Quantum theory of spin transfer and spin pumping in collinear antiferromagnets and ferrimagnets

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Antiferromagnets are promising candidates as active components in spintronic applications. They share features with ferrimagnets in that opposing spin orientations exist in two or more sublattices. Spin-transfer torque and spin pumping are essential ingredients in antiferromagnetic and ferrimagnet spintronics. This paper develops an out-of-equilibrium quantum theory of the spin dynamics of collinear magnets containing many spins coupled to normal metal reservoirs. At equilibrium, the spins are parallel or antiparallel to the easy axis. The theory, therefore, covers collinear antiferromagnets and ferrimagnets. We focus on the resulting semiclassical spin dynamics. The dissipation in the spin dynamics is enhanced due to spin pumping. Spin accumulations in the normal metals induce deterministic spin-transfer torques on the magnet. Additionally, each electron's discrete spin angular momentum causes stochastic fluctuating torques on the antiferromagnet or ferrimagnet. We derive these fluctuating torques. The fluctuation-dissipation theorem holds at high temperatures, including the effects of spin pumping. At low temperatures, we derive shot noise contributions to the fluctuations.

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I. INTRODUCTION

Spin-transfer torque (STT) and spin pumping (SP) are essential ingredients in the generation and detection of spin currents and are central components in modern spintronics research and devices [1]. The use of magnetic insulators enables signal propagation without moving charges and could provide low-dissipation and ultrafast memory devices [2]. Initially, much of spintronic research focused on the study of STT [3–5] and SP [6–8] in ferromagnets (FMs). Subsequently, this included also studies on fluctuations [9–11] and pumped magnon condensates [12–14].

Unlike FMs, whose macroscopically apparent magnetic properties have been known for thousands of years, antiferromagnets (AFMs) carry zero net magnetic moments and were elusive for some time. Even after their discovery, AFMs were believed to have few potential applications [15] and were disregarded in the early days of spintronics research. Recent theoretical and experimental findings have highlighted the potential of using AFMs in spintronics applications, thus starting the field of antiferromagnetic spintronics. Key discoveries were the robustness of AFMs to external magnetic perturbations and the high resonance frequency of antiferromagnetic material [16,17]. The prediction [18] and subsequent experimental detection [19] of an STT in AFMs sparked a massive interest in using AFMs as the active component in spintronics devices [16,20]. Moreover, it was predicted that contrary to what was believed, antiferromagnets are as efficient in pumping spin currents as FMs [21]. This effect was later experimentally detected in the easy-axis AFM MnF_2 [22]. These discoveries opened up the possibility of utilizing AFMs in spintronic applications, enabling the possible fabrication of stray-field-free devices operating in the THz regime [16,23], allowing for much faster device operation than in FMs.

In recent years, the spin dynamics in AFMs have been explored extensively, including the effects of disorder [24], generation of spin-Hall voltages [25], and the properties of antiferromagnetic skyrmions [26]. The spin dynamics of ferrimagnetic materials have also been studied [27]. Phenomenological models of intra- and cross-lattice torques were introduced in [28]. Reference [29] further discusses the competition between intra- and cross-sublattice spin pumping in specific models of antiferromagnets.

As in antiferromagnets, ferrimagnets have opposing magnetic moments. However, these moments have different magnitudes, resulting in a net magnetization. These features result in rich spin dynamics ranging from behavior reminiscent of antiferromagnets to ferromagnets. A prime example of a ferrimagnet is yttrium-iron-garnet (YIG). The low-energy magnon modes in YIG resemble modes in ferromagnets.

In the study of nonequilibrium effects, the Keldysh path integral approach to nonequilibrium quantum field theory is a powerful tool in the study of nonequilibrium systems beyond linear response [30,31]. Although most of the research on STT and SP utilized a semiclassical approach, some studies have used the Keldysh framework in the study of spin dynamics in FMs out of equilibrium [10,11,32–34]. Moreover, the Keldysh method was recently used to formalize a fully quantum mechanical theory of STT and SP, including the effects of quantum fluctuations [35]. These fluctuations have become increasingly relevant with the development of new devices operating in the low-temperature regime. Nevertheless, applying the Keldysh method to derive microscopic relations for SP, STT, and fluctuating torques in an AFM system is lacking.

In this paper, we extend the approach of Ref. [35], which examined a ferromagnet in the macrospin approximation coupled to normal metals featuring spin and charge accumulation, to a similar system but instead featuring a collinear magnet with many individual spins coupled at different sublattices to normal metals. Our study thus covers antiferromagnets, ferrimagnets, and ferromagnets. We derive the spin dynamics using a fully quantum mechanical Keldysh nonequilibrium approach. We find expressions for the spin-transfer torque, spin-pumping-induced Gilbert damping, and fluctuating fields, including low-temperature shot-noise contributions. The Gilbert damping and fluctuations contain both interlattice and intralattice terms. Using Onsager reciprocal relations, we relate the spin-pumping and spin-transfer coefficients. Our results enhance the knowledge of the microscopic expressions of STT and SP and fluctuating torques in antiferromagnets and ferrimagnets coupled to normal metals in the low-energy regime, where quantum fluctuations become essential.

The subsequent sections of this paper are structured as follows. In Sec. II, we introduce the model employed for the itinerant electrons in the normal metals, the localized magnetic moments in the antiferromagnetic or ferrimagnet, and the electron-magnon coupling between them. We then present the key findings of this paper in Sec. III, including microscopic definitions of the spin-transfer torque, spin pumping, and fluctuating torques in many spin magnets, being an antiferromagnetic, ferrimagnet, or ferromagnet. The derivation of an effective magnon action, achieved by integrating fermionic degrees of freedom resulting from the interaction with normal metals, is detailed in Sec. IV. The evaluation of this effective action is then provided in Sec. V. Finally, Sec. VI concludes the paper.

II. MODEL

We consider a bipartite collinear magnet coupled to an arbitrary number of normal metal reservoirs. The magnet can represent an antiferromagnet, a ferrimagnet, or a ferromagnet. The total Hamiltonian is

$$\hat{H} = \hat{H}_e + \hat{H}_{em} + \hat{H}_m \tag{1}$$

in terms of the Hamiltonian describing the electrons in the normal metal $\hat{H}_e$, the Hamiltonian describing the interaction between the electrons and the magnet $\hat{H}_{em}$, and the Hamiltonian of the magnet $\hat{H}_m$.

The Hamiltonian of the electrons combined with the Hamiltonian representing the interaction between the electrons and the magnet is

$$\hat{H}_e + \hat{H}_{em} = \int d\mathbf{r}\hat{\psi}^\dagger\left[H_e + \hbar^{-1}\sum_i u_i\boldsymbol{\sigma}\cdot\hat{\mathbf{S}}_i\right]\hat{\psi}\,, \tag{2}$$

where $\hat{\psi}^\dagger = (\hat{\psi}_\uparrow^\dagger, \hat{\psi}_\downarrow^\dagger)$ is the spatially dependent two-component itinerant electron field operator, and $\boldsymbol{\sigma}$ is the vector of Pauli matrices in the 2×2 spin space. In the Hamiltonian (2), $u_i(\boldsymbol{r})$ represents the spatially dependent exchange interaction between the localized spin at site i and the itinerant electrons. This interaction is localized around spin i inside the magnet. The sum over the localized spins i consists of a sum over sites in sublattice $\mathcal{A}$ and sublattice $\mathcal{B}$, i.e., $\sum_i \ldots \to \sum_a \ldots + \sum_b \ldots$. The localized spin operator $\hat{\boldsymbol{S}}_i$ has a total spin angular momentum $S_i = \hbar\sqrt{s_i(s_i+1)}$ where s_i is the (unitless) spin quantum number of the localized spin, such that $\hat{\boldsymbol{S}}_i^{\,2} = \hbar^2 s_i(s_i+1)$. For large s_i, the difference between $S_i/\hbar$ and s_i is a first-order correction, and we can approximate $S_i \approx \hbar s_i$.

The spin-independent part of the single-particle electron Hamiltonian is

$$H_e = -\frac{\hbar^2}{2m}\nabla^2 + V_c\,, \tag{3}$$

where V_c is the spatially dependent charge potential.

In the classical limit of the magnet, the spins at sublattice $\mathcal{A}$ are along a certain direction and the spins at sublattice $\mathcal{B}$ are along the opposite direction in the ground state. We will consider the semiclassical spin dynamics near the instantaneous classical direction of the spins that we let be along the z direction and adiabatically adjust the evolution of the small deviation [10,11,35]. In the following, it is constructive to expand the interaction term to the second order in the magnet creation/annihilation operators using a Holstein-Primakoff transformation,

$$\hat{H}_{em} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2\,, \tag{4}$$

where $\hat{H}_0$ is the interaction with the classical magnetic ground state and $\hat{H}_1$ ($\hat{H}_2$) is the interaction term to the first (second) order. The classical ground-state contribution to the interaction is then

$$\hat{H}_0 = \int d\mathbf{r}\hat{\psi}^\dagger V_s\sigma_z\hat{\psi}\,, \tag{5}$$

where the magnitude of the spatially dependent spin potential experienced by the itinerant electrons is

$$V_s(\boldsymbol{r}) = \sum_a s_a u_a(\boldsymbol{r}) - \sum_b s_b u_b(\boldsymbol{r})\,, \tag{6}$$

and oscillates rapidly with the staggered field.

In the macrospin approximation, $\sum_i \boldsymbol{S}_i$ can be treated as a giant spin in ferromagnets. Then, $u_i(\boldsymbol{r})$ becomes the effective exchange interaction. Reference [35] shows how the electronic Hamiltonian $\hat{H}_e$ combined with the electron-magnon Hamiltonian to zeroth-order $\hat{H}_0$ become particularly transparent in ferromagnet-normal metal systems in terms of the scattering states of the itinerant electrons for the macrospin dynamics. We generalize this approach to magnet-normal metal systems with individual localized spins. In this picture, the electronic Hamiltonian remains simple, as in Ref. [35],

$$\hat{H}_e + \hat{H}_0 = \sum_{s\alpha}\epsilon_\alpha \hat{c}_{s\alpha}^\dagger\hat{c}_{s\alpha}\,, \tag{7}$$

where $\hat{c}_{s\alpha}$ annihilates an electron with spin s ($s=\uparrow$ or $s=\downarrow$). The quantum number $\alpha = \kappa n\epsilon$ captures the lead κ, the transverse waveguide mode n, and the electron energy ϵ. The electron energy consists of a transverse contribution ϵ_n and a longitudinal contribution $\epsilon(k) = k^2/2m$, where k is the longitudinal momentum, such that $\epsilon = \epsilon_n + \epsilon(k)$. The eigenenergy is spin degenerate, since the leads are paramagnetic. Furthermore, we consider identical leads such that the eigenenergy is independent of the lead index. The system setup is shown in Fig. 1 for the case of two leads. In Eq. (7) and similar expressions to follow, the sum over the scattering states implies that $\sum_\alpha X_{s\alpha} = \sum_{\kappa n}\int_{\epsilon_n}^\infty d\epsilon X_{s\kappa n}(\epsilon)$. In the scattering approach, the field operator is

$$\hat{\psi}_s = \sum_\alpha \hat{c}_{s\alpha}\psi_{s\alpha}\,, \tag{8}$$

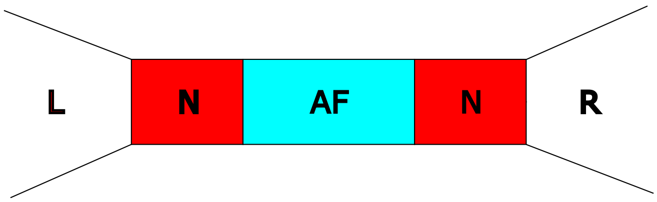

FIG. 1. An antiferromagnet (AF) with conductors (N) on either side connected to a right (R) and a left (L) lead.

where $\psi_{s\alpha}(\boldsymbol{r})$ is the wave function of a scattering state of spin s and quantum number α.

The Hamiltonian of the antiferromagnet is [i denotes a site at sublattice $\mathcal{A}$ ($i = a$) or $\mathcal{B}$ ($i = b$)]

$$H_m = \hbar^{-2}\sum_{ij} J_{ij}\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - K\hbar^{-2}\sum_i (\hat{\mathbf{S}}_i \cdot z)^2 + \gamma\mu_0 \sum_a \boldsymbol{H}^A_a \cdot \hat{\boldsymbol{S}}_a + \gamma\mu_0 \sum_b \boldsymbol{H}^B_b \cdot \hat{\boldsymbol{S}}_b , \quad (9)$$

where J_{ij} is the symmetric exchange interaction, $K > 0$ is the easy-axis anisotropy energy, and $\gamma = g^*\mu_B/\hbar$ is the (absolute value of) the effective gyromagnetic ratio, where g^* is the effective Landé g factor and μ_B is the Bohr magneton. In Eq. (9), $\boldsymbol{H}^{A,B}_i$ is the external magnetic field in units of Am^{-1} at lattice site $i = \{a, b\}$, and μ_0 is the vacuum permeability, which appears because we are employing SI units. In reality, $\boldsymbol{H}^A = \boldsymbol{H}^B$ in the presence of a uniform external magnetic field. However, to illustrate and understand the physics, we allow the external fields at sublattices $\mathcal{A}$ and $\mathcal{B}$ to differ, and to depend on the lattice site.

We consider the low-energy excitations from the semiclassical ground state of the staggered spin orientation. To this end, we carry out a Holstein-Primakoff expansion to the second order in magnon excitations at each sublattice $\mathcal{A}$ and $\mathcal{B}$ described via the annihilation operators $\hat{a}_a$ and $\hat{b}_b$ as detailed in Appendix A. Introducing the raising/lowering fields as $H_\pm = H_x \pm iH_y$, the magnon Hamiltonian becomes

$$\begin{aligned} H_m = {} & E_0 + \sum_a E^A_a \hat{a}^\dagger_a \hat{a}_a + \sum_b E^B_b \hat{b}^\dagger_b \hat{b}_b \\ & + 2\sum_{aa'} J_{aa'}\sqrt{s_a s_{a'}}\,\hat{a}^\dagger_a \hat{a}_{a'} + 2\sum_{bb'} J_{bb'}\sqrt{s_b s_{b'}}\,\hat{b}^\dagger_b \hat{b}_{b'} \\ & + 2\sum_{ab} J_{ab}\sqrt{s_a s_b}[\hat{a}_a \hat{b}_b + \hat{a}^\dagger_a \hat{b}^\dagger_b] \\ & + \gamma\mu_0\hbar \sum_a \sqrt{\frac{s_a}{2}}\big[H^A_{a-}\hat{a}_a + H^A_{a+}\hat{a}^\dagger_a\big] \\ & + \gamma\mu_0\hbar \sum_b \sqrt{\frac{s_b}{2}}\big[H^B_{b-}\hat{b}^\dagger_b + H^B_{b+}\hat{b}_b\big] , \end{aligned} \quad (10)$$

where the classical ground-state energy E_0 is

$$\begin{aligned} E_0 = {} & \sum_{aa'} s_a s_{a'} J_{aa'} + \sum_{bb'} s_b s_{b'} J_{bb'} - 2\sum_{ab} s_a s_b J_{ab} \\ & - 2K\sum_i s_i^2 + \hbar\mu_0 \sum_a s_a H^A_{az} - \hbar\mu_0 \sum_b s_b H^B_{bz} , \end{aligned} \quad (11)$$

and is disregarded in the following;

$$\begin{aligned} E^{A(B)}_{a(b)} = {} & 2\sum_{b(a)} s_{b(a)} J_{ab} - \sum_{a'(b')} s_{a'(b')} J_{a(b)a'(b')} \\ & + 2s_{a(b)}K \mp \hbar\gamma\mu_0 H^{A(B)}_{a(b)z} \end{aligned} \quad (12)$$

is the energy of a local excitation, where the upper sign holds for sites on sublattice $\mathcal{A}$ and the lower sign holds for sites on sublattice $\mathcal{B}$.

In the scattering basis of the electronic states, the corrections to the antiferromagnetic ground-state electron-magnon interaction to quadratic order in the magnet operators becomes $\hat{H}_{em} - \hat{H}_0 = \hat{H}_1 + \hat{H}_2$. The first-order contribution of electron-magnon interaction is

$$\begin{aligned} \hat{H}_1 = {} & \sum_{a\alpha\beta}\sqrt{\frac{2}{s_a}}\big[\hat{a}_a \hat{c}^\dagger_{\downarrow\alpha} W^{\alpha\beta}_{a\downarrow\uparrow}\hat{c}_{\uparrow\beta} + \hat{a}^\dagger_a \hat{c}^\dagger_{\uparrow\alpha} W^{\alpha\beta}_{a\uparrow\downarrow}\hat{c}_{\downarrow\beta}\big] \\ & + \sum_{b\alpha\beta}\sqrt{\frac{2}{s_b}}\big[\hat{b}^\dagger_b \hat{c}^\dagger_{\downarrow\alpha} W^{\alpha\beta}_{b\downarrow\uparrow}\hat{c}_{\uparrow\beta} + \hat{b}_b \hat{c}^\dagger_{\uparrow\alpha} W^{\alpha\beta}_{b\uparrow\downarrow}\hat{c}_{\downarrow\beta}\big] , \end{aligned} \quad (13)$$

and describes the spin-flip scattering of the itinerant electrons associated with creating or annihilating localized magnons. The dimensionless matrix W_i is governed by the exchange potential $u_i(\boldsymbol{r})$ and the scattering states wave functions $\psi_{s\alpha}$,

$$W^{\alpha\beta}_{iss'} = \int d\boldsymbol{r}\,\psi^*_{s\alpha}(\boldsymbol{r}) s_i u_i(\boldsymbol{r}) \psi_{s'\beta}(\boldsymbol{r}) , \quad (14)$$

and is Hermitian, $W^{\alpha\beta}_{i\uparrow\downarrow} = [W^{\beta\alpha}_{i\downarrow\uparrow}]^*$. The electron-magnon interaction that is second order in the magnon operators is

$$\begin{aligned} \hat{H}_2 = {} & -\sum_{a\alpha\beta}\frac{\hat{a}^\dagger_a \hat{a}_a}{s_a}\big[\hat{c}^\dagger_{\uparrow\alpha} W^{\alpha\beta}_{a\uparrow\uparrow}\hat{c}_{\uparrow\beta} - \hat{c}^\dagger_{\downarrow\alpha} W^{\alpha\beta}_{a\downarrow\downarrow}\hat{c}_{\downarrow\beta}\big] \\ & + \sum_{b\alpha\beta}\frac{\hat{b}^\dagger_b \hat{b}_b}{s_b}\big[\hat{c}^\dagger_{\uparrow\alpha} W^{\alpha\beta}_{b\uparrow\uparrow}\hat{c}_{\uparrow\beta} - \hat{c}^\dagger_{\downarrow\alpha} W^{\alpha\beta}_{b\downarrow\downarrow}\hat{c}_{\downarrow\beta}\big] , \end{aligned} \quad (15)$$

where the matrix elements are defined in Eq. (14). We note that our electron-magnon interaction is isotropic in spin space and will give rise to zeroth-, first-, and second-order magnon terms in the Hamiltonian, i.e., $\hat{H}_0$, $\hat{H}_1$, and $\hat{H}_2$, respectively. This is in contrast to the model used in Refs. [12,32], where only the first-order term $\hat{H}_1$ is considered.

Finally, in normal metal reservoirs, the occupation of the state is

$$\langle c^\dagger_{s'\alpha} c_{s\beta}\rangle = \delta_{\alpha\beta} n_{ss'\alpha} , \quad (16)$$

where the 2×2 out-of-equilibrium distribution is

$$\begin{aligned} n_{ss'\alpha} = {} & \tfrac{1}{2}[f_{\kappa\uparrow}(\epsilon_\alpha) + f_{\kappa\downarrow}(\epsilon_\alpha)]\delta_{ss'} \\ & + \tfrac{1}{2}[f_{\kappa\uparrow}(\epsilon_\alpha) - f_{\kappa\downarrow}(\epsilon_\alpha)]\mathbf{u}_\kappa \cdot \boldsymbol{\sigma}_{ss'} , \end{aligned} \quad (17)$$

allowing for a (lead-dependent) spin accumulation in the direction of the unit vector $\boldsymbol{u}_k$. $f_\uparrow$ and $f_\downarrow$ are general distribution functions for spin-up and spin-down particles, which generally differ for elastic or inelastic transport [35]. In equilibrium, the distribution function only depends on energy,

$$f^{\mathrm{eq}}_{\kappa\uparrow}(\epsilon) = f^{\mathrm{eq}}_{\kappa\downarrow}(\epsilon) = f(\epsilon - \mu_0) , \quad (18)$$

where f is the equilibrium Fermi-Dirac distribution and μ_0 is the equilibrium chemical potential.

In inelastic transport, the spin and charge accumulations μ^C and μ^S correspond to chemical potential in a (spin-dependent) Fermi-Dirac function,

$$f^{\text{in}}_{\kappa\uparrow}(\epsilon) = f\big(\epsilon - \mu_0 - \mu^C_\kappa - \mu^S_\kappa/2\big), \tag{19a}$$

$$f^{\text{in}}_{\kappa\downarrow}(\epsilon) = f\big(\epsilon - \mu_0 - \mu^C_\kappa + \mu^S_\kappa/2\big). \tag{19b}$$

For notational simplicity, we define the chemical potentials

$$\mu_{\kappa\uparrow} = \mu_0 + \mu^C_\kappa + \mu^S_\kappa/2, \tag{20a}$$

$$\mu_{\kappa\downarrow} = \mu_0 + \mu^C_\kappa - \mu^S_\kappa/2. \tag{20b}$$

In the limit of small charge and spin accumulations compared to the Fermi level, it can be derived that

$$\mu^C_\kappa + \frac{\mu^S_\kappa}{2} = \int d\epsilon\big[f^{\text{in}}_{\kappa\uparrow}(\epsilon) - f(\epsilon)\big]. \tag{21}$$

In the elastic regime, the distribution function cannot generally be described as a Fermi-Dirac function. The distribution function is instead given as a linear combination of Fermi-Dirac functions in the connected reservoirs [35],

$$f^{el}_{s\kappa}(\epsilon) = \sum_l R_{s\kappa l} f(\epsilon - \mu_l), \tag{22}$$

where the index l runs over the reservoirs, and $R_{s\kappa l}$ is the lead and spin-dependent transport coefficient for reservoir l. The transport coefficients satisfy

$$\sum_l R_{s\kappa l} = 1. \tag{23}$$

In the elastic transport regime, it is advantageous to define the *effective* charge and spin accumulations through

$$\mu^C_\kappa + \frac{\mu^S_\kappa}{2} = \int d\epsilon\big[f^{\text{el}}_{\kappa\uparrow}(\epsilon) - f(\epsilon)\big]. \tag{24}$$

The elastic and inelastic transport regime results in different results for the fluctuations in the magnetization dynamics of the magnet.

Having specified the model for the system in consideration, we proceed by presenting the main results of the paper.

III. MAIN RESULTS: EQUATIONS OF MOTION

This section presents the main results of our paper. Our primary result is the derivation of a Landau–Lifshitz–Gilbert–Slonczewski (LLGS) equation for the localized (normalized) spins $\boldsymbol{m}_i = \boldsymbol{S}_i/S_i$ in a general magnet coupled to normal metal reservoirs,

$$\partial_t \boldsymbol{m}_i = \boldsymbol{\tau}^{\text{b}}_i + \boldsymbol{\tau}^{\text{f}}_i + \boldsymbol{\tau}^{\text{sp}}_i + \boldsymbol{\tau}^{\text{stt}}_i \tag{25}$$

valid for low-energy excitations when the equilibrium magnetization is parallel (antiparallel) to the z axis. The bulk antiferromagnet torque $\boldsymbol{\tau}^{\text{b}}_i$ for a site $i = \{a, b\}$ arises from contributions of anisotropy, exchange coupling, and external fields, and reads

$$\boldsymbol{\tau}^{\text{b}}_i = -\boldsymbol{z} \times (\hbar^{-1} E_i \boldsymbol{m}_i + \gamma\mu_0 \boldsymbol{H}_i), \tag{26}$$

where E_i is the energy of a local excitation and $\boldsymbol{H}_i$ is the applied field. Hence, the bulk torque remains unaffected by the presence of normal metal reservoirs and the associated spin- and charge accumulations.

The spin-transfer torque $\boldsymbol{\tau}^{\text{stt}}_i$ is induced by spin accumulation in the normal metals, and can be expressed as follows:

$$\boldsymbol{\tau}^{\text{stt}}_i = \hbar^{-1} \sum_\kappa \big[\beta^I_{i\kappa} \boldsymbol{z} \times \boldsymbol{\mu}^S_\kappa - \beta^R_{i\kappa} \boldsymbol{z} \times \big(\boldsymbol{z} \times \boldsymbol{\mu}^S_\kappa\big)\big]. \tag{27}$$

In Eq. (27), the superscripts "R, I" denote the real and imaginary part. The site and lead-dependent coefficients $\beta_{i\kappa}$ are expressed in terms of the microscopic scattering matrix elements defined in Eq. (14) evaluated at the Fermi energy,

$$\beta_{i\kappa} = -\frac{2i}{s_i} \sum_n W^{\kappa n \kappa n}_{i\uparrow\downarrow}, \tag{28}$$

and can be calculated numerically for any particular system configuration.

The spin-pumping torque $\boldsymbol{\tau}^{\text{sp}}_i$ contains contributions from both sublattices and is given by

$$\boldsymbol{\tau}^{\text{sp}}_i = \sum_j \big[\alpha^R_{ij} \boldsymbol{z} \times \partial_t \boldsymbol{m}_j + \alpha^I_{ij} \boldsymbol{z} \times (\boldsymbol{z} \times \partial_t \boldsymbol{m}_j)\big], \tag{29}$$

where j runs over all sites and α_{ij} is expressed in the low-energy limit using the scattering matrix elements evaluated at the Fermi energy,

$$\alpha_{ij} = \frac{2\pi}{\sqrt{s_i s_j}} \sum_{\kappa\lambda nm} W^{\kappa n \lambda m}_{i\downarrow\uparrow} W^{\lambda m \kappa n}_{j\uparrow\downarrow}, \tag{30}$$

and $\alpha^{R(I)}$ denotes the real (imaginary) part of the matrix. Using the Onsager reciprocal relations in Appendix B, we find that the spin-transfer torque and spin pumping are related in the case of the most relevant case of a single reservoir,

$$\sum_j \alpha_{ij} = \beta_i. \tag{31}$$

Finally, the fluctuating torque $\boldsymbol{\tau}^{\text{f}}_i$ is expressed in terms of a fluctuating transverse field $\boldsymbol{H}^{\text{f}}_i$,

$$\boldsymbol{\tau}^{\text{f}}_i = -\gamma\mu_0 \boldsymbol{z} \times \boldsymbol{H}^{\text{f}}_i. \tag{32}$$

The fluctuating field exhibit interlattice and intralattice correlators $\langle H_{\mu i} H_{\nu j}\rangle$, where $\mu, \nu = \{x, y\}$,

$$2\sqrt{s_i s_j}\gamma^2\mu_0^2\big\langle H^{\text{f}}_{xi} H^{\text{f}}_{xj}\big\rangle = \text{Im}\Sigma^K_{ij} + 4\text{Im}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{33a}$$

$$2\sqrt{s_i s_j}\gamma^2\mu_0^2\big\langle H^{\text{f}}_{xi} H^{\text{f}}_{yj}\big\rangle = -\text{Re}\Sigma^K_{ij} - 4\text{Re}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{33b}$$

$$2\sqrt{s_i s_j}\gamma^2\mu_0^2\big\langle H^{\text{f}}_{yi} H^{\text{f}}_{yj}\big\rangle = \text{Im}\Sigma^K_{ij} - 4\text{Im}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{33c}$$

where the time arguments t and t' of the fields and the relative time argument $(t - t')$ of the self energies are omitted for simplicity. The self-energy Σ is due to charge and longitudinal spin accumulations in the normal metals and is nonzero even in equilibrium. It is conveniently written as a product of a frequency-dependent quantity $\pi(\omega)$ and a site and scattering states dependent quantity σ_{ij} [35],

$$\Sigma^K_{ij}(\omega) = \frac{i}{\hbar} \sum_{\kappa\lambda} \sigma_{ij\kappa\lambda} \pi_{\kappa\lambda}(\omega), \tag{34}$$

with

$$\pi_{\kappa\lambda}(\omega) = -2\int d\epsilon[2n_{\uparrow\uparrow\kappa}(\epsilon)n_{\downarrow\downarrow\lambda}(\epsilon+\hbar\omega) - n_{\uparrow\uparrow\kappa}(\epsilon) - n_{\downarrow\downarrow\lambda}(\epsilon+\hbar\omega)], \tag{35a}$$

$$\sigma_{ij\kappa\lambda} = \frac{2\pi}{\sqrt{s_i s_j}}\sum_{nm} W^{\kappa n\lambda m}_{i\downarrow\uparrow} W^{\lambda m\kappa n}_{j\uparrow\downarrow}. \tag{35b}$$

Conversely, the self-energy matrices $\tilde{\Sigma}_{\uparrow\downarrow}$ are due to transverse spin accumulation in the normal metals and, as a result, vanish in equilibrium. Analogous to the decomposition in Eq. (34), we write

$$\tilde{\Sigma}^K_{\uparrow\downarrow ij} = -\frac{i}{\hbar}\sum_{\kappa\lambda}\tilde{\sigma}_{\uparrow\downarrow ij\kappa\lambda}\tilde{\pi}_{\kappa\lambda}(\omega), \tag{36}$$

where

$$\tilde{\pi}_{\uparrow\downarrow}(\omega) = -4\int d\epsilon n_{\uparrow\downarrow\kappa}(\epsilon)n_{\uparrow\downarrow\lambda}(\epsilon+\hbar\omega), \tag{37a}$$

$$\tilde{\sigma}_{\uparrow\downarrow ij\kappa\lambda} = -\frac{\pi}{\sqrt{s_i s_j}}\sum_{nm} W^{\kappa n\lambda m}_{i\downarrow\uparrow} W^{\lambda m\kappa n}_{j\downarrow\uparrow}. \tag{37b}$$

The noise matrices $\pi(\omega)$ and $\tilde{\pi}(\omega)$ are similar to what was found in Ref. [35], and are calculated in the equilibrium, elastic, and inelastic transport regime in Sec. V B. Crucially, the shot noise differs on various sites, due to the site dependence of σ and $\tilde{\sigma}$. At equilibrium, the fluctuation-dissipation theorem holds, e.g.,

$$2s_i\gamma^2\mu_0^2\langle H^{\mathrm{f}}_{\mu i}H^{\mathrm{f}}_{\nu i}\rangle = \delta_{\mu\nu}\alpha_{ii}4k_BT\xi\left(\frac{\hbar\omega}{2k_BT}\right), \tag{38}$$

where $\xi(x) = x\coth x$.

In the next section, we discuss the Keldysh action of the model presented in this section and derive an effective action by integrating out the fermionic degrees of freedom.

IV. KELDYSH THEORY AND EFFECTIVE ACTION

In this section, we derive the semiclassical spin dynamics by using an out-of-equilibrium path integral formalism [30]. We introduce the closed contour action S and the partition function Z,

$$Z = \int D[\bar{a}a\bar{b}b\bar{c}_\uparrow c_\uparrow \bar{c}_\downarrow c_\downarrow]e^{iS/\hbar}. \tag{39}$$

The action S consists of contributions from the localized magnetic excitations a and b, and the spin-up $c_\uparrow$ and spin-down electrons $c_\downarrow$ from the scattering states. We will integrate out the fermion operators and get an effective action for the magnetic excitations a and b, which includes effective transverse and longitudinal fields that arise from the charge and spin accumulations in the normal metals.

We follow Ref. [30] and replace the fields in Eq. (39) with "±" fields residing on the forward and backward part of the Schwinger-Keldysh contour. The action in the ± basis is given in Appendix C. These fields are not independent of each other and can be Keldysh rotated into a new basis that takes into account the coupling between them. The rotated fields have the advantage of suggesting a transparent physical interpretation, corresponding to the semiclassical equations and quantum corrections.

A. Keldysh action

For magnons, the classical (cl) and quantum (q) fields are defined linear combinations of the ± fields, as described in detail in Appendix C. In Keldysh space, it is convenient to also introduce the matrices

$$\gamma^q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{cl} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{40}$$

The Keldysh rotated magnon action becomes

$$\begin{aligned} S_m = {} & \sum_{at}\bar{a}^q_a(i\hbar\partial_t - E^A_a)a^{cl}_a + \sum_{bt}\bar{b}^q_b(i\hbar\partial_t - E^B_b)b^{cl}_b \\ & + \sum_{at}a^q_a(i\hbar\partial_t - E^A_a)\bar{a}^{cl}_a + \sum_{bt}b^q_b(i\hbar\partial_t - E^B_b)\bar{b}^{cl}_b \\ & - 2\sum_{aa't}\sqrt{s_a s_{a'}}J_{aa'}[\bar{a}^q_a a^{cl}_{a'} + \text{H.c.}] \\ & - 2\sum_{bb't}\sqrt{s_b s_{b'}}J_{bb'}[\bar{b}^q_b b^{cl}_{b'} + \text{H.c.}] \\ & - 2\sum_{abt}\sqrt{s_a s_b}J_{ab}[\bar{a}^q_a\bar{b}^{cl}_b + \bar{a}^{cl}_a\bar{b}^q_b + \text{H.c.}] \\ & - \gamma\mu_0\hbar\sum_{at}\sqrt{s_a}[H^A_{a-}a^q_a + H^A_{a+}\bar{a}^q_a] \\ & - \gamma\mu_0\hbar\sum_{bt}\sqrt{s_b}[H^B_{b-}\bar{b}^q_b + H^B_{b+}b^q_b], \end{aligned} \tag{41}$$

where we wrote the time integral as a sum for compact notation. In Eq. (41), H.c. denotes the Hermitian conjugate of the previous term. The fermion action becomes

$$S_e + S_0 = \sum_{st}\bar{C}_s\gamma^{cl}(i\hbar\partial_t - \epsilon)C_s, \tag{42}$$

where we introduced vector notation for the 1/2 fields, $\bar{C}_s = (\bar{c}^1_{s\alpha}, \bar{c}^2_{s\alpha})$, and ϵ is a diagonal matrix containing the single-particle energies of the electrons. The Keldysh rotated first-order electron-magnon interaction is

$$\begin{aligned} S_1 = {} & -\sum_{\substack{at\\ \alpha\beta}}\frac{1}{\sqrt{s_a}}[a^{cl}_a W^{\alpha\beta}_{a\downarrow\uparrow}\bar{C}_{\downarrow\alpha}\gamma^{cl}C_{\uparrow\beta} + a^q_a W^{\alpha\beta}_{a\downarrow\uparrow}\bar{C}_{\downarrow\alpha}\gamma^q C_{\uparrow\beta} + \text{H.c.}] \\ & - \sum_{\substack{bt\\ \alpha\beta}}\frac{1}{\sqrt{s_b}}[\bar{b}^{cl}_a W^{\alpha\beta}_{b\downarrow\uparrow}\bar{C}_{\downarrow\alpha}\gamma^{cl}C_{\uparrow\beta} + \bar{b}^q_b W^{\alpha\beta}_{b\downarrow\uparrow}\bar{C}_{\downarrow\alpha}\gamma^q C_{\uparrow\beta} + \text{H.c.}], \end{aligned} \tag{43}$$

and, finally, the second-order term reads

$$S_2 = \sum_{\substack{at\\ \alpha\beta}} \frac{1}{2s_a}\big[W^{\alpha\beta}_{a\uparrow\uparrow}(\bar{A}_a\gamma^{cl}A_a\bar{C}_{\uparrow\alpha}\gamma^{cl}C_{\uparrow\beta} + \bar{A}_a\gamma^{q}A_a\bar{C}_{\uparrow\alpha}\gamma^{q}C_{\uparrow\beta}) - W^{\alpha\beta}_{a\downarrow\downarrow}(\bar{A}_a\gamma^{cl}A_a\bar{C}_{\downarrow\alpha}\gamma^{cl}C_{\downarrow\beta} + \bar{A}_a\gamma^{q}A_a\bar{C}_{\downarrow\alpha}\gamma^{q}C_{\downarrow\beta})\big]$$
$$- \sum_{\substack{bt\\ \alpha\beta}} \frac{1}{2s_b}\big[W^{\alpha\beta}_{b\uparrow\uparrow}(\bar{B}_b\gamma^{cl}B_b\bar{C}_{\uparrow\alpha}\gamma^{cl}C_{\uparrow\beta} + \bar{B}_b\gamma^{q}B_b\bar{C}_{\uparrow\alpha}\gamma^{q}C_{\uparrow\beta}) - W^{\alpha\beta}_{b\downarrow\downarrow}(\bar{B}_b\gamma^{cl}B_b\bar{C}_{\downarrow\alpha}\gamma^{cl}C_{\downarrow\beta} + \bar{B}_b\gamma^{q}B_b\bar{C}_{\downarrow\alpha}\gamma^{q}C_{\downarrow\beta})\big], \quad (44)$$

where the magnon q/cl operators are consolidated in vectors $\bar{A}_a$ and $\bar{B}_b$. The Keldysh rotated action proves to be well suited for the computation of an effective magnon action, a topic we delve into in the following section.

B. Integrating out the fermionic degrees of freedom

For the itinerant electrons in the normal metal and antiferromagnet, the total effective electron action is $S_{e,\text{tot}} = S_e + S_{em}$, and can be expressed as

$$S_{e,\text{tot}} = \sum_{ss'tt'} \bar{C}_{s,t}G^{-1}_{ss',tt'}C_{s',t'}\,, \quad (45)$$

where the interacting Green's function G is given in terms of the noninteracting Green's function G_0 and interaction terms as

$$G^{-1} = G_0^{-1} + \tilde{W}_1 + \tilde{W}_2\,. \quad (46)$$

Here, $\tilde{W}_1$ contains the first-order magnon operators on both sublattices,

$$\tilde{W}_1 = \delta(t-t')\big[W_1^A + W_1^B\big], \quad (47)$$

where W_1^A and W_1^B are spin-flip operators,

$$W_1^A = -\sum_{xa}\frac{1}{\sqrt{s_a}}\gamma^x\big[W_{a\uparrow\downarrow}\bar{a}^x_a\sigma_+ + W_{a\downarrow\uparrow}a^x_a\sigma_-\big], \quad (48a)$$

$$W_1^B = -\sum_{xb}\frac{1}{\sqrt{s_b}}\gamma^x\big[W_{b\uparrow\downarrow}b^x_b\sigma_+ + W_{b\downarrow\uparrow}\bar{b}^x_b\sigma_-\big]. \quad (48b)$$

In Eq. (48), the variable $x = \{cl, q\}$ represents a Keldysh space index, and $\sigma_\pm$ are the usual raising and lowering Pauli matrices. Similarly, $\tilde{W}_2$ contains the magnon operators to quadratic order for both sublattices,

$$\tilde{W}_2 = \delta(t-t')\big[W_2^A - W_2^B\big], \quad (49)$$

with W_2^A and W_2^B given by

$$W_2^A = \sum_{axy}\frac{1}{2s_a}\bar{a}^x_a\gamma^x a^y_a\gamma^y\begin{pmatrix} W_{a\uparrow\uparrow} & 0\\ 0 & -W_{a\downarrow\downarrow}\end{pmatrix}, \quad (50a)$$

$$W_2^B = \sum_{bxy}\frac{1}{2s_b}\bar{b}^x_b\gamma^x b^y_b\gamma^y\begin{pmatrix} W_{b\uparrow\uparrow} & 0\\ 0 & -W_{b\downarrow\downarrow}\end{pmatrix}, \quad (50b)$$

where the spin structure is explicitly written out as a matrix. The matrices $W_1^{A(B)}$ and $W_2^{A(B)}$ have a structure in the scattering states space from $W_{a(b)}$, spin space from the Pauli matrices, and Keldysh space from γ^x. The inverse free electron Green's function G_0^{-1} from Eq. (46) has the conventional causality structure in Keldysh space, with a retarded (R), advanced (A), and Keldysh (K) component,

$$G_0^{-1} = \begin{pmatrix} [G_0^R]^{-1} & [G_0^K]^{-1}\\ 0 & [G_0^A]^{-1}\end{pmatrix}, \quad (51)$$

and has equilibrium components that are diagonal in both spin space and in the scattering states space,

$$\big[G_0^{-1}\big]^{R(A)}_{\alpha\beta,ss'} = \delta_{\alpha\beta}\delta_{ss'}\delta(t-t')(i\hbar\partial_t - \epsilon_\alpha \pm i\delta)\,, \quad (52)$$

where the upper sign corresponds to the retarded component, while the lower sign is applicable to the advanced component. The Keldysh component includes information about the distribution function, and will be discussed below, when we Fourier transform the Green's functions.

From the effective electron action in Eq. (45), it is evident that the partition function of $S_{e,\text{tot}}$ takes on a Gaussian form with respect to the fermionic operators. Hence, the fermionic integral in the partition function can be evaluated exactly, with an inconsequential proportionality constant being disregarded,

$$\int D[C]e^{iS_{e,\text{tot}}/\hbar} = e^{\text{Tr}\left[\ln\left[1+G_0\tilde{W}_1+G_0\tilde{W}_2\right]\right]}. \quad (53)$$

In Eq. (53), we have used the shorthand notation for the functional integral measure of all fermionic states, $D[C] = D[\bar{C}_\uparrow C_\uparrow \bar{C}_\downarrow C_\downarrow]$. We have absorbed a normalization constant into the functional integral measure for simplicity. We note that as a consequence of the continuity of the time coordinate and scattering states energy that we are employing, the unit matrix is a delta function in time and energy, $1 \equiv \delta(t-t')\delta(\epsilon_\alpha - \epsilon_\beta)$, and thus quantities inside the logarithm carries dimension $\text{J}^{-1}\text{s}^{-1}$. The trace, on the other hand, is an integral operator with unit Js. As long as one interprets the logarithm in terms of its Taylor expansion, this does not lead to any problems, as the exponent of Eq. (53) becomes dimensionless for all terms in the expansion. The exponent is interpreted as an additional contribution to the magnon action,

$$\frac{i}{\hbar}S_{\text{eff}} = \text{Tr}[\ln[1 + G_0\tilde{W}_1 + G_0\tilde{W}_2]]\,. \quad (54)$$

The way forward is to treat this interaction as a perturbation, expanding the logarithm in first and second-order contributions and disregarding higher-order terms,

$$S_{\text{eff}} \approx -i\hbar\text{Tr}[G_0\tilde{W}_1] - i\hbar\text{Tr}[G_0\tilde{W}_2] + \frac{i\hbar}{2}\text{Tr}[G_0\tilde{W}_1G_0\tilde{W}_1]\,. \quad (55)$$

To evaluate the trace in these terms, it is convenient to Fourier transform all quantities from the time domain to the energy domain. This diagonalizes the noninteracting Green's functions, making calculations much more straightforward.

C. Fourier representation

The paper employs the Fourier transform convention defined in Appendix D. In Fourier space, the fermion equilibrium Green's function components are particularly simple,

$$[G_0]^{R(A)}_{\alpha\beta,ss'}(\omega) = \delta_{\alpha\beta}\delta_{ss'}(\hbar\omega - \epsilon \pm i\delta)^{-1}, \tag{56}$$

where $\delta > 0$ is an infinitesimal quantity ensuring convergence. The Keldysh component accounts for nonequilibrium phenomena though the spin-dependent distribution $n_{ss'\alpha}$ defined in Eq. (17),

$$[G_0]^{K}_{\alpha\beta,ss'}(\omega) = -2\pi i\delta_{\alpha\beta}\delta(\hbar\omega - \epsilon_\alpha)[\delta_{ss'} - 2n_{ss'\alpha}]. \tag{57}$$

The Keldysh component has off-diagonal terms in spin space if the distribution function $n_{ss'\alpha}$ has off-diagonal elements, i.e., if there is a transverse spin accumulation in the normal metals.

V. NONEQUILIBRIUM SPIN DYNAMICS

Having derived the effective action as expressed in Eq. (55), we proceed by evaluating the traces and delving into the resultant terms. The discussion unveils effective longitudinal and transverse fields, which we ascribe to spin-transfer torque and spin pumping originating from the normal metal reservoirs.

A. First-order contribution

Evaluating the trace in the first-order term in Eq. (55) corresponds to summing over the diagonal elements in spin space and Keldysh space, integrating over both time variables, and summing over the space of scattering states, we find

$$\begin{aligned} -i\hbar\mathrm{Tr}[G_0\tilde{W}_1] = &-\sum_{a\alpha}\frac{2}{\sqrt{s_a}}W^{\alpha\alpha}_{a\uparrow\downarrow}n_{\downarrow\uparrow\alpha}\int dt\,\bar{a}^q_a(t)\\ &-\sum_{a\alpha}\frac{2}{\sqrt{s_a}}W^{\alpha\alpha}_{a\downarrow\uparrow}n_{\uparrow\downarrow\alpha}\int dt\,a^q_a(t)\\ &-\sum_{b\alpha}\frac{2}{\sqrt{s_a}}W^{\alpha\alpha}_{b\uparrow\downarrow}n_{\downarrow\uparrow\alpha}\int dt\,b^q_b(t)\\ &-\sum_{b\alpha}\frac{2}{\sqrt{s_a}}W^{\alpha\alpha}_{b\downarrow\uparrow}n_{\uparrow\downarrow\alpha}\int dt\,\bar{b}^q_b(t). \end{aligned} \tag{58}$$

Here, we have used the general Green's function identity $G^R(t,t) + G^A(t,t) = 0$ [30], and written the time integration explicitly. Comparing the first-order contribution in Eq. (58) with the magnon action in Eq. (41), we observe that the first-order effect of the spin accumulation in the normal metal is equivalent to an effective deterministic transverse magnetic field $\boldsymbol{H}^{\rm stt}_i$, which act on a localized spin at site $i = \{a, b\}$ in the antiferromagnet. The "stt" superscript indicates that this field will take the form of a spin-transfer torque, which will be elaborated on below. The magnitudes of these effective transverse fields are given by

$$\gamma\mu_0 H^{\rm stt}_{i-} = \frac{2}{s_i\hbar}\sum_\alpha W^{\alpha\alpha}_{i\downarrow\uparrow}n_{\uparrow\downarrow\alpha}, \tag{59a}$$

$$\gamma\mu_0 H^{\rm stt}_{i+} = \frac{2}{s_i\hbar}\sum_\alpha W^{\alpha\alpha}_{i\uparrow\downarrow}n_{\downarrow\uparrow\alpha}, \tag{59b}$$

which implies that the Cartesian components read

$$\gamma\mu_0 H^{\rm stt}_{ix} = \frac{2}{s_i\hbar}\sum_\alpha \mathrm{Re}[W^{\alpha\alpha}_{i\uparrow\downarrow}n_{\downarrow\uparrow\alpha}], \tag{60a}$$

$$\gamma\mu_0 H^{\rm stt}_{iy} = \frac{2}{s_i\hbar}\sum_\alpha \mathrm{Im}[W^{\alpha\alpha}_{i\uparrow\downarrow}n_{\downarrow\uparrow\alpha}]. \tag{60b}$$

Recalling that the spin accumulation is given by Eqs. (24) and (21), we write the effective fields from Eq. (60) in the conventional spin-transfer torque form,

$$\gamma\mu_0\boldsymbol{H}^{\rm stt}_i = \frac{1}{\hbar}\sum_\kappa\left[\beta^R_{i\kappa}\boldsymbol{z}\times\boldsymbol{\mu}^S_\kappa + \beta^I_{i\kappa}\boldsymbol{z}\times\left(\boldsymbol{z}\times\boldsymbol{\mu}^S_\kappa\right)\right], \tag{61}$$

where the appearance of $\boldsymbol{z}$ is a consequence of our theory being restricted to small deviations for the equilibrium magnetization $\pm\boldsymbol{z}$. This results in the spin-transfer torque given in Eq. (27). In Eq. (61), the superscripts "R" and "I" denote the real and imaginary parts and the lead- and site-dependent constants $\beta_{i\kappa}$ have been introduced as sums over the transverse modes of the scattering matrix elements,

$$\beta_{i\kappa} = -\frac{2i}{s_i}\sum_n W^{\kappa n\kappa n}_{i\uparrow\downarrow}, \tag{62}$$

and where we have assumed that the transverse spin distribution functions $n_{\uparrow\downarrow}$ and $n_{\downarrow\uparrow}$ are only significant close to the Fermi surface, such that the scattering states matrix elements are well approximated by their value at the Fermi surface. The expression for the spin-transfer field in Eq. (61) is valid in both the elastic and inelastic regime, and vanishes in equilibrium. We note that the coefficient $\beta_{i\kappa}$, for $i = \{a, b\}$, depends not only on the potential at lattice site i but also indirectly of all lattice sites on both sublattices through the scattering states.

To the lowest order, the sublattice magnetizations are parallel and antiparallel to the z axis, $\boldsymbol{m}_A \approx \boldsymbol{z}$ and $\boldsymbol{m}_B \approx -\boldsymbol{z}$. Thus, to the lowest order in the magnon operators, the expressions for the transverse fields are ambiguous, and we can write the transverse field in Eq. (61) in terms of $\boldsymbol{m}_A$ or $\boldsymbol{m}_B$. To the lowest order in the magnon operators, the Keldysh technique cannot be used to identify which sublattice the transverse fields in Eq. (58) originate from.

B. Second-order contribution

The second-order contribution in Eq. (55) has contributions from $\tilde{W}_2$,

$$S_{21} = -i\hbar\mathrm{Tr}[G_0\tilde{W}_2], \tag{63}$$

as well as a contribution from $\tilde{W}_1$,

$$S_{22} = \frac{i\hbar}{2}\mathrm{Tr}[G_0\tilde{W}_1G_0\tilde{W}_1]. \tag{64}$$

Proceeding in a manner analogous to the treatment of the first-order term, the trace in S_{21} is evaluated,

$$S_{21} = -\sum_{a\alpha} \frac{\pi}{s_a}\big[W^{\alpha\alpha}_{a\uparrow\uparrow}(1-2n_{\uparrow\uparrow\alpha}) - W^{\alpha\alpha}_{a\downarrow\downarrow}(1-2n_{\downarrow\downarrow\alpha})\big]\int dt \bar{A}_a(t)\gamma^q A_a(t)$$
$$+\sum_{b\alpha} \frac{\pi}{s_b}\big[W^{\alpha\alpha}_{b\uparrow\uparrow}(1-2n_{\uparrow\uparrow\alpha}) - W^{\alpha\alpha}_{b\downarrow\downarrow}(1-2n_{\downarrow\downarrow\alpha})\big]\int dt \bar{B}_b(t)\gamma^q B_b(t)\,. \tag{65}$$

From Eq. (41), it is apparent that the second-order terms in S_{21} are equivalent with a longitudinal magnetic field, with magnitude

$$\gamma\mu_0 H^{A21}_{iz} = -\frac{\pi}{\hbar s_i}\sum_{\alpha}\big[W^{\alpha\alpha}_{i\uparrow\uparrow}(1-2n_{\uparrow\uparrow\alpha}) - W^{\alpha\alpha}_{i\downarrow\downarrow}(1-2n_{\downarrow\downarrow\alpha})\big], \tag{66}$$

which, in this reference frame, renormalizes the energies of localized magnon excitations. However, such longitudinal magnetic fields should not affect the spin dynamics since they, in the instantaneous reference field, correspond to contributions to the total free energy proportional to $\mathbf{S}_i^2$.

The final contribution S_{22} to the effective action contains interlattice and intralattice terms and can be written compactly by introducing a field $d_i = \{a_a, \bar{b}_b\}$ and summing over the two field components, i.e., $\sum_i d_i = \sum_a a_a + \sum_b \bar{b}_b$,

$$S_{22} = \int dt dt' \sum_{xx'ij} \frac{i\hbar}{2\sqrt{s_i s_j}} \mathrm{Tr}[G_{0,\uparrow\downarrow}(t',t)W_{i\downarrow\uparrow}\gamma^x G_{0,\uparrow\downarrow}(t,t')\gamma^{x'}W_{j\downarrow\uparrow}]d_i^x(t)d_j^{x'}(t')$$
$$+\int dt dt' \sum_{xx'ij} \frac{i\hbar}{2\sqrt{s_i s_j}} \mathrm{Tr}[G_{0,\downarrow\uparrow}(t',t)W_{i\uparrow\downarrow}\gamma^x G_{0,\downarrow\uparrow}(t,t')\gamma^{x'}W_{j\uparrow\downarrow}]\bar{d}_i^x(t)\bar{d}_j^{x'}(t')$$
$$+\int dt dt' \sum_{xx'ij} \frac{i\hbar}{2\sqrt{s_i s_j}} \mathrm{Tr}[G_{0,\downarrow\downarrow}(t',t)W_{i\downarrow\uparrow}\gamma^x G_{0,\uparrow\uparrow}(t,t')\gamma^{x'}W_{j\uparrow\downarrow}]d_i^x(t)\bar{d}_j^{x'}(t')$$
$$+\int dt dt' \sum_{xx'ij} \frac{i\hbar}{2\sqrt{s_i s_j}} \mathrm{Tr}[G_{0,\uparrow\uparrow}(t',t)W_{i\uparrow\downarrow}\gamma^x G_{0,\downarrow\downarrow}(t,t')\gamma^{x'}W_{j\downarrow\uparrow}]\bar{d}_i^x(t)d_j^{x'}(t'), \tag{67}$$

where the trace is taken only over the 2×2 Keldysh space and the space of scattering states α. The interlattice terms, i.e., $d = d'$, are discussed in Ref. [35] for a macrospin ferromagnet. Here, we summarize this discussion and highlight the addition of the interlattice terms not present in the macrospin ferromagnet.

Evaluating the trace in the first and second line of Eq. (67), we note that only the Keldysh component has off-diagonal elements in spin space, and find a contribution only from $x = x' = q$,

$$\tilde{S}^{qq}_{22} = \hbar\int dt dt' \sum_{ij}\big[d_i^q(t)\tilde{\Sigma}^K_{\uparrow\downarrow ij}(t,t')d_j^q(t')\big], \tag{68a}$$

$$\tilde{S}^{\bar{q}\bar{q}}_{22} = \hbar\int dt dt' \sum_{ij}\big[\bar{d}_i^q(t)\tilde{\Sigma}^K_{\downarrow\uparrow ij}(t,t')\bar{d}_j^q(t')\big], \tag{68b}$$

where the self-energies are

$$\tilde{\Sigma}^K_{\uparrow\downarrow ij}(t-t') = -\frac{2i}{\hbar^2\sqrt{s_i s_j}}\sum_{\alpha\beta} n_{\uparrow\downarrow\alpha}n_{\uparrow\downarrow\beta}W^{\alpha\beta}_{i\downarrow\uparrow}W^{\beta\alpha}_{j\downarrow\uparrow}e^{i(\epsilon_\alpha-\epsilon_\beta)(t-t')/\hbar}, \tag{69a}$$

$$\tilde{\Sigma}^K_{\downarrow\uparrow ij}(t-t') = -\frac{2i}{\hbar^2\sqrt{s_i s_j}}\sum_{\alpha\beta} n_{\downarrow\uparrow\alpha}n_{\downarrow\uparrow\beta}W^{\alpha\beta}_{i\uparrow\downarrow}W^{\beta\alpha}_{j\uparrow\downarrow}e^{i(\epsilon_\alpha-\epsilon_\beta)(t-t')/\hbar}\,. \tag{69b}$$

The reasoning behind identifying this self-energy as a Keldysh component is that it couples the quantum components of the fields, see Eq. (68). The terms in Eq. (68) do not have a direct analog in the magnon action in Eq. (41), and interpreting these will be the subject of Sec. V C. The self-energies in Eq. (69) are invariant under a joint time and lattice site reversal, i.e., $\tilde{\Sigma}_{ij}(t-t') = \tilde{\Sigma}_{ji}(t'-t)$. Moreover, due to the properties $n_{\uparrow\downarrow} = n^*_{\downarrow\uparrow}$ and $W^{\alpha\beta}_{i\uparrow\downarrow} = [W^{\beta\alpha}_{i\downarrow\uparrow}]^*$, we see that the self-energies are related by $\tilde{\Sigma}^K_{\uparrow\downarrow ij}(t-t') = -[\tilde{\Sigma}^K_{\downarrow\uparrow ij}(t-t')]^*$, which will be important later.

Disregarding terms of the order k_BT/ϵ_F and μ_s/ϵ_F [35], we find that the Fourier-transformed self-energy becomes

$$\tilde{\Sigma}^K_{\uparrow\downarrow ij}(\omega) = -\frac{i}{\hbar}\sum_{\kappa\lambda}\tilde{\sigma}_{\uparrow\downarrow ij\kappa\lambda}\tilde{\pi}_{\kappa\lambda}(\omega)\,, \tag{70}$$

where

$$\tilde{\pi}_{\kappa\lambda}(\omega) = -4\int d\epsilon n_{\uparrow\downarrow\kappa}(\epsilon)n_{\uparrow\downarrow\lambda}(\epsilon+\omega), \quad \tilde{\sigma}_{\uparrow\downarrow ij\kappa\lambda} = -\frac{\pi}{\sqrt{s_i s_j}}\sum_{nm} W^{\kappa n\lambda m}_{i\downarrow\uparrow}W^{\lambda m\kappa n}_{j\downarrow\uparrow}, \tag{71}$$

and where the matrix elements W are evaluated at the Fermi surface. This is a straightforward generalization of the macrospin ferromagnet case, with the addition of shot-noise contributions from interlattice and intralattice interactions between different lattice sites. We can evaluate the quantity $\tilde{\pi}_{\kappa\lambda\uparrow\downarrow}(\omega)$ by using Eq. (17),

$$\tilde{\pi}_{\uparrow\downarrow\kappa\lambda}(\omega) = -u_{\kappa-}u_{\lambda-}\int d\epsilon[f_{\uparrow\kappa}(\epsilon)-f_{\downarrow\kappa}(\epsilon)][f_{\uparrow\lambda}(\epsilon+\hbar\omega)-f_{\downarrow\lambda}(\epsilon+\hbar\omega)], \tag{72}$$

where we introduced the conventional "lowering" vector $u_- = u_x - iu_y$. This can be computed in equilibrium, elastic, and inelastic scattering cases, and results exactly similar to those in Ref. [35].

We now turn our attention to the third and fourth lines of the second-order action in Eq. (67). The contributions from the two lines are equal, which is evident from interchanging summation indices and rearranging terms. Their total contribution to the action S_{22} can be split into contributions $S_{22}^{\bar{q}q}$, $S_{22}^{\bar{q}cl}$, and $S_{22}^{\bar{cl}q}$. The contribution S^{clcl} vanishes, due to the quantity $G^R(t'-t)G^R(t-t')$ being nonzero only for $t=t'$, which has measure zero, and similarly for G^A. This ensures that the action satisfies the general requirement $S[\phi^{cl},\phi^q=0]=0$ [30]. Introducing, for notational convenience, the vector $\bar{D}_i = (\bar{d}^{cl} \quad \bar{d}^q)$, we find

$$S_{22}^{\bar{q}q} + S_{22}^{\bar{q}cl} + S_{22}^{\bar{cl}q} = \hbar\int dtdt'\sum_{ij}\bar{D}_i(t)\hat{\Sigma}_{ij}(t-t')D_j(t'), \tag{73}$$

where the self-energy matrix has structure in Keldysh space and in the sublattice space,

$$\hat{\Sigma}_{ij}(t-t') = \begin{pmatrix} 0 & \Sigma^A(t-t') \\ \Sigma^R(t-t') & \Sigma^K(t-t')\end{pmatrix}_{ij}, \tag{74}$$

and its components are given by

$$\Sigma^K_{ij}(t-t') = \frac{2i}{\sqrt{s_i s_j}\hbar^2}\sum_{\alpha\beta}(n_{\uparrow\uparrow\alpha}+n_{\downarrow\downarrow\beta}-2n_{\uparrow\uparrow\alpha}n_{\downarrow\downarrow\beta})W^{\alpha\beta}_{i\downarrow\uparrow}W^{\beta\alpha}_{j\uparrow\downarrow}e^{i(\epsilon_\alpha-\epsilon_\beta)(t-t')/\hbar}, \tag{75a}$$

$$\Sigma^R_{ij}(t-t') = \frac{2i}{\sqrt{s_i s_j}\hbar^2}\theta(t-t')\sum_{\alpha\beta}(n_{\uparrow\uparrow\alpha}-n_{\downarrow\downarrow\beta})W^{\alpha\beta}_{i\downarrow\uparrow}W^{\beta\alpha}_{j\uparrow\downarrow}e^{i(\epsilon_\alpha-\epsilon_\beta)(t-t')/\hbar}, \tag{75b}$$

$$\Sigma^A_{ij}(t'-t) = -\frac{2i}{\sqrt{s_i s_j}\hbar^2}\theta(t-t')\sum_{\alpha\beta}(n_{\uparrow\uparrow\alpha}-n_{\downarrow\downarrow\beta})W^{\alpha\beta}_{i\downarrow\uparrow}W^{\beta\alpha}_{j\uparrow\downarrow}e^{i(\epsilon_\alpha-\epsilon_\beta)(t-t')/\hbar}. \tag{75c}$$

The Keldysh component of this self-energy has the symmetry $[\Sigma^K_{ij}(t-t')]^* = -\Sigma^K_{ji}(t'-t)$. Imperatively, as a consequence of this symmetry, the quantities

$$\Sigma^K_{ij}(t-t') - \Sigma^K_{ji}(t'-t) = 2\mathrm{Re}\big[\Sigma^K_{ij}(t-t')\big], \tag{76}$$

$$i\Sigma^K_{ij}(t-t') + i\Sigma^K_{ji}(t'-t) = -2\mathrm{Im}\big[\Sigma^K_{ij}(t-t')\big], \tag{77}$$

are real numbers, which will be important in the next section. We proceed by a similar analysis to what was done with $\tilde{\Sigma}$, writing it in terms of a shot-noise matrix. We assume that the matrices W can be approximated by their value on the Fermi surface, and write

$$\Sigma^K_{ij}(\omega) = \frac{i}{\hbar}\sum_{\kappa\lambda}\sigma_{ij\kappa\lambda}\pi_{\kappa\lambda}(\omega), \tag{78}$$

where we introduced the matrices

$$\pi_{\kappa\lambda}(\omega) = -2\int d\epsilon[2n_{\uparrow\uparrow\kappa}(\epsilon)n_{\downarrow\downarrow\lambda}(\epsilon+\hbar\omega) - n_{\uparrow\uparrow\kappa}(\epsilon) - n_{\downarrow\downarrow\lambda}(\epsilon+\hbar\omega)], \quad \sigma_{ij\kappa\lambda} = \frac{2\pi}{\sqrt{s_i s_j}}\sum_{nm}W^{\kappa n\lambda m}_{i\downarrow\uparrow}W^{\lambda m\kappa n}_{j\uparrow\downarrow}. \tag{79}$$

The matrix $\pi(\omega)$ can also be evaluated in equilibrium, and for elastic and inelastic scattering, and the results are again exactly similar to those in Ref. [35].

Comparing with the magnetic action in Eq. (41), we notice that the terms with the retarded and advanced self-energies are equivalent with longitudinal fields, which we in the following will show consists of dissipative Gilbert-like terms and nondissipative field-like terms. Fourier transforming and applying the identity (D5), the retarded and advanced self-energies from Eqs. (75b) and (75c) become

$$\Sigma^{R,A}_{ij} = \frac{-2}{\sqrt{s_i s_j}\hbar} \sum_{\alpha\beta} \frac{n_{\uparrow\uparrow\alpha} - n_{\downarrow\downarrow\beta}}{\hbar\omega + \epsilon_\alpha - \epsilon_\beta \pm i\delta} W^{\alpha\beta}_{i\downarrow\uparrow} W^{\beta\alpha}_{j\uparrow\downarrow} . \tag{80}$$

This self-energy has equilibrium contributions as well as nonequilibrium contributions; however, the nonequilibrium contributions scale as $\mu_{\uparrow\uparrow}/\epsilon_F$ and $\mu_{\downarrow\downarrow}/\epsilon_F$ and are disregarded in the following. The equilibrium part of Eq. (80) becomes particularly transparent when expanding to first order in the frequency ω,

$$\Sigma^{R/A}_{\uparrow\downarrow ij}(\omega) \approx \Sigma^{R/A}_{\uparrow\downarrow ij}(\omega = 0) \pm i\omega\alpha_{ij} , \tag{81}$$

where we introduced the frequency-independent matrix element

$$\alpha_{ij} = \frac{2\pi}{\sqrt{s_i s_j}} \sum_{\alpha\beta} [-f'(\epsilon_\alpha)]\delta(\epsilon_\alpha - \epsilon_\beta) W^{\alpha\beta}_{i\downarrow\uparrow} W^{\beta\alpha}_{j\uparrow\downarrow} , \tag{82}$$

which can be approximated to

$$\alpha_{ij} = \frac{2\pi}{\sqrt{s_i s_j}} \sum_{\kappa\lambda nm} W^{\kappa n\lambda m}_{i\downarrow\uparrow} W^{\lambda m\kappa n}_{j\uparrow\downarrow} , \tag{83}$$

where the scattering states matrix elements are evaluated at the Fermi surface. We note from the identity $[W^{\alpha\beta}_{i\uparrow\downarrow}]^* = W^{\beta\alpha}_{i\downarrow\uparrow}$ that α is a Hermitian matrix in the space of lattice sites, i.e., $[\alpha_{ij}]^* = \alpha_{ji}$. The zeroth-order term in frequency is

$$\big[S^{\bar{q}cl}_{22} + S^{\bar{cl}q}_{22}\big]_0 = \hbar \sum_{ij} \int d\omega \bar{d}^q_i(\omega)\Sigma^R_{\uparrow\downarrow ij}(0) d^{cl}_j(\omega) + \hbar \sum_{ij} \int d\omega \bar{d}^{cl}_i(\omega)\Sigma^A_{\uparrow\downarrow ij}(0) d^q_j(\omega) , \tag{84}$$

which is a constant longitudinal field that plays no role in the instantaneous reference frame, as discussed above.

The first-order term in frequency is finite even in equilibrium,

$$\big[S^{\bar{q}cl}_{22} + S^{\bar{cl}q}_{22}\big]_1 = \hbar \sum_{ij} \alpha_{ij} \int dt \bar{D}_i \gamma^q \partial_t D_j , \tag{85}$$

and takes the form of a Gilbert damping term, including both interlattice and intralattice contributions. The spin-transfer torque coefficient α and the spin-pumping coefficient β are related to each other as a consequence of the Onsager reciprocal relations [36]. In Appendix B we derive this relation, which is given in Eq. (B11), and derive an optical theorem relating the scattering matrices, given in Eq. (B12).

Summarizing this section, we have found that the corrections to the magnon action S_m in the presence of spin and charge accumulations in surrounding normal metals is $S_1 + S_{21} + S^{\bar{q}cl}_{22} + S^{\bar{cl}q}_{22} + \tilde{S}^{qq}_{22} + S^{\bar{q}q}_{22}$, and found that the first three of these contributions appear like magnetic fields and (in the low-frequency limit) like Gilbert-like damping terms in the effective magnon action. Importantly, we find both longitudinal and transverse fields in the general case. The last two contributions to the action consist of coupled quantum fields and are the result of purely quantum effects. These terms are the subject of the next section.

C. Fluctuating fields

From the effective action in the last section, we were able to associate the (q, cl) and (cl, q) terms with longitudinal fields by comparing them with the magnon action in Eq. (41). Now, we must address the issue of how to interpret the (q, q) terms, which lack an analog in the action described in Eq. (41). In this section, we derive fluctuating forces from these terms by employing a Hubbard-Stratonovich (HS) transformation on the quadratic fields in the effective action, introducing auxiliary fields in the process. Commencing with the contribution from the term $S^{\bar{q}q}_{22}$, we introduce the complex auxiliary field $h^{\bar{q}q}_i$ (in units of inverse second) via a conventional Hubbard–Stratonovich transformation,

$$\begin{aligned} e^{iS^{\bar{q}q}_{22}/\hbar} &= \exp\bigg[\int dt dt' \sum_{ij} \bar{d}^q_i(t) i\Sigma^K_{ij}(t-t') d^q_j(t')\bigg] \\ &= \frac{1}{\det\big[-i\Sigma^K\big]} \int \prod_i D[h^{\bar{q}q}_i] \exp\bigg[i\int dt \sum_i \big(h^{\bar{q}q}_i(t)\bar{d}^q_i(t) + \text{H.c.}\big) - \int dt dt' \sum_{ij} \bar{h}^{\bar{q}q}_i(t)\big[-i\Sigma^K_{ij}(t-t')\big]^{-1} h^{\bar{q}q}_j(t')\bigg], \end{aligned} \tag{86}$$

where a shorthand notation for the measure was introduced as $D[h^{\bar{q}q}_i] = \Pi_k\{d[\text{Im}h^{\bar{q}q}_i(t_k)]d[\text{Re}h^{\bar{q}q}_i(t_k)]/\pi\}$, where k is the index used to order the discretization of the time coordinate. From the Gaussian form of Eq. (86), the correlators of the auxiliary field can be identified as

$$\langle h_i^{\bar{q}q}(t)\rangle = 0, \tag{87a}$$
$$\langle h_i^{\bar{q}q}(t)h_j^{\bar{q}q}(t')\rangle = 0, \tag{87b}$$
$$\langle \bar{h}_i^{\bar{q}q}(t)h_j^{\bar{q}q}(t')\rangle = -i\Sigma_{ij}^K(t-t'). \tag{87c}$$

The second term in the exponent is quadratic in the new fields, and gives no contribution to the magnon action, while the first term is linear in the magnon field d_i and is interpreted as an effective transverse field in the magnon action.

The contribution from the terms $\tilde{S}_{22}^{qq}+\tilde{S}_{22}^{\bar{q}\bar{q}}$ is HS transformed by performing an unconventional transformation in the two complex fields $\tilde{h}_i^{qq}$ and $\overline{\tilde{h}_i^{qq}}$ separately,

$$e^{i\tilde{S}_{22}/\hbar} = \frac{1}{\sqrt{\det[-2i\tilde{\Sigma}^K_{\downarrow\uparrow}]}\sqrt{\det[-2i\tilde{\Sigma}^K_{\uparrow\downarrow}]}}\int\prod_i D[\tilde{h}_i^{qq}]\exp\Bigg[i\int dt\sum_i\left(\overline{\begin{pmatrix}\tilde{h}^{qq}(t) & \overline{\tilde{h}^{qq}}(t)\end{pmatrix}}_i\begin{pmatrix}d(t)\\ \bar{d}(t)\end{pmatrix}_i+\text{H.c}\right).$$
$$-\int dt dt'\sum_{ij}\overline{\begin{pmatrix}\tilde{h}^{qq}(t) & \overline{\tilde{h}^{qq}}(t)\end{pmatrix}}_i\begin{pmatrix}0 & -i\tilde{\Sigma}^K_{\downarrow\uparrow}(t-t')\\ -i\tilde{\Sigma}^K_{\uparrow\downarrow}(t-t') & 0\end{pmatrix}^{-1}_{ij}\begin{pmatrix}\tilde{h}^{qq}(t')\\ \overline{\tilde{h}^{qq}}(t')\end{pmatrix}_j\Bigg], \tag{88}$$

again interpreting the exponent as an effective action including the field $\tilde{h}_i^{qq}$, which has the correlators

$$\langle \tilde{h}_i^{qq}(t)\rangle = 0, \tag{89a}$$
$$\langle \tilde{h}_i^{qq}(t)\tilde{h}_j^{qq}(t')\rangle = -i\tilde{\Sigma}^K_{\uparrow\downarrow ij}(t-t'), \tag{89b}$$
$$\langle \overline{\tilde{h}_i^{qq}}(t)\overline{\tilde{h}_j^{qq}}(t')\rangle = -i\tilde{\Sigma}^K_{\downarrow\uparrow ij}(t-t'), \tag{89c}$$
$$\langle \overline{\tilde{h}_i^{qq}}(t)\tilde{h}_j^{qq}(t')\rangle = 0. \tag{89d}$$

We remark that the unconventional form of the Hubbard-Stratonovich decoupling leads to nonzero correlators for equal fields, as opposed to the conventional approach where the nonzero correlators involve one field being the complex conjugate of the other. The fields $h^{\bar{q}q}$ and $\tilde{h}^{qq}$ are interpreted as fluctuating transverse fields with, in general, different amplitudes depending on the lattice site, but with correlators between lattice sites. Comparing the effective action in Eqs. (86) and (88) with the magnon action in Eq. (41), the components of the total fluctuating field $\boldsymbol{H}^{\text{f}}$ can be identified as

$$\gamma\mu_0 H^{\text{f}}_{+,i} = -\frac{1}{\sqrt{s_i}}\left[2\tilde{h}_i^{qq}+h_i^{\bar{q}q}\right], \tag{90a}$$
$$\gamma\mu_0 H^{\text{f}}_{-,i} = -\frac{1}{\sqrt{s_i}}\left[2\overline{\tilde{h}}_i^{qq}+\bar{h}_i^{\bar{q}q}\right]. \tag{90b}$$

In this expression, the factor of 2 arises from the unconventional nature of the Hubbard-Stratonovich transformation in Eq. (88). The correlators between the Cartesian components of the fluctuating field can be calculated using Eqs. (89) and (90),

$$2\sqrt{s_is_j}\gamma^2\mu_0^2\langle H^{\text{f}}_{xi}H^{\text{f}}_{xj}\rangle = \text{Im}\Sigma^K_{ij}+4\text{Im}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{91a}$$
$$2\sqrt{s_is_j}\gamma^2\mu_0^2\langle H^{\text{f}}_{xi}H^{\text{f}}_{yj}\rangle = -\text{Re}\Sigma^K_{ij}-4\text{Re}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{91b}$$
$$2\sqrt{s_is_j}\gamma^2\mu_0^2\langle H^{\text{f}}_{yi}H^{\text{f}}_{yj}\rangle = \text{Im}\Sigma^K_{ij}-4\text{Im}\tilde{\Sigma}_{\uparrow\downarrow ij}, \tag{91c}$$

from which we conclude that the correlators in the fluctuating field $\boldsymbol{H}^{\text{f}}$ are real numbers. In Eq. (91), we omitted the time arguments for notational simplicity. Furthermore, it is evident that for $i=j$ and $t=t'$, the correlators in Eqs. (91a) and (91c) are positive, aligning with the conditions expected for representing the variance of a real field.

D. Equations of motion

After HS decoupling the qq components, the effective action reads

$$\begin{aligned}S_{\text{eff}} = &-\gamma\hbar\mu_0\int dt\Bigg[\sum_a\sqrt{s_a}\left(H^{\text{stt}}_{a+}+H^{\text{f}}_{a+}\right)\bar{a}^q_a(t)\\ &+\sum_b\sqrt{s_b}\left(H^{\text{stt}}_{b-}+H^{\text{f}}_{b-}\right)\bar{b}^q_b(t)+\text{H.c.}\Bigg]\\ &+\hbar\int dt\Bigg[\sum_{aa'}\beta_{aa'}\bar{a}^q_a\partial_t a^{cl}_a+\sum_{ab}\beta_{ab}a^q_a\partial_t b^{cl}_b\\ &+\sum_{ba}\beta_{ba}\bar{b}^q_b\partial_t\bar{a}^{cl}_a+\sum_{bb'}\beta_{bb'}\bar{b}^q_b\partial_t b^{cl}_{b'}+\text{H.c.}\Bigg].\end{aligned} \tag{92}$$

Having cast the total action S_m+S_{eff} in a form that is linear in the quantum fields $\bar{a}^q$ and $\bar{b}^q$ and their complex conjugates, we can integrate over these fields in the partition function, producing the functional delta function imposing the semi-classical equations of motion for the fields a^{cl} and b^{cl} [30]. Using $a^{cl}_a = S_{+a}/(\hbar\sqrt{s_a})$ and $\bar{b}^{cl}_b = S_{+b}/(\hbar\sqrt{s_b})$ in the semi-classical limit, we find the coupled equations of motion,

$$\begin{aligned}i\partial_t S_{i+} = &\hbar^{-1}E_iS_{i+}+s_i\hbar\mu_0\gamma\left(H_{i+}+H^{\text{f}}_{i+}+H^{\text{stt}}_{i+}\right)\\ &-\sum_j\beta_{ij}\partial_t S_{j+},\end{aligned} \tag{93}$$

as well as its complex conjugated counterpart. Both in the definition of this field and in Eq. (93), the upper sign holds for sublattice $\mathcal{A}$, and the lower sign holds for sublattice $\mathcal{B}$. We find the Cartesian components by taking the real and imaginary parts and divide with $\hbar s_i$ to find an equation for the vector $\boldsymbol{m}_i = \boldsymbol{S}_i/(\hbar s_i)$,

$$\partial_t\boldsymbol{m}_i = \boldsymbol{\tau}^{\text{b}}_i+\boldsymbol{\tau}^{\text{f}}_i+\boldsymbol{\tau}^{\text{sp}}_i+\boldsymbol{\tau}^{\text{stt}}_i, \tag{94}$$

where

$$\boldsymbol{\tau}_i^{\mathrm{b}} = -\boldsymbol{z} \times (\hbar^{-1} E_i \boldsymbol{m}_i + \gamma\mu_0 \boldsymbol{H}_i), \tag{95a}$$

$$\boldsymbol{\tau}_i^{\mathrm{f}} = -\gamma\mu_0 \boldsymbol{z} \times \boldsymbol{H}_i^{\mathrm{f}}, \tag{95b}$$

$$\boldsymbol{\tau}_i^{\mathrm{stt}} = -\gamma\mu_0 \boldsymbol{z} \times \boldsymbol{H}_i^{\mathrm{stt}}, \tag{95c}$$

$$\boldsymbol{\tau}_i^{\mathrm{sp}} = \sum_j \mathrm{Re}\beta_{ij} \boldsymbol{z} \times \partial_t \boldsymbol{m}_j + \sum_j \mathrm{Im}\beta_{ij} \boldsymbol{z} \times (\boldsymbol{z} \times \partial_t \boldsymbol{m}_j), \tag{95d}$$

is microscopic expressions for the bulk torque τ^{b}, the fluctuating torque $\boldsymbol{\tau}^{\mathrm{f}}$, the spin-pumping torque $\boldsymbol{\tau}^{\mathrm{sp}}$, and the spin-transfer torque $\boldsymbol{\tau}^{\mathrm{stt}}$.

VI. CONCLUSIONS

In this paper, we have presented a general quantum theory of spin dynamics in magnet-normal metal systems, generalizing earlier results to a general antiferromagnetic or ferrimagnetic bipartite lattice. Spin and charge accumulations in the normal metals influence the magnetization dynamics in the magnet through spin-transfer torque, and the damping is enhanced due to spin pumping, including both inter- and intralattice contributions. We derived expressions for transverse fluctuating fields arising due to the electron magnon interactions. These fields have contributions from equilibrium terms as well as charge and spin accumulation in the normal metals. We found site-dependent shot noise contributions that are nonnegligible at low temperatures.

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APPENDIX A: HOLSTEIN-PRIMAKOFF TRANSFORMATION

In this Appendix, we discuss the transformations used to diagonalize the magnon Hamiltonian of Eq. (9). To go from the SU(2) spin operators to bosonic annihilation and creation operators, we employ the Holstein-Primakoff transformation [37,38] at sublattices $\mathcal{A}$ and $\mathcal{B}$ and expand to the lowest order in the bosonic operators, assuming the antiferromagnet is close to the Néel state, i.e., that all spins on sublattice $\mathcal{A}$ ($\mathcal{B}$) is close to being parallel (antiparallel) to the z direction. At sublattice $\mathcal{A}$, we expand

$$\hat{S}_{a+} = \hbar\sqrt{2s_a}\left(1 - \frac{\hat{a}_a^\dagger \hat{a}_a}{2s_a}\right)^{1/2} \hat{a}_a \approx \hbar\sqrt{2s_a}\hat{a}_a, \tag{A1}$$

$$\hat{S}_{a-} = \hbar\sqrt{2s_a}\hat{a}_a^\dagger\left(1 - \frac{\hat{a}_a^\dagger \hat{a}_a}{2s_a}\right)^{1/2} \approx \hbar\sqrt{2s_a}\hat{a}^\dagger, \tag{A2}$$

$$\hat{S}_{az} = \hbar(s_a - \hat{a}_a^\dagger \hat{a}_a), \tag{A3}$$

where a_a annihilates a localized magnon and s_a is the total spin at lattice site a. In the expansion of the square roots in Eqs. (A1) and (A2), we assumed $s_a \gg 1$ and expanded the square root to lowest order in $1/s_A$. We have employed the standard raising and lowering spin operators, defined as $S_\pm = S_x \pm iS_y$.

Similarly, at sublattice $\mathcal{B}$, we expand

$$\hat{S}_{b+} = \hbar\sqrt{2s_b}\hat{b}_b^\dagger\left(1 - \frac{\hat{b}_b^\dagger \hat{b}_b}{2s_b}\right) \approx \sqrt{2s_b}\hat{b}_b^\dagger, \tag{A4}$$

$$\hat{S}_{b-} = \hbar\sqrt{2s_b}\left(1 - \frac{\hat{b}_b^\dagger \hat{b}_b}{2s_b}\right)\hat{b}_b \approx \sqrt{2s_b}\hat{b}_b, \tag{A5}$$

$$\hat{S}_{bz} = \hbar(-s_b + \hat{b}_b^\dagger \hat{b}_b), \tag{A6}$$

where $\hat{b}$ annihilates a localized spin-up magnon.

APPENDIX B: RELATING SPIN-TRANSFER TORQUE AND SPIN-PUMPING COEFFICIENTS

We relate the spin-transfer pumping coefficients defined in Eq. (82) to the spin-transfer coefficients found in Eq. (62) in the case of one normal metal reservoir using the Onsager reciprocal relations [36]. We start by defining the pumped spin current (in units of electrical current, i.e., ampere) into normal metal as the change in total spin inside the antiferromagnetic due to spin pumping, i.e.,

$$\boldsymbol{I}^S = -\frac{e}{\hbar}\sum_j S_j \boldsymbol{\tau}_j^{\mathrm{sp}}. \tag{B1}$$

The appearance of $S_j = \hbar\sqrt{s_j(s_j+1)}$ is due to the way we have defined the torques in the main text, causing them to have the dimension of inverse time. The dynamics of the localized magnetic moment $\boldsymbol{\mu}_j = -\gamma S_j \boldsymbol{m}_j$ and the spin current are driven by the external effective field $\boldsymbol{H}_{\mathrm{eff}}$ and the spin accumulation $\boldsymbol{\mu}^S$, which are the thermodynamic forces in our system. In linear response, we can then write the equations for the spin dynamics and the spin current in matrix form,

$$\begin{pmatrix} -\gamma S_i \partial_t \boldsymbol{m}_i \\ \boldsymbol{I}^S \end{pmatrix} = \begin{pmatrix} L_{ij}^{mm} & L_i^{ms} \\ L_j^{sm} & L^{ss} \end{pmatrix} \begin{pmatrix} \mu_0 \boldsymbol{H}_j^{\mathrm{eff}} \\ \boldsymbol{\mu}^S/e \end{pmatrix}, \tag{B2}$$

where the matrix elements 3×3 tensors that effectively apply the relevant cross products to make Eq. (B2) consistent with the Landau-Lifshitz equation, and where we use the Einstein summation convention for repeated Latin indices.

1. Identifying L^{sm}

Inserting the spin-pumping torque from Eq. (29), the spin current becomes

$$\boldsymbol{I}^S = -X_j \partial_t \boldsymbol{m}_j, \tag{B3}$$

where we defined the 3×3 matrix X_j as

$$X_j = \frac{e}{\hbar} S_j \sum_i \left[\alpha_{ij}^R \tilde{O} + \alpha_{ij}^I \tilde{O}^2\right], \tag{B4}$$

and the 3×3 matrix $\tilde{O}$ implements the cross product $\boldsymbol{z} \times \boldsymbol{v} = \tilde{O}\boldsymbol{v}$ and can be defined in terms of the Levi-Civita tensor. The LLG equation in the absence of spin accumulation (causing the spin-transfer torque to vanish) reads

$$(1 - \alpha_b \tilde{O})\partial_t \boldsymbol{m}_i = \tilde{O}\big(-\gamma\mu_0 \boldsymbol{H}_i^{\mathrm{eff}}\big), \tag{B5}$$

where α_b is the (bulk) Gilbert damping constant. Hence, we identify

$$L_j^{sm} = \gamma X_j \tilde{O}(1-\alpha_b\tilde{O})^{-1} . \tag{B6}$$

2. Identifying L^{ms}

Inserting the spin-transfer torque from Eq. (27) into the LLGS equation in the absence of an effective field, we find

$$\partial_t \boldsymbol{m}_i = \hbar^{-1}(1-\alpha_b\tilde{O})^{-1}\big[\beta_i^I\tilde{O} - \beta_i^R\tilde{O}^2\big]\boldsymbol{\mu}^S , \tag{B7}$$

meaning that we can identify the linear response coefficient L^{ms} as (no Einstein summation)

$$L_i^{ms} = -\frac{S_i\gamma e}{\hbar}(1-\alpha_b\tilde{O})^{-1}\big[\beta_i^I\tilde{O} - \beta_i^R\tilde{O}^2\big]. \tag{B8}$$

3. Deriving relations from the Onsager reciprocal relations

We are now looking to employ Onsager's reciprocal relation,

$$\big[L_i^{sm}(\{-\boldsymbol{m}_j\})\big]^T = L_i^{ms}(\{\boldsymbol{m}_j\}) , \tag{B9}$$

where the superscript T indicates a matrix transpose in the 3×3 Cartesian space. Using the matrix identity $\tilde{O}^3 = -\tilde{O}$, we find that Eq. (B9) implies that

$$\beta_j^I\tilde{O} - \beta_j^R\tilde{O}^2 = \sum_i\big[\alpha_{ij}^I\tilde{O} - \alpha_{ij}^R\tilde{O}^2\big]. \tag{B10}$$

This equality is satisfied if

$$\beta_j = \sum_i \alpha_{ij} , \tag{B11}$$

which generalizes the result from Ref. [35]. Inserting the definitions of these coefficients in the low-temperature limit, we find that

$$\sum_n W_{j\uparrow\downarrow}^{nn} = i\pi\sum_{inm} W_{i\downarrow\uparrow}^{nm}W_{j\uparrow\downarrow}^{mn} , \tag{B12}$$

which we classify as a generalized optical theorem, since in the diagonal case $i=j$, we can rewrite the imaginary part of this to

$$\mathrm{Im}\left[\sum_n W_{i\uparrow\downarrow}^{nn}\right] = \pi\sum_{inm}\big|W_{i\downarrow\uparrow}^{nm}\big|^2 , \tag{B13}$$

which is reminiscent of the optical theorem in wave scattering theory.

APPENDIX C: CONTOUR FIELDS AND KELDYSH ROTATIONS

In this Appendix, we show how the action can be written in the $\pm$ basis, and introduce the Keldysh rotated fields, which differ in the case of fermionic and bosonic fields. In the $\pm$ field basis, the action of the scattering (electron) states, corresponding to the Hamiltonian in Eq. (7), reads

$$\begin{aligned} S_e + S_0 &= \sum_s \int_{-\infty}^{\infty} dt\,\bar{c}_s^+(i\hbar\partial_t - \epsilon)c_s^+ \\ &\quad - \sum_s \int_{-\infty}^{\infty} dt\,\bar{c}_s^-(i\hbar\partial_t - \epsilon)c_s - \\ &= \sum_{s\xi t} \bar{c}_s^\xi(i\hbar\partial_t - \epsilon)c_s^\xi , \end{aligned} \tag{C1}$$

where now c_s is a vector containing the scattering fields, $\bar{c}_s$ denotes its complex conjugate, and ϵ is a diagonal matrix containing all energy eigenvalues of the scattering states. In the final line, we have written the time integration as a sum for concise notation. Additionally, we introduced the sum over "$\pm$" fields as a sum over $\xi = \{+,-\}$, with an implicit negative sign before the "$-$" field, i.e., $\sum_\xi \ldots^\xi = \ldots^+ - \ldots^-$. A similar notation will also be used for the magnon fields below. The negative sign ($\xi = -$) in the integral in Eq. (C1) and in the other actions below originates from reversing the integration limits on the backward contour. The magnon action is

$$\begin{aligned} S_m &= \sum_{\xi abt}\big[\bar{a}_a^\xi\big(i\hbar\partial_t - E_{ab}^A\big)a_a^\xi + \bar{b}_b^\xi\big(i\hbar\partial_t - E_{ab}^B\big)b_b^\xi\big] \\ &\quad - 2\sum_{aa'} J_{aa'}\sqrt{s_a s_{a'}}\,\bar{a}_a^\xi a_{a'}^\xi - 2\sum_{bb'} J_{bb'}\sqrt{s_b s_{b'}}\,\bar{b}_b^\xi b_{b'}^\xi \\ &\quad - 2\sum_{\xi abt} J_{ab}\sqrt{s_a s_b}\big[a_a^\xi b_b^\xi + \bar{a}_a^\xi\bar{b}_b^\xi\big] \\ &\quad - \gamma\mu_0\hbar\sum_{\xi at}\sqrt{\frac{s_a}{2}}\big[H_{a-}^A a_a^\xi + H_{a+}^A\bar{a}_a^\xi\big] \\ &\quad - \gamma\mu_0\hbar\sum_{\xi bt}\sqrt{\frac{s_b}{2}}\big[H_{b-}^B\bar{b}_b^\xi + H_{b+}^B b_b^\xi\big] . \end{aligned} \tag{C2}$$

The first-order electron-magnon interaction is

$$\begin{aligned} S_1 &= -\sum_{\substack{\xi at\\ \alpha\beta}}\sqrt{\frac{2}{s_a}}\big[a_a^\xi\bar{c}_{\downarrow\alpha}^\xi W_{a\downarrow\uparrow}^{\alpha\beta}c_{\uparrow\beta}^\xi + \bar{a}_a^\xi\bar{c}_{\uparrow\alpha}^\xi W_{a\uparrow\downarrow}^{\alpha\beta}c_{\downarrow\beta}^\xi\big] \\ &\quad - \sum_{\substack{\xi bt\\ \alpha\beta}}\sqrt{\frac{2}{s_b}}\big[\bar{b}_b^\xi\bar{c}_{\downarrow\alpha}^\xi W_{b\downarrow\uparrow}^{\alpha\beta}c_{\uparrow\beta}^\xi + b_b^\xi\bar{c}_{\uparrow\alpha}^\xi W_{b\uparrow\downarrow}^{\alpha\beta}c_{\downarrow\beta}^\xi\big] , \end{aligned} \tag{C3}$$

and the second-order term is

$$\begin{aligned} S_2 &= \sum_{\substack{\xi at\\ \alpha\beta}}\frac{1}{s_a}\bar{a}_a^\xi a_a^\xi\big[\bar{c}_{\uparrow\alpha}^\xi W_{a\uparrow\uparrow}^{\alpha\beta}c_{\uparrow\beta}^\xi - \bar{c}_{\downarrow\alpha}^\xi W_{a\downarrow\downarrow}^{\alpha\beta}c_{\downarrow\beta}^\xi\big] \\ &\quad - \sum_{\substack{\xi bt\\ \alpha\beta}}\frac{1}{s_b}\bar{b}_b^\xi b_b^\xi\big[\bar{c}_{\uparrow\alpha}^\xi W_{b\uparrow\uparrow}^{\alpha\beta}c_{\uparrow\beta}^\xi - \bar{c}_{\downarrow\alpha}^\xi W_{b\downarrow\downarrow}^{\alpha\beta}c_{\downarrow\beta}^\xi\big] . \end{aligned} \tag{C4}$$

For a general bosonic field ϕ, the classical (cl) and quantum (q) fields are defined as [30]

$$\phi^{cl/q} = \frac{1}{\sqrt{2}}(\phi^+ \pm \phi^-), \quad \bar{\phi}^{cl/q} = \frac{1}{\sqrt{2}}(\bar{\phi}^+ \pm \bar{\phi}^-) . \tag{C5}$$

In our case, we have $\phi = \{a, b\}$. The upper (lower) sign holds for the classical (quantum) fields. For a fermionic field c, the rotated fields are denoted by 1 and 2, and defined as

$$c^{1/2} = \frac{1}{\sqrt{2}}(c^+ \pm c^-), \qquad \bar{c}^{1/2} = \frac{1}{\sqrt{2}}(\bar{c}^+ \mp \bar{c}^-). \quad \text{(C6)}$$

For fermions, $\bar{c}$ and c are independent variables, not related by complex conjugation.

APPENDIX D: FOURIER TRANSFORM

For a general function of relative time $t - t'$, we define the Fourier transform between the relative time domain and the energy domain as

$$f(\omega) = \int_{-\infty}^{\infty} d(t-t') e^{i\omega(t-t')} f(t-t'), \quad \text{(D1)}$$

$$f(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} f(\omega). \quad \text{(D2)}$$

The delta function can be represented as

$$\delta(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}, \quad \text{(D3)}$$

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(t-t') e^{i\omega(t-t')}. \quad \text{(D4)}$$

Finally, we note a frequently employed identity,

$$-i \int_{-\infty}^{\infty} d(t-t') e^{i\omega(t-t')} \theta(t-t') = (\omega + i\delta)^{-1}, \quad \text{(D5)}$$

$$i \int_{-\infty}^{\infty} d(t-t') e^{i\omega(t-t')} \theta(t'-t) = (\omega - i\delta)^{-1}, \quad \text{(D6)}$$

where δ is an infinitesimal positive quantity.

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