\Gamma^h_{\rm R}\left|\bar{g}^r_{12}\right|^2,\\
T_{eh} &= \Gamma^e_{\rm L}\Gamma^h_{\rm R}\left|F^r_{12}\right|^2, \quad T_{he} = \Gamma^h_{\rm L}\Gamma^e_{\rm R}\left|\bar{F}^r_{12}\right|^2,
\end{aligned}
\tag{4}
$$

where g^r_{12} ($\bar{g}^r_{12}$) and F^r_{12} ($\bar{F}^r_{12}$) are the normal and anomalous (or pair amplitude) components of the intersuperconductor retarded Green's function, obtained from G^r [128]. Interestingly, the CAR processes $T_{eh(he)}$ are directly determined by the squared modulus of the intersuperconductor pair amplitudes F^r_{12}. We note that, while the pair amplitudes F^r_{12} and $\bar{F}^r_{12}$ are not directly measurable, their modulo respectively determines the finite value of the nonlocal probabilities T_{eh} and T_{he}, thus facilitating the detection of these emergent pairings.

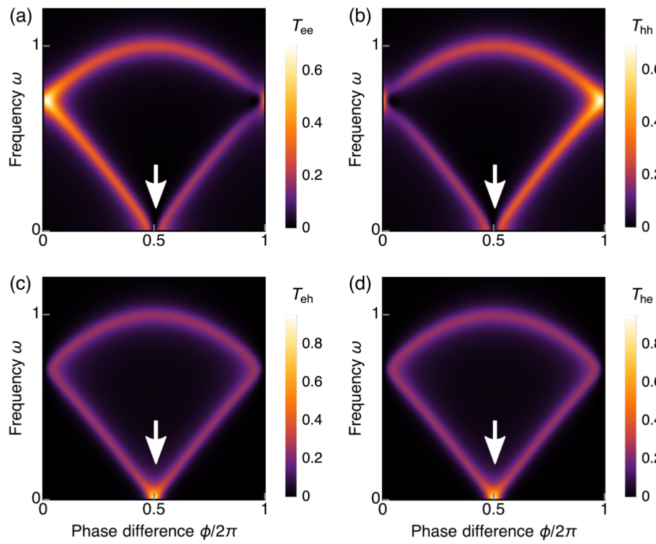

FIG. 4. Electron tunneling (top row) and crossed Andreev reflection (bottom) processes as a function of ω and ϕ. Parameters: $\Delta = 0.5$, $\epsilon = 0$, $\Gamma_j = 0.1$, and $t_0 = 0.5$.

Under general circumstances, F^r_{12} includes both symmetric even-ω and antisymmetric odd-ω terms and the symmetric part vanishes at $\epsilon = 0$ for any ϕ; see Eqs. (2). Thus the CAR amplitudes have the potential to directly probe the antisymmetric intersuperconductor odd-ω pairing. However, as shown above, the CAR processes $T_{eh(he)}$ are always accompanied by electron tunnelings $T_{ee(hh)}$. Therefore, even if $T_{eh(he)}$ directly probes odd-ω pairs, their total effect in the nonlocal conductance can be masked if $T_{ee(hh)}$ are larger. For this reason, to directly detect intersuperconductor odd-ω pairing, a regime where $T_{ee(hh)} \ll T_{eh(he)}$ is needed. Even though this regime might sound challenging to find, we now demonstrate that it is in fact possible. To show this, we consider $\phi_1 = -\phi/2$, $\phi_2 = \phi/2$ and assume symmetric couplings to the leads $\Gamma_j = \Gamma$. Then, for $\epsilon = 0$, g^r_{12} and the antisymmetric pair amplitude $F^{r,-}_{12}$ are given by [128]

$$
\begin{aligned}
g^r_{12} &= -4t_0\left\{(\Gamma - 2i\omega)^2 + 4t_0^2 + 4\Delta^2 e^{-i\phi}\right\}/D,\\
F^{r,-}_{12} &= 16it_0\Delta(2\omega + i\Gamma)\sin(\phi/2)/D,
\end{aligned}
\tag{5}
$$

where $D = 16t_0^4 + [4\Delta^2 + (\Gamma - 2i\omega)^2]^2 + 8t_0^2(\Gamma - 2i\omega)^2 + 32t_0^2\Delta^2\cos(\phi)$, $\bar{g}^r_{12}(\phi) = -g^r_{12}(-\phi)$, and $\bar{F}^r_{12}(\phi) = F^r_{12}(\phi)$. We note that F^r_{12} can be obtained from Eqs. (2) by replacing $\omega \to \omega + i0^+ + i\Gamma/2$. Now, we can exploit the fact that the energy of the ABSs at $\epsilon = 0$ and $\phi = \pi$ is given by $|\omega_\pm| = |t_0 - \Delta|$, which clearly vanishes for $t_0 = \Delta$. In this regime we have $|g^r_{12}|/|F^{r,-}_{12}| \approx \omega/(2\Delta) \ll 1$ for low frequencies. Thus it is possible to obtain a regime where the antisymmetric pair amplitude is larger than the normal contribution. Hence, in this regime, $T_{eh(he)}$ are expected to be larger than $T_{ee(hh)}$ and constitute the main contribution to the nonlocal conductance, whose finite value indicates a direct evidence of intersuperconductor odd-ω pairing.

To visualize the above argument, in Fig. 4 we plot ET and CAR processes as a function of ϕ and ω at $\epsilon = 0$. The most important feature is that, at high frequencies, ET processes $T_{ee(hh)}$ acquire large values near $\phi = 0, 2\pi$ but are vanishing small at low ω near $\phi = \pi$, in line with the discussion presented above. Interestingly, the CAR processes $T_{eh(he)}$ acquire large values around $\phi = \pi$ at low frequencies but smaller values at higher frequencies. The finite values of these CAR processes directly probe the formation of induced odd-ω pairs. Of particular relevance here are the values around $\phi = \pi$ and low ω, because, at such points, CAR dominates over ET and it thus determines the nonlocal conductance. We have verified that this behavior also holds for JJs with more than two superconductors but in the weak tunneling regime, thus supporting the direct detection of proximity-induced intersuperconductor odd-ω pairing in a nonlocal transport measurement. Hence, despite being an induced effect, the nonlocal odd-ω pairs determine CAR processes by simply tuning the superconducting phases in multisuperconductor JJs.

VI. CONCLUSIONS

In conclusion, we have studied multisuperconductor Josephson junctions and found that intersuperconductor even- and odd-ω Cooper pairs can be generated, controlled, and detected by virtue of the superconducting phases. We found that even-ω pairing vanishes when the phase differences between two superconductors is π, thus leaving odd-ω pairing as the only type of intersuperconductor pair correlations. While this finding is exact for Josephson junctions with two superconductors, it is only valid at weak couplings between superconductors in junctions with more than two superconductors. Due to the vanishing of even-ω pairing, only odd-ω pairs contribute to CAR processes, whose finite values directly probe the presence of odd-ω Cooper pairs.

Given the advances in the fabrication of superconducting heterostructures, including a promising tunability of CAR processes [133], we expect that the physics discussed here could be soon realized in multiterminal Josephson junctions [117,122,134–136] and in superconducting quantum dots [137–146]. Of particular relevance are Refs. [117,122, 134–136] because they have already demonstrated the fabrication of multisuperconductor Josephson junctions and the control of several superconducting phases. In this regard, our work offers an entirely different route for the generation, control, and detection of odd-ω Cooper pairs that might be even possible to explore using already existing experimental techniques.

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[1] A. Acín, I. Bloch, H. Buhrman, T. Calarco, C. Eichler, J. Eisert, D. Esteve, N. Gisin, S. J. Glaser, F. Jelezko, S. Kuhr, M. Lewenstein, M. F. Riedel, P. O. Schmidt, R. Thew, A. Wallraff, I. Walmsley, and F. K. Wilhelm, The quantum technologies roadmap: A european community view, New J. Phys. **20**, 080201 (2018).
[2] V. L. Berezinskii, New model of the anisotropic phase of superfluid ^{3}He, Zh. Eksp. Teor. Fiz. **20**, 628 (1974) [JETP Lett. **20**, 287 (1974)].
[3] T. R. Kirkpatrick and D. Belitz, Disorder-induced triplet superconductivity, Phys. Rev. Lett. **66**, 1533 (1991).
[4] A. Balatsky and E. Abrahams, New class of singlet superconductors which break the time reversal and parity, Phys. Rev. B **45**, 13125 (1992).
[5] D. Belitz and T. R. Kirkpatrick, Even-parity spin-triplet superconductivity in disordered electronic systems, Phys. Rev. B **46**, 8393 (1992).
[6] E. Abrahams, A. Balatsky, D. J. Scalapino, and J. R. Schrieffer, Properties of odd-gap superconductors, Phys. Rev. B **52**, 1271 (1995).
[7] P. Coleman, A. Georges, and A. Tsvelik, Reflections on the one-dimensional realization of odd-frequency pairing, J. Phys.: Condens. Matter **9**, 345 (1997).
[8] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Odd triplet superconductivity and related phenomena in superconductor-ferromagnet structures, Rev. Mod. Phys. **77**, 1321 (2005).
[9] Y. Tanaka, M. Sato, and N. Nagaosa, Symmetry and topology in superconductors -odd-frequency pairing and edge states-, J. Phys. Soc. Jpn. **81**, 011013 (2012).
[10] J. Linder and A. V. Balatsky, Odd-frequency superconductivity, Rev. Mod. Phys. **91**, 045005 (2019).
[11] J. Cayao, C. Triola, and A. M. Black-Schaffer, Odd-frequency superconducting pairing in one-dimensional systems, Eur. Phys. J. Spec. Top. **229**, 545 (2020).
[12] C. Triola, J. Cayao, and A. M. Black-Schaffer, The role of odd-frequency pairing in multiband superconductors, Ann. Phys. (NY) **532**, 1900298 (2020).
[13] Y. Tanaka and S. Tamura, Theory of surface andreev bound states and odd-frequency pairing in superconductor junctions, J. Supercond. Novel Magn. **34**, 1677 (2021).
[14] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Long-range proximity effects in superconductor-ferromagnet structures, Phys. Rev. Lett. **86**, 4096 (2001).
[15] A. Kadigrobov, R. I. Shekhter, and M. Jonson, Quantum spin fluctuations as a source of long-range proximity effects in

diffusive ferromagnet-superconductor structures, Europhys. Lett. **54**, 394 (2001).
[16] Y. Tanaka and A. A. Golubov, Theory of the proximity effect in junctions with unconventional superconductors, Phys. Rev. Lett. **98**, 037003 (2007).
[17] T. Yokoyama, Y. Tanaka, and A. A. Golubov, Manifestation of the odd-frequency spin-triplet pairing state in diffusive ferromagnet/superconductor junctions, Phys. Rev. B **75**, 134510 (2007).
[18] Y. Asano, Y. Tanaka, and A. A. Golubov, Josephson effect due to odd-frequency pairs in diffusive half metals, Phys. Rev. Lett. **98**, 107002 (2007).
[19] Y. Tanaka, A. A. Golubov, S. Kashiwaya, and M. Ueda, Anomalous Josephson effect between even- and odd-frequency superconductors, Phys. Rev. Lett. **99**, 037005 (2007).
[20] Y. Tanaka, Y. Tanuma, and A. A. Golubov, Odd-frequency pairing in normal-metal/superconductor junctions, Phys. Rev. B **76**, 054522 (2007).
[21] M. Eschrig, T. Löfwander, T. Champel, J. C. Cuevas, J. Kopu, and G. Schön, Symmetries of pairing correlations in superconductor–ferromagnet nanostructures, J. Low Temp. Phys. **147**, 457 (2007).
[22] T. Yokoyama, Y. Tanaka, and A. A. Golubov, Theory of pairing symmetry inside the abrikosov vortex core, Phys. Rev. B **78**, 012508 (2008).
[23] Y. Asano, A. A. Golubov, Y. V. Fominov, and Y. Tanaka, Unconventional surface impedance of a normal-metal film covering a spin-triplet superconductor due to odd-frequency cooper pairs, Phys. Rev. Lett. **107**, 087001 (2011).
[24] Y. Asano and Y. Tanaka, Majorana fermions and odd-frequency cooper pairs in a normal-metal nanowire proximity-coupled to a topological superconductor, Phys. Rev. B **87**, 104513 (2013).
[25] A. M. Black-Schaffer and A. V. Balatsky, Proximity-induced unconventional superconductivity in topological insulators, Phys. Rev. B **87**, 220506(R) (2013).
[26] H. Ebisu, K. Yada, H. Kasai, and Y. Tanaka, Odd-frequency pairing in topological superconductivity in a one-dimensional magnetic chain, Phys. Rev. B **91**, 054518 (2015).
[27] Y. V. Fominov, Y. Tanaka, Y. Asano, and M. Eschrig, Odd-frequency superconducting states with different types of meissner response: Problem of coexistence, Phys. Rev. B **91**, 144514 (2015).
[28] F. Crépin, P. Burset, and B. Trauzettel, Odd-frequency triplet superconductivity at the helical edge of a topological insulator, Phys. Rev. B **92**, 100507(R) (2015).
[29] P. Burset, B. Lu, G. Tkachov, Y. Tanaka, E. M. Hankiewicz, and B. Trauzettel, Superconducting proximity effect in three-dimensional topological insulators in the presence of a magnetic field, Phys. Rev. B **92**, 205424 (2015).
[30] Z. Huang, P. Wölfle, and A. V. Balatsky, Odd-frequency pairing of interacting majorana fermions, Phys. Rev. B **92**, 121404(R) (2015).
[31] O. Kashuba, B. Sothmann, P. Burset, and B. Trauzettel, Majorana stm as a perfect detector of odd-frequency superconductivity, Phys. Rev. B **95**, 174516 (2017).
[32] J. Cayao and A. M. Black-Schaffer, Odd-frequency superconducting pairing and subgap density of states at the edge of a two-dimensional topological insulator without magnetism, Phys. Rev. B **96**, 155426 (2017).
[33] A. V. Balatsky, S. S. Pershoguba, and C. Triola, Odd-frequency pairing in conventional Josephson junctions, arXiv:1804.07244.
[34] B. Lu, P. Burset, Y. Tanuma, A. A. Golubov, Y. Asano, and Y. Tanaka, Influence of the impurity scattering on charge transport in unconventional superconductor junctions, Phys. Rev. B **94**, 014504 (2016).
[35] P. Burset, B. Lu, S. Tamura, and Y. Tanaka, Current fluctuations in unconventional superconductor junctions with impurity scattering, Phys. Rev. B **95**, 224502 (2017).
[36] F. Keidel, P. Burset, and B. Trauzettel, Tunable hybridization of Majorana bound states at the quantum spin Hall edge, Phys. Rev. B **97**, 075408 (2018).
[37] C. Fleckenstein, N. T. Ziani, and B. Trauzettel, Conductance signatures of odd-frequency superconductivity in quantum spin Hall systems using a quantum point contact, Phys. Rev. B **97**, 134523 (2018).
[38] J. Cayao and A. M. Black-Schaffer, Odd-frequency superconducting pairing in junctions with rashba spin-orbit coupling, Phys. Rev. B **98**, 075425 (2018).
[39] S. V. Bakurskiy, Y. V. Fominov, A. F. Shevchun, Y. Asano, Y. Tanaka, M. Y. Kupriyanov, A. A. Golubov, M. R. Trunin, H. Kashiwaya, S. Kashiwaya, and Y. Maeno, Local impedance on a rough surface of a chiral p-wave superconductor, Phys. Rev. B **98**, 134508 (2018).
[40] S.-Y. Hwang, P. Burset, and B. Sothmann, Odd-frequency superconductivity revealed by thermopower, Phys. Rev. B **98**, 161408(R) (2018).
[41] S. Tamura and Y. Tanaka, Theory of the proximity effect in two-dimensional unconventional superconductors with rashba spin-orbit interaction, Phys. Rev. B **99**, 184501 (2019).
[42] S.-I. Suzuki, A. A. Golubov, Y. Asano, and Y. Tanaka, Effects of phase coherence on local density of states in superconducting proximity structures, Phys. Rev. B **100**, 024511 (2019).
[43] A. Tsintzis, A. M. Black-Schaffer, and J. Cayao, Odd-frequency superconducting pairing in Kitaev-based junctions, Phys. Rev. B **100**, 115433 (2019).
[44] C. Triola and A. M. Black-Schaffer, Odd-frequency superconductivity induced by nonmagnetic impurities, Phys. Rev. B **100**, 144511 (2019).
[45] D. Takagi, S. Tamura, and Y. Tanaka, Odd-frequency pairing and proximity effect in Kitaev chain systems including a topological critical point, Phys. Rev. B **101**, 024509 (2020).
[46] P. Dutta, F. Parhizgar, and A. M. Black-Schaffer, Finite bulk Josephson currents and chirality blockade removal from interorbital pairing in magnetic Weyl semimetals, Phys. Rev. B **101**, 064514 (2020).
[47] D. Kuzmanovski, R. S. Souto, and A. V. Balatsky, Odd-frequency superconductivity near a magnetic impurity in a conventional superconductor, Phys. Rev. B **101**, 094505 (2020).
[48] D. Kuzmanovski, A. M. Black-Schaffer, and J. Cayao, Suppression of odd-frequency pairing by phase disorder in a nanowire coupled to majorana zero modes, Phys. Rev. B **101**, 094506 (2020).
[49] F. Keidel, S.-Y. Hwang, B. Trauzettel, B. Sothmann, and P. Burset, On-demand thermoelectric generation of equal-spin cooper pairs, Phys. Rev. Res. **2**, 022019(R) (2020).

[50] F. L. N. Santos, V. Perrin, F. Jamet, M. Civelli, P. Simon, M. C. O. Aguiar, E. Miranda, and M. J. Rozenberg, Odd-frequency superconductivity in dilute magnetic superconductors, Phys. Rev. Res. **2**, 033229 (2020).
[51] R. Seoane Souto, D. Kuzmanovski, and A. V. Balatsky, Signatures of odd-frequency pairing in the Josephson junction current noise, Phys. Rev. Res. **2**, 043193 (2020).
[52] T. Savander, S. Tamura, C. Flindt, Y. Tanaka, and P. Burset, Thermoelectric detection of Andreev states in unconventional superconductors, Phys. Rev. Res. **2**, 043388 (2020).
[53] N. T. Ziani, C. Fleckenstein, L. Vigliotti, B. Trauzettel, and M. Sassetti, From fractional solitons to majorana fermions in a paradigmatic model of topological superconductivity, Phys. Rev. B **101**, 195303 (2020).
[54] S. Ikegaya, S. Tamura, D. Manske, and Y. Tanaka, Anomalous proximity effect of planar topological Josephson junctions, Phys. Rev. B **102**, 140505(R) (2020).
[55] S. Mishra, S. Tamura, A. Kobayashi, and Y. Tanaka, Impact of impurity scattering on odd-frequency spin-triplet pairing near the edge of the Kitaev chain, Phys. Rev. B **103**, 024501 (2021).
[56] R. Nakai, K. Nomura, and Y. Tanaka, Edge-induced pairing states in a Josephson junction through a spin-polarized quantum anomalous Hall insulator, Phys. Rev. B **103**, 184509 (2021).
[57] S. Matsubara, Y. Tanaka, and H. Kontani, Generation of odd-frequency surface superconductivity with spontaneous spin current due to the zero-energy Andreev bound state, Phys. Rev. B **103**, 245138 (2021).
[58] S. Pal and C. Benjamin, Exciting odd-frequency equal-spin triplet correlations at metal-superconductor interfaces, Phys. Rev. B **104**, 054519 (2021).
[59] T. Löthman, C. Triola, J. Cayao, and A. M. Black-Schaffer, Disorder-robust p-wave pairing with odd-frequency dependence in normal metal–conventional superconductor junctions, Phys. Rev. B **104**, 094503 (2021).
[60] L. G. Johnsen and J. Linder, Spin injection and spin relaxation in odd-frequency superconductors, Phys. Rev. B **104**, 144513 (2021).
[61] S. Tamura, S. Hoshino, and Y. Tanaka, Generalization of spectral bulk-boundary correspondence, Phys. Rev. B **104**, 165125 (2021).
[62] S.-I. Suzuki, T. Hirai, M. Eschrig, and Y. Tanaka, Anomalous inverse proximity effect in unconventional superconductor junctions, Phys. Rev. Res. **3**, 043148 (2021).
[63] Y. Tanaka, T. Kokkeler, and A. Golubov, Theory of proximity effect in $s + p$-wave superconductor junctions, Phys. Rev. B **105**, 214512 (2022).
[64] D. Takagi, M. T. Mercaldo, Y. Tanaka, and M. Cuoco, Odd-frequency pairing in a nonunitary p-wave superconductor with multiple majorana fermions, Phys. Rev. B **105**, 224506 (2022).
[65] T. H. Kokkeler, Y. Tanaka, and A. A. Golubov, Spin-projected charge conductance in SNN junctions with noncentrosymmetric superconductors, Phys. Rev. Res. **5**, L012022 (2023).
[66] S. Tamura, Y. Tanaka, and T. Yokoyama, Generation of polarized spin-triplet cooper pairings by magnetic barriers in superconducting junctions, Phys. Rev. B **107**, 054501 (2023).
[67] T. Mizushima, S. Tamura, K. Yada, and Y. Tanaka, Odd-frequency pairs and anomalous proximity effect in nematic and chiral states of superconducting topological insulators, Phys. Rev. B **107**, 064504 (2023).
[68] T. Kokkeler, A. Golubov, S. Bergeret, and Y. Tanaka, Proximity effect of time-reversal symmetry broken noncentrosymmetric superconductors, Phys. Rev. B **108**, 094503 (2023).
[69] Y. Tanaka, S. Tamura, and J. Cayao, Theory of majorana zero modes in unconventional superconductors, arXiv:2402.00643.
[70] A. M. Black-Schaffer and A. V. Balatsky, Odd-frequency superconducting pairing in multiband superconductors, Phys. Rev. B **88**, 104514 (2013).
[71] A. V. Burmistrova, I. A. Devyatov, A. A. Golubov, K. Yada, and Y. Tanaka, Theory of tunneling spectroscopy of multiband superconductors, J. Phys. Soc. Jpn. **82**, 034716 (2013).
[72] B. Sothmann, S. Weiss, M. Governale, and J. König, Unconventional superconductivity in double quantum dots, Phys. Rev. B **90**, 220501(R) (2014).
[73] L. Komendová, A. V. Balatsky, and A. M. Black-Schaffer, Experimentally observable signatures of odd-frequency pairing in multiband superconductors, Phys. Rev. B **92**, 094517 (2015).
[74] Y. Asano and A. Sasaki, Odd-frequency cooper pairs in two-band superconductors and their magnetic response, Phys. Rev. B **92**, 224508 (2015).
[75] P. Burset, B. Lu, H. Ebisu, Y. Asano, and Y. Tanaka, All-electrical generation and control of odd-frequency s-wave cooper pairs in double quantum dots, Phys. Rev. B **93**, 201402(R) (2016).
[76] H. Ebisu, B. Lu, J. Klinovaja, and Y. Tanaka, Theory of time-reversal topological superconductivity in double Rashba wires: symmetries of Cooper pairs and Andreev bound states, Prog. Theor. Exp. Phys. **2016**, 083I01 (2016).
[77] L. Komendová and A. M. Black-Schaffer, Odd-frequency superconductivity in Sr_2RuO_4 measured by Kerr rotation, Phys. Rev. Lett. **119**, 087001 (2017).
[78] Y. Asano, A. Sasaki, and A. A. Golubov, Dirty two-band superconductivity with interband pairing order, New J. Phys. **20**, 043020 (2018).
[79] J. Schmidt, F. Parhizgar, and A. M. Black-Schaffer, Odd-frequency superconductivity and meissner effect in the doped topological insulator Bi_2Se_3, Phys. Rev. B **101**, 180512(R) (2020).
[80] S. Tamura, S. Nakosai, A. M. Black-Schaffer, Y. Tanaka, and J. Cayao, Bulk odd-frequency pairing in the superconducting Su-Schrieffer-Heeger model, Phys. Rev. B **101**, 214507 (2020).
[81] F. Parhizgar and A. M. Black-Schaffer, Diamagnetic and paramagnetic Meissner effect from odd-frequency pairing in multiorbital superconductors, Phys. Rev. B **104**, 054507 (2021).
[82] P. Dutta, F. Parhizgar, and A. M. Black-Schaffer, Superconductivity in spin-3/2 systems: Symmetry classification, odd-frequency pairs, and Bogoliubov Fermi surfaces, Phys. Rev. Res. **3**, 033255 (2021).
[83] C. Triola and A. V. Balatsky, Odd-frequency superconductivity in driven systems, Phys. Rev. B **94**, 094518 (2016).
[84] J. Cayao, C. Triola, and A. M. Black-Schaffer, Floquet engineering bulk odd-frequency superconducting pairs, Phys. Rev. B **103**, 104505 (2021).
[85] T. Kuhn, B. Sothmann, and J. Cayao, Floquet engineering Higgs dynamics in time-periodic superconductors, Phys. Rev. B **109**, 134517 (2024).

[86] J. Cayao and A. M. Black-Schaffer, Exceptional odd-frequency pairing in non-hermitian superconducting systems, Phys. Rev. B **105**, 094502 (2022).

[87] R. S. Keizer, S. T. Gönnenwein, T. M. Klapwijk, G. Miao, G. Xiao, and A. Gupta, A spin triplet supercurrent through the half-metallic ferromagnet CrO_2, Nature (London) **439**, 825 (2006).

[88] I. Sosnin, H. Cho, V. T. Petrashov, and A. F. Volkov, Superconducting phase coherent electron transport in proximity conical ferromagnets, Phys. Rev. Lett. **96**, 157002 (2006).

[89] J. Robinson, J. Witt, and M. Blamire, Controlled injection of spin-triplet supercurrents into a strong ferromagnet, Science **329**, 59 (2010).

[90] D. Sprungmann, K. Westerholt, H. Zabel, M. Weides, and H. Kohlstedt, Evidence for triplet superconductivity in Josephson junctions with barriers of the ferromagnetic Heusler alloy Cu_2MnAl, Phys. Rev. B **82**, 060505(R) (2010).

[91] M. S. Anwar, F. Czeschka, M. Hesselberth, M. Porcu, and J. Aarts, Long-range supercurrents through half-metallic ferromagnetic CrO_2, Phys. Rev. B **82**, 100501(R) (2010).

[92] T. S. Khaire, M. A. Khasawneh, W. P. Pratt, and N. O. Birge, Observation of spin-triplet superconductivity in Co-based Josephson junctions, Phys. Rev. Lett. **104**, 137002 (2010).

[93] M. Anwar and J. Aarts, Inducing supercurrents in thin films of ferromagnetic CrO_2, Supercond. Sci Technol. **24**, 024016 (2011).

[94] C. Klose, T. S. Khaire, Y. Wang, W. P. Pratt, N. O. Birge, B. J. McMorran, T. P. Ginley, J. A. Borchers, B. J. Kirby, B. B. Maranville, and J. Unguris, Optimization of spin-triplet supercurrent in ferromagnetic Josephson junctions, Phys. Rev. Lett. **108**, 127002 (2012).

[95] M. Anwar, M. Veldhorst, A. Brinkman, and J. Aarts, Long range supercurrents in ferromagnetic CrO_2 using a multilayer contact structure, Appl. Phys. Lett. **100**, 052602 (2012).

[96] A. Di Bernardo, Z. Salman, X. L. Wang, M. Amado, M. Egilmez, M. G. Flokstra, A. Suter, S. L. Lee, J. H. Zhao, T. Prokscha, E. Morenzoni, M. G. Blamire, J. Linder, and J. W. A. Robinson, Intrinsic paramagnetic meissner effect due to *s*-wave odd-frequency superconductivity, Phys. Rev. X **5**, 041021 (2015).

[97] A. Di Bernardo, S. Diesch, Y. Gu, J. Linder, G. Divitini, C. Ducati, E. Scheer, M. G. Blamire, and J. W. Robinson, Signature of magnetic-dependent gapless odd frequency states at superconductor/ferromagnet interfaces, Nat. Commun. **6**, 8053 (2015).

[98] J. A. Krieger, A. Pertsova, S. R. Giblin, M. Döbeli, T. Prokscha, C. W. Schneider, A. Suter, T. Hesjedal, A. V. Balatsky, and Z. Salman, Proximity-induced odd-frequency superconductivity in a topological insulator, Phys. Rev. Lett. **125**, 026802 (2020).

[99] V. Perrin, F. L. N. Santos, G. C. Ménard, C. Brun, T. Cren, M. Civelli, and P. Simon, Unveiling odd-frequency pairing around a magnetic impurity in a superconductor, Phys. Rev. Lett. **125**, 117003 (2020).

[100] S.-P. Chiu, C. Tsuei, S.-S. Yeh, F.-C. Zhang, S. Kirchner, and J.-J. Lin, Observation of triplet superconductivity in $CoSi_2/TiSi_2$ heterostructures, Sci. Adv. **7**, eabg6569 (2021).

[101] S.-P. Chiu, V. Mishra, Y. Li, F.-C. Zhang, S. Kirchner, and J.-J. Lin, Tuning interfacial two-component superconductivity in $CoSi_2/TiSi_2$ heterojunctions via $TiSi_2$ diffusivity, Nanoscale **15**, 9179 (2023).

[102] D. Chakraborty and A. M. Black-Schaffer, Interplay of finite-energy and finite-momentum superconducting pairing, Phys. Rev. B **106**, 024511 (2022).

[103] D. Chakraborty and A. M. Black-Schaffer, Quasiparticle interference as a direct experimental probe of bulk odd-frequency superconducting pairing, Phys. Rev. Lett. **129**, 247001 (2022).

[104] P. V. Leksin, N. N. Garif'yanov, I. A. Garifullin, Y. V. Fominov, J. Schumann, Y. Krupskaya, V. Kataev, O. G. Schmidt, and B. Büchner, Evidence for triplet superconductivity in a superconductor-ferromagnet spin valve, Phys. Rev. Lett. **109**, 057005 (2012).

[105] M. G. Flokstra, T. C. Cunningham, J. Kim, N. Satchell, G. Burnell, P. J. Curran, S. J. Bending, C. J. Kinane, J. F. K. Cooper, S. Langridge, A. Isidori, N. Pugach, M. Eschrig, and S. L. Lee, Controlled suppression of superconductivity by the generation of polarized cooper pairs in spin-valve structures, Phys. Rev. B **91**, 060501(R) (2015).

[106] N. Banerjee, J. Robinson, and M. G. Blamire, Reversible control of spin-polarized supercurrents in ferromagnetic Josephson junctions, Nat. Commun. **5**, 4771 (2014).

[107] A. Iovan, T. Golod, and V. M. Krasnov, Controllable generation of a spin-triplet supercurrent in a Josephson spin valve, Phys. Rev. B **90**, 134514 (2014).

[108] W. M. Martinez, W. P. Pratt, and N. O. Birge, Amplitude control of the spin-triplet supercurrent in $s/f/s$ Josephson junctions, Phys. Rev. Lett. **116**, 077001 (2016).

[109] J. A. Glick, V. Aguilar, A. B. Gougam, B. M. Niedzielski, E. C. Gingrich, R. Loloee, W. P. Pratt, Jr., and N. O. Birge, Phase control in a spin-triplet squid, Sci. Adv. **4**, eaat9457 (2018).

[110] J. C. Cuevas, A. Martín-Rodero, and A. Levy Yeyati, Hamiltonian approach to the transport properties of superconducting quantum point contacts, Phys. Rev. B **54**, 7366 (1996).

[111] J. C. Cuevas and M. Fogelström, Quasiclassical description of transport through superconducting contacts, Phys. Rev. B **64**, 104502 (2001).

[112] Y. Avishai, A. Golub, and A. D. Zaikin, Tunneling through an anderson impurity between superconductors, Phys. Rev. B **63**, 134515 (2001).

[113] J. Bauer, A. Oguri, and A. Hewson, Spectral properties of locally correlated electrons in a Bardeen–Cooper–Schrieffer superconductor, J. Phys.: Condens. Matter **19**, 486211 (2007).

[114] B. van Heck, S. Mi, and A. R. Akhmerov, Single fermion manipulation via superconducting phase differences in multiterminal Josephson junctions, Phys. Rev. B **90**, 155450 (2014).

[115] T. Yokoyama and Y. V. Nazarov, Singularities in the Andreev spectrum of a multiterminal Josephson junction, Phys. Rev. B **92**, 155437 (2015).

[116] R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, Multi-terminal Josephson junctions as topological matter, Nat. Commun. **7**, 11167 (2016).

[117] E. Strambini, S. D'ambrosio, F. Vischi, F. Bergeret, Y. V. Nazarov, and F. Giazotto, The ω-squipt as a tool to

phase-engineer Josephson topological materials, Nat. Nanotechnol. **11**, 1055 (2016).
[118] M. Amundsen, J. A. Ouassou, and J. Linder, Analytically determined topological phase diagram of the proximity-induced gap in diffusive n-terminal Josephson junctions, Sci. Rep. **7**, 40578 (2017).
[119] H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Weyl nodes in Andreev spectra of multiterminal Josephson junctions: Chern numbers, conductances, and supercurrents, Phys. Rev. B **97**, 035443 (2018).
[120] M. Houzet and J. S. Meyer, Majorana-Weyl crossings in topological multiterminal junctions, Phys. Rev. B **100**, 014521 (2019).
[121] K. Sakurai, M. T. Mercaldo, S. Kobayashi, A. Yamakage, S. Ikegaya, T. Habe, P. Kotetes, M. Cuoco, and Y. Asano, Nodal Andreev spectra in multi-Majorana three-terminal Josephson junctions, Phys. Rev. B **101**, 174506 (2020).
[122] N. Pankratova, H. Lee, R. Kuzmin, K. Wickramasinghe, W. Mayer, J. Yuan, M. G. Vavilov, J. Shabani, and V. E. Manucharyan, Multiterminal Josephson effect, Phys. Rev. X **10**, 031051 (2020).
[123] V. Kornich, H. S. Barakov, and Y. V. Nazarov, Fine energy splitting of overlapping Andreev bound states in multiterminal superconducting nanostructures, Phys. Rev. Res. **1**, 033004 (2019).
[124] V. Fatemi, A. R. Akhmerov, and L. Bretheau, Weyl Josephson circuits, Phys. Rev. Res. **3**, 013288 (2021).
[125] L. Peralta Gavensky, G. Usaj, and C. A. Balseiro, Multiterminal Josephson junctions: A road to topological flux networks, Europhys. Lett. **141**, 36001 (2023).
[126] G. D. Mahan, *Many-Particle Physics* (Springer Science & Business Media, New York, 2013).
[127] A. Zagoskin, *Quantum Theory of Many-Body Systems: Techniques and Applications* (Springer, Berlin, 2014).
[128] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.109.205406 for details on the calculation of the Green's functions, nonlocal conductance, and realization in realistic finite length superconductors. See also Refs. [147–151] therein.
[129] The dependence of the intersuperconductor pair amplitudes on the phase difference between two superconductors has been discussed before but only in the weak tunneling limit [33], a regime that does not reveal the formation of ABSs in the pair amplitudes.
[130] We have verified that higher order in t_0 produces finite values of both even- and odd-ω pair amplitudes between next nearest neighbor superconductors.
[131] J. Cayao, P. Dutta, P. Burset, and A. M. Black-Schaffer, Phase-tunable electron transport assisted by odd-frequency Cooper pairs in topological Josephson junctions, Phys. Rev. B **106**, L100502 (2022).
[132] S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge, UK, 1997).
[133] A. Bordin, G. Wang, C.-X. Liu, S. L. D. ten Haaf, N. van Loo, G. P. Mazur, D. Xu, D. van Driel, F. Zatelli, S. Gazibegovic, G. Badawy, E. P. A. M. Bakkers, M. Wimmer, L. P. Kouwenhoven, and T. Dvir, Tunable crossed Andreev reflection and elastic cotunneling in hybrid nanowires, Phys. Rev. X **13**, 031031 (2023).
[134] A. W. Draelos, M.-T. Wei, A. Seredinski, H. Li, Y. Mehta, K. Watanabe, T. Taniguchi, I. V. Borzenets, F. Amet, and G. Finkelstein, Supercurrent flow in multiterminal graphene Josephson junctions, Nano Lett. **19**, 1039 (2019).
[135] G. V. Graziano, M. Gupta, M. Pendharkar, J. T. Dong, C. P. Dempsey, C. Palmstrøm, and V. S. Pribiag, Selective control of conductance modes in multi-terminal Josephson junctions, Nat. Commun. **13**, 5933 (2022).
[136] M. Coraiola, D. Z. Haxell, D. Sabonis, H. Weisbrich, A. E. Svetogorov, M. Hinderling, S. C. ten Kate, E. Cheah, F. Krizek, R. Schott, W. Wegscheider, J. C. Cuevas, W. Belzig, and F. Nichele, Phase-engineering the Andreev band structure of a three-terminal Josephson junction,, Nat. Commun. **14**, 6784 (2023).
[137] J. A. Van Dam, Y. V. Nazarov, E. P. Bakkers, S. De Franceschi, and L. P. Kouwenhoven, Supercurrent reversal in quantum dots, Nature (London) **442**, 667 (2006).
[138] L. Hofstetter, S. Csonka, J. Nygård, and C. Schönenberger, Cooper pair splitter realized in a two-quantum-dot y-junction, Nature (London) **461**, 960 (2009).
[139] S. De Franceschi, L. Kouwenhoven, C. Schönenberger, and W. Wernsdorfer, Hybrid superconductor–quantum dot devices, Nat. Nanotechnol. **5**, 703 (2010).
[140] T. Dirks, T. L. Hughes, S. Lal, B. Uchoa, Y.-F. Chen, C. Chialvo, P. M. Goldbart, and N. Mason, Transport through Andreev bound states in a graphene quantum dot, Nat. Phys. **7**, 386 (2011).
[141] E. J. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber, and S. De Franceschi, Spin-resolved Andreev levels and parity crossings in hybrid superconductor–semiconductor nanostructures, Nat. Nanotechnol. **9**, 79 (2014).
[142] J.-D. Pillet, V. Benzoni, J. Griesmar, J.-L. Smirr, and C. O. Girit, Nonlocal Josephson effect in Andreev molecules, Nano Lett. **19**, 7138 (2019).
[143] R. Zitko, Superconducting quantum dot and the sub-gap states, in *Spintronics XI*, edited by H. Jaffrès, H.-J. Drouhin, J.-E. Wegrowe, and M. Razeghi (SPIE, Bellingham, WA, 2018).
[144] A. Bargerbos, M. Pita-Vidal, R. Žitko, L. J. Splitthoff, L. Grünhaupt, J. J. Wesdorp, Y. Liu, L. P. Kouwenhoven, R. Aguado, C. K. Andersen, A. Kou, and B. van Heck, Spectroscopy of spin-split Andreev levels in a quantum dot with superconducting leads, Phys. Rev. Lett. **131**, 097001 (2023).
[145] T. Dvir, G. Wang, N. van Loo, C.-X. Liu, G. P. Mazur, A. Bordin, S. L. D. ten Haaf, J.-Y. Wang, D. van Driel, F. Zatelli, X. Li, F. K. Malinowski, S. Gazibegovic, G. Badawy, E. P. A. M. Bakkers, M. Wimmer, and L. P. Kouwenhoven, Realization of a minimal Kitaev chain in coupled quantum dots, Nature (London) **614**, 445 (2023).
[146] A. Bordin, X. Li, D. van Driel, J. C. Wolff, Q. Wang, S. L. D. ten Haaf, G. Wang, N. van Loo, L. P. Kouwenhoven, and T. Dvir, Crossed Andreev reflection and elastic co-tunneling in a three-site Kitaev chain nanowire device, arXiv:2306.07696.
[147] J. C. Cuevas, A. Levy Yeyati, and A. Martín-Rodero, Kondo effect in normal-superconductor quantum dots, Phys. Rev. B **63**, 094515 (2001).
[148] P. Burset, W. J. Herrera, and A. L. Yeyati, Microscopic theory of Cooper pair beam splitters based on carbon nanotubes, Phys. Rev. B **84**, 115448 (2011).

[149] P. Trocha and J. Barnaś, Spin-polarized Andreev transport influenced by Coulomb repulsion through a two-quantum-dot system, Phys. Rev. B **89**, 245418 (2014).
[150] M. Žonda, P. Zalom, T. Novotný, G. Loukeris, J. Bätge, and V. Pokorný, Generalized atomic limit of a double quantum dot coupled to superconducting leads, Phys. Rev. B **107**, 115407 (2023).
[151] V. V. Baran, E. J. P. Frost, and J. Paaske, Surrogate model solver for impurity-induced superconducting subgap states, Phys. Rev. B **108**, L220506 (2023).