Bound on approximating non-Markovian dynamics by tensor networks in the time domain

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The spin-boson (SB) model plays a central role in studies of dissipative quantum dynamics, due to bothits conceptual importance and relevance to a number of physical systems. Here, we provide rigorous bounds of the computational complexity of the SB model for the physically relevant case of a zero temperature ohmic bath. We start with the description of the bosonic bath via its Feynman-Vernon influence functional (IF), which is a tensor on the space of the trajectory of an impurity spin. By expanding the kernel of the IF via a sum of decaying exponentials, we obtain an analytical approximation of the continuous bath by a finite number of damped bosonic modes. We bound the error induced by restricting bosonic Hilbert spaces to a finite-dimensional subspace with small boson numbers, which yields an analytical form of a matrix-product state (MPS) representation of the IF. We show that the MPS bond dimension D scales polynomially in the error on physical observables as well as in the evolution time T, $D \propto T^4/\epsilon^2$. This bound indicates that the SB model can be efficiently simulated using a polynomial in time-computational resources.

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I. INTRODUCTION

Every quantum system, irrespective of its nature, interacts with an environment. While in some cases environment dynamics can be described in the Markovian approximation, there is a broad class of problems where this is not sufficient, and non-Markovian effects are essential. Examples of such phenomena range from nonequilibrium transport in quantum dots [1,2] to micromechanical resonators [3] and light-harvesting complexes [4,5].

An archetypical model of non-Markovian quantum dynamics is the spin-boson (SB) model, describing a spin-$\frac{1}{2}$ coupled to a harmonic bosonic bath. As discussed in the seminal work of Leggett *et al.* [6], this model exhibits a rich variety of dynamical regimes, including a dissipative phase transition. Despite a variety of analytical results obtained in various limits, the exact description of the dynamics of a system in the strong coupling limit, which is needed to model realistic physical systems, remained an outstanding challenge.

To address this challenge, a variety of numerical methods has been developed. In one direction, Chin *et al.* [7] used a chain mapping of a continuous bosonic bath, afterward modeled by tDMRG techniques. Another family of approaches is based on truncating the Feynman-Vernon influence functional (IF) or a related object, augmented reduced density tensor. While an early quasiadiabatic propagator path integral (QUAPI) algorithm [8] had exponential scaling with the memory time, more recently, efficient schemes based on tensor-networks compression have been introduced [9–14], see also Ref. [15] for the improved QUAPI algorithm. In another direction, it was argued [16–18] that the IF can be approximated by a bath of a finite number of auxiliary bosons or by a finitely many hierarchical equations of motion, which also leads to a drastic boost of computational efficiency [19]. It was concluded that the system can be simulated efficiently. We note in passing that related ideas were developed in the context of fermionic quantum impurity models [20–25] and for interacting environments [26,27].

Despite these remarkable developments, no theoretical bounds on the complexity of simulating the SB model exist, and the goal of this paper is to fill this gap. Focusing on the case of azero-temperature ohmic bath, we introduce a number of (damped) auxiliary bosons approximating the original bath. As a main result, we show that the Feynman-Vernon IF, as well as the evolution of the system are efficiently simulated by the auxiliary modes. We provide an analytical expression for the matrix-product state (MPS) approximation of the IF and prove that the bond dimension scales polynomially with time:

$$D \sim \frac{(\omega_c T)^4}{\epsilon^2}, \tag{1}$$

where ϵ is the error for the physical observables, i.e., the density matrix of the spin at time T.

Finding a representation in terms of the auxiliary modes is equivalent to approximating certain correlation functions in terms of a finite sum of decaying exponents. Each exponent leads to a single bosonic mode, which then can be truncated to a finite dimensional subspace. The decomposition of an arbitrary function in terms of decaying exponents is a well-known problem in the theory of signal processing and mathematical physics, which can be efficiently solved numerically with the help of Prony's method [28]. For our purposes, we use another

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approach by Beylikin and Monzon [29], which allows for analytical estimates, see the Supplemental Material [30] for the review.

II. IF DESCRIPTION OF THE SB MODEL

The main object of our interest will be a SB model consisting of a spin impurity coupled to a bosonic bath at zero temperature. Its Hamiltonian reads

$$H = H_{\rm b} + H_{\rm int} + H_{\rm s}, \tag{2}$$

with

$$H_{\rm b} = \sum_k \omega_k a_k^\dagger a_k, \tag{3}$$

$$H_{\rm int} = c_k(a_k^\dagger + a_k)\sigma_z. \tag{4}$$

Our method is suitable for arbitrary spin Hamiltonian $H_{\rm s}(t)$, which is not specified.

We consider discrete time dynamics, obtained by the Trotterization of the original model. Theevolution operator over a small time step Δt is approximately given by

$$U_{\rm total} = \exp[i\Delta t(H_{\rm b} + H_{\rm int})]\exp[i\Delta t H_{\rm s}(t_k)] + O(\Delta t^2). \tag{5}$$

As we are interested in continuous dynamics, we assume $\Delta t \sim \epsilon_1 = \frac{\epsilon}{T}$. It is useful to introduce shorthand notation for the exponentials in the above expression:

$$\exp[i\Delta t(H_{\rm b} + H_{\rm int})] = U_{\rm b}(a, a^\dagger|\sigma_z), \tag{6}$$

$$\exp[i\Delta t H_{\rm s}(t_k)] = U^{(k)}. \tag{7}$$

For a fixed evolution time $T = N\Delta t$, we can integrate out the bosonic degrees of freedom defining a discrete analog of the Feynman-Vernon IF for the fixed trajectory of a spin:

$$\mathcal{I}_{s_N,\bar{s}_N,\ldots,s_1,\bar{s}_1} = {}_{\rm bath}\langle 0|\prod_{k=1}^{N-1} U^\dagger_{{\rm b},\bar{s}_k}\rho\prod_{k=1}^{N-1} U_{{\rm b},s_k}|0\rangle_{\rm bath}, \tag{8}$$

where $U_{{\rm b},s_k} \stackrel{\rm def}{=} \langle s_k|U_{\rm b}(a, a^\dagger|\sigma_z)|s_k\rangle$. Then we rewrite the time-evolved density matrix in terms of IF:

$$\tilde{\rho}_{s_N,\bar{s}_N} = \sum_{s_k,\bar{s}_k=1,2} \mathcal{I}_{s_N,\bar{s}_N,\ldots,s_1,\bar{s}_1}\prod_{k=1}^{N-1}\left[U^{(k)}_{s_k,s_{k+1}}U^{(k)\,\dagger}_{\bar{s}_{k+1},\bar{s}_k}\right]\rho_{s_1,\bar{s}_1}. \tag{9}$$

The IF of the spin trajectories $\mathcal{I}_{s_N,\bar{s}_N,\ldots,s_1,\bar{s}_1} \equiv \mathcal{I}_{\{\mathbf{s},\bar{\mathbf{s}}\}}$ can be computed explicitly [8]:

$$\mathcal{I}_{\{\mathbf{s},\bar{\mathbf{s}}\}} = \exp\left\{-\sum_{i\leqslant j}[(s_i - \bar{s}_i)(s_j\eta_{i,j} - \eta^\star_{i,j}\bar{s}_j)]\right\}, \tag{10}$$

where $\eta_{i,j} = \kappa_{i,j} + i\phi_{i,j}$ is a known function:

$$\eta_{i,j} = \eta_{i-j} = 4\int_0^\infty d\omega J(\omega)\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}\exp(i\omega t_{i-j}), \quad i > j \tag{11}$$

$$\eta_{i,i} = 2\int_0^\infty d\omega J(\omega)\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}, \tag{12}$$

with $t_{i-j} = (i-j)\Delta t$. We note in passing that our analysis is applicable to the case of a finite Δt, but we will mostly be interested in the limit $\Delta t \to 0$. Below, we will concentrate on the case of an ohmic bath, for which the spectral density takes the following form:

$$J(\omega) = \alpha\omega\exp\left(-\frac{\omega}{\omega_c}\right). \tag{13}$$

III. MPS REPRESENTATION FOR THE IF VIA AUXILIARY BOSONS

To represent the IF as an MPS, we will use the approach of Ref. [31], which we briefly review below. This approach provides an MPO representation of an exponent of a quasilocal Hamiltonian. For our needs, we will apply it to the exponent of an action, considering the variables as commuting operators. We introduce the action functional $\mathcal{I} = e^{\mathcal{S}}$:

$$\mathcal{S} = -\sum_{i\leqslant j}[(s_i - \bar{s}_i)(s_j\eta_{i,j} - \eta^\star_{i,j}\bar{s}_j)], \tag{14}$$

where the function $\eta_{i,j}$ decays polynomially, as $\frac{1}{[\Delta t(i-j)]^2}$ for the ohmic case. Let us fix site i and split the action $\mathcal{S}$ into a sum of three different terms:

$$\mathcal{S} = \mathcal{S}_i^L + \mathcal{S}_i^R + \sum_{m=1}^K h^L_{i,m}h^R_{i,m}, \tag{15}$$

where $\{\mathcal{S}_i^L, h^L_{i,m}\}$ depend only on the variables to the left of site i, and $\{\mathcal{S}_i^R, h^R_{i,m}\}$ depend on the variables to the right of site i.

Suppose the existence of the following relation [31]:

$$\begin{pmatrix}\mathcal{S}^R_{i-1}\\ \mathbf{h}^R_{i-1}\\ \mathbb{1}_{i-1}\end{pmatrix} = \begin{pmatrix}\mathbb{1} & \mathbf{C}^i & D^i\\ 0 & \mathbf{A}^i & \mathbf{B}^i\\ 0 & 0 & \mathbb{1}\end{pmatrix}\begin{pmatrix}\mathcal{S}^R_i\\ \mathbf{h}^R_i\\ \mathbb{1}_i\end{pmatrix}. \tag{16}$$

Here, $\mathbf{h}^R_i$ is a row of numbers $h^R_{i,m}$, $\mathbf{A}^i$ is a $K\times K$ matrix, $\mathbf{C}^i$ and $\mathbf{B}^i$ are a1 $\times K$ column and a$K\times 1$ row, respectively, and D^i is a scalar. First, the above relation yields an MPS representation for the action $\mathcal{S}$. We start from the site $N+1$, for which $\mathcal{S}^R_{N+1} = 0$. Then we move to the left, using the recurrence relationin Eq. (16),

$$\mathbf{h}^R_{i-1} = \mathbf{A}^i\mathbf{h}^R_i + \mathbf{B}^i, \tag{17}$$

$$\mathcal{S}^R_{i-1} = \mathcal{S}^R_i + \mathbf{C}^i\mathbf{h}^R_i + D^i. \tag{18}$$

$\mathcal{S}^R_0$ is nothing but the total action $\mathcal{S}$.

To represent an exponent of the total action $e^{\mathcal{S}} = \exp(\mathcal{S}^R_0)$ in terms of MPS, we introduce n bosonic modes:

$$\mathbf{a} = \{a_1, \ldots a_K\}^T, \tag{19}$$

$$\mathbf{a}^\dagger = \{a_1^\dagger, \ldots a_K^\dagger\}, \tag{20}$$

with the standard commutation relations:

$$[a_i, a_j^\dagger] = \delta_{i,j}. \tag{21}$$

We also introduce coherent states:

$$|\mathbf{h}^R_i\rangle = \exp\left(\sum_{m=1}^K h^R_{i,m}a^\dagger_m\right)|0\rangle. \tag{22}$$

The operationsin Eqs. (17) and (18) can be written in terms of the operator:

$$M^i = \exp(D^i)\exp[(\mathbf{a}^\dagger \cdot \mathbf{B}^i)]\exp\{[\mathbf{a}^\dagger \cdot \ln(\mathbf{A}^i)\mathbf{a}]\} \times \exp[(\mathbf{C}^i \cdot \mathbf{a})], \tag{23}$$

which implicitly depends on $s_i, \bar{s}_i$. Indeed, this operator can be equivalently defined by its action on the coherent states:

$$M^i|\mathbf{h}\rangle = \exp[D^i + (\mathbf{C}^i \cdot \mathbf{h})]|\mathbf{B}^i + \mathbf{A}^i\mathbf{h}\rangle. \tag{24}$$

Applying it to the state $\exp(\mathcal{S}_i^R)|\mathbf{h}_i^R\rangle$, we get

$$M^i \exp\left(\mathcal{S}_i^R\right)\left|\mathbf{h}_i^R\right\rangle = \exp\left(\mathcal{S}_{i-1}^R\right)\left|\mathbf{h}_{i-1}^R\right\rangle. \tag{25}$$

Finally, the MPS representation of the IF is provided by the formula:

$$I_{\mathbf{s},\bar{\mathbf{s}}} = \langle 0|M^1_{s_1,\bar{s}_1}, \ldots M^N_{s_N,\bar{s}_N}|0\rangle. \tag{26}$$

Formally, this yields an MPS representation of the IF. Next, we will show that the IF of an ohmic bath can be well approximated by choosing a finite K. We will further truncate operator $M^i_{s_i,\bar{s}_i}$ to a finite-dimensional boson subspace which will give rise to an MPS with a finite bond dimension. Due to the rapid decay of matrix elements with the growing number of excited bosons, this will introduce a controlled error.

IV. EXPONENTIALLY DECAYING INTERACTION

Applying this approach to the IF in Eq. (10), we introduce two species of bosonic operators $\mathbf{a}, \bar{\mathbf{a}}$ as well as the second set of $\mathbf{A}^i, \mathbf{B}^i, \mathbf{C}^i$ matrices: $\bar{\mathbf{A}}^i, \bar{\mathbf{B}}^i, \bar{\mathbf{C}}^i$, corresponding to forward and backward variables of the IF.

For our purposes, it is sufficient to assume that the matrix $\mathbf{A}^i$ does not depend on i or s variables. We also assume it to be diagonalizable:

$$\mathbf{A} = \mathrm{Diag}[\exp(-\Omega_1\Delta t), \ldots, \exp(-\Omega_K\Delta t)], \tag{27}$$

$$\bar{\mathbf{A}} = \mathrm{Diag}[\exp(-\Omega_1^\star\Delta t), \ldots, \exp(-\Omega_K^\star\Delta t)], \tag{28}$$

with $\Omega_k = \gamma_k + i\omega_k$, $\gamma_k > 0$. Furthermore, we choose $\mathbf{B}, \mathbf{C}$ to be linear in s:

$$\mathbf{C}^i = (s_i - \bar{s}_i)\{\lambda_1\Delta t, \ldots, \lambda_K\Delta t\}, \tag{29}$$

$$\bar{\mathbf{C}}^i = -(s_i - \bar{s}_i)\{\lambda_1^\star\Delta t, \ldots, \lambda_K^\star\Delta t\}, \tag{30}$$

$$\mathbf{B}^i = s_i\{\lambda_1\Delta t, \ldots, \lambda_K\Delta t\}^T, \tag{31}$$

$$\bar{\mathbf{B}}^i = \bar{s}_i\{\lambda_1^\star\Delta t, \ldots, \lambda_K^\star\Delta t\}^T, \tag{32}$$

$$D^i = \eta_{i,i}(s_i - \bar{s}_i)^2. \tag{33}$$

This will lead to the following ansatz for the IF:

$$\tilde{I}_{\{\mathbf{s},\bar{\mathbf{s}}\}} = \exp\left\{-\Delta t^2 \sum_{0\leqslant i\leqslant j\leqslant N} (s_i - \bar{s}_i)[s_j\tilde{\eta}_{i-j} - (\tilde{\eta})^\star_{i-j}\bar{s}_j]\right\}, \tag{34}$$

with

$$\tilde{\eta}_{i-j} = \sum_{k=1}^{K} \lambda_k^2 \exp[-(i-j)\Delta t\Omega_k]. \tag{35}$$

Each term in the expansion in Eq. (35) gives rise to two bosonic modes, yielding an infinite-dimensional MPS approximation of the IF. We emphasize that, despite the polynomial decay in time of the kernel η in Eq. (11), it can be approximated by a sum of exponentials, some of which have sufficiently small decay rates. Below, we provide an explicit decomposition with alogarithmically growing number of terms K in Eq. (43).

We note that, for a real $\lambda = |\lambda|$, the same MPS could be generated by an auxiliary quantum channel, see the Supplemental Material [30]. For a complex λ, the corresponding IF does not describe any physical bath. However, our methods to estimate the errors for the observables do not require λ to be real and positive; therefore, we keep it arbitrary.

Truncation of the infinite-dimensional MPS in Eq. (26) to a finite-dimensional one involves two types of errors. First, errors occur due to the truncation of the bosonic modes to a finite-dimensional subspace. The second type of error appears due to the inaccuracy of approximating function η_{i-j} by the finite sum in Eq. (35). Before discussing the approximation by a finite sum of exponentials, let us briefly explain the intuition behind the estimate of the errors of the first kind, see the Supplemental Material [30] for a rigorous analysis.

Let us change perspective for a moment and consider the dynamics of bosons in the environment of the spin mode. The evolution of bosons is a competition between the driving force $\lambda_k s_i a_k^\dagger + \lambda_k^\star s_i \bar{a}_k^\dagger$, which creates the bosons with rate λ and the overall decay $-\gamma_k(a_k^\dagger a_k + \bar{a}_k^\dagger \bar{a}_k)$, which damps the wave function component with n bosons with rate γn. As a result, processes that involve significantly more than $\frac{|\lambda_k|^2}{\gamma_k^2}$ bosons in the system are strongly suppressed. In fact, below, we derive expressions for λ_k, γ_k and prove (see the Supplemental Material [30]) that the amplitude to have n bosons is suppressed as $4\nu_\star^{n/2}$, with some $\nu_\star < 1$. This in turn explains why the states with ahigh number of excitations could be neglected. We conclude that the problem of approximating of the IF in Eq. (10) in terms of an MPS is equivalent to expanding function η_{i-j} in Eq. (11) in terms of a sum of exponentials in Eq. (35).

V. MPS REPRESENTATION FOR THE BOSONIC BATH

To approximate a power-law function η by a finite sum of exponentials, we follow the approach of Ref. [29]. Let us start with an integral representation in the continuous time limit:

$$\eta^0(t) = \alpha \int_0^\infty \omega \exp\left(-\frac{\omega}{\omega_c}\right) e^{i\omega t} d\omega, \tag{36}$$

$$\eta^0_{i,j} = \eta^0[\Delta t(i-j)], \tag{37}$$

with $t = \Delta t(i-j)$. In the Supplemental Material [30], by performing a change of variables followed by discretization, we show that this integral is well approximated by a sum:

$$\eta^0(t) \to \alpha\chi \sum_{k=-\infty}^{\infty} \omega_k^2 \exp\left(-\frac{\omega_k}{\omega_c}\right)\exp(i\omega_k t), \tag{38}$$

where points ω_k are situated on a contour in the complex plane $\omega_k = \omega_c \exp(k\chi + \frac{i\pi}{4})$. This sum may be restricted to a finite number of terms because of the fast decay of the tails.

We prove in the Supplemental Material [30] that $\eta^0(t)$ may be approximated by a sum of $\sim \ln^2(\frac{\alpha\omega_c T}{\epsilon})$ exponentials of the form:

$$\eta^0(t) = \sum_{k=-N_{\epsilon_1}}^{M_{\epsilon_1}} \lambda_k^2 \exp(-\Omega_k t) + \delta\eta(t), \tag{39}$$

with

$$\lambda_k^2 = i\alpha\omega_c^2\chi \exp\left\{-\frac{1+i}{\sqrt{2}}\exp\left[\left(k+\frac{1}{2}\right)\chi\right]\right\} \times \exp\left[2\left(k+\frac{1}{2}\right)\chi\right], \tag{40}$$

$$\Omega_k = \omega_c \exp\left[\left(k+\frac{1}{2}\right)\chi\right]\frac{1-i}{\sqrt{2}}. \tag{41}$$

Thanks to the fact that the function is smooth enough, the discretization step χ grows logarithmically with epsilon $\chi \sim \frac{1}{\ln(\frac{\alpha\omega_c T}{\epsilon})}$. The discrepancy $\delta\eta(t)$ is given by

$$\int_{t=0}^{T} |\delta\eta(t)| < \epsilon_1 = \frac{\epsilon}{T}. \tag{42}$$

The total number of modes $K = M_\epsilon + N_\epsilon$ scales as

$$K \sim \ln^2\left(\frac{\alpha\omega_c T}{\epsilon}\right), \tag{43}$$

see the Supplemental Material [30] for details.

For finite but small Δt, the IF can be approximated in an MPS form in Eq. (26):

$$M_{s_i,\bar{s}_i} = \exp[-\Lambda(s_i-\bar{s}_i)^2] \prod_{k=-N_\epsilon}^{M_\epsilon} \exp\left(\mathcal{M}^k_{s_i,\bar{s}_i}\Delta t\right), \tag{44}$$

$$\mathcal{M}^k_{s_i,\bar{s}_i} = -\Omega_k\left[a_k^\dagger + \frac{(s_i-\bar{s}_i)}{\sqrt{2}}\frac{\lambda_k}{\Omega_k}\right]\left(a_k - \sqrt{2}s_i\frac{\lambda_k}{\Omega_k}\right) - \Omega_k^\star\left[\bar{a}_k^\dagger - \frac{(s_i-\bar{s}_i)}{\sqrt{2}}\frac{\lambda_k^\star}{\Omega_k^\star}\right]\left(\bar{a}_k - \sqrt{2}\bar{s}_i\frac{\lambda_k^\star}{\Omega_k^\star}\right). \tag{45}$$

$$\Lambda = \eta_{0,0} + \sum_{k=-N_\epsilon}^{M_\epsilon} \mathrm{Re}\left(\frac{\lambda_k^2}{\Omega_k}\right)\Delta t. \tag{46}$$

Thus, operator $M_{s,\bar{s}}$ is a product of a bosonic MPS and an operator $\exp[-\Lambda(s_i-\bar{s}_i)^2]$ acting on a single site. The value of Λ may be estimated via the Euler-Maclaurin formula, and it is important to note that it remains positive and of the order of ϵ [32].

VI. BOUND ON THE ERROR FOR SPIN DYNAMICS

To estimate the effects of the inaccuracy in Eq. (39), let us examine the dynamics provided by the MPS in Eqs. (44)–(46). We fix spin dynamics, specified by unitary operators $U^{(i)}$ acting on the two-dimensional space. The system dynamics may be computed in terms of a product of operators acting on both bosonic and spin degrees of freedom:

$$\mathcal{U}^{(i)}_{s_{i+1},\bar{s}_{i+1}|s_i,\bar{s}_i} = M_{s_i,\bar{s}_i} \otimes \left\{U^{(i)}_{s_{i+1},s_i} \otimes [U^{(i)}]^\star_{\bar{s}_{i+1},\bar{s}_i}\right\}. \tag{47}$$

The density matrix at time $T = N\Delta t$ reads

$$\rho_{s_N,\bar{s}_N}(T) = {}_{\text{bosons}}\langle 0|\left[\prod_{i=1}^{N}\mathcal{U}^{(i)}\right]|0\rangle_{\text{bosons}}\ \rho_{s_1,\bar{s}_1}. \tag{48}$$

An important property of the operator $\mathcal{U}^{(i)}$ is that it does not increase the norm of the vectors. Indeed, let us note that the eigenvalues of quadratic operators $\exp(\mathcal{M}^k_{s_i,\bar{s}_i})$ are <1. Consequently, the eigenvalues of $M^\dagger_{s,\bar{s}}M_{s,\bar{s}}$ are also <1. One can conclude that any bounded vector $|v\rangle$ propagating in time remains bounded:

$$\left|\prod_{i=n_1}^{n_2}\mathcal{U}^{(i)}|v\rangle\right|^2 \leqslant \langle v|v\rangle. \tag{49}$$

Now we are in a position to estimate the error induced by the inaccuracy of approximation of the quadratic kernel $\eta_{i,j} = \eta(i-j)$ for the IF of the formin Eq. (10). Suppose that we have found an MPS representation for a kernel η^{ϵ_1} such that it is close to the actual kernel in the L_1 norm:

$$\sum_{j=0}^{N}|\eta(j) - \eta^{\epsilon_1}(j)| = |\delta\eta|_{L_1} \leqslant \frac{\epsilon_1}{4} = \frac{\epsilon}{4T}. \tag{50}$$

The first-point correlator:

$$\delta\rho_T = \sum_{i<j}\langle(s_i-\bar{s}_i)(s_j\delta\eta_{i,j} - \delta\eta^\star_{i,j}\bar{s}_j)\rangle, \tag{51}$$

where we define the correlators as

$$\left\langle\prod_{i\in I}s_i\prod_{j\in J}\bar{s}_j\right\rangle \stackrel{\text{def}}{=} \sum_{s_a,\bar{s}_a}\prod_{i\in I}s_i\prod_{j\in J}\bar{s}_j\prod_{k=1}^{N-1}\mathcal{U}^{(k)}_{s_{k+1},\bar{s}_{k+1}|s_k,\bar{s}_k}\rho_{s_1,\bar{s}_1}. \tag{52}$$

The property in Eq. (49) guarantees that each correlator is bounded by 1, and so the total error is bounded as

$$|\delta\rho_T| \leqslant 4T|\delta\eta|_{L_1} \leqslant \epsilon. \tag{53}$$

A similar analysis, for a slightly more general case, was provided in Ref. [33].

To sum up, there are two sources of errors. One type of error is due to an inaccuracy in the approximation of the function $\eta_{i,j}$ in Eq. (39). This error is related to the number of bosonic modes K in Eq. (43). Another source of errors stems from the truncation of the bosonic modes to a finite-dimensional subspace. We show in the Supplemental Material [30] that, due to the decay of bosonic modes, the bosonic wave function decays with the number of bosonic excitations n as $\nu_\star^n$. This allows us to restrict the total number of bosonic excitations as $n_\star \sim \frac{\ln[\frac{(\omega_c T)^2}{\epsilon}]}{\ln(\nu_\star^{-1})}$. Combining these two results, after some combinatorics, we arrive at the main result in Eq. (1).

VII. CONCLUDING REMARKS

In this paper, we develop a nonperturbative approach to the SB model, by introducing an analytical method to approximate a zero-temperature ohmic bosonic bath by a number of damped oscillators, with a decay rate in Eq. (41) that controls

the memory of the corresponding mode and the coupling strength in Eq. (46). For a fixed error ϵ, the required number of bosonic modes scales polylogarithmically in ϵ and evolution time T. This explicit construction enables a homogeneous MPS approximation for the Feynman-Vernon functional. We prove that the bond dimension scales at most polynomially with ϵ and T in Eq. (1). Previous numerical results of Ref. [9] also suggest the polynomial scaling of bond dimension $D \sim T^q$, with non universal $q \in [1, 2]$. Note that our MPS description of the Feynman-Vernon IF provides an accurate description of the dynamics of the system independent of local spin dynamics and coupling strength; thus, it is natural to expect that the bond dimension required for a specific choice of the dynamics of the spin may exhibit a better scaling. Apart from theoretical interest, our construction may have a practical applications: It can be used as a starting point for numerical calculations, as our analytic MPS can be further compressed using singular value decomposition.

Although we proved the bounds for the errors, there are general mathematical constraints on the IF. One way to formulate the restriction is to say that the spin dynamicsin Eq. (9) should provide a completely positive trace-preserving (CPTP) map [34], for any unitaries $U^{(k)}$. It seems that our MPS approximation does not in general satisfy this requirement, but it has a sufficiently small deviation from it. It is still unclear to us whether it is possible to provide a physical MPS approximation with the same bond dimensions.

An interesting future direction is to provide estimates for the IF of a finite-temperature bath with generic spectral functions, including cases of subohmic and superohmic baths. For instance, our analysis may be carried over [30] to the case of a finite-temperature bath, yielding the same bound in Eq. (1). Finally, let us note that our analysis can be extended to a wider class of IFs, including anon-Gaussian one. This can be achieved by including spin dependence of **A** matrices in Eq. (23).

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