




Beyond mean-field corrections to the blueshift of a driven-dissipative exciton-polariton condensate

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In the absence of vortices or phase slips, the phase dynamics of exciton-polariton condensates was shown to map onto the Kardar-Parisi-Zhang (KPZ) equation, which describes the stochastic growth of a classical interface. This implies that the coherence of such nonequilibrium quasicondensates decays in space and time following stretched exponentials, characterized by KPZ universal critical exponents. In this work, we focus on the time evolution of the average phase of a one-dimensional exciton-polariton condensate in the KPZ regime and determine the frequency of its evolution, which is given by the blueshift, i.e., the nonequilibrium analog of the chemical potential. We determine the stochastic corrections to the blueshift within Bogoliubov linearized theory and find that while this correction physically originates from short scale effects, and depends both on density and phase fluctuations, it can still be related to the effective large-scale KPZ parameters. Using numerical simulations of the full dynamics, we investigate the dependence of these blueshift corrections on both noise and interaction strength, and compare the results to the Bogoliubov prediction. Our finding contributes both to the close comparison between equilibrium and nonequilibrium condensates, and to the theoretical understanding of the KPZ mapping.

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I. INTRODUCTION

Exciton-polaritons (EPs) are bosonic quasiparticles that arise from the strong coupling between light and matter [1]. They are typically formed in a quantum well embedded in an optical microcavity, from the interaction between quantum well excitons and cavity photons [2,3]. The system undergoes unavoidable losses due to the leakage of cavity photons, which have to be compensated by a continuous laser drive in order to sustain a nonequilibrium steady state of EPs. Remarkably, a Bose-Einstein condensate (BEC) can form in these driven-dissipative conditions, as was first experimentally demonstrated in [4], and yields a wide range of novel physical phenomena [5,6]. Among them, recent studies established a connection with the Kardar-Parisi-Zhang (KPZ) universality class [7–13], known in classical statistical physics to describe the scale-invariant properties of kinetically roughening interfaces and characterized by a superdiffusive universal exponent [14]. This connection was first observed in experiments for a one-dimensional (1D) EP condensate in Ref. [15], which reported the expected stretched exponential decay of the first-order correlation function, settling a major difference with coherence properties of 1D equilibrium quasicondensates [16,17], which display power-law (or simple exponential) decay at zero (or finite temperature). Another notable feature of 1D EP condensates arises from the compact nature of the phase, which, in contrast with the unbounded interface, allows for the generation of phase defects in space and time in the form of quantized 2π jumps often referred to as space-time vortices [9,15,18].

In this work, we focus on a weakly interacting 1D EP Bose-Einstein condensate in the KPZ regime, where the phase

is defect free, and we investigate the blueshift of the energy spectrum. The dominant contributions to this energy shift, signature of the U(1) symmetry breaking of the condensation transition, are the polariton-polariton and exciton reservoir-polariton mean-field interactions [19,20]. Our aim is to determine the beyond mean-field corrections to the blueshift induced by fluctuations. The interest is twofold: on the one hand, it provides a further characterization of the difference between equilibrium and nonequilibrium condensates, and on the other hand, it deepens the understanding of the KPZ mapping. Let us elaborate on these two points. For an equilibrium quasi-BEC, the quantity analogous to the blueshift is the chemical potential, and its corrections associated to quantum and thermal fluctuations are well known [21] (see also [22]). In nonequilibrium conditions, because the drive and loss are intrinsically stochastic processes, the dominant fluctuations at weak interactions are of stochastic nature. We determine their effect on the blueshift, both through an analytical expression obtained within Bogoliubov approximation, and through numerical simulations of the full nonlinear stochastic dynamics.

The KPZ mapping shows that within certain general assumptions, the dynamics of the condensate phase follows the KPZ equation which describes the stochastic growth of a classical interface [14]. The KPZ height field $h(x, t)$ is known to behave at late times at a given space point as $h(t) \sim v_\infty t + (\Gamma t)^{1/3} \chi$, where Γ is a nonuniversal parameter, $v_\infty = \lim_{t \rightarrow +\infty} \langle \partial_t h \rangle$ is the asymptotic deterministic velocity of the interface [23], and χ is a stochastic variable, whose distribution is non-Gaussian and is sensitive to the global geometry of the growth, or equivalently to its initial conditions. For a flat

interface, the distribution of χ is a Tracy-Widom GOE (Gaussian orthogonal ensemble) distribution, while for a curved interface (droplet growth) it is a Tracy-Widom GUE (Gaussian unitary ensemble) distribution, and finally for the stationary growth it is the Baik-Rains distribution [24]. This unveils a striking connection with random matrix theory where these distributions first emerged. The KPZ mapping for the condensate implies that, once the mean-field evolution is subtracted, the unwrapped phase should also evolve as $\tilde{\theta}(t) \sim -\Omega_\infty t + (\Gamma t)^{1/3} \chi$ where the minus sign is added to define $\Omega_\infty > 0$ (see below). While much effort has been devoted to determining the statistics of the random variable χ in exciton-polariton systems [10,11,15], the linear term has received no attention yet. In particular, it was shown that the distinct distributions expected for a KPZ growth in the different geometries also arise for the phase fluctuations of exciton-polaritons, which is a definite evidence for the validity of the mapping. In contrast, nothing is known on Ω_∞ , although this linear term should induce corrections, originating in the nonlinear KPZ behavior, to the dominant mean-field contribution to the blueshift. This is the issue addressed in this work. We show that the stochastic corrections to the blueshift are both due to density and phase fluctuations and hence go beyond the effective KPZ dynamics, where only phase fluctuations are relevant.

The paper is organized as follows. The model for the stochastic dynamics of the EP condensate is presented in Sec. II A, and the link between the phase dynamics and the KPZ physics is detailed in Sec. II B. We discuss in Sec. II C the connections between blueshift stochastic fluctuations and chemical potential quantum corrections. We determine in Sec. III the expression for the blueshift stochastic correction obtained within the Bogoliubov approximation. Our results from numerical simulations of the full dynamics are then presented in Sec. IV, where we determine the dependence of this correction on the noise amplitude and on the interaction strength. Finally, Sec. V offers some concluding remarks.

II. MODEL AND OBSERVABLES

A. Model for the dynamics of a 1D exciton-polariton condensate

The dynamics of a weakly interacting 1D exciton-polariton condensate subjected to incoherent pumping can be modeled by a generalized Gross-Pitaevskii equation (gGPE) for the condensate wave function Φ , coupled to a rate equation for the dynamics of the density of the excitonic reservoir n_R , as [25]

$$i\hbar\partial_t\Phi = \left[\mathcal{F}^{-1} \left[\epsilon_{\hat{k}} - \frac{i\hbar}{2}\gamma_\ell(\hat{k}) \right] + \frac{i\hbar R}{2}n_R + \hbar g|\Phi|^2 + 2\hbar g_R n_R \right] \Phi + \hbar\xi, \quad (1)$$

$$\partial_t n_R = P - (\gamma_R + R|\Phi|^2)n_R, \quad (2)$$

where \mathcal{F}^{-1} denotes the inverse Fourier transform, with $\epsilon_{\hat{k}} = \hbar^2 k^2 / (2m)$ the dispersion relation on the lower-polariton branch, and $\gamma_\ell(\hat{k}) = \gamma_0 + \gamma_2 k^2$ the k -dependent polariton loss rate around the condensation momentum $k = 0$, both within the quadratic approximation. The momentum dependence of

the loss rate is typically observed in experimental realizations of polariton condensates [15]. m denotes the polariton mass, g represents the polariton-polariton interaction strength, and g_R the exciton-polariton one. The excitonic reservoir is incoherently driven at pumping rate P , scatters with polaritons with amplitude R , and dissipates through other decay channels at rate γ_R . The complex noise ξ is chosen of amplitude σ with zero mean $\langle \xi \rangle = 0$ and a covariance $\langle \xi(x, t) \xi^*(x', t') \rangle = 2\sigma \delta(x - x') \delta(t - t')$.

B. Mapping to the KPZ equation and asymptotic behavior of the phase

It was realized that the dynamics of the phase of the driven-dissipative condensate maps within some conditions onto the KPZ equation [7,8]. In order to show this, one writes the condensate wave function Φ in density-phase representation $\Phi = \sqrt{n}e^{i\theta}$ and deduces from the gGPE (1) the coupled dynamical equations for the phase $\theta(x, t)$ and the density $n(x, t)$ fields as

$$\begin{aligned} \partial_t n = & -\frac{\hbar}{m}(\partial_x n \partial_x \theta + n \partial_x^2 \theta) \\ & + \gamma_2 \left(\frac{1}{2} \partial_x^2 n - \frac{(\partial_x n)^2}{4n} - n(\partial_x \theta)^2 \right) \\ & - \gamma_0 n + R n_R n - 2\sqrt{\sigma n} \text{Im}(\xi e^{-i\theta}), \end{aligned} \quad (3)$$

$$\begin{aligned} \partial_t \theta = & \frac{\hbar}{2m} \left(\frac{1}{2} \frac{\partial_x^2 n}{n} - \frac{1}{4} \left(\frac{\partial_x n}{n} \right)^2 - (\partial_x \theta)^2 \right) \\ & + \frac{\gamma_2}{2} \left(\frac{\partial_x n \partial_x \theta}{n} + \partial_x^2 \theta \right) \\ & - gn - 2g_R n_R + \sqrt{\frac{\sigma}{n}} \text{Re}(\xi e^{-i\theta}). \end{aligned} \quad (4)$$

Then one expands the condensate and reservoir densities, as well as the condensate phase, in terms of small stochastic corrections around their mean-field noise-independent values as

$$n(x, t) = n_0 - \tilde{n}(x, t), \quad (5)$$

$$n_R(x, t) = n_{R_0} + \tilde{n}_R(x, t), \quad (6)$$

$$\theta(x, t) = \theta_0(t) + \tilde{\theta}(x, t), \quad (7)$$

where the minus sign in (5) is added to define $\langle \tilde{n} \rangle > 0$ in the following, and where $n_0 = \gamma_R(P/P_{\text{th}} - 1)/R$, $n_{R_0} = \gamma_0/R$, with the pump threshold for condensation $P_{\text{th}} = \gamma_R \gamma_0 / R$. The mean-field component of the phase evolves with time as $\theta_0(t) = -\Omega_0 t$, where $\Omega_0 = gn_0 + 2g_R n_{R_0} > 0$. This is a good approximation for the blueshift of the weakly interacting EP condensate from which the interaction strength can be estimated [19,20].

Moreover, one assumes that both density and reservoir fluctuations evolve adiabatically compared to the phase fluctuations $\tilde{\theta}$, such that one can neglect their temporal variations $\partial_t \tilde{n} \approx 0$, $\partial_t \tilde{n}_R \approx 0$. This yields for the reservoir density fluctuations $\tilde{n}_R = -2\frac{g_i}{R}\tilde{n}$ with $g_i = (R n_{R_0})^2 / 2P$, and allows one to extract from (3) the expression for the condensate density

fluctuations \tilde{n} in terms of derivatives of \tilde{n} and of $\tilde{\theta}$, which are then inserted into Eq. (4) for the phase dynamics.

By assuming that the density profile remains smooth, such that its spatial variations can be neglected, one finds that the phase is governed by the following effective dynamical equation:

$$\partial_t \tilde{\theta} = v \partial_x^2 \tilde{\theta} + \frac{\lambda}{2} (\partial_x \tilde{\theta})^2 + \sqrt{D} \eta, \quad (8)$$

which is the KPZ equation [14], where η is a real Gaussian noise of zero mean $\langle \eta \rangle = 0$ and delta correlations in space and time $\langle \eta(x, t) \eta(x', t') \rangle = 2\delta(x - x')\delta(t - t')$. Its parameters are given in terms of the gGPE microscopic parameters by

$$v = \frac{\gamma_2}{2} + \frac{\hbar g_e}{2mg_i}, \quad (9)$$

$$\lambda = -\frac{\hbar}{m} + \gamma_2 \frac{g_e}{g_i}, \quad (10)$$

$$D = \frac{\sigma}{2n_0} \left[1 + \left(\frac{g_e}{g_i} \right)^2 \right], \quad (11)$$

where the effective polariton-polariton interaction strength within the adiabatic approximation is denoted $g_e = g - 4 \frac{g_R g_i}{R}$.

The KPZ dynamics of the phase determines the coherence properties of the EP condensate. These properties are encoded in the first-order correlation function of the condensate, defined as

$$g^{(1)}(x, t) = \frac{\langle \Phi^*(x, t) \Phi(0, 0) \rangle}{\sqrt{\langle n(x, t) \rangle \langle n(0, 0) \rangle}}, \quad (12)$$

where the average $\langle \cdot \rangle$ is performed over realizations of the stochastic dynamics (1). If one neglects the density-phase correlations, and assumes constant density-density correlations, then one obtains within a cumulant expansion [15] that

$$-2 \log[|g^{(1)}(x, t)|] \simeq \langle (\tilde{\theta}(x, t) - \tilde{\theta}(0, 0))^2 \rangle \equiv C_{\theta\theta}(x, t), \quad (13)$$

where $C_{\theta\theta}$ is the connected two-point correlation function of the phase. We adopt in the following the shorthand notation $\mathcal{G}^{(1)}(x, t) \equiv -2 \log[|g^{(1)}(x, t)|]$. For the KPZ dynamics of the height field $h(x, t)$, the connected correlation function $C_{hh}^{\text{KPZ}}(x, t)$ is known to take a scaling form given by

$$C_{hh}^{\text{KPZ}}(x, t) = C_0 t^{2/3} g_{\text{KPZ}} \left(y_0 \frac{x}{t^{2/3}} \right), \quad (14)$$

from which we read the 1D KPZ universal growth and roughness critical exponents $\beta_{\text{KPZ}} = \frac{1}{3}$ and $\alpha_{\text{KPZ}} = \frac{1}{2}$, respectively. C_0, y_0 are nonuniversal constants, and g_{KPZ} is the KPZ universal scaling function, which was calculated exactly in [26]. It has the following asymptotic behaviors: $g_{\text{KPZ}}(y) \rightarrow g_0$, with g_0 finite, when $y \rightarrow 0$ and $g_{\text{KPZ}}(y) \sim |y|$ when $y \rightarrow \infty$. Hence, the correlation function behaves as power laws both at coinciding space points $C_{hh}^{\text{KPZ}}(0, t) \sim t^{2\beta_{\text{KPZ}}} \equiv t^{2/3}$ and equal times $C_{hh}^{\text{KPZ}}(x, 0) \sim |x|^{2\alpha_{\text{KPZ}}} \equiv |x|$.

The linearized version of the KPZ equation (8) (setting $\lambda = 0$) is the Edwards-Wilkinson (EW) equation. Its correspond-

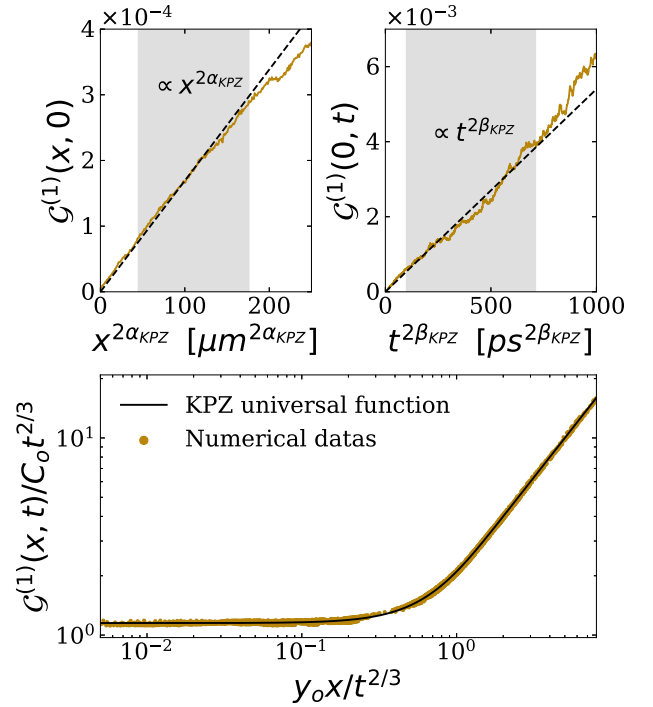


FIG. 1. KPZ scaling regime: (upper left panel) spatial scaling of $\mathcal{G}^{(1)}(x, 0)$ as a function of x , indicated by the gray shade in-between $[60 \mu\text{m}, 220 \mu\text{m}]$; (upper right panel) temporal scaling of $\mathcal{G}^{(1)}(0, t)$ as a function of $t^{2/3}$, indicated by the grey shade in-between $[10^3 \text{ ps}, 2 \times 10^4 \text{ ps}]$; (lower panel) collapse of $\mathcal{G}^{(1)}(x, t)$ onto the KPZ universal scaling function g_{KPZ} . The parameters used on this figure are $\gamma_R = 4.5\gamma_0$, and $\sigma = 10^{-2}\gamma_0/2$. The system size is set to $2000 \mu\text{m}$. Other parameters are specified in Sec. IV.

ing height field connected correlation function $C_{hh}^{\text{EW}}(x, t)$ takes a similar scaling form

$$C_{hh}^{\text{EW}}(x, t) = C_1 t^{1/2} g_{\text{EW}} \left(y_1 \frac{x}{t^{1/2}} \right), \quad (15)$$

where C_1, y_1 are nonuniversal constants. The universal exponents $\beta_{\text{EW}} = \frac{1}{4}$, $\alpha_{\text{EW}} = \frac{1}{2}$, and the universal scaling function g_{EW} define the EW universality class [27,28].

It follows from (13) that if the phase of the condensate obeys a KPZ dynamics at least on certain time and length scales, with an effective nonlinearity $\lambda \neq 0$, then the coherence of the condensate endows the scaling form

$$\mathcal{G}^{(1)}(x, t) \simeq C_{hh}^{\text{KPZ}}(x, t) \sim \begin{cases} t^{2/3} & x = 0, \\ |x| & t = 0 \end{cases} \quad (16)$$

in the space-time window where the KPZ dynamics is relevant. This behavior has been recently reported in experiments [15], which confirms the emergence of KPZ universality in exciton-polariton condensates. We also find a KPZ regime in our numerical simulations, reported in Sec. IV (and Fig. 1).

As mentioned in the Introduction, another hallmark of KPZ universality is the probability distribution of the reduced height fluctuations χ defined from the long time behavior at a given space point as $h(t) \sim v_\infty t + (\Gamma t)^{1/3} \chi$. In the absence

of phase defects, $\tilde{\theta}$ can be safely unwrapped into a continuous variable, which is expected in the KPZ regime to follow the asymptotic form

$$\tilde{\theta}(x_0, t) \underset{t \rightarrow +\infty}{\sim} -\Omega_\infty t + (\Gamma t)^{1/3} \chi. \quad (17)$$

The first term describes the nonuniversal linear growth of the phase fluctuations through the asymptotic frequency defined as $\Omega_\infty = -\lim_{t \rightarrow +\infty} \langle \partial_t \tilde{\theta} \rangle$. In the second term, the power law $t^{1/3}$ gives rise to the KPZ universal scaling law for $C_{\theta\theta}$ in time, and the distribution of the fluctuations χ was shown to exhibit the Tracy-Widom and Baik-Rains forms expected for the KPZ universality class, where the curved interface geometry in particular can be emulated for the phase through a confinement potential [11].

In this work, we focus on the correction Ω_∞ , due to stochastic fluctuations, to the mean-field blueshift Ω_0 . We investigate its dependence on the system parameters using Bogoliubov theory and numerical simulations of Eqs. (1) and (2). Let us first elaborate on the link between the blueshift in nonequilibrium condensates and the analogous quantity defined at equilibrium.

C. Connection with equilibrium quasicondensates

For equilibrium weakly interacting Bose-Einstein condensates, in homogeneous conditions and with contact interactions $v(x-x') = \hbar g \delta(x-x')$, the mean-field time evolution of the wave function $\Phi(t) = \sqrt{n} e^{i\theta(t)}$ is only contained in the phase and is linear in time $\theta(t) = -\frac{\mu_0}{\hbar} t$, where the blueshift $\mu_0/\hbar = gn_0$ is the condensate chemical potential. The beyond mean-field quantum corrections to the blueshift at weak interactions have been calculated for a three-dimensional (3D) Bose gas by Lee, Huang, and Yang [21] within the Bogoliubov approximation in terms of the scattering length of the fluid. Reference [22] provides an alternative derivation of such corrections based on an equation-of-motion approach, yielding the following expression for the chemical potential $\mu/\hbar = gn_0 + 2g\langle \tilde{n} \rangle_Q + g\langle \tilde{m} \rangle_Q$, where $\langle \cdot \rangle_Q$ denotes the quantum average over the ground-state wave function, $\langle \tilde{n} \rangle_Q$ is the noncondensed particle density corresponding to the quantum depletion at zero temperature, and $\langle \tilde{m} \rangle_Q$ is the anomalous density. The above calculation has also been extended to low-dimensional quasicondensates [29–32].

For the nonequilibrium 1D EP condensate, one also finds that, at the mean-field level, the phase linearly decreases with time with the blueshift $\Omega_0 = gn_0 + 2g_R n_{R0}$, which is analogous to the mean-field expression for the chemical potential μ_0 . The total blueshift $\Omega = \Omega_0 + \Omega_\infty$ can thus be interpreted as the equivalent of the chemical potential for the equilibrium condensate. However, since the EP condensate is out of equilibrium, Ω cannot be defined by the usual thermodynamics relation $\partial E/\partial N|_{T,V}$, but instead has to be calculated from the dynamics $\Omega = -\langle \partial_t \theta \rangle$ similarly to Ref. [22]. The term Ω_∞ is hence analogous to the quantum Lee-Huang-Yang corrections at equilibrium, except that here it originates from stochastic fluctuations.

III. BOGOLIUBOV APPROXIMATION IN THE ADIABATIC LIMIT OF THE EXCITONIC RESERVOIR

In this section, we derive an expression for the blueshift correction Ω_∞ using the linearized Bogoliubov approximation.

A. Linearized dynamics and equal-time correlation functions

Since the time evolution of the reservoir dynamics is typically much slower than the timescales of the condensate dynamics, we neglect it in the following. Within this adiabatic approximation, we express the reservoir density in terms of the condensate density and hence close the set of coupled equations (3) and (4). Linearizing these equations gives the coupled dynamics for the fluctuations of density δn and phase $\delta\theta$ to first order in the noise. This reads as in Fourier space

$$\begin{pmatrix} \delta n/2n_0 \\ \delta\theta \end{pmatrix}(k, \omega) = G(k, \omega) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad (18)$$

where $G(k, \omega) = (i\omega - \mathcal{L}_{\text{rot}})^{-1}$ is the Green function, \mathcal{L}_{rot} is the adiabatic Bogoliubov matrix

$$\mathcal{L}_{\text{rot}} = \begin{pmatrix} \Gamma_{\hat{k}} - g_i n_0 & \epsilon_{\hat{k}} \\ -(\epsilon_{\hat{k}} + 2g_e n_0) & \Gamma_{\hat{k}} + g_i n_0 \end{pmatrix} \quad (19)$$

with $\Gamma_{\hat{k}} = \frac{\Delta\gamma(\hat{k})}{2} - g_i n_0$, $\Delta\gamma(\hat{k}) = Rn_{R0} - \gamma_\ell(\hat{k})$ and where the noise satisfies $\langle \xi_i \rangle = 0$, $\langle \xi_1 \xi_2 \rangle = 0$, $\langle \xi_i(k, \omega) \xi_j(k', \omega') \rangle = 4\pi^2 \frac{\sigma}{n_0} \delta(k+k') \delta(\omega+\omega')$.

Within our model for $\gamma_\ell(\hat{k})$ and $\epsilon_{\hat{k}}$ (see Sec. II A), the Bogoliubov matrix of Eq. (19) simplifies to

$$\mathcal{L}_{\text{rot}} = \begin{pmatrix} -2g_i n_0 - \frac{\gamma_2}{2} k^2 & \frac{\hbar k^2}{2m} \\ -\frac{\hbar k^2}{2m} - g_e n_0 & -\frac{\gamma_2}{2} k^2 \end{pmatrix}. \quad (20)$$

From Eq. (18), we find the equal-time correlation functions, expressed in terms of both the gGPE microscopic parameters and the effective viscosity ν and nonlinearity λ of the KPZ mapping [Eqs. (9) and (10), respectively]

$$\langle \delta n(k) \delta n(-k) \rangle = \frac{2\sigma n_0}{\gamma_2} \frac{2k^2 + \kappa_h^2}{(k^2 + \kappa_h^2)(k^2 + \kappa_d^2)}, \quad (21)$$

$$\langle \delta n(k) \delta\theta(-k) \rangle = -\frac{\sigma \kappa_h^2 n_0 g_i \lambda}{\nu \gamma_2^2 \kappa_d^2 (k^2 + \kappa_h^2)(k^2 + \kappa_d^2)}, \quad (22)$$

$$\begin{aligned} \langle \delta\theta(k) \delta\theta(-k) \rangle &= \frac{\langle \delta n(k) \delta n(-k) \rangle}{4n_0^2} + \frac{\sigma \kappa_h^2}{\nu n_0 \gamma_2^2 \kappa_d^2} \\ &\times \frac{2\nu g_i n_0 k^2 + 2n_0^2 (g_i^2 + g_e^2)}{k^2 (k^2 + \kappa_h^2) (k^2 + \kappa_d^2)}, \end{aligned} \quad (23)$$

where we introduced two momentum scales κ_h and κ_d representing the nonequilibrium healing length and dissipation scale, respectively, defined by

$$\kappa_h = \sqrt{\frac{8\nu g_i n_0}{\hbar^2/m^2 + \gamma_2^2}}, \quad \kappa_d = \sqrt{\frac{2g_e n_0}{\gamma_2}}, \quad (24)$$

with typically $\kappa_h > \kappa_d$. Note that the choice of $\gamma_\ell(\hat{k})$ we made regulates the large-momentum behavior of Eqs. (21) and (23) through the effective loss rate $\Gamma_{\hat{k}}$, similarly to the frequency-dependent amplification introduced in Ref. [33]. Generic expressions of the equal-time correlation functions for unspecified $\gamma_\ell(\hat{k})$ and $\epsilon_{\hat{k}}$ are given in Appendix A. The effect of alternative choices for the polariton loss rate $\gamma_\ell(\hat{k})$ is discussed in Appendix B.

In addition to $C_{\theta\theta}(x) \equiv C_{\theta\theta}(x, 0)$ defined in Eq. (13), we introduce the density-density and density-phase equal-time correlation functions $C_{nn}(x) = \langle \delta n(x) \delta n(0) \rangle - \langle \delta n(0)^2 \rangle$ and $C_{n\theta}(x) = \langle \delta n(x) \delta \theta(0) \rangle$, respectively. Within Bogoliubov approximation, we find from Eqs. (21)–(23) that

$$C_{nn}(x) = \frac{\sigma n_0}{\gamma_2} \left[\frac{\kappa_h}{\kappa_h^2 - \kappa_d^2} (e^{-\kappa_h|x|} - 1) + \left(2 - \frac{\kappa_h^2}{\kappa_h^2 - \kappa_d^2} \right) \frac{e^{-\kappa_d|x|} - 1}{\kappa_d} \right], \quad (25)$$

$$C_{n\theta}(x) = \sigma \lambda \kappa_h^2 \frac{e^{-\kappa_h|x|/\kappa_h} - e^{-\kappa_d|x|/\kappa_d}}{4\gamma_2 \nu (\kappa_h^2 - \kappa_d^2)}, \quad (26)$$

$$C_{\theta\theta}(x) = -\frac{C_{nn}}{2n_0^2} + \frac{e^{-\kappa_d|x|} - 1}{\kappa_d/\mathcal{K}(d,d)} - \frac{e^{-\kappa_h|x|} - 1}{\kappa_h/\mathcal{K}(h,h)} + \frac{D}{\nu \kappa_d^2} |x|, \quad (27)$$

where ν, λ, D are the KPZ parameters introduced in Eqs. (9)–(11), and with the momentum matrix elements

$$\mathcal{K}^{(i,j)} = \frac{\kappa_h^2}{\kappa_h^2 - \kappa_d^2} \left[\frac{D\kappa_d^2}{\nu \kappa_i \kappa_j} - \frac{\sigma}{\gamma_2 n_0} \right]. \quad (28)$$

Let us comment on the phase-phase correlation function (27). We observe that at long distances, it exhibits the scaling $C_{\theta\theta} \propto |x|$, which reveals the deep connection between the condensate phase dynamics and stochastically growing interfaces (see Appendix C), leading to $|g^{(1)}(x, 0)| \sim e^{-|x|}$. This scaling indeed corresponds to the expected one for the equal-time EW correlation (15) with $\alpha = \frac{1}{2}$. The same equilibrium spatial scaling is obtained for the KPZ dynamics, merely reflecting that the stationary KPZ interface is a Brownian in space. This spatial scaling coincides with the one of 1D equilibrium quasicondensates at finite temperature [17], as mentioned in the Introduction. It should also be noted that $C_{\theta\theta}$ explicitly depends on C_{nn} . While being irrelevant at large distances as mentioned above, this plays an important role when $x \rightarrow 0$ and will be crucial in the following.

B. Blueshift corrections

We obtain the blueshift $\Omega_\infty = -\langle \partial_t \tilde{\theta} \rangle$ by taking the average of Eq. (4) over the stochastic fluctuations. This average involves on the right-hand side spatial derivatives of two-point correlation functions of the phase and density, which we approximate by their expressions within Bogoliubov theory (25)–(27). For instance, we have $\langle (\partial_x \tilde{\theta})^2 \rangle \approx \langle (\partial_x \delta \theta)^2 \rangle = \frac{1}{2} \lim_{x \rightarrow 0} \partial_x^2 C_{\theta\theta}(x)$. Hence, the blueshift correction reads as

$$\Omega_\infty \underset{x \rightarrow 0}{=} \frac{\hbar}{4m} \left[\frac{\partial_x^2 C_{nn}}{2n_0^2} + \partial_x^2 C_{\theta\theta} \right] + \gamma_2 \frac{\partial_x^2 C_{n\theta}}{2n_0} - g_e \langle \tilde{n} \rangle_S, \quad (29)$$

where $\langle \tilde{n} \rangle_S \equiv \langle \tilde{n} \rangle$ is the noncondensate density arising from stochastic fluctuations. Similarly to Ω_∞ , it is obtained from the dynamics of the density fluctuations \tilde{n} by inserting the expansions (5)–(7) into Eq. (3), performing the adiabatic approximation of the reservoir dynamics. Assuming stationarity $\partial_t \tilde{n} = 0$ and taking the average, we can extract $\langle \tilde{n} \rangle_S$ in terms of spatial derivatives of two-point correlation functions. Following the same steps as above and approximating these correlation functions by Eqs. (25)–(27) gives

$$\langle \tilde{n} \rangle_S \underset{x \rightarrow 0}{=} \frac{\gamma_2}{4g_i} \left[\frac{\partial_x^2 C_{nn}}{2n_0^2} + \partial_x^2 C_{\theta\theta} \right] - \frac{\hbar \partial_x^2 C_{n\theta}}{2g_i m n_0}. \quad (30)$$

Replacing $\langle \tilde{n} \rangle_S$ in Eq. (29), one finds an expression of Ω_∞ which can be conveniently expressed in terms of the KPZ parameters λ and ν as

$$\Omega_\infty \underset{x \rightarrow 0}{=} -\frac{\lambda}{4} \left[\frac{\partial_x^2 C_{nn}}{2n_0^2} + \partial_x^2 C_{\theta\theta} \right] + \nu \frac{\partial_x^2 C_{n\theta}}{n_0}. \quad (31)$$

It is instructive to compare this expression, obtained from the coupled density-phase dynamics (18), with the one ensuing from the KPZ equation of phase fluctuations (8) alone. For the latter, one can follow the same procedure, i.e., one first linearizes the dynamics (18), which yields the EW equation, then computes the associated correlation function $C_{\theta\theta}^{\text{EW}}(x)$, and finally defines Ω_∞ as $\Omega_\infty = -\lim_{t \rightarrow +\infty} \langle \partial_t \tilde{\theta} \rangle$ from the full KPZ dynamics (8). One then finds $\Omega_\infty \underset{x \rightarrow 0}{=} -\frac{\lambda}{4} \partial_x^2 C_{\theta\theta}^{\text{EW}}(x)$.

While this term corresponds to the second one in (31), additional terms enter the latter expression, which thus does not depend on the phase only. This means that, even though the spatiotemporal large-scale properties of the quasicondensate are satisfactorily described by the effective KPZ dynamics (8) (see Fig. 1 and Refs. [10,11,15]), the correlations of the density, and the coupling between density and phase bring additional contributions to Ω_∞ .

Moreover, let us emphasize that the expression for Ω_∞ is sensitive to the microscopic details of the condensate dynamics stemming from the short-distance properties of both density and phase correlation functions (25)–(27). In fact, both C_{nn} and $C_{\theta\theta}$ contributions to (31) are needed in order to make Ω_∞ well defined in the UV regime. Indeed, taking the second spatial derivative of $C_{\theta\theta}(x)$, one finds that it diverges for $x \rightarrow 0$. While this is also the case for $C_{hh}^{\text{EW,KPZ}}(x)$, the particularity here is that the diverging part comes from the first term in the right-hand side of (27), i.e.,

$$\partial_x^2 C_{\theta\theta} \underset{x \rightarrow 0}{=} -\frac{\partial_x^2 C_{nn}}{2n_0^2} \Big|_{x=0} + \mathcal{K}^{(h,d)} (\kappa_h - \kappa_d), \quad (32)$$

where the density contribution reads as

$$-\frac{\partial_x^2 C_{nn}}{2n_0^2} \underset{x \rightarrow 0}{=} \frac{\sigma}{2n_0 \gamma_2} \left[4 \frac{\Lambda}{\pi} - 2\kappa_d \frac{\kappa_h^2}{\kappa_h + \kappa_d} \right] \quad (33)$$

with Λ an arbitrary UV cutoff. A similar divergence is also predicted for 1D quasi-BECs at equilibrium, where the standard regularization is to consider that Bogoliubov theory is applied on a lattice of spacing $\ell = \pi/\Lambda$. As formulated in Ref. [34], the theory is valid as long as the parameter $\langle (\partial_x \delta \theta)^2 \rangle \propto 1/n_0 \ell^3$ is small in 1D. Adapted to the nonequilibrium dynamics, the same argument implies that $\langle (\partial_x \delta \theta)^2 \rangle \underset{x \rightarrow 0}{=} \frac{1}{2} \partial_x^2 C_{\theta\theta}(x) \propto \frac{\sigma}{n_0 \gamma_2 \ell}$ must remain a small parameter. As a

complement, we show in Appendix C that the criterion for the applicability of Bogoliubov theory on the driven-dissipative BEC can, as for its equilibrium counterpart, also be generalized to higher dimensions. In fact, we show in Sec. IV B that the dependence of the gradients of correlation functions on the short length scale ℓ is not an artifact of the linearization of (3) and (4), but also emerges at the level of the gGPE from the full nonlinear dynamics of Eqs. (1) and (2).

In the one-dimensional case for closed systems, because of the Mermin-Wagner-Hohenberg theorem, $\langle \tilde{n} \rangle_Q$ and $\langle \tilde{m} \rangle_Q$ both diverge in the IR. The two diverging terms are of opposite signs and cancel out, leading to a finite chemical potential correction [34]. The same mechanism operates here, the diverging contributions in the density and phase correlations precisely compensate in (31), such that the blueshift correction is finite, and reads as, within the Bogoliubov approximation,

$$\Omega_\infty = -\frac{\lambda D}{4\nu(\kappa_h^{-1} + \kappa_d^{-1})}. \quad (34)$$

Let us comment on this result. The correction Ω_∞ is found to be proportional to the ratio of KPZ effective parameters $\frac{\lambda D}{\nu}$ and in particular to the noise strength σ through D . Additionally, as already emphasized, Ω_∞ depends on the microscopic details of the condensate through both the nonequilibrium healing length and dissipation length. It is interesting to remark that also the linewidth of extended lasers depends on two-point correlation functions [35–38], where D is the Schawlow-Townes linewidth enhanced by the Henry factor g_e/g_i [39,40].

Let us discuss the sign of this blueshift correction. For the quasicondensate to be stable, the mass m and the effective coupling g_e must have the same sign [3]. This implies that $\nu > 0$, while λ can be either positive or negative, implying that the sign of Ω_∞ is fully controlled by the KPZ nonlinearity λ . This is consistent with the interpretation of Ω_∞ as the average growth velocity of the growing interface $\tilde{\theta}$. In typical experimental realizations of stable exciton-polariton condensates, $m < 0$ and $\lambda < 0$ [3,15] so that Ω_∞ is positive.

Let us now comment on the correction $\langle \tilde{n} \rangle_S$ to the condensate density. Replacing (25)–(27) in (30), we obtain

$$\langle \tilde{n} \rangle_S = \frac{Dn_0}{2\nu\kappa_d^2(\kappa_h^{-1} + \kappa_d^{-1})} + \frac{\sigma}{\gamma_2(\kappa_h + \kappa_d)}, \quad (35)$$

from which we read $\langle \tilde{n} \rangle_S > 0$ for any parameters. The situation is analogous to the Bogoliubov occupation of three-dimensional BECs $\langle \tilde{n} \rangle_Q$, which can only take positive values, thus depleting the condensate, and reflecting the fact that the condensate chemical potential is slightly higher than its mean-field value. There are, however, two main differences with the equilibrium case. First, $\langle \tilde{n} \rangle_S$ is well defined in 1D while $\langle \tilde{n} \rangle_Q$ diverges because of large phase fluctuations. Second, $\langle \tilde{n} \rangle_S$ is not proportional to λ and cannot change sign contrary to Ω_∞ . Recalling that Ω_∞ represents an analogous chemical potential correction for the driven-dissipative condensate, this implies that, at variance with equilibrium BECs, one could in principle reach a parameter regime for which the stochastic depletion $\langle \tilde{n} \rangle_S$ is accompanied by a decrease of the total chemical potential. This situation cannot be realized for

systems at equilibrium, where these two quantities are related by a state equation.

In order to test the predictions (34) and (35) ensuing from the Bogoliubov linearized dynamics, we now perform numerical simulations of the full nonlinear dynamics (1) and (2) and extract the numerical estimate for Ω_∞ . We study in particular its dependence on the KPZ noise strength D through the gGPE noise σ , and we also vary the coupling strength g to make the nonlinearity λ of the effective phase dynamics (8) gradually decrease to zero.

IV. NUMERICAL SIMULATIONS

Numerical resolutions of Eqs. (1) and (2) are performed using a combination of split-step method and first-order Euler scheme [41]. The parameters are chosen to reproduce the typical experimental conditions in GaAs microcavities [15]: $\hbar\gamma_0 = 48.5 \mu\text{eV}$, $\hbar\gamma_2 = 1.6 \times 10^4 \mu\text{eV} \mu\text{m}^2$, $\gamma_R = 0.45\gamma_0$, $R = 8.8 \times 10^{-4} \mu\text{m ps}^{-1}$, $P = 1.1P_{\text{th}}$, $\hbar g_R = 6 \times 10^{-4} \mu\text{m meV}$. The system size is $500 \mu\text{m}$ with lattice parameter $dx = 3.9 \mu\text{m}$. The polariton mass $m = -3.3 \times 10^{-6}m_e$ is negative to obtain a stable condensate despite the attractive effective interaction between polaritons [3]. The nominal value of the noise strength σ is $\sigma_N = \frac{1}{2}\gamma_0$ and is controlled by the reservoir [5,25]. In order to test the validity of the expression (34), we consider the noise amplitude as a free parameter and we allow us to vary it independently of γ_0 . The polariton-polariton interaction strength g is set to zero in Sec. IV D and increased in Sec. IV E.

A. Large-scale behavior: KPZ scaling regime

For the chosen parameters, a KPZ regime was reported both in experiments and numerical simulations [15]. In order to check this, we first compute the first-order correlation function $g^{(1)}(x, t)$ of the condensate. The results for $\mathcal{G}^{(1)}(x, t)$ are shown in Fig. 1 for a system of length $2000 \mu\text{m}$ and a smaller exciton lifetime $\gamma_R = 4.5\gamma_0$, to enlarge the scaling windows for the sake of illustration. We observe the expected scaling behavior (16) of both the equal-time and equal-space correlations with the KPZ exponents. Moreover, the whole set of space-time data points of $\mathcal{G}^{(1)}(x, t)$ lying in the KPZ window perfectly collapses onto the exact KPZ universal scaling function. This confirms that the phase follows a KPZ dynamics in this space-time window.

B. Short-scale behavior: Dependence on lattice spacing

We established in Sec. III B that, within the Bogoliubov approximation, the second derivative of the two-point correlation functions of the phase and density of the condensate both diverge at short distance, which requires the introduction of a UV cutoff $\Lambda \sim 1/\ell$. We here study this aspect, computing numerically C_{mn} and $C_{\theta\theta}$ from the full dynamics (1) and (2) and for different values of the lattice spacing dx . The results obtained for $dx \approx 0.24 \mu\text{m}$ are shown on Fig. 2 (upper panel) and are found to be accurately described by Eqs. (25) and (26) from Bogoliubov approximation.

Second derivatives evaluated at coincident points $x = 0$ of both C_{mn} and $C_{\theta\theta}$ are shown for different lattice spacings on Fig. 2 (lower panel) together with the predictions

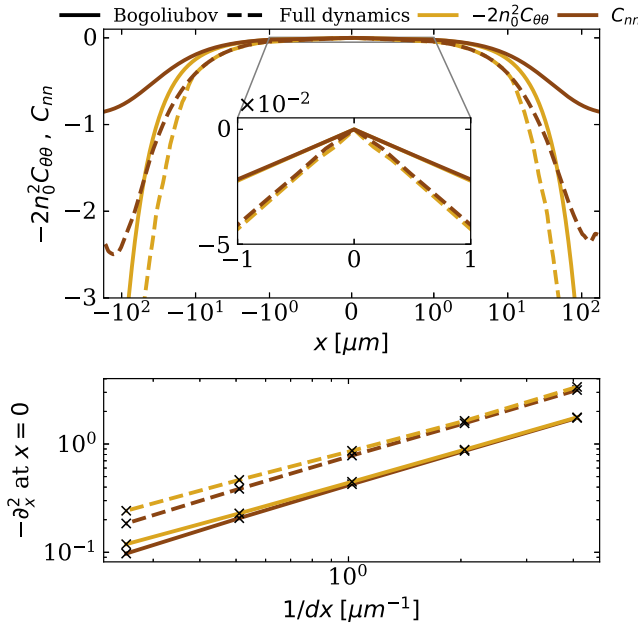


FIG. 2. (Upper panel) Spatial profile of $C_{nn}(x)$ (brown) and $-2n_0^2 C_{\theta\theta}(x)$ (orange) for $dx = \ell = 0.24 \mu\text{m}$ from Bogoliubov theory (solid line) and numerical resolution of the full dynamics (dashed line). (Lower panel) Second derivatives evaluated at $x = 0$ as a function of the inverse lattice spacing $1/dx$, same lines and color code as in the upper panel.

of Eqs. (33) and (32), where we set $\ell = dx$. We show that at $x = 0$, $2n_0^2 \partial_x^2 C_{\theta\theta}$ closely follows $-\partial_x^2 C_{nn}$ and evolves as $1/dx$, confirming the relations derived within Bogoliubov approximation. In fact, results obtained from the nonlinear dynamics merely differ from Eqs. (33) and (32) by a prefactor. Hence, this divergence is not a mere artifact of the Bogoliubov approximation, but is intrinsic to the nonequilibrium gGPE. The divergences also compensate in the full dynamics such that the overall blueshift correction Ω_∞ is observed to be independent of the lattice spacing, as found in Bogoliubov theory [Eq. (34)].

C. Method to extract Ω_∞

In order to extract Ω_∞ , we use the following procedure. The condensate dynamics is solved numerically until the BEC reaches its nonequilibrium steady state at time t_0 , from which the first-order correlation function $g^{(1)}$ and phase fluctuations trajectories defined as $\tilde{\theta}(t + t_0) = [\theta(0, t + t_0) - \theta(0, t_0)] - [\theta_0(t + t_0) - \theta_0(t_0)]$, with $\theta_0(t)$ the mean-field expression, are recorded. The coherence $g^{(1)}$ is used to check for the presence of the KPZ regime and determine the relevant length scales and timescales in which it develops. The blueshift correction Ω_∞ is extracted in the time window in which the KPZ scaling is observed (see inset Fig. 3). This can be done in two ways. The first method is to perform a nonlinear fit of the average of the phase trajectories, using the asymptotic behavior (17) as fitting ansatz. A second method is to reconstruct $\Omega_\infty = -(\partial_t \tilde{\theta})$ from the dynamical equation of the phase (4), by computing each of its contributions separately. An example of numerical measurement of Ω_∞ following both of these procedures is shown on Fig. 3. We verified that

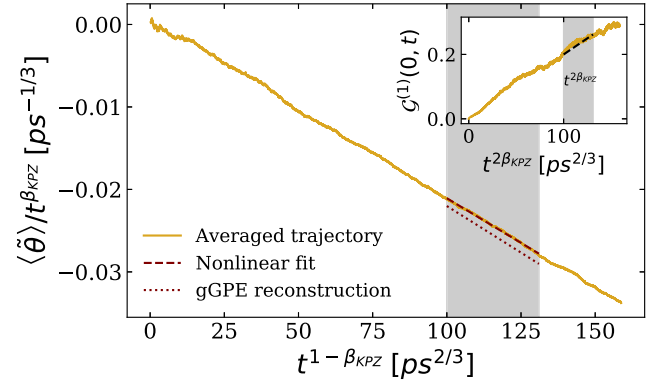


FIG. 3. Determination of Ω_∞ , corresponding to the slope of the average phase trajectory $\langle \tilde{\theta} \rangle$ in the time window where the KPZ scaling is observed in time, for $\sigma = 0.1\sigma_N$. This window is determined from the $g^{(1)}(0, t)$ function, shown in inset (shaded area [10^3 ps, 1.5×10^3 ps]). The result from the nonlinear fit and from the reconstruction of Eq. (4) are shown in dashed and dotted lines, respectively.

the value of Ω_∞ extracted from the nonlinear fit or from the reconstruction of the gGPE coincides within numerical precision in all cases. This consistency check being fulfilled, we only show data coming from the reconstruction of the gGPE in the following. The result can then be compared with the effective KPZ phase dynamics (8), in order to assess the importance of the different terms neglected in the mapping. Note that $\langle \cdot \rangle$ represents here an ensemble average over numerical realizations of the noise, discarding any phase trajectories containing defects. We checked that these defects rarely occur for the chosen parameters. In addition, we verified that the adiabatic approximation for the reservoir density holds in the parameter regime considered. In particular, the relative error committed by estimating the average fluctuation of the reservoir density as $\langle \tilde{n}_R \rangle = -2 \frac{g_i}{R} \langle \tilde{n} \rangle$ is small, such that the condensate depletion contribution to Ω_∞ can be estimated as $\langle g\tilde{n} + 2g_R \tilde{n}_R \rangle \approx -g_e \langle \tilde{n} \rangle_S$, as written in Eq. (29).

D. Blueshift corrections as a function of the noise strength

The activation of phase defects is exponentially sensitive to the noise strength σ [9, 18]. In order to study the dependence on the noise of the blueshift correction while maintaining a defect-free condensate, σ is varied from $10^{-2}\sigma_N$ to σ_N with the integration time step $dt = 0.076$ ps. The results are shown in Fig. 4, and compared with Bogoliubov theory [Eq. (34)]. We find that the numerical estimate for the blueshift correction exhibits a proportionality in the noise strength σ consistent with the prediction (34), with $\Omega_\infty \approx 5 \times 10^{-3} \Omega_0$ in typical experimental conditions. It is, however, larger than the Bogoliubov prediction (34) by an approximate 3.5 factor. This difference cannot be explained by finite-size effects on the asymptotic growth velocity of the interface [23], as we checked that Ω_∞ is already converged for condensates of length $100 \mu\text{m}$. Examining the values of the terms entering in the numerical reconstruction from Eq. (4), we find that the deviation from the theory is instead traced back to density fluctuations, whose contribution dominates the blueshift

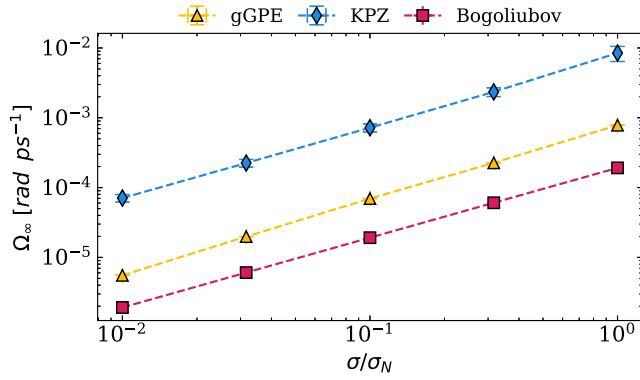


FIG. 4. Ω_∞ as a function of the noise strength σ extracted from the reconstruction of the full dynamics (4) (orange triangle), of the effective KPZ dynamics (8) (blue diamonds), and of the Bogoliubov prediction (34) (purple squares).

correction $g_e \langle \tilde{n} \rangle_S \approx 0.7 \times \Omega_\infty$. We show on Fig. 5 the condensate depletion from both numerical simulation and the Bogoliubov calculation (35). Even though both follow the same trend, the former is larger than the latter by a factor ≈ 2 , hence explaining the result of Fig. 4.

Moreover, we also compare in Fig. 4 the value of Ω_∞ extracted from the full phase dynamics (4) and from the effective KPZ phase dynamics (8). Although this second estimate is proportional to σ as well, its actual values largely differ from both gGPE reconstruction and Eq. (34). This difference originates from the diverging contribution to $\langle (\partial_x \tilde{\theta})^2 \rangle$, regularized by dx in the numerics [see Eq. (32) and Fig. 2], which is removed when considering all the other contributions. This numerically confirms the need of including both density and density-phase terms to correctly describe blueshift corrections.

E. Influence of the interaction strength on blueshift corrections

In this section, the polariton-polariton interaction strength g is varied from 0 to $g_{\max} \simeq 2.6 \times g_R$ in order to vary the KPZ

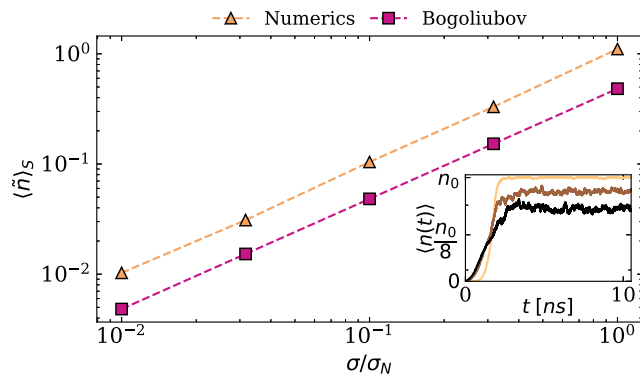


FIG. 5. Stochastic depletion $\langle \tilde{n} \rangle_S$ as a function of the noise strength σ from numerical resolution of the full dynamics (1) (orange triangle) and the Bogoliubov prediction (35) (purple squares). Inset: average density as a function of time for increasing noise (from light to dark $\sigma/\sigma_N = \{10^{-2}, 0.5, 1\}$), where the vertical scale is logarithmic from 0 to $n_0/8$ and linear above. The stationary density $\langle n(t) \rangle$ coincides with the mean-field density n_0 for vanishing noise.

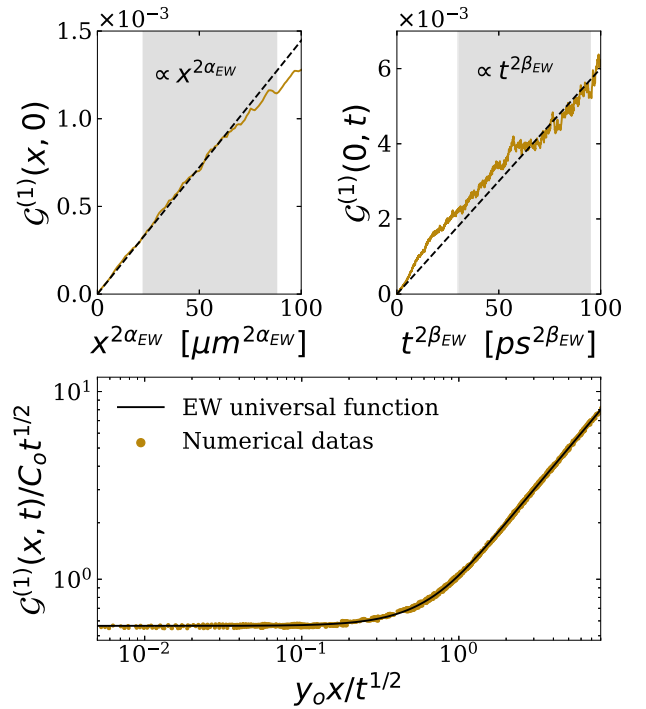


FIG. 6. EW scaling regime reached when $g = g_{\max}$: (upper left panel) spatial scaling of $\mathcal{G}^{(1)}(x, 0)$ as a function of x , indicated by the gray shade in-between $[22 \mu\text{m}, 88 \mu\text{m}]$; (upper right panel) temporal scaling of $\mathcal{G}^{(1)}(0, t)$ as a function of $t^{1/2}$, indicated by the gray shade in-between $[9 \times 10^2 \text{ps}, 9 \times 10^3 \text{ps}]$; (lower panel) collapse of $\mathcal{G}^{(1)}(x, t)$ onto the EW scaling universal function g_{EW} .

nonlinearity. Note that this also induces a change of the other KPZ parameters ν , D , and of the length κ_h , such that the effect of the KPZ nonlinearity on Ω_∞ cannot be isolated by varying the gGPE microscopic parameters. The maximal value of g studied is chosen such that λ vanishes, and the Edwards-Wilkinson limit of the effective phase dynamics (8) is reached. The noise strength is fixed to $\sigma = 0.1\sigma_N$ to avoid any defect and, as the interaction strength increases, the integration time step is gradually decreased from $dt = 0.076 \text{ps}$ to 0.010ps to keep the phase dynamics well resolved in time. Note that $g_e < 0$ for all these values of g , such that the condensate does not exhibit any modulation instability. When the KPZ nonlinearity λ vanishes for $g = g_{\max}$, the scaling form of $\mathcal{G}^{(1)}(x, t)$ changes from C_{hh}^{KPZ} [Eq. (14)] to C_{hh}^{EW} [Eq. (15)]. We show in Fig. 6 that the EW scaling regime is indeed reached when $g = g_{\max}$. For this value, as the KPZ scaling is not visible anymore, Ω_∞ is computed within the temporal window of the EW scaling.

According to the Bogoliubov prediction (34), the blueshift correction Ω_∞ should vary linearly with λ until it vanishes in the EW regime. However, since varying g also induces a variation of ν and D , the global behavior of (34) is more complex. We show in Fig. 7 the variation of the blueshift correction computed both from the Bogoliubov prediction (34) and from reconstruction of the full dynamics (4), as a function of the effective KPZ nonlinearity λ . We observe that it decreases with λ , in an algebraic way not too far from linear. This indicates that the variation of λ is a dominant contribution to

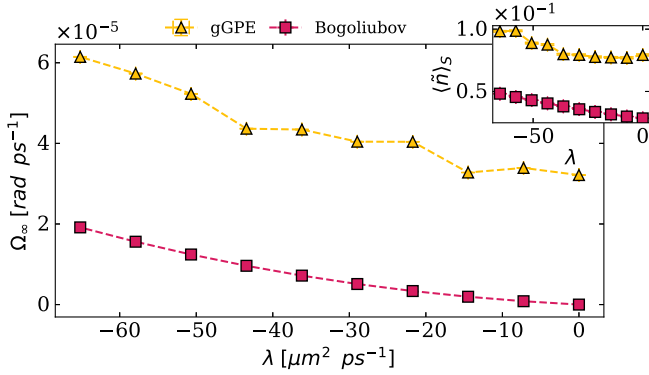


FIG. 7. Ω_∞ as a function of the KPZ nonlinearity λ extracted from the reconstruction of the full dynamics (4) (orange triangle) and from the Bogoliubov prediction (34) (purple squares). Inset: condensate depletion $\langle \bar{n} \rangle_s$ as a function of λ .

the behavior of Ω_∞ when varying the interaction strength g . Moreover, for the Bogoliubov prediction, it vanishes as expected when $\lambda = 0$ and the EW limit is reached. The blueshift correction Ω_∞ evaluated from the full gGPE qualitatively exhibits the same decay with λ as predicted from Eq. (34), but it does not vanish at $\lambda = 0$. As in Sec. IV D, the shift between these two curves can be explained by the discrepancy between the predicted and numerically observed condensate depletion (shown in the inset of Fig. 7).

F. Experimental observability

We restricted so far our theoretical analysis to idealized parameter conditions, in particular small noise levels, in order to damp the emergence of defects in the phase. Under these conditions, the overall stochastic correction to the blueshift Ω_∞ is found to be very small compared to the mean-field contribution Ω_0 , typically of order $\Omega_\infty/\Omega_0 \simeq 5 \times 10^{-3}$, which makes it hardly observable in actual experiments. From the Bogoliubov expression (34), this correction increases linearly with the noise strength, and also with g_R as $\sim g_R^{3/2}$.

The noise level can be increased in experiments in a controllable fashion by increasing the temperature of the cryostat. This leads to a growth of the population of the phonons of the semiconductor material. Such phonons are coupled to the cavity polaritons via the exciton-phonon coupling, and act as an additional bath for the polaritons [42,43]. We performed additional simulations up to twice the nominal noise. In this regime, we found that the KPZ universal properties are resilient to the presence of some defects and for $\sigma = 2\sigma_0$ we obtain $\Omega_\infty/\Omega_0 \simeq 10^{-2}$, which becomes experimentally detectable. For higher values of the noise strength, the number of phase defects rapidly increases as shown in [9,18], making it difficult to distinguish between the linear evolution of the average unwrapped phase caused by blueshift corrections Ω_∞ and the evolution arising from phase defects.

The interaction strength with the reservoir g_R can be also experimentally controlled to some extent by increasing the number of quantum wells in the optical microcavities. However, in our parameter setting, increasing g_R readily leads to modulation instabilities for which the condensate exhibits

many phase defects. It should be emphasized that the estimated values for Ω_∞ as well as the appearance of modulation instability depend on the whole set of parameters of the model. In our simulations, we have chosen parameters close to the experiment of Ref. [15], but by changing the design of the experimental sample and conditions, one could explore a large parameter space, which may allow one to increase blueshift corrections without triggering the modulational instability.

V. CONCLUSIONS

We showed that the blueshift of a 1D exciton-polariton condensate fluctuates around its mean-field value, resulting in a nonzero average correction Ω_∞ . This correction originates from the nonlinear stochastic dynamics of the driven-dissipative condensate and was studied in its defect-free KPZ phase. Using Bogoliubov theory, we determined an analytical expression for Ω_∞ , which, although sensitive to short-distance properties of the condensate correlation functions, can surprisingly be summarized in terms of both the KPZ effective parameters and characteristic length scales of the condensate. We then performed numerical simulations of the full dynamics, varying both the noise and the interaction strength, to check this prediction. We highlighted in particular that spatial variations of the density, which can be safely neglected to study the long-distance properties of the phase, give in contrast an important contribution to the blueshift correction.

With idealized parameter conditions, in particular small noise levels, the overall stochastic correction to the blueshift is very small. However, we verified that our predictions remain valid for larger noise strength up to twice the nominal noise, for which it could become detectable. As a consequence, blueshift corrections could in principle be accessed in experimental platforms where phonon-polariton interaction strength is sufficiently strong. In outlook, it would be interesting to investigate how to enhance this correction in other ways, for example, exploring other regions of the parameter space, in order to make it more easily observable in experiments.

ACKNOWLEDGMENTS

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APPENDIX A: BOGOLIUBOV THEORY

We start from the dynamical equations for the condensate and reservoir equations (1) and (2), setting $\hbar = 1$ for simplicity of notations:

$$i\partial_t \Phi = \left[\mathcal{F}^{-1} \left[\epsilon_{\hat{k}} - \frac{i}{2} \gamma_{\ell}(\hat{k}) \right] + \frac{iR}{2} n_R + g|\Phi|^2 + 2g_R n_R \right] \Phi + \xi, \quad (\text{A1})$$

$$\partial_t n_R = P - (\gamma_R + R|\Phi|^2) n_R, \quad (\text{A2})$$

where $\langle \xi \rangle = 0$ and $\langle \xi(x, t) \xi^*(x', t') \rangle = 2\sigma \delta(x - x') \delta(t - t')$. In the adiabatic approximation of the reservoir $\partial_t n_R \approx 0$, Eq. (A2) reduces to $n_R = P/(\gamma_R + R|\Phi|^2)$. Expanding the wave function around its fluctuations $\Phi = e^{-i\Omega_0 t} [\sqrt{n_0} + \delta\Phi(x, t)]$ and injecting it into Eq. (A1) leads at zeroth order to $n_{R0} = \gamma_0/R$ and $\Omega_0 = gn_0 + 2g_R n_{R0}$. At first order in $\delta\Phi$, Eq. (A1) yields the linear dynamics

$$i\partial_t \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix} = \mathcal{F}^{-1}[\mathcal{L}] \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix} + \sqrt{\sigma} \begin{pmatrix} \xi \\ -\xi^* \end{pmatrix}, \quad (\text{A3})$$

where we introduced $\tilde{\xi} = \xi e^{-i\Omega_0 t}$ and the Bogoliubov matrix

$$\mathcal{L} = \begin{pmatrix} \epsilon_{\hat{k}} + g_e n_0 + i\Gamma_{\hat{k}} & g_e n_0 - ig_i n_0 \\ -g_e n_0 - ig_i n_0 & -(\epsilon_{\hat{k}} + g_e n_0) - i\Gamma_{\hat{k}} \end{pmatrix}, \quad (\text{A4})$$

with $\Gamma_{\hat{k}} = \frac{\Delta\gamma(\hat{k})}{2} - g_i n_0$, $\Delta\gamma(\hat{k}) = Rn_{R0} - \gamma_\ell(\hat{k})$, and $g_i = (Rn_{R0})^2/2P$.

We introduce the rotation matrix

$$\mathcal{A} = \frac{1}{2\sqrt{n_0}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \quad (\text{A5})$$

such that

$$\begin{pmatrix} \delta n/2n_0 \\ \delta\theta \end{pmatrix} = \mathcal{A} \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix}. \quad (\text{A6})$$

The dynamics of the density and phase is given by

$$(\partial_t - \mathcal{F}^{-1}[\mathcal{L}_{\text{rot}}]) \begin{pmatrix} \delta n/2n_0 \\ \delta\theta \end{pmatrix} = \sqrt{\frac{\sigma}{n_0}} \begin{pmatrix} \text{Im}(\tilde{\xi}) \\ -\text{Re}(\tilde{\xi}) \end{pmatrix} \quad (\text{A7})$$

with

$$\mathcal{L}_{\text{rot}} = \begin{pmatrix} \Gamma_{\hat{k}} - g_i n_0 & \epsilon_{\hat{k}} \\ -(\epsilon_{\hat{k}} + 2g_e n_0) & \Gamma_{\hat{k}} + g_i n_0 \end{pmatrix} \quad (\text{A8})$$

whose eigenvalues read as

$$\omega_{\pm} = -i\Gamma_{\hat{k}} \pm \sqrt{\epsilon_{\hat{k}}(\epsilon_{\hat{k}} + 2g_e n_0) - g_i^2 n_0^2}. \quad (\text{A9})$$

From Eq. (A7), and after integrating over the frequency ω , we obtain the equal-time correlation functions

$$\langle \delta n(k) \delta n(-k) \rangle = \frac{-\sigma n_0}{\Gamma_{\hat{k}}} \frac{\epsilon_{\hat{k}}^2 + \Delta\gamma(\hat{k})^2/4}{E_{\hat{k}}^2 + \Gamma_{\hat{k}}^2 - g_i^2 n_0^2} - \frac{\sigma n_0}{\Gamma_{\hat{k}}}, \quad (\text{A10})$$

$$\langle \delta\theta(k) \delta\theta(-k) \rangle = \frac{\langle \delta n(k) \delta n(-k) \rangle}{4n_0^2} + \frac{\sigma}{\Gamma_{\hat{k}} n_0} \times \frac{\Gamma_{\hat{k}} g_i n_0 - \epsilon_{\hat{k}} g_e - g_e^2 n_0^2}{E_{\hat{k}}^2 + \Gamma_{\hat{k}}^2 - g_i^2 n_0^2}, \quad (\text{A11})$$

$$\langle \delta n(k) \delta\theta(-k) \rangle = -\frac{\sigma n_0}{\Gamma_{\hat{k}}} \frac{g_i \epsilon_{\hat{k}} + g_e \Delta\gamma(\hat{k})}{E_{\hat{k}}^2 + \Gamma_{\hat{k}}^2 - g_i^2 n_0^2}, \quad (\text{A12})$$

where $E_{\hat{k}} = \sqrt{\epsilon_{\hat{k}}(\epsilon_{\hat{k}} + 2g_e n_0)}$.

APPENDIX B: EFFECT OF THE k -DEPENDENT DISSIPATION

The momentum dependence of the loss rate $\gamma_\ell(\hat{k})$ affects the large-momentum behavior of the Bogoliubov

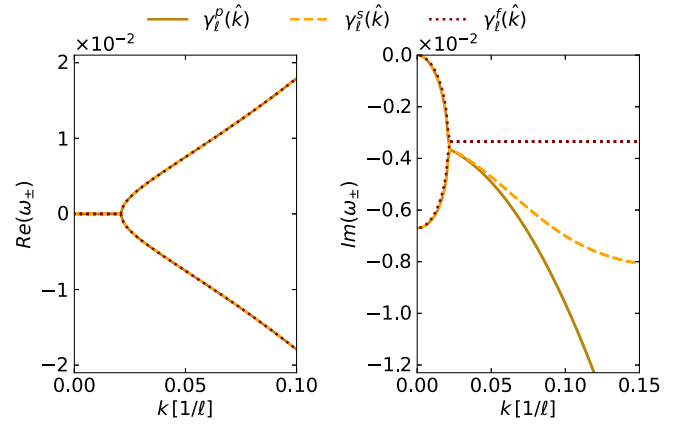


FIG. 8. Real (left panel) and imaginary (right panel) parts of the Bogoliubov spectrum (A9) for different losses $\gamma_\ell(\hat{k})$. Red dotted line: flat dissipation $\gamma_\ell^f(\hat{k}) = \gamma_0$. Brown plain line: parabolic dissipation $\gamma_\ell^p(\hat{k}) = \gamma_0 + \gamma_2 k^2$. Orange dashed line: saturated dissipation $\gamma_\ell^s(\hat{k})$ defined in Eq. (B1), with $\gamma_{\text{sat}} = 55 \mu\text{eV}$.

eigenvalues (A9) through the effective loss rate $\Gamma_{\hat{k}}$, similarly to the frequency-dependent amplification of Ref. [33]. This is illustrated on Fig. 8 below, where the Bogoliubov eigenvalues are shown for three experimentally relevant loss rates. While the momentum-independent loss rate $\gamma_\ell^f(\hat{k}) = \gamma_0$ is most of the time sufficient to describe the phenomenology of EP BECs, it is rather observed that the exciton-polariton linewidth broadens quadratically at small k before saturating to γ_{sat} at large momentum, which is modeled by the following saturated loss rate:

$$\gamma_\ell^s(\hat{k}) = \gamma_0 \frac{1 + \alpha}{e^{-\beta k^2} + \alpha}, \quad (\text{B1})$$

where $\alpha = \frac{\gamma_0}{\gamma_{\text{sat}} - \gamma_0}$ and $\beta = \frac{\gamma_2}{\gamma_0}(1 + \alpha)$ [15]. Expanding (B1) at small k , the polariton linewidth can be approximated by a parabola $\gamma_\ell^s(\hat{k}) \approx \gamma_\ell^p(\hat{k}) = \gamma_0 + \gamma_2 k^2$.

Let us now discuss the consequences of the shape of the loss rate on blueshift corrections. Replacing $\epsilon_{\hat{k}} = \hbar^2 k^2/2m$ and $\gamma_\ell^f(\hat{k})$ in Eqs. (A10)–(A12), we recover the equal-time correlations given in [33]. Inserting them in Eq. (31) we find that their UV diverging contributions (discussed in the main text and Ref. [33]) compensate such that $\Omega_\infty = \kappa_h D$ remains well defined. This leads to the estimate $\Omega_\infty \approx \Omega_0/10$ for $\gamma_\ell^f(\hat{k})$. A similar value is expected from the saturated linewidth $\gamma_\ell^s(\hat{k})$. In this work, as in experimental realizations of EP condensates, the pump P extends over a large number of sites. As a consequence, the exciton reservoir, hence the condensate, are occupied in the vicinity of the momentum $k = 0$ only. This implies that the parabolic linewidth $\gamma_\ell^p(\hat{k})$ is the most relevant to describe losses in the condensate.

APPENDIX C: BLUESHIFT CORRECTIONS IN HIGHER DIMENSIONS

The Bogoliubov approximation developed in Sec. III and Appendix A can be extended to higher dimensions by replacing $\partial_x \rightarrow \nabla$ and $k \rightarrow \mathbf{k}$. Note that for $\epsilon_{\hat{k}} = \hbar^2 k^2/2m$ and independently of the specific choice of $\gamma_\ell(\hat{k})$, the

connected phase correlation function $C_{\theta\theta}(\mathbf{r})$ contains a term $\propto \int \frac{dk}{(2\pi)^d} \frac{e^{ikr}-1}{k^2}$. This term yields an Edwards-Wilkinson contribution $C^{\text{EW}}(\mathbf{r}, 0)$ to the phase spatial correlations, of the form $C^{\text{EW}}(\mathbf{r}, 0) \sim |\mathbf{r}|^{2\alpha}$ where $\alpha = \frac{2-d}{2}$ is the EW spatial exponent. Similarly in time, one would find $C^{\text{EW}}(0, t) \sim t^{2\beta}$ with $\beta = \frac{2-d}{4}$. In a one-dimensional system, this gives a stretched exponential decay of the first-order correlation function $|g^{(1)}(r, t)|^2 \propto e^{-\sqrt{t}}$ followed for a finite-size system by Schawlow-Townes exponential decay $|g^{(1)}(r, t)|^2 \propto e^{-t}$ [13]. As commented in the main text, a similar exponential decay is obtained for equilibrium condensates at finite temperature [17].

In addition, the UV divergences mentioned in Sec. III for $\gamma_\ell(\hat{\mathbf{k}}) = \gamma_0 + \gamma_2 \hat{\mathbf{k}}^2$ become more severe when increasing the spatial dimension. Bogoliubov theory remains valid provided $\langle (\nabla \delta\theta)^2 \rangle \propto \frac{\sigma}{n_0 \gamma_2 \ell^d}$ is a small parameter. In fact, if we first set $\gamma_2 = 0$ and perform again the calculation, we recover the same condition for the validity of Bogoliubov theory as in Ref. [34], namely, that $\langle (\ell \nabla \delta\theta)^2 \rangle \propto \frac{\sigma}{n_0 \ell^d}$ is a small parameter, where the driven-dissipative nature of the condensate explicitly appears through the noise strength σ . Nevertheless, Ω_∞ remains well defined in the UV regime for $d \leq 3$ and follows the general expression

$$\Omega_{\infty, d} = -\frac{\lambda D}{2v} \kappa_h^2 \kappa_d^2 \int \frac{dk/(2\pi)^d}{(k^2 + \kappa_h^2)(k^2 + \kappa_d^2)}. \quad (\text{C1})$$

Setting $d = 1$ in Eq. (C1), we recover Eq. (34) of the main text.

In $d > 1$, the linearized theory cannot capture any effect from the KPZ physics. This is because the KPZ fixed point becomes genuinely nonperturbative in $d > 1$ [44]. It was shown that the perturbation theory in λ fails to all orders to access the KPZ fixed point [45]. Thus, the results obtained from Bogoliubov theory in $d > 1$ cannot be related to KPZ physics. We still express them for convenience in terms of the KPZ parameters, although this should not be interpreted as resulting from nonlinear KPZ physics.

For $d = 2$ we find by parity that $\Omega_\infty^{d=2} = 0$, thus the stochastic dynamics gives no correction to the blueshift within the Bogoliubov approximation. The case $d = 3$ is of fundamental interest for EP BECs, as the driven-dissipative condensate is predicted to recover some characteristics of its equilibrium counterpart [46]. For example, it was shown in Ref. [33] within Bogoliubov approximation that the IR divergence of the momentum distribution, which is a manifestation of the Mermin-Wagner-Hohenberg theorem for $d < 2$, is suppressed for $d = 3$. Setting $d = 3$ in Eq. (C1), we find $\Omega_\infty^{d=3} = \frac{\lambda D}{8\pi v} \frac{\kappa_h \kappa_d}{\kappa_h^{-1} + \kappa_d^{-1}}$. In typical parameter regimes, this blueshift correction is approximately 500 times smaller than for $d = 1$. This is consistent with the idea that fluctuations are enhanced in low-dimensional systems.

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