Emergent SU(3) symmetry in a four-leg spin tube

Edmond Orignac
ENSL, CNRS, Laboratoire de Physique, F-69342 Lyon, France

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We consider an antiferromagnetic four-leg spin-1/2 tube using Abelian and non-Abelian bosonization. We show that in the limit of weak interchain coupling, the most relevant interaction gives rise to an emergent SU(3) symmetry, broken only by marginal interactions that can be canceled by diagonal interchain couplings. We discuss the low-energy spectrum in the semiclassical limit and using a mapping to a trimerized SU(3) spin chain. We establish that the correlation functions of ferroquadrupolar operators can be used to reveal the emergent symmetry.

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I. INTRODUCTION

Emergent symmetry [1] is a symmetry that is obtained when restricting to a particular subspace of the Hilbert space, such as the low-energy subspace, and it differs from the symmetry of the full Hamiltonian. A well-known example is furnished by quantum critical points of one-dimensional many-body systems, where scale and conformal invariance [2] determine the low-energy spectrum and the low-energy long-wavelength response functions. In such a case, only a finite number of operators are relevant at the critical point, and when the system is tuned to the critical point, all the irrelevant symmetry-breaking fields are canceled by the renormalization-group flow. As an example of unexpected emergent symmetry, a critical point described by the $SU(3)_1$ Wess-Zumino-Novikov-Witten [2,3] model has been proposed [4] in a spin-2 chain model in which only SU(2) symmetry should be present. However, the critical character of the correlations in that model is still being debated [5].

Cases with a fully gapped or partially gapped interacting system with emergent symmetry are more scarce. One example is a two-leg spin-1/2 ladder with biquadratic interaction that shows SO(4) symmetry [6]. A mechanism for emergent symmetry in gapped systems is dynamical symmetry enlargement (DSE) symmetry, which is produced by a marginally relevant renormalization [7] flow in which the relevant parameters converge asymptotically to those of the SO(2N) Gross-Neveu model [8]. In the two-leg Hubbard ladder, DSE has been predicted from SU(2) in the lattice model to SO(8) in the low-energy theory at half-filling [7,9,10] and from SU(2) in the lattice model to SO(6) at low energy away from half-filling [11,12]. A DSE from SU(4) $\sim$ SO(6) to SO(8) has also been found in a generalization of the Hubbard model with SU(4) symmetry [13], and a DSE from SU(2) to SO(6) in a model of zigzag carbon nanotubes [14]. In all these models, DSE allows us to take advantage of the integrability of the Gross-Neveu model [15,16] to obtain form-factor expansions [17] of the correlation and response functions in the two-leg Hubbard ladder [9,12].

In the present paper, we wish to propose an example of emergent symmetry in a four-leg spin tube system. Spin-1/2 tube systems [18–21] are made of antiferromagnetic spin-1/2 chains with a transverse coupling satisfying periodic boundary conditions, whereas in planar spin ladders the transverse coupling obeys open boundary conditions [22,23]. Experimentally, three-leg spin tubes were proposed in $Na_2V_3O_7$ [24], $CsCrF_4$ [25], and $[(CuCl_2tachH)_3Cl]Cl_2$, and four-leg spin tubes were proposed in Sul-CuC_{14} [26–30].

Planar antiferromagnetic spin-1/2 ladder systems exhibit an even-odd alternation of ground-state magnetic properties: ladders with an odd number of legs present a ground state with quasi-long-range order and a gap branch of linearly dispersing excitations, while ladders with an even number of legs present a ground state with short-range order and gapped excitations [22,23,31]. Such a result is analogous to the alternation between short-range order for integer spin and quasi-long-range order for half-odd integer spin in antiferromagnetic spin chains [32,33], and it can be understood in terms of a topological contribution to the action [34,35]. In spin tubes with an odd number of legs, the periodic character of the transverse interaction can modify the nature of the ground state [18,19,36,37]. In contrast with the gapless three-leg ladder, the three-leg spin tube presents short-range order and a spin gap [18,19] as a result of frustration in the rung direction. With an even number of legs, the transverse interaction is not frustrating, and the spin gap phase of the tube is analogous to that of the planar ladder. In the case of the four-leg spin tube, series expansion studies have confirmed the presence of a spin gap, in the limit of strong rung coupling, but they found a richer excitation spectrum than in the two-leg ladder [21].

In the present manuscript, we consider the four-leg spin-1/2 tube at weak coupling using bosonization [31,38] and conformal field theory methods. We find that although the ground state has the same gap and short-range order as in a planar four-leg ladder, the excitation spectrum presents an emergent SU(3) symmetry. Going beyond the spectrum, we also show that some ferroquadrupolar [39] (or nematic) order parameters can reveal the emergent symmetry via their correlation functions. The microscopic model is introduced in Sec. II, in Sec. III the non-Abelian bosonization is used to reveal the emergent symmetry, and in Sec. IV a more detailed

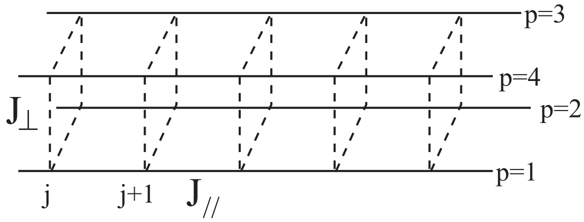

FIG. 1. Interactions in the four-leg tube. Solid lines represent the interactions along the chains, dashed lines represent the interactions along the rung. Chain indices are indicated on the right-hand site.

Abelian bosonization treatment allows us to describe the operators whose correlations reveal the SU(3) symmetry of the low-energy theory. We present our conclusions in Sec. V.

II. MODEL AND HAMILTONIAN

We consider a four-leg spin tube made of four antiferromagnetic spin-$1/2$ chains with intrachain exchange interaction $J_\parallel$ and interchain exchange $J_\perp$. Its Hamiltonian reads

$$H = \sum_{\substack{j=1,N \\ p=1,4}} J_\parallel \mathbf{S}_{j,p} \cdot \mathbf{S}_{j+1,p} + J_\perp S_{j,p} \cdot S_{j,p+1}, \quad (1)$$

with the identification $\mathbf{S}_{j,5} = \mathbf{S}_{j,1}$. The interactions are represented in Fig. 1. We see that the interchain exchange interaction can be rewritten

$$J_\perp \sum_{j=1,N} (\mathbf{S}_{j,1} + \mathbf{S}_{j,3}) \cdot (\mathbf{S}_{j,2} + \mathbf{S}_{j,4}) \quad (2)$$

or

$$\sum_{j=1,N} \frac{J_\perp}{2} [(\mathbf{S}_{j,1} + \mathbf{S}_{j,2} + \mathbf{S}_{j,3} + \mathbf{S}_{j,4})^2 - (\mathbf{S}_{j,1} + \mathbf{S}_{j,3})^2 - (\mathbf{S}_{j,2} + \mathbf{S}_{j,4})^2]. \quad (3)$$

When the squares are decoupled, $J_\parallel = 0$, the spins on the odd and on the even chains add up forming either a spin 0 or a spin 1 state. When at least one of the pairs of spins is in the singlet state, the rung energy (3) vanishes. When both pairs form a triplet, the rung energy is $-2J_\perp$ when the two triplets combine into a singlet, $-J_\perp$ when they combine into a triplet, and $J_\perp$ when they combine into an $S = 2$ quintuplet [21]. When a small $J_\parallel \ll J_\perp$ is introduced, the ground state remains the singlet state formed of two triplets on the diagonals on the square. The lowest energy magnon band results from triplets generated by the pair of triplets on the diagonals of the square. Two magnon bands of higher energy are formed from one diagonal in the triplet state and the other diagonal in the singlet state. Finally, a singlet excitation from both diagonals in the singlet state and an $S = 2$ excitation formed from both diagonals in the triplet state can be obtained [21].

In the opposite limit of $J_\parallel \ll J_\perp$, we consider the model using non-Abelian [40] and Abelian [38] bosonization. The first approach takes full advantage of the symmetries of the model, while the second approach gives a more detailed picture of the relevant observables.

III. NON-ABELIAN BOSONIZATION APPROACH

Using non-Abelian bosonization in the limit of $J_\perp = 0$, the Hamiltonian of the four decoupled chains reads

$$\mathcal{H}_0 = \sum_{p=1}^{4} \frac{2\pi u}{3} \int dx (\mathbf{J}_{R,p} \cdot \mathbf{J}_{R,p} + \mathbf{J}_{L,p} \cdot \mathbf{J}_{L,p}), \quad (4)$$

where $u = \frac{\pi}{2} J_\parallel a$ is the velocity of spin excitations, with a the lattice spacing. The operators $J_{\nu,p}$ ($\nu = R, L$) are the $\mathrm{SU}(2)_1$ currents [40] of a Wess-Zumino-Novikov-Witten (WZNW) model [2,3,41]. Each of these models has central charge $c = 1$. The spin operators on chain j are represented [40] as

$$\mathbf{S}_{j,p} = \mathbf{J}_{R,p}(ja) + \mathbf{J}_{L,p}(ja) + \lambda(-)^j \mathbf{n}_p(ja), \quad (5)$$

where the current operators $\mathbf{J}_{\nu,p}$ of momentum $q \sim 0$ have scaling dimension 1 while the staggered spin operators $\mathbf{n}_p$ of momentum $q \sim \frac{\pi}{a}$ are $\mathrm{SU}(2)_1$ WZNW spin-1/2 primaries with scaling dimension 1/2. The coefficient λ is known quantitatively in XXZ spin-1/2 chains [42–44]. The most relevant contribution in the renormalization-group sense is given by the staggered operators in (2) and reads

$$\mathcal{H}_{\mathrm{int},b} = \frac{J_\perp \lambda^2}{a} \int dx (\mathbf{n}_1 + \mathbf{n}_3) \cdot (\mathbf{n}_2 + \mathbf{n}_4), \quad (6)$$

while the current operators contribute a marginal interaction,

$$\mathcal{H}_{\mathrm{int},f} = \frac{J_\perp}{a} \int dx \sum_{\nu,\nu'=R,L} (\mathbf{J}_{\nu,1} + \mathbf{J}_{\nu,3}) \cdot (\mathbf{J}_{\nu',2} + \mathbf{J}_{\nu',4}), \quad (7)$$

to the full bosonized Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int},f} + \mathcal{H}_{\mathrm{int},b}$. The relevant interaction in Eq. (6) gives rise to a gap $\Delta \sim J_\perp \lambda^2$ in the excitation spectrum, and the marginal interaction in Eq. (7) can yield logarithmic corrections to Δ. There are only two distinct phases, depending on the sign of $J_\perp$. They can be understood qualitatively by taking the limit $|J_\perp/J_\parallel| \to +\infty$ in the lattice model. For $J_\perp > 0$, spin singlets are formed on the square plaquettes (see Fig. 1), and spin-spin correlations decay exponentially with distance. For $J_\perp < 0$, the spins of the square plaquette add up to form an effective spin 2. The antiferromagnetic chain of spin-2 is known to form a Haldane gap [32] in which spin-spin correlations are also decaying exponentially with distance. In the rest of the section, we will explain how the SU(3) symmetry emerges in the gapped phases. The form of the interaction in Eqs. (6) and (7) hints that a coset construction [2]

$$\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_1 \sim \mathrm{SU}(2)_2 \times \mathrm{Ising} \quad (8)$$

can be used to rewrite the model in terms of operators belonging to

$$\mathrm{Ising}_{\mathrm{odd}} \times \mathrm{Ising}_{\mathrm{even}} \times \mathrm{SU}(2)_{2,\mathrm{even}} \times \mathrm{SU}(2)_{2,\mathrm{odd}}, \quad (9)$$

where *odd* indicates that the coset construction is applied to the operators with an odd chain index, and *even* that it is applied to operators with an even chain index. The magnetic degrees of freedom are described by the $\mathrm{SU}(2)_2$ WZNW models, and the remaining nonmagnetic degrees of freedom [45] by the Ising models. Combining together the magnetic degrees

TABLE I. Conformal weights of the right-moving scaling fields in the coset $A(2,2) = \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_2/\mathrm{SU}(2)_4$ coset seen as an $\mathcal{N}=1$ superconformal field theory.

Operator	Weight	Sector
$\phi_{(1,1)R}$	0	Neveu-Schwarz (NS)
$\phi_{(2,1)R}$	$\frac{3}{8}$	Ramond (R)
$\phi_{(3,1)R}$	1	NS
$\phi_{(1,2)R}$	$\frac{1}{16}$	R
$\phi_{(2,2)R}$	$\frac{1}{16}$	NS
$\phi_{(3,2)R}$	$\frac{9}{16}$	R
$\phi_{(1,3)R}$	$\frac{1}{6}$	NS
$\phi_{(2,3)R}$	$\frac{1}{24}$	R

of freedom, a second coset construction [46,47],

$$\mathrm{SU}(2)_{2,\mathrm{even}} \times \mathrm{SU}(2)_{2,\mathrm{odd}} \sim \frac{\mathrm{SU}(2)_2 \times \mathrm{SU}(2)_2}{\mathrm{SU}(2)_4} \times \mathrm{SU}(2)_4, \tag{10}$$

yields the final representation

$$\mathrm{Ising}_{\mathrm{odd}} \times \times \mathrm{Ising}_{\mathrm{even}} \times \frac{\mathrm{SU}(2)_2 \times \mathrm{SU}(2)_2}{\mathrm{SU}(2)_4} \times \mathrm{SU}(2)_4, \tag{11}$$

in which the coset $A(2,2) = \frac{\mathrm{SU}(2)_2\times \mathrm{SU}(2)_2}{\mathrm{SU}(2)_4}$ is of central charge $2\times\frac{3}{2} - 2 = 1$ and belongs to the $\mathcal{N}=1$ superconformal minimal series [2]. The conformal weights h_{rs} of the primary operators ϕ_{rs} of the coset are given by [2]

$$h_{rs} = \frac{(3r-2s)^2 - 1}{48} + \frac{1-(-)^{r-s}}{32}. \tag{12}$$

We have $h_{rs} = h_{4-r,6-s}$ so we only need $s \leqslant 3$ in the Kac Table I.

Now, if we consider the staggered spin operators, according to Eq. (8), the sum of two spin-1/2 primaries (odd or even) in $\mathrm{SU}(2)_1$ can be written as the product of a spin-1/2 primary in $\mathrm{SU}(2)_2$ times an Ising operator [2]. Since a spin j primary in $\mathrm{SU}(2)_k$ has scaling dimension $2\frac{j(j+1)}{k+2}$, the spin-1/2 operators in $\mathrm{SU}(2)_1$ have dimension 1/2 while the spin-1/2 operators in $\mathrm{SU}(2)_2$ have dimension 3/8. The Ising operator has dimension 1/8 and can be taken as an Ising disorder operator giving

$$\mathbf{n}_1 + \mathbf{n}_3 \sim \mu_{\mathrm{odd}}\mathbf{N}_{\mathrm{odd}}, \tag{13}$$

$$\mathbf{n}_2 + \mathbf{n}_4 \sim \mu_{\mathrm{even}}\mathbf{N}_{\mathrm{even}}, \tag{14}$$

allowing us to rewrite the most relevant interaction as

$$\mathcal{H}_{\mathrm{int},b} = \frac{J_\perp \lambda^2}{a}\mu_{\mathrm{odd}}\mu_{\mathrm{even}}\mathbf{N}_{\mathrm{odd}}\cdot\mathbf{N}_{\mathrm{even}}. \tag{15}$$

Now, let us consider $\mathbf{N}_{\mathrm{odd}}\cdot\mathbf{N}_{\mathrm{even}}$ of scaling dimension 3/4. Using the second coset construction, Eq. (10), we can rewrite it as a sum of the product of one operator of the superconformal theory by one operator of the $\mathrm{SU}(2)_4$ theory. Since both operators are of spin 1/2, their product yields operators of spin 0 (identity) or of spin 1 in $\mathrm{SU}(2)_4$. The spin-1 primaries in $\mathrm{SU}(2)_4$ have scaling dimension 2/3, so the operator in the superconformal theory must be of dimension $3/4 - 2/3 = 1/12$. Looking up the Kac table I, it is identified as $\Phi_{(23)}(z,\bar{z}) = \phi_{(23)R}(z)\phi_{(23)L}(\bar{z})$. The operator multiplying the identity in $\mathrm{SU}(2)_4$ has to be of dimension 3/4 and the operator in the superconformal theory with matching dimension [2] is $\Phi_{(21)}(z,\bar{z})$. Thus, we can write

$$\mathbf{N}_{\mathrm{odd}}\cdot\mathbf{N}_{\mathrm{even}} \sim \Phi_{(21)} + \Phi_{(23)}\Phi^{(1)}_{\mathrm{SU}(2)_4}, \tag{16}$$

where $\Phi^{(1)}$ is an SU(2) invariant combination of spin-1 primaries in $\mathrm{SU}(2)_4$. Now, it is known that there exists a conformal embedding [2] $\mathrm{SU}(3)_1 \subset \mathrm{SU}(2)_4$ such that the three $\mathrm{SU}(2)_4$ currents plus the five spin-2 primaries of $\mathrm{SU}(2)_4$ can be written as eight $\mathrm{SU}(3)_1$ currents and the spin-1 primaries can be expressed using the $\mathrm{SU}(3)_1$ primaries in the fundamental representation of SU(3). The interaction (16) can thus be rewritten using only $\mathrm{SU}(3)_1$ operators [48]. This implies that the most relevant interactions, Eq. (6), are giving rise to a gapful ground state in which the symmetry is enlarged from SU(2) to SU(3). In particular, the excited states above the ground state belong to irreducible representations of SU(3). Moreover, some operators transforming according to different irreducible representations of SU(2) can belong to the same irreducible representation of SU(3) and thus exhibit identical correlation functions. Another model having a SU(3) symmetric low-energy spectrum, albeit less realistic than the four-leg tube, is a two-leg ladder made of two spin-1 chains described by the Takhtajan-Babujian Hamiltonian [49,50], whose low-energy excitations are described by the $\mathrm{SU}(2)_2$ Wess-Zumino-Novikov-Witten model [51], and coupled by an exchange interaction. The coset decomposition, Eq. (10), yields the interchain interaction (without Ising disorder fields) and a spectrum with SU(3) symmetry is obtained. Of course, the marginal current-current interaction in Eq. (7) involves only the $\mathrm{SU}(2)_4$ current and none of the spin-2 primaries, and it lowers the symmetry of the full model back to SU(2). However, such a marginal perturbation is expected from perturbation theory to give only corrections $O[J_\perp^2 \ln(J_\parallel/J_\perp)/J_\parallel]$ to the gaps [45] to the excited states, so that for weak coupling, the degeneracy lifting in the spectrum is at a much lower scale than the spin gap $\Delta = O(J_\perp)$. Beyond perturbation theory, the correction from the marginal terms can be estimated by the following renormalization-group argument. If the gap to some excited state is Δ_n, its dependence on the scale ℓ is given by

$$\Delta_n = J_\parallel e^{-\ell}\delta_n\left(\frac{J_\perp}{J_\parallel}e^{\ell}, \frac{J_\perp}{J_\parallel + \mathcal{C}J_\perp\ell}\right), \tag{17}$$

where the dimensionless gap δ_n depends on the dimensionless relevant and marginal couplings, and $\mathcal{C} = O(1)$ is a prefactor entering the marginal flow equation. Renormalizing to the scale $\ell^* = \ln(J_\parallel/|J_\perp|)$, we find that the gap behaves as

$$\Delta_n = J_\perp\delta_n\left(1, \frac{J_\perp}{J_\parallel + \mathcal{C}J_\perp \ln(J_\parallel/|J_\perp|)}\right), \tag{18}$$

and since $\ln(J_\parallel/|J_\perp|) \ll J_\parallel/|J_\perp|$, δ_n can be expanded as a Taylor series. We note that the logarithmic corrections have been resummed in the denominator, and the first correction is then $O(J_\perp^2/J_\parallel) \ll |J_\perp|$ provided $J_\perp \ll J_\parallel$. For $J_\perp/J_\parallel$ sufficiently small, the approximate SU(3) symmetry is preserved. The emergent SU(3) symmetry can be contrasted with the one obtained by DSE in the two-leg Hubbard ladder at half-filling [7]. In the latter case, the coupling constants are all marginally relevant, and under the renormalization-group flow, they flow towards the line that corresponds to the SO(8) Gross-Neveu model. In our case, there are both marginal and relevant couplings. The initial values of the relevant couplings are already on the SU(3) symmetric manifold, and the marginal couplings are driving the flow away from the symmetric manifold. However, their growth under the renormalization group being slow, the renormalized low-energy Hamiltonian always remain close to a Hamiltonian with SU(3) symmetry. In fact, by adding a diagonal rung interaction

$$-\frac{J_\perp}{2}\sum_j[(\mathbf{S}_{j,1}+\mathbf{S}_{j,3})\cdot(\mathbf{S}_{j+1,2}+\mathbf{S}_{j+1,4}) + (\mathbf{S}_{j,2}+\mathbf{S}_{j,4})\cdot(\mathbf{S}_{j+1,1}+\mathbf{S}_{j+1,2})] \quad (19)$$

to the lattice Hamiltonian, Eq. (1), the marginal interaction is entirely canceled [45] and the SU(3) breaking interactions are irrelevant. In such a model, the SU(3) symmetry in the low-energy spectrum is easier to characterize in exact diagonalizations [52]. Another consequence of Eq. (16) is that since (Ising)^2 [53–56], the superconformal $c = 1$ theory [57], and the SU(3)$_1$ theory [58] admit Abelian bosonization [38,59] representations, one can use Abelian bosonization to recover Eq. (16) and express all operators in terms of boson fields. This will be the object of Sec. IV. In the present section, we recall briefly the results obtained in Ref. [57]. Both $\phi_{(23)R}$ and $\phi_{(21)R}$ belong to the Ramond sector, and their bosonized expression is [46,57]

$$\phi_{(23)R}(z) = e^{\frac{i\Phi_R(z)}{2\sqrt{3}}}, \quad (20)$$

$$\phi_{(21)R}(z) = e^{\frac{i\sqrt{3}\Phi_R(z)}{2}} \quad (21)$$

for

$$H_R = v\int\frac{dx}{4\pi}(\nabla\Phi_R)^2. \quad (22)$$

Similar expressions hold for the antiholomorphic fields with Φ_L in the place of Φ_R. This leads to a bosonized representation

$$\mathbf{N}_{\rm odd}\cdot\mathbf{N}_{\rm even}\sim\cos(\sqrt{3}\phi_c)+\cos\left(\frac{\phi_c}{\sqrt{3}}\right)\Phi^{(1)}_{\mathrm{SU}(2)_4}, \quad (23)$$

where $\phi_c = (\Phi_R + \Phi_L)/2$. In Eq. (23), $\Phi^{(1)}$ is a combination of left- and right-moving spin-1 primary fields that is invariant under a global SU(2) rotation. To obtain expressions for the spin operators $\mathbf{N}_{\rm odd}$ or $\mathbf{N}_{\rm even}$ themselves, we note that they must be the product of an operator in $A(2,2)$ by a primary operator of spin 1/2. Matching the scaling dimensions gives a dimension $3/8 - 1/4 = 1/8$. According to Table I, there are two possible operators $\Phi_{(1,2)}$ and $\Phi_{(2,2)}$ with the required dimension. Both of them are twisted fields that do not have a representation in terms of a boson field [57]. Moreover, the spin-1/2 primary operators in SU(2)$_4$ cannot be expressed [2] in terms of the operators of SU(3)$_1$. For that reason, the SU(3) symmetry of the low-energy theory is not apparent in the spin-spin correlation functions. However, if we take a tensor $N^a_{\rm odd}N^b_{\rm even}$ which is rewritten as the product of an operator in $A(2,2)$ by a spin-1 primary in SU(2)$_4$, that is, a SU(3)$_1$ primary, its correlation functions can reflect the underlying SU(3) symmetry. This suggests to consider symmetric tensor products, that is, quadrupolar (nematic) order parameters to detect the SU(3) symmetry of the model. Such nematic correlations are accessible in experimental systems by resonant inelastic x-ray scattering measurements [60]. To conclude that section, we note that an alternative coset representation applicable to our model is given by $\mathrm{SU}(2)_1^4\sim\mathrm{SU}(2)_4\times G_4$ with $G_4=\mathbb{Z}_2\times\mathrm{TIM}\times\mathbb{Z}_3$ a tensor product of minimal models [61,62]. While it leads to the same conclusion concerning the SU(3) symmetry of the low-energy Hamiltonian, it treats the odd and even ladders in a less symmetrical way since it is built from successive tensor products $\mathrm{SU}(2)_n\times\mathrm{SU}(2)_1$. This forces us to choose first a pair of spin chains and apply the coset representation (8), then decide in which order the remaining two spin chains are used to form coset representations of the tricritical Ising model (TIM) and of the three-state clock model $\mathbb{Z}_3$. We thus end up with two nonequivalent representations for the nonmagnetic degrees of freedom of our model. Such a representation would in fact be more convenient in a case in which the rung exchange interaction has reflection symmetry only around one of the diagonals of the tube.

IV. ABELIAN BOSONIZATION

In Abelian bosonization [38,63–65], the decoupled chains have the Hamiltonian (24)

$$\mathcal{H}_0=\sum_{j=1}^4\int\frac{dx}{2\pi}u[(\pi\Pi_j)^2+(\partial_x\phi_j)^2], \quad (24)$$

where $[\phi_j(x),\Pi_k(x')] = i\delta_{jk}\delta(x-x')$ and $u=\frac{\pi}{2}J_\parallel a$. Meanwhile, the SU(2)$_1$ currents are

$$J^+_{\nu,p}(x)=\left(J^x_\nu+iJ^y_\nu\right)(x)=\frac{1}{2\pi a}e^{-i\sqrt{2}(\theta_p-r_\nu\phi_p)(x)}, \quad (25)$$

$$J^z_{\nu,p}=\frac{1}{2\pi\sqrt{2}}[r_\nu\pi\Pi_p-\partial_x\phi_p], \quad (26)$$

with $r_R = 1$ and $r_L = -1$ and $\partial_x\theta_p = \pi\Pi_p$, and the spin-1/2 primaries are

$$n^+_p(x)=\left(n^x_p+in^y_p\right)(x)=e^{-i\sqrt{2}\theta_p(x)}, \quad (27)$$

$$n^z_p(x)=\sin\sqrt{2}\phi_p(x), \quad (28)$$

$$\epsilon_p(x)=\cos\sqrt{2}\phi_p(x), \quad (29)$$

where $\epsilon(x)$ is the dimerization operator, such that $\mathbf{S}_{j,p}\cdot\mathbf{S}_{j+1,p}\sim\frac{1}{4}[(\pi\Pi_P)^2+(\partial_x\phi_p)^2]+(-)^j\bar{\lambda}\epsilon_p(ja)$. The coefficient $\bar{\lambda}$ has been determined in the case of XXZ spin-1/2 chains [44,66].

A. Hamiltonian in Abelian bosonization

1. Derivation of the low-energy Hamiltonian

Introducing [67,68]

$$\theta_{o,r}=\frac{1}{\sqrt{2}}(\theta_1+r\theta_3)\ \phi_{o,r}=\frac{1}{\sqrt{2}}(\phi_1+r\phi_3), \quad (30)$$

$$\theta_{e,r}=\frac{1}{\sqrt{2}}(\theta_2+r\theta_4)\ \phi_{o,r}=\frac{1}{\sqrt{2}}(\phi_2+r\phi_4), \quad (31)$$

the Hamiltonian of the decoupled chains becomes

$$\mathcal{H}_0 = \sum_{\substack{\nu=e,o\\ r=\pm}} \int \frac{dx}{2\pi} u[(\pi \Pi_{\nu,r})^2 + (\partial_x \phi_{\nu,r})^2], \tag{32}$$

and it can be rewritten in terms of Majorana fermions [6,45] as

$$\mathcal{H}_0 = -i\frac{u}{2} \sum_{\nu=e,o j=0,1,2,3} \int dx(\zeta_{R,\nu,j}\partial_x\zeta_{R,\nu,j} - \zeta_{L,\nu,j}\partial_x\zeta_{L,\nu,j}), \tag{33}$$

where we have defined ($\nu = e, o$)

$$\frac{1}{\sqrt{2}}(\zeta_{R,\nu,1} + i\zeta_{R,\nu,2}) = \frac{e^{i(\theta_{\nu+}-\phi_{\nu+})}}{\sqrt{2\pi\alpha}}\eta_{\nu+}, \tag{34}$$

$$\frac{1}{\sqrt{2}}(\zeta_{R,\nu,3} + i\zeta_{R,\nu,0}) = \frac{e^{i(\theta_{\nu-}-\phi_{\nu-})}}{\sqrt{2\pi\alpha}}\eta_{\nu-}, \tag{35}$$

$$\frac{1}{\sqrt{2}}(\zeta_{L,\nu,1} + i\zeta_{L,\nu,2}) = \frac{e^{i(\theta_{\nu+}+\phi_{\nu+})}}{\sqrt{2\pi\alpha}}\eta_{\nu+}, \tag{36}$$

$$\frac{1}{\sqrt{2}}(\zeta_{L,\nu,3} + i\zeta_{L,\nu,0}) = \frac{e^{i(\theta_{\nu-}+\phi_{\nu-})}}{\sqrt{2\pi\alpha}}\eta_{\nu-}, \tag{37}$$

with $\{\eta_{\nu r}, \eta_{\nu' r'}\} = 2\delta_{\nu\nu'}\delta_{rr'}$ Majorana fermion operators that ensure anticommutation of fermions with different ν or r indices [59]. Introducing the corresponding Ising order and disorder parameters [53–56,69,70], the most relevant interaction becomes (see Appendix A for details)

$$\begin{aligned}\mathcal{H}_{\text{int},b} &= \frac{J_\perp\lambda^2}{a}\int dx(\mathbf{n}_1 + \mathbf{n}_3)\cdot(\mathbf{n}_2 + \mathbf{n}_4)\\ &= -\frac{J_\perp\lambda^2}{a}\int dx\mu_{e,0}\mu_{o,0}\left[\sum_{j=1}^{3}\mu_{o,j}\mu_{e,j}\prod_{\substack{1\leqslant k\leqslant 3\\ k\neq j}}\sigma_{o,k}\sigma_{e,k}\right].\end{aligned} \tag{38}$$

We now pair differently the Majorana fermion operators entering the Hamiltonian (33) to form new Dirac fermions and define new boson fields ϑ_j, φ_j such that

$$\begin{aligned}\psi_{R,j} &= \frac{1}{\sqrt{2}}(\zeta_{R,e,j} + i\zeta_{R,o,j}) = \frac{e^{i\vartheta_j - i\varphi_j}}{\sqrt{2\pi\alpha}}\eta_j,\\ \psi_{L,j} &= \frac{1}{\sqrt{2}}(\zeta_{L,e,j} + i\zeta_{L,o,j}) = \frac{e^{i\vartheta_j + i\varphi_j}}{\sqrt{2\pi\alpha}}\eta_j.\end{aligned} \tag{39}$$

We can express products of Ising order and disorder operators in terms of the new fields using [69,70]

$$\cos\varphi_j = \mu_{e,j}\mu_{o,j} \sin\varphi_j = i\sigma_{e,j}\sigma_{o,j}\eta_{e,j}\eta_{o,j}, \tag{40}$$

$$\cos\vartheta_j = \sigma_{e,j}\mu_{o,j}i\eta_j\eta_{e,j} \sin\vartheta_j = \mu_{e,j}\sigma_{o,j}i\eta_j\eta_{o,j}. \tag{41}$$

In Eqs. (39) and (40), $\eta_j, \eta_{e/o,j}$ are Majorana fermion operators normalized by $\eta_j^2 = \eta_{e/o,j}^2 = 1$. Using Eqs. (40), we rewrite the interchain coupling in the form

$$\begin{aligned}\mathcal{H}_{\text{int},b} = \frac{J_\perp\lambda^2}{a}\int dx\cos\varphi_0[&\cos(\varphi_1+\varphi_2-\varphi_3)\\ &+\cos(\varphi_3+\varphi_1-\varphi_2)+\cos(\varphi_2+\varphi_3-\varphi_1)\\ &-3\cos(\varphi_1+\varphi_2+\varphi_3)],\end{aligned} \tag{42}$$

while the Hamiltonian of the decoupled chains reads

$$\mathcal{H}_0 = \sum_{j=0}^{3}\int\frac{dx}{2\pi}u[(\partial_x\vartheta_j)^2 + (\partial_x\varphi_j)^2]. \tag{43}$$

To make the SU(3) symmetry apparent, we introduce the linear combinations of the boson fields [58,71]

$$\begin{pmatrix}\varphi_c\\ \varphi_a\\ \varphi_b\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} \tag{44}$$

to obtain

$$\begin{aligned}\mathcal{H}_{\text{int},b} = \frac{J_\perp\lambda^2}{a}\int dx\cos\varphi_0\bigg[&2\cos\left(\frac{\varphi_c}{\sqrt{3}} - \sqrt{\frac{2}{3}}\varphi_b\right)\cos\sqrt{2}\varphi_a\\ &+\cos\left(\frac{\varphi_c}{\sqrt{3}} + 2\sqrt{\frac{2}{3}}\varphi_b\right) - 3\cos\sqrt{3}\varphi_c\bigg],\end{aligned} \tag{45}$$

with the Hamiltonian of the decoupled chains

$$\mathcal{H}_0 = \sum_{\nu=0,a,b,c}\int\frac{dx}{2\pi}u[(\partial_x\vartheta_\nu)^2 + (\partial_x\varphi_\nu)^2]. \tag{46}$$

In Eq. (45), the fields all have scaling dimension 1, yielding a spin gap $\Delta \sim J_\perp\lambda^2$ as in planar ladders [18,31,45,67], and long-range ordering for the fields $\varphi_{0,a,b,c}$. The interchain interaction, Eq. (45), is minimized by $\langle\varphi_{0,a,b}\rangle = 0$, and $\pm\langle\varphi_c\rangle/\sqrt{3} = \pi - \arccos(1/\sqrt{3})$. As a consequence, exponentials of any dual field $\vartheta_{0,a,b,c}$ have autocorrelation functions decaying exponentially with distance [38].

2. *Symmetries of the low-energy Hamiltonian*

Equation (45) is expressed in terms of the Dirac fermion operators (39) in the form

$$\begin{aligned}\mathcal{H}_{\text{int},b} &= \frac{J_\perp\lambda^2}{a}\int dx\cos\varphi_0\left[\sum_j(e^{i\sqrt{3}\varphi_c}e^{-2i\varphi_j} + \text{H.c.}) - 3\cos\sqrt{3}\varphi_c\right]\\ &= 2\pi J_\perp\lambda^2\int dx\cos\varphi_0\left[i\sum_j(e^{i\sqrt{3}\varphi_c}\psi^\dagger_{R,j}\psi_{L,j} - \text{H.c.})\right]\\ &\quad - \frac{3J_\perp\lambda^2}{a}\int dx\cos\varphi_0\cos\sqrt{3}\varphi_c.\end{aligned} \tag{47}$$

According to Eq. (44), the total fermion density is

$$-\frac{1}{\pi}\partial_x(\varphi_1+\varphi_2+\varphi_3) = -\frac{\sqrt{3}}{\pi}\partial_x\varphi_c, \tag{48}$$

so that any SU(3) rotation $U^\dagger\psi_{r,j}U = U_{jj'}\psi_{rj'}$ with $r = R, L$ leaves invariant φ_c. Then, since $\sum_j \psi^\dagger_{R,j}\psi_{L,j}$ is also invariant, the interchain interaction (47) is invariant. An alternative derivation of Eq. (47) that does not rely on Majorana fermions and Ising order and disorder operators is shown in

Appendix B. Although it allows us to establish the SU(3) symmetry of the interaction, it is less convenient to derive bosonized representations of observables.

In terms of conformal field theory, in Eq. (45) the fields $e^{i(\sqrt{2}\varphi_a \pm \sqrt{2/3}\varphi_b)}$ and $e^{i\sqrt{8/3}\varphi_b}$ have scaling dimension $2/3$, which matches [2] the scaling dimension of $\mathrm{SU}(2)_4$ primary fields of spin 1. They correspond to the operator $\mathrm{Tr}(g)$ in the $\mathrm{SU}(3)_1$ WZNW model. Using Eq. (16), the fields $e^{i\sqrt{3}\varphi_c}$ and $e^{i\varphi_c/\sqrt{3}}$ are, respectively, identified with the operators $\phi_{(21)}$ and $\phi_{(23)}$ of the superconformal coset in complete agreement with Ref. [57] and Eq. (23).

Now, let us briefly discuss the other symmetries of Eq. (45). In Eq. (45), the sign of $J_\perp$ can be absorbed by making $\varphi_0 \to \varphi_0 + \pi$ or $\varphi_c \to \varphi_c + \pi\sqrt{3}$. The interaction is a periodic function of the fields and even in φ_0 and φ_a. It is also invariant under the simultaneous sign change of φ_c and φ_b. It has periodicity under translations

$$\varphi_c \to \varphi_c + \frac{2\pi}{\sqrt{3}} n_c, \tag{49}$$

$$\varphi_b \to \varphi_b - \frac{\pi}{\sqrt{6}} n_c + \pi\sqrt{\frac{3}{2}} n_b, \tag{50}$$

$$\varphi_a \to \varphi_a + \frac{\pi}{\sqrt{2}}(n_b + n_c + 2n_a), \tag{51}$$

with n_a, n_b, n_c integers. Finally, it is invariant under the $\frac{2\pi}{3}$ rotation

$$\varphi_a = -\frac{1}{2}\varphi_a' - \frac{\sqrt{3}}{2}\varphi_b', \tag{52}$$

$$\varphi_b = \frac{\sqrt{3}}{2}\varphi_a' - \frac{1}{2}\varphi_b', \tag{53}$$

which amounts to a circular permutation of $\varphi_{1,2,3}$.

B. SU(3) currents and conserved quantities

Having derived the bosonized Hamiltonian of the four-leg tube, we now turn to the generators of SU(3) symmetry. Their density and currents are given by the SU(3) right- and left-moving currents. We will first discuss the $\mathrm{SU}(2)_4$ currents, and then we will turn to the spin-2 primaries.

1. SU(2) currents

We first turn our attention to the $\mathrm{SU}(2)_4$currents. The sum of the right-moving currents in odd and even chains is expressed in terms of Majorana fermions as [45]

$$J^a_{R1} + J^a_{R3} = -\frac{i}{2}\epsilon_{abc}\zeta_{R,o,b}\zeta_{R,o,c}, \tag{54}$$

$$J^a_{R2} + J^a_{R4} = -\frac{i}{2}\epsilon_{abc}\zeta_{R,e,b}\zeta_{R,e,c}, \tag{55}$$

so we can rewrite their sum using

$$\Psi_R = \begin{pmatrix} \psi_{R,1} \\ \psi_{R,2} \\ \psi_{R,3} \end{pmatrix} \tag{56}$$

in the form

$$\sum_{n=1}^{4} J^x_{R,n} = \Psi_R^\dagger \Lambda^7 \Psi_R, \tag{57}$$

$$\sum_{n=1}^{4} J^y_{R,n} = -\Psi_R^\dagger \Lambda^5 \Psi_R, \tag{58}$$

$$\sum_{n=1}^{4} J^z_{R,n} = \Psi_R^\dagger \Lambda^2 \Psi_R, \tag{59}$$

where $\Lambda^{2,5,7}$ are Gell-Mann matrices [72,73]. Similar relations hold for the left-moving currents $J^{x,y,z}_{L,n}$. The matrices $(\Lambda^7, -\Lambda^5, \Lambda^2)$ generate a spin-1 su(2) subalgebra of the su(3) algebra [73] engendered by the full set of Gell-Mann matrices. With the unitary transformation

$$\begin{pmatrix} \psi_{R,1} \\ \psi_{R,2} \\ \psi_{R,3} \end{pmatrix} = e^{i\frac{\pi}{4}\Lambda_1} e^{i\frac{\pi}{4}(\Lambda_3 - \sqrt{3}\Lambda_8)} \begin{pmatrix} \bar{\psi}_{R,1} \\ \bar{\psi}_{R,-1} \\ \bar{\psi}_{R,0} \end{pmatrix}, \tag{60}$$

we can write

$$\sum_{n=1}^{4} J^x_{R,n} = \bar{\Psi}_R^\dagger(x) \frac{\Lambda_5 - \Lambda_7}{\sqrt{2}} \bar{\Psi}_R(x), \tag{61}$$

$$\sum_{n=1}^{4} J^y_{R,n} = -\bar{\Psi}_R^\dagger(x) \frac{\Lambda_4 + \Lambda_6}{\sqrt{2}} \bar{\Psi}_R(x), \tag{62}$$

$$\sum_{n=1}^{4} J^z_{R,n} = -\bar{\Psi}_R^\dagger(x) \Lambda_3 \bar{\Psi}_R(x), \tag{63}$$

and recover (up to a $\pi/2$ rotation around the z axis) the expression of the spin currents in terms of $\mathrm{SU}(3)_1$ operators [71] obtained when considering the bilinear-biquadratic spin-1 chain [74] at the Uimin-Lai-Sutherland [75–77] critical point. Bosonizing the $\bar{\Psi}$ fermions, and introducing fields $\bar{\varphi}_{a,b,c}$ and their duals $\bar{\vartheta}_{a,b,c}$ as in Eqs. (39)–(44), we find

$$-\frac{1}{\pi\sqrt{2}}\left(\sum_{p=1}^{4} \partial_x \phi_p\right) = -\frac{\sqrt{2}}{\pi}\partial_x \bar{\varphi}_a, \tag{64}$$

allowing us to relate the total magnetization with $\partial_x \bar{\varphi}_a$. Similarly, the total magnetization current is related with $\partial_x \bar{\vartheta}_a$. After performing the $\frac{\pi}{2}$ rotation around the z-axis, we find the bosonized expression

$$\sum_{n=1}^{4} \left(J^x_{R,n} + iJ^y_{R,n}\right) = \frac{e^{-i\frac{\bar{\vartheta}_a - \bar{\varphi}_a}{\sqrt{2}}}}{\pi\alpha\sqrt{2}}\Big[e^{-i\sqrt{\frac{3}{2}}(\bar{\vartheta}_b - \bar{\varphi}_b)}\eta_1\eta_0 + e^{i\sqrt{\frac{3}{2}}(\bar{\vartheta}_b - \bar{\varphi}_b)}\eta_0\eta_{-1}\Big], \tag{65}$$

which recovers the coset representation [78] $\mathrm{SU}(2)_4 \sim \mathrm{U}(1) \times \mathbb{Z}_4$ of the $\mathrm{SU}(2)_4$ currents, with U(1) a free $c = 1$ bosonic theory and $\mathbb{Z}_4$ the four-state clock model [79] with $c = 1$. The right-moving parafermion field of dimension $3/4$ is given by

$$\psi_{R,\mathbb{Z}_4} \sim e^{-i\sqrt{\frac{3}{2}}(\bar{\vartheta}_b - \bar{\varphi}_b)}\eta_1\eta_0 + e^{i\sqrt{\frac{3}{2}}(\bar{\vartheta}_b - \bar{\varphi}_b)}\eta_0\eta_{-1}. \tag{66}$$

Using that coset decomposition, we obtain [78] the spin-1 primary operators of $SU(2)_4$ in the form

$$\Phi^{(1)}_{[11]} \sim e^{i\sqrt{2}\bar{\varphi}_a}\sigma_2, \tag{67}$$

$$\Phi^{(1)}_{[00]} \sim \varepsilon^{(1)}, \tag{68}$$

where σ_2 is the spin field of dimension 1/6, and $\varepsilon^{(1)}$ is the thermal operator of the $\mathbb{Z}_4$ clock model. We can identify $\sigma_2 \sim \cos(\sqrt{2/3}\bar{\varphi}_a)$ and $\varepsilon^{(1)} \sim \cos(\sqrt{8/3}\bar{\varphi}_a)$ by comparing with (45). In the ground state of the four-leg tube, the $\mathbb{Z}_4$ degrees of freedom exhibit long-range ordering. If we consider the spin-1/2 $SU(2)_4$ primaries, we have

$$\Phi^{(1/2)}_{[1/2,1/2]} \sim \sigma_1 e^{i\frac{\bar{\varphi}_a}{\sqrt{2}}}, \tag{69}$$

$$\Phi^{(1/2)}_{[1/2,-1/2]} \sim \mu_1 e^{i\frac{\bar{\vartheta}_a}{\sqrt{2}}}, \tag{70}$$

where σ_1 and μ_1 are the spin field of dimension 1/8 of the $\mathbb{Z}_4$ clock model and its dual. Since φ_a is ordered, the field $e^{i\frac{\bar{\vartheta}_a}{\sqrt{2}}}$ in the second line has short-range order. The SU(2) symmetry then implies that σ_1 is also short-range-ordered, and μ_1 must be long-range-ordered.

2. Spin-2 primaries

The five remaining $SU(3)_1$ currents are

$$\Psi_R^\dagger \Lambda^{1,3,4,6,8}\Psi_R, \tag{71}$$

but substituting (39) in the above expression shows that it depends on products $i\zeta_{R,o,\alpha}\zeta_{R,e,\beta}$ in the original decoupled chains basis. Hence, these operators are not local operators in the initial lattice model. However, since

$$J^a_{R,1} - J^a_{R,3} = i\zeta_{R,o,0}\zeta_{R,o,a}, \tag{72}$$

$$J^a_{R,2} - J^a_{R,4} = i\zeta_{R,e,0}\zeta_{R,e,a}, \tag{73}$$

we have

$$\begin{aligned}&\big(J^1_{R,1} - J^1_{R,3}\big)\big(J^2_{R,2} - J^2_{R,4}\big) + \big(J^2_{R,1} - J^2_{R,3}\big)\big(J^1_{R,2} - J^1_{R,4}\big)\\ &\quad = i\zeta_{R,o,0}\zeta_{R,e,0}\Psi_R^\dagger\Lambda^1\Psi_R,\end{aligned} \tag{74}$$

$$\begin{aligned}&\big(J^1_{R,1} - J^1_{R,3}\big)\big(J^1_{R,2} - J^1_{R,4}\big) - \big(J^2_{R,1} - J^2_{R,3}\big)\big(J^2_{R,2} - J^2_{R,4}\big)\\ &\quad = i\zeta_{R,o,0}\zeta_{R,e,0}\Psi_R^\dagger\Lambda^3\Psi_R,\end{aligned} \tag{75}$$

$$\begin{aligned}&\big(J^1_{R,1} - J^1_{R,3}\big)\big(J^3_{R,2} - J^3_{R,4}\big) + \big(J^3_{R,1} - J^3_{R,3}\big)\big(J^1_{R,2} - J^1_{R,4}\big)\\ &\quad = i\zeta_{R,o,0}\zeta_{R,e,0}\Psi_R^\dagger\Lambda^4\Psi_R,\end{aligned} \tag{76}$$

$$\begin{aligned}&\big(J^2_{R,1} - J^2_{R,3}\big)\big(J^3_{R,2} - J^3_{R,4}\big) + \big(J^3_{R,1} - J^3_{R,3}\big)\big(J^2_{R,2} - J^2_{R,4}\big)\\ &\quad = i\zeta_{R,o,0}\zeta_{R,e,0}\Psi_R^\dagger\Lambda^6\Psi_R,\end{aligned} \tag{77}$$

$$\begin{aligned}&\big(J^1_{R,1} - J^1_{R,3}\big)\big(J^1_{R,2} - J^1_{R,4}\big) + \big(J^2_{R,1} - J^2_{R,3}\big)\big(J^2_{R,2} - J^2_{R,4}\big)\\ &\qquad - 2\big(J^3_{R,1} - J^3_{R,3}\big)\big(J^3_{R,2} - J^3_{R,4}\big)\\ &\quad = i\sqrt{6}\zeta_{R,o,0}\zeta_{R,e,0}\Psi_R^\dagger\Lambda^8\Psi_R,\end{aligned} \tag{78}$$

showing that tensor products of current differences are expressible with the $SU(2)_4$ spin-2 primaries.

3. Conserved quantities

If we turn to globally conserved quantities, the isospin $I_3 = I_{3,R} + I_{3,L}$ in the SU(3) theory is given by

$$I_3 = \frac{1}{2}\int dx \sum_\nu \bar{\Psi}_\nu^\dagger(x)\Lambda_3\bar{\Psi}_\nu(x), \tag{79}$$

and identifies with half the total spin of the lattice system. The other two components of the spin also give rise to conserved quantities, but they do not commute with I_3. But, the second conserved quantity is the hypercharge

$$Y = \frac{1}{\sqrt{3}}\int dx \sum_\nu \bar{\Psi}_\nu^\dagger(x)\Lambda^8\bar{\Psi}_\nu(x), \tag{80}$$

which is a nonlocal quantity in the original spin variables. Therefore, although the low-energy excited states are classified by irreducible representations of SU(3) and possess both isospin I_3 and hypercharge Y, only the former can be determined from local observables of the lattice model. In particular, the group SU(3) possesses two nonequivalent irreducible representations [80] of dimension 3, called $\mathbf{3}$ and $\bar{\mathbf{3}}$ with opposite isospins and hypercharges. But since only I_3 can be measured from the total spin, those representations appear as two SU(2) spin-1 triplets in the spectrum. More generally, the irreducible representations of SU(3) decompose into direct sums of irreducible representations of SU(2) of given total spin. In the presence of SU(3) symmetry, degeneracies between states of different total spin are obtained. For the sake of concreteness, let us consider two elementary examples. If we take the tensor product [80] of SU(3) representations,

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \tag{81}$$

seen as an SU(2) tensor product $1 \otimes 1 = 0 \oplus 1 \oplus 2$, the one-dimensional representation of SU(3) identifies with the SU(2) spin singlet, while the eight-dimensional representation is reducible into the direct sum of SU(2) spin-1 and spin-2 representations. When the SU(3) symmetry is present, spin-1 and spin-2 states forming the eight representations are degenerate in energy. If we now take the tensor product

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \tag{82}$$

interpreted in terms of SU(2) representations, we have again the tensor product $1 \otimes 1 = 0 \oplus 1 \oplus 2$. Since the $\bar{\mathbf{3}}$ representation of SU(3) has to be identified with the spin-1 representation of SU(2), the $\mathbf{6}$ representation is reducible into a sum of spin-2 and a spin-0 representation of SU(2). So when the SU(3) symmetry is present, a degeneracy between spin-2 and spin-0 states is observed. So, even though the hypercharge cannot be measured from the local spin observables, the SU(3) symmetry manifests itself in the form of apparently accidental degeneracies in the spectrum. Moreover, considering the degeneracies of states containing two triplet excitations can reveal the representation of SU(3) to which the triplet belongs, and thus indirectly characterize their hypercharge.

C. Excited states

1. Soliton and antisolitons

In the semiclassical limit, excitations above the ground state take the form of solitons that interpolate between the different minima of Eq. (45). The fields $\varphi_{0,a,b,c}(x)$ take different limits $\varphi_{0,a,b,c}(\pm\infty)$ as $x \to \pm\infty$ such that the potential Eq. (45) has the same limit for $x \to \pm\infty$. Introducing the notation

$$\Delta\varphi_\nu = \int_{-\infty}^{+\infty} dx \partial_x \varphi_\nu(x) = \varphi_\nu(+\infty) - \varphi_\nu(-\infty), \quad (83)$$

we define the charge Q, isospin I_3, and hypercharge Y of a soliton given by

$$Q = -\frac{\sqrt{3}}{\pi}\Delta\varphi_c, \quad (84)$$

$$I_3 = -\frac{1}{\pi\sqrt{2}}\Delta\varphi_a, \quad (85)$$

$$Y = -\frac{\sqrt{2}}{\pi\sqrt{3}}\Delta\varphi_b. \quad (86)$$

2. Magnetic solitons and antisolitons

Let us first consider the semiclassical limit of (45) and search for the quantum numbers of solitons and antisolitons. If we consider solitons in which $\Delta\varphi_0 = \varphi_0(+\infty) - \varphi_0(-\infty) = \pi$, equating the limits at $\pm\infty$ of the potential yields

$$\begin{aligned} &\sqrt{3}\Delta\varphi_c = (2n_c + 1)\pi, \\ &\frac{\Delta\varphi_c}{\sqrt{3}} - \sqrt{2}\Delta\varphi_a - \sqrt{\frac{2}{3}}\Delta\varphi_b = (2n_1 + 1)\pi, \\ &\frac{\Delta\varphi_c}{\sqrt{3}} + \sqrt{2}\Delta\varphi_a - \sqrt{\frac{2}{3}}\Delta\varphi_b = (2n_{-1} + 1)\pi, \\ &\frac{\Delta\varphi_c}{\sqrt{3}} + 2\sqrt{\frac{2}{3}}\Delta\varphi_b = (2n_0 + 1)\pi, \end{aligned} \quad (87)$$

where $n_{c,-1,0,1}$ are integers, to ensure that the potential has the same limit at $\pm\infty$. Combining the above equations yields $n_c = n_0 + n_1 + n_{-1} + 1$ and

$$I_3 = \tfrac{1}{2}(n_1 - n_{-1}), \quad Y = \tfrac{1}{3}(n_1 + n_{-1} - 2n_0). \quad (88)$$

To minimize Q, we have to set $n_0 + n_1 + n_{-1} = -2$ ($Q = 1$) or $n_0 + n_1 + n_{-1} = -1$ ($Q = -1$). In the first case, $n_k = 0, n_{j\neq k} = -1$, we have charge $Q = 1$ and isospin and hypercharge $(I_3, Y) \in \{(-1/2, 1/3), (1/2, 1/3), (0, -2/3)\}$. The rotation (52) can be used to generate all of them starting, for instance, with the one of isospin 0 and hypercharge $-2/3$. These solitons carry the same quantum numbers as the fermions $\psi_{R/L,j}$ ($j = 1, 2, 3$) but they also carry the topological charge associated with φ_0, so the bosonized form of their creation operator contains a factor $e^{-i(\vartheta_j \pm \vartheta_0)}$. Given their quantum numbers, the solitons transform in the **3** representation of SU(3).

In the second case, we must set $n_k = -1, n_{j\neq k} = 0$, for $k = -1, 0, 1$ to find antisolitons with SU(3) isospin and hypercharge $(I_3, Y) \in \{(1/2, -1/3), (1/2, -1/3), (0, 2/3)\}$. They carry quantum numbers as the antifermions $\psi_{R/L,j}$ ($j = 1, 3$), as well as the topological charge associated with φ_0, so the bosonized expression of the antisoliton creation operator contains $e^{i\vartheta_j \pm \vartheta_0}$. The antisolitons transform in the $\bar{\mathbf{3}}$ representation of SU(3). In terms of spin, since $S^z = 2I_3$, the solitons and the antisolitons give rise to two degenerate branches of gapped spin-1 excitations. Equations (72) show that the Matsubara response functions of current differences contain contributions from solitons and antisolitons that give rise to sharp peaks in the dynamical structure factor after analytic continuation. Topological excitations with different Q, I_3, Y might also exist at the semiclassical level, and would correspond, for instance, to bound states of solitons and/or antisolitons (breathers) [81]. However, it is unclear which of these bound states persists at the fully quantum level. In the case of the integrable quantum sine-Gordon model [15,82,83], it is known that the number of bound states depends on the Tomonaga-Luttinger exponent. As the Tomonaga-Luttinger exponent increases, the number of breathers decreases, and beyond a critical value, solitons and antisolitons do not form bound states. In our case, the interaction Eq. (45) does not seem to lead to an integrable model, and the breather stability remains an open question. We can only state that if breather excitations exist, they must organize in SU(3) multiplets.

3. Trimerized SU(3) spin chain

To form a more accurate image of the magnetic solitons and antisolitons, we need to return to the original quantum Hamiltonian. We will only assume that the fields φ_0 and φ_c having long-range order can be replaced by their expectation value in Eq. (45), and the resulting low-energy Hamiltonian reduces to the bosonized Hamiltonian of a trimerized SU(3) spin chain [58,71],

$$H = \sum_n (J + \delta J_n) \sum_{a=1}^{8} \lambda_n^a \lambda_{n+1}^a, \quad (89)$$

$$J_n = J + \delta J\left(e^{i[\frac{2\pi}{3}n - \frac{\langle\varphi_c\rangle}{\sqrt{3}}]} + e^{-i[\frac{2\pi}{3}n - \frac{\langle\varphi_c\rangle}{\sqrt{3}}]}\right), \quad (90)$$

where the SU(3) spins are in the **3** representation, J is chosen [75–77] to reproduce the excitation velocity u, and $\delta J \ll J$ is proportional to $J_\perp\langle\cos\varphi_0\rangle$. In that improved approximation, only the fields carrying nonmagnetic degrees of freedom are treated semiclassically. For $\langle\varphi_c\rangle = 0$ and $\delta J < 0$, the periodic pattern satisfies $0 < J_{3n} < J_{3n+2} = J_{3n+1}$, and we can consider as a strong-coupling fixed point a trimerized chain made of independent groups of 3 SU(3) spins that form a singlet in the ground state, as shown in Fig. 2(a). For $\langle\varphi_c\rangle = 0$ and $\delta J > 0$, the periodic pattern satisfies $0 < J_{3n+2} = J_{3n+1} < J_{3n}$, so we can take strong-coupling fixed point pairs of spins on the strong bond forming an effective spin in the $\mathbf{3}^*$ representation. We obtain a chain in which spins in the representation **3** and $\bar{\mathbf{3}}$ alternate and the spectrum is gapped [84,85]. We can picture the ground state as the spontaneous formation of singlet pairs with spins in the **3** and $\bar{\mathbf{3}}$ representation. Let us discuss first the case of $\delta J < 0$. When solitons are present, $\langle\varphi_0\rangle$ is shifted by π and $\langle\varphi_c\rangle/\sqrt{3}$ is shifted by $\pi/3$, so that the trimerization pattern shifts by two lattice spacings when moving from $-\infty$ to $+\infty$. A dimer defect is introduced somewhere along the chain, giving rise [see Fig. 2(b)] to a spin in the $\bar{3}$ representation. With $\langle\varphi_c\rangle/\sqrt{3}$ shifted by $-\pi$, a single SU(3) spin in

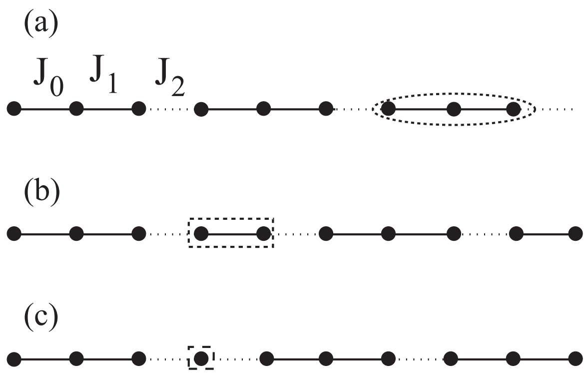

FIG. 2. (a) The exchange coupling in the trimerized SU(3) spin chain. $J_0 = J_1$ are the string links, and J_2 is the weak link. The ellipse represents 3 SU(3) spin forming a single state. (b) Domain wall in which the trimerization pattern has been shifted by one lattice spacing. A SU(3) dimer (indicated by a rectangle) is formed. It allows the formation of an excitation belonging to the $\bar{\mathbf{3}}$ representation. (c) Domain wall in which the pattern has been shifted by two lattice spacings. An isolated SU(3) spin is present.

the $\mathbf{3}$ is present [see Fig. 2(c)]. With $\delta J > 0$, solitons create defects $\bar{\mathbf{3}} - \mathbf{3} - \mathbf{3} - \bar{\mathbf{3}}$ that give rise to an unpaired spin in the $\mathbf{3}$ representation and antisoliton defects $\mathbf{3} - \bar{\mathbf{3}} - \bar{\mathbf{3}} - \mathbf{3}$ that give to an unpaired spin in the $\bar{\mathbf{3}}$ representation. We can now return to the question of soliton/antisoliton bound states. If we consider pairs of solitons in the $\mathbf{3}$ representation, of antisolitons in the $\bar{\mathbf{3}}$ representation, or a soliton antisoliton pair, we need to consider the tensor products [80]

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \tag{91}$$

$$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \mathbf{3} \oplus \bar{\mathbf{6}}, \tag{92}$$

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \tag{93}$$

In the simplest case, the only bound states of two solitons are antisolitons, the only bound states of two antisolitons are solitons, and there are no soliton-antisoliton bound states, so that solitons and antisolitons are the only excitations. The representations $\mathbf{6}, \bar{\mathbf{6}}, \mathbf{8}$ then correspond to excitations in the continuum formed of unbound soliton/antisoliton pairs. In terms of SU(2) representations, only two degenerate branches of gapped triplet excitations are present besides the continuum.

In a slightly more complicated case, soliton-antisolitons bound states (breathers) in the $\mathbf{8}$ representation are also present. In terms of SU(2) representation, the $\mathbf{8}$ representation gives a gapped branch of spin-2 excitations degenerate with a gapped branch of spin-1 excitations. To characterize the breather-soliton, breather-antisoliton, and breather-breather bound states, we need the tensor products [80]

$$\mathbf{3} \otimes \mathbf{8} = \mathbf{3} \oplus \mathbf{6} \oplus \mathbf{15}, \tag{94}$$

$$\bar{\mathbf{3}} \otimes \mathbf{8} = \bar{\mathbf{3}} \oplus \bar{\mathbf{6}} \oplus \bar{\mathbf{15}}, \tag{95}$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{27}, \tag{96}$$

indicating that the bound state of a soliton (antisoliton) with a breather is a soliton (antisoliton), and bound states of breathers are breathers. In terms of the trimerized SU(3) chain, the excitations of a trimer are obtained by considering the tensor products

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}, \tag{97}$$

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \tag{98}$$

$$\mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10}, \tag{99}$$

and they would allow for both signs of δJ a delocalized excitation in the $\mathbf{8}$ representation of SU(3).

4. Nonmagnetic excitations

In addition to excitations possessing SU(3) spin and hypercharge, we can also have excitations involving only φ_0 and φ_c. For instance, when only φ_c is varying, the potential reduces to

$$J_\perp \lambda^2 [3 \cos(\varphi_c/\sqrt{3}) - 3 \cos \sqrt{3}\varphi_c], \tag{100}$$

and it allows for short kinks interpolating from $\pi - \arccos(1/\sqrt{3})$ to $\pi + \arccos(1/\sqrt{3})$ and long kinks from $\arccos(1/\sqrt{3}) - \pi$ to $\pi - \arccos(1/\sqrt{3})$. It is also possible to have kinks where $\Delta\langle\varphi_0\rangle = \pi$ and $\Delta\langle\varphi_c\rangle = \pm\pi\sqrt{3}$. All those kinks are SU(3) singlets and possess a noninteger charge Q. If they survive in the quantum limit, they give rise to branches of gapped spin-singlet excitations.

D. Observables

We would like to determine the observables that make the SU(3) symmetry of the model apparent. Since the spin-1/2 primaries in $SU(2)_4$ are in the twisted sector [2], they cannot be realized with $SU(3)_1$ primaries. Thus, we need to consider operators containing the product of two spin-1/2 primaries that can be expressed in terms of $SU(2)_4$ spin 1 primaries that are also $SU(3)_1$ primaries. Obvious candidates are the vector chiralities [39] $(\mathbf{n}_1 \pm \mathbf{n}_3) \times (\mathbf{n}_2 \pm \mathbf{n}_3)$, and the nematic order parameter $2Q^{ab}_{\pm\pm} = (n^a_1 \pm n^a_3)(n^b_2 \pm n^b_4) + (n^b_1 \pm n^b_3)(n^a_2 \pm n^a_4) - \delta_{ab}(\mathbf{n}_1 \pm \mathbf{n}_3) \cdot (\mathbf{n}_2 \pm \mathbf{n}_4)/3$. We can also consider operators [60] formed from the product of a dimerization operator by a staggered magnetization such as $(\epsilon_1 \pm \epsilon_3)(n^a_2 \pm n^a_4) + (\epsilon_2 \pm \epsilon_4)(n^a_1 \pm n^a_3)$. All these operators transform according to spin-1 or spin-2 representations of SU(2), and in a one-dimensional spin gapped system, their correlation functions decay exponentially with distance. *A priori*, the correlation functions of operators transforming in a spin-2 representation of SU(2) should not be related to those of operators transforming in a spin-1 representation. However, we will see that the enlargement of the symmetry to SU(3) makes the nematic operator and the product of dimerization by staggered magnetization transform in the same representation of SU(3) so that their correlation functions become proportional to each other.

1. Symmetric case

Let us first consider the case with both symmetric combinations, $(\mathbf{n}_1 + \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)$ and $2Q^{ab}_{++} = (n^a_1 + n^a_3)(n^b_2 + n^b_4) + (n^b_1 + n^b_3)(n^a_2 + n^a_4) - \delta_{ab}(\mathbf{n}_1 + \mathbf{n}_3) \cdot (\mathbf{n}_2 + \mathbf{n}_4)/3$. Using the expression in terms of Ising order and disorder operators, we find

$$\begin{aligned}(n_2+n_4)^1(n_1+n_3)^2 &= (\mu_{e1}\sigma_{o1})(\sigma_{e2}\mu_{o2})(\sigma_{e3}\sigma_{o3})(\mu_{e0}\mu_{o0})\\ &\quad\times(i\eta_{e2}\eta_{e3})(i\eta_{o3}\eta_{o1}), \qquad (101)\\ &= -i\eta_1\eta_2\sin\vartheta_1\cos\vartheta_2\sin\varphi_3\cos\varphi_0,\end{aligned}$$

and similarly, exchanging e and o indices,

$$\begin{aligned}(n_2+n_4)^2(n_1+n_3)^1 &= (\mu_{o1}\sigma_{e1})(\sigma_{o2}\mu_{e2})(\sigma_{e3}\sigma_{o3})(\mu_{e0}\mu_{o0})\\ &\quad\times(i\eta_{o2}\eta_{o3})(i\eta_{e3}\eta_{e1}),\\ &= +i\eta_1\eta_2\cos\vartheta_1\sin\vartheta_2\sin\varphi_3\cos\varphi_0,\end{aligned} \tag{102}$$

yielding

$$\begin{aligned}Q^{12}_{++} &= \sin(\vartheta_2-\vartheta_1)\sin\varphi_3\cos\varphi_0 i\eta_1\eta_2,\\ &\quad\times[(\mathbf{n}_2+\mathbf{n}_4)\times(\mathbf{n}_1+\mathbf{n}_3)]^3\\ &= -\sin(\vartheta_2+\vartheta_1)\sin\varphi_3\cos\varphi_0 i\eta_1\eta_2.\end{aligned} \tag{103}$$

The other components are obtained by circular permutations. For the diagonal components of the nematic order parameter, we find

$$Q^{11}_{++}-Q^{22}_{++} = -\sin(\varphi_2-\varphi_1)\sin\varphi_3\cos\varphi_0, \tag{104}$$

$$\begin{aligned}Q^{33}_{++} = &-\cos\varphi_0[\cos(\varphi_2+\varphi_3-\varphi_1)+\cos(\varphi_3+\varphi_1-\varphi_2)\\ &-2\cos(\varphi_1+\varphi_2-\varphi_3)].\end{aligned} \tag{105}$$

Now, let us write Q^{ab}_{++} in terms of fermion operators. Using $\varphi_3=\sqrt{3}\varphi_c-\varphi_1-\varphi_2$, we can show that

$$Q^{12}_{++} \sim i\cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_1\Psi_L-\text{H.c.}\big], \tag{106}$$

$$Q^{23}_{++} \sim i\cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_6\Psi_L-\text{H.c.}\big], \tag{107}$$

$$Q^{13}_{++} \sim i\cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_4\Psi_L-\text{H.c.}\big], \tag{108}$$

$$Q^{11}_{++}-Q^{12}_{++} \sim i\cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_3\Psi_L-\text{H.c.}\big], \tag{109}$$

$$Q^{33}_{++} \sim i\cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_8\Psi_L-\text{H.c.}\big], \tag{110}$$

showing that the nematic order parameter transforms according to the **8** representation of SU(3). Now, if we turn to

$$\begin{aligned}&(n_2+n_4)^1(\epsilon_1+\epsilon_3)+(n_1+n_3)^1(\epsilon_2+\epsilon_4)\\ &\quad= -\cos(\vartheta_2-\vartheta_3)\cos\varphi_1\cos\varphi_0 i\eta_2\eta_3,\end{aligned} \tag{111}$$

and similar expressions obtained by circular permutations, we find

$$\begin{aligned}&(n_2+n_4)^1(\epsilon_1+\epsilon_3)+(n_1+n_3)^1(\epsilon_2+\epsilon_4)\\ &\quad\sim \cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_7\Psi_L+\text{H.c.}\big],\end{aligned} \tag{112}$$

$$\begin{aligned}&(n_2+n_4)^2(\epsilon_1+\epsilon_3)+(n_1+n_3)^2(\epsilon_2+\epsilon_4)\\ &\quad\sim \cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_5\Psi_L+\text{H.c.}\big],\end{aligned} \tag{113}$$

$$\begin{aligned}&(n_2+n_4)^3(\epsilon_1+\epsilon_3)+(n_1+n_3)^3(\epsilon_2+\epsilon_4)\\ &\quad\sim \cos\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi_R^\dagger\Lambda_2\Psi_L+\text{H.c.}\big],\end{aligned} \tag{114}$$

showing that these operators also transform according to the **8** representation of SU(3). If we consider their correlation functions, since $\cos\varphi_0$, $\sin\varphi_3$, and $\cos\varphi_3$ are all long-range-ordered, their exponential decay is determined by the one of $e^{i(\vartheta_i-\vartheta_j)}$. As a result, they must present the same correlation length as Q^{ab}_{++}, and at long distance, the correlation functions are proportional to each other. The difference in amplitude results from the different expectation values $\langle\cos\varphi_3\rangle \neq \langle\sin\varphi_3\rangle$ and the different prefactors λ and $\bar{\lambda}$. This proportionality is a first sign of the hidden SU(3) symmetry of the model. Moreover, in the case in which stable breathers belonging to the **8** representation exist, an excited state containing a single breather will present a nonvanishing matrix element with the ground state for one of the eight operators we have just identified. Calling q that operator, the Fourier transform of its ground-state correlator $\langle\{q(x,t),q(0,0)\}\rangle$ contains a contribution

$$|\langle B,k|q|0\rangle|^2\delta\big(\omega-\sqrt{(uk)^2+m_B^2}\big), \tag{115}$$

separate from any continuum. The dynamical structure factors of the operators Q^{ab}_{++} and $(\mathbf{n_2}+\mathbf{n_4})(\epsilon_1+\epsilon_3)+(\mathbf{n_1}+\mathbf{n_2})(\epsilon_2+\epsilon_4)$ then show sharp peaks associated with the breathers. In the lattice model, the operators to consider are

$$\begin{aligned}Q^{ab}_{++}(n) \sim \tfrac{1}{2}\big[&(S_{n,1}+S_{n,3}-S_{n-1,1}-S_{n-1,3})^a\\ &\times(S_{n,2}+S_{n,4}-S_{n-1,2}-S_{n-1,4})^b\\ &+(S_{n,2}+S_{n,4}-S_{n-1,2}-S_{n-1,4})^a\\ &\times(S_{n,1}+S_{n,3}-S_{n-1,1}-S_{n-1,3})^b\\ &-\tfrac{2}{3}(\mathbf{S}_{n,1}+\mathbf{S}_{n,3}-\mathbf{S}_{n-1,1}-\mathbf{S}_{n-1,3})\\ &\times\cdot(\mathbf{S}_{n,2}+\mathbf{S}_{n,4}-\mathbf{S}_{n-1,2}-\mathbf{S}_{n-1,4})\delta_{ab}\big]\end{aligned} \tag{116}$$

for the ferroquadrupolar order parameter, where $\mathbf{S}_{n,p}-\mathbf{S}_{n-1,p}$ is used to filter out the contribution from $\mathbf{J}_{R,p}+\mathbf{J}_{L,p}$ in Eq. (5) and retain only $\mathbf{n}_p$. Similarly, for the vector operator $(\epsilon_1+\epsilon_3)(\mathbf{n}_2+\mathbf{n}_4)+(\epsilon_2+\epsilon_4)(\mathbf{n}_1+\mathbf{n}_3)$, the lattice expression is

$$\begin{aligned}&(\epsilon_1+\epsilon_3)(\mathbf{n}_2+\mathbf{n}_4)+(\epsilon_2+\epsilon_4)(\mathbf{n}_1+\mathbf{n}_3)\\ &\quad\sim[\mathbf{S}_{n,1}\cdot(\mathbf{S}_{n+1,1}-\mathbf{S}_{n-1,1})+\mathbf{S}_{n,3}\cdot(\mathbf{S}_{n+1,3}-\mathbf{S}_{n-1,3})]\\ &\quad\times(\mathbf{S}_{n,2}+\mathbf{S}_{n,4}-\mathbf{S}_{n-1,2}-\mathbf{S}_{n-1,4})\\ &\quad+[\mathbf{S}_{n,2}\cdot(\mathbf{S}_{n+1,2}-\mathbf{S}_{n-1,2})+\mathbf{S}_{n,4}\cdot(\mathbf{S}_{n+1,4}-\mathbf{S}_{n-1,4})]\\ &\quad\times(\mathbf{S}_{n,1}+\mathbf{S}_{n,3}-\mathbf{S}_{n-1,1}-\mathbf{S}_{n-1,3}).\end{aligned} \tag{117}$$

Turning to $(\mathbf{n}_1+\mathbf{n}_3)\times(\mathbf{n}_2+\mathbf{n}_4)$, we find

$$\begin{aligned}&(n_1+n_3)^1(n_2+n_4)^2-(n_1+n_3)^2(n_2+n_4)^1\\ &\quad=\sin(\vartheta_1+\vartheta_2)\sin\varphi_3\cos\varphi_0 i\eta_1\eta_2,\end{aligned} \tag{118}$$

$$\begin{aligned}&(n_1+n_3)^2(n_2+n_4)^3-(n_1+n_3)^3(n_2+n_4)^2\\ &\quad=\sin(\vartheta_2+\vartheta_3)\sin\varphi_1 i\cos\varphi_0\eta_2\eta_3,\end{aligned} \tag{119}$$

$$\begin{aligned}&(n_1+n_3)^3(n_2+n_4)^1-(n_1+n_3)^1(n_2+n_4)^3\\ &\quad=\sin(\vartheta_1+\vartheta_3)\sin\varphi_2\cos\varphi_0 i\eta_3\eta_1,\end{aligned} \tag{120}$$

allowing us to rewrite

$$\begin{aligned}[(\mathbf{n}_1+\mathbf{n}_3)\times(\mathbf{n}_2+\mathbf{n}_4)]_j &= \sin(\sqrt{3}\vartheta_c-\vartheta_j)\sin\varphi_j\cos\varphi_0\frac{i}{2}\epsilon_{jkl}\eta_j\eta_l \\ &\sim \frac{1}{4}[e^{i\sqrt{3}\vartheta_c}(\psi^\dagger_{Rj}-\psi^\dagger_{Lj})-\text{H.c.}]\cos\varphi_0 \end{aligned} \tag{121}$$

showing that these operators transform in the **3** and $\bar{\mathbf{3}}$ representations of SU(3). However, since they do not shift φ_0, their matrix elements between the ground state and states containing a soliton or an antisoliton vanish. On the lattice, those operators can be written

$$\begin{aligned}(\mathbf{n}_1+\mathbf{n}_3)\times(\mathbf{n}_2+\mathbf{n}_4) &\sim (\mathbf{S}_{n+1,1}+\mathbf{S}_{n+1,3}-\mathbf{S}_{n,1}-\mathbf{S}_{n,3}) \\ &\quad\times(\mathbf{S}_{n+1,2}+\mathbf{S}_{n+1,4}-\mathbf{S}_{n,2}-\mathbf{S}_{n,4}).\end{aligned} \tag{122}$$

The other symmetric operators are $\overleftrightarrow{Q}_{--}=(\mathbf{n}_1-\mathbf{n}_3)\otimes(\mathbf{n}_2-\mathbf{n}_4)$, $(\mathbf{n}_2-\mathbf{n}_4)(\epsilon_1-\epsilon_3)+(\mathbf{n}_1-\mathbf{n}_3)(\epsilon_2-\epsilon_4)$, and $(\mathbf{n}_1-\mathbf{n}_3)\times(\mathbf{n}_2-\mathbf{n}_4)$. The expression of Q_{--} is deduced from that of Q_{++} by the duality transformation (see Appendix A) $\mu_{o,j}\leftrightarrow\sigma_{o,j}$ and $\mu_{e,j}\leftrightarrow\sigma_{e,j}$. As a result, its bosonized expression is given by the change of variable $\varphi_j\to\frac{\pi}{2}-\varphi_j$ and $\vartheta_j\to\frac{\pi}{2}-\vartheta_j$. Under such duality,

$$\psi_{Rj}\to\psi^\dagger_{Rj}, \tag{123}$$

$$\psi_{Lj}\to-\psi^\dagger_{Lj}, \tag{124}$$

and $\sqrt{3}\varphi_c\to\frac{3\pi}{2}-\sqrt{3}\varphi_c$. We then obtain from Eq. (106)

$$Q^{12}_{--}\sim\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_1\Psi_L+\text{H.c.}\big], \tag{125}$$

$$Q^{23}_{--}\sim\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_6\Psi_L+\text{H.c.}\big], \tag{126}$$

$$Q^{13}_{--}\sim\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_4\Psi_L+\text{H.c.}\big], \tag{127}$$

$$Q^{11}_{--}-Q^{12}_{--}\sim\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_3\Psi_L+\text{H.c.}\big], \tag{128}$$

$$Q^{33}_{--}\sim\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_8\Psi_L+\text{H.c.}\big]. \tag{129}$$

Applying the same argument to (112), we obtain

$$\begin{aligned}&(n_2-n_4)^1(\epsilon_1-\epsilon_3)+(n_1-n_3)^1(\epsilon_2-\epsilon_4) \\ &\sim i\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_7\Psi_L-e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_L\Lambda_7\Psi_R\big],\end{aligned} \tag{130}$$

$$\begin{aligned}&(n_2-n_4)^2(\epsilon_1-\epsilon_3)+(n_1-n_3)^2(\epsilon_2-\epsilon_4) \\ &\sim i\sin\varphi_0\big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_5\Psi_L-e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_L\Lambda_5\Psi_R\big],\end{aligned} \tag{131}$$

$$\begin{aligned}&(n_2+n_4)^3(\epsilon_1+\epsilon_3)+(n_1+n_3)^3(\epsilon_2+\epsilon_4) \\ &\sim i\sin\varphi_0\Big[e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_R\Lambda_2\Psi_L-e^{-i\sqrt{3}\varphi_c}\Psi^\dagger_L\Lambda_2\Psi_R\Big],\end{aligned} \tag{132}$$

so both $\overleftrightarrow{Q}_{--}$ and $(\mathbf{n}_2-\mathbf{n}_4)(\epsilon_1-\epsilon_3)+(\mathbf{n}_1-\mathbf{n}_3)(\epsilon_2-\epsilon_4)$ transform in the **8** representation of SU(3). Because of the presence of the factor $\sin\varphi_0$, the correlation function of operators Q^{ab}_{--} is shorter than that of the operators Q^{ab}_{++}. We also find that

$$\begin{aligned}&[(\mathbf{n}_1-\mathbf{n}_3)\times(\mathbf{n}_2-\mathbf{n}_4)]_j \\ &\sim\tfrac{1}{4}\big[e^{i\sqrt{3}\vartheta_c}(\psi^\dagger_{Rj}+\psi^\dagger_{Lj})+\text{H.c.}\big]\sin\varphi_0,\end{aligned} \tag{133}$$

so $(\mathbf{n}_1-\mathbf{n}_3)\times(\mathbf{n}_2-\mathbf{n}_4)$ is a linear combination of operators transforming in the **3** and $\bar{\mathbf{3}}$ representations. Lattice expressions can be obtained on the model of Eqs. (116), (117), and (122). Explicitly,

$$\begin{aligned}Q^{ab}_{--}(n)\sim\tfrac{1}{2}\big[&(S_{n,1}-S_{n,3}-S_{n-1,1}+S_{n-1,3})^a \\ &\times(S_{n,2}-S_{n,4}-S_{n-1,2}+S_{n-1,4})^b \\ &+(S_{n,2}-S_{n,4}-S_{n-1,2}+S_{n-1,4})^a \\ &\times(S_{n,1}-S_{n,3}-S_{n-1,1}+S_{n-1,3})^b \\ &-\tfrac{2}{3}(\mathbf{S}_{n,1}-\mathbf{S}_{n,3}-\mathbf{S}_{n-1,1}+\mathbf{S}_{n-1,3}) \\ &\times\cdot(\mathbf{S}_{n,2}-\mathbf{S}_{n,4}-\mathbf{S}_{n-1,2}+\mathbf{S}_{n-1,4})\delta_{ab}\big],\end{aligned} \tag{134}$$

$$\begin{aligned}&(\epsilon_1-\epsilon_3)(\mathbf{n}_2-\mathbf{n}_4)+(\epsilon_2-\epsilon_4)(\mathbf{n}_1-\mathbf{n}_3) \\ &\sim[\mathbf{S}_{n,1}\cdot(\mathbf{S}_{n+1,1}-\mathbf{S}_{n-1,1})-\mathbf{S}_{n,3}\cdot(\mathbf{S}_{n+1,3}-\mathbf{S}_{n-1,3})] \\ &\times(\mathbf{S}_{n,2}-\mathbf{S}_{n,4}-\mathbf{S}_{n-1,2}+\mathbf{S}_{n-1,4}) \\ &+[\mathbf{S}_{n,2}\cdot(\mathbf{S}_{n+1,2}-\mathbf{S}_{n-1,2})-\mathbf{S}_{n,4}\cdot(\mathbf{S}_{n+1,4}-\mathbf{S}_{n-1,4})] \\ &\times(\mathbf{S}_{n,1}-\mathbf{S}_{n,3}-\mathbf{S}_{n-1,1}+\mathbf{S}_{n-1,3}),\end{aligned} \tag{135}$$

$$\begin{aligned}(\mathbf{n}_1-\mathbf{n}_3)\times(\mathbf{n}_2-\mathbf{n}_4) &\sim(\mathbf{S}_{n+1,1}-\mathbf{S}_{n+1,3}-\mathbf{S}_{n,1}+\mathbf{S}_{n,3}) \\ &\quad\times(\mathbf{S}_{n+1,2}-\mathbf{S}_{n+1,4}-\mathbf{S}_{n,2}+\mathbf{S}_{n,4}).\end{aligned} \tag{136}$$

2. *Asymmetric case*

We now turn to combinations of operators that are symmetric on one pair of legs and antisymmetric on the other pair. We begin with operators symmetric on the even legs, and antisymmetric on the odd legs. For the operator $Q^{ab}_{-+}=(n_1-n_3)^a(n_2+n_4)^b+(n_1-n_3)^b(n_2+n_4)^a$, we find

$$Q^{12}_{-+}=\sin(\varphi_1-\varphi_2)\cos\vartheta_3\sin\vartheta_0 i\eta_3\eta_0, \tag{137}$$

$$Q^{23}_{-+}=\sin(\varphi_2-\varphi_3)\cos\vartheta_1\sin\vartheta_0 i\eta_1\eta_0, \tag{138}$$

$$Q^{31}_{-+}=\sin(\varphi_3-\varphi_1)\cos\vartheta_2\sin\vartheta_0 i\eta_2\eta_0, \tag{139}$$

$$Q^{11}_{-+}-Q^{22}_{-+}=\sin(\vartheta_1-\vartheta_2)\cos\vartheta_3\sin\vartheta_0\eta_1\eta_2\eta_3\eta_0, \tag{140}$$

$$\begin{aligned}&Q^{11}_{-+}+Q^{22}_{-+}-2Q^{33}_{-+} \\ &=\tfrac{1}{2}[\sin(\vartheta_1+\vartheta_3-\vartheta_2)+\sin(\vartheta_2+\vartheta_3-\vartheta_1) \\ &\quad-2\sin(\vartheta_1+\vartheta_2-\vartheta_3)]\sin\vartheta_0\eta_1\eta_2\eta_3\eta_0,\end{aligned} \tag{141}$$

$$\begin{aligned}\text{Tr}(Q_{-+})=\tfrac{1}{4}[&3\sin(\vartheta_1+\vartheta_2+\vartheta_3)+\sin(\vartheta_1+\vartheta_2-\vartheta_3) \\ &+\sin(\vartheta_3+\vartheta_1-\vartheta_2)+\sin(\vartheta_2+\vartheta_3-\vartheta_1)] \\ &\times\sin\vartheta_0\eta_1\eta_2\eta_3\eta_0,\end{aligned} \tag{142}$$

which can be rewritten

$$Q^{12}_{-+} \sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_1\Psi_L + \text{H.c.}\big]\eta_1\eta_2\eta_3\eta_0, \tag{143}$$

$$Q^{23}_{-+} \sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_6\Psi_L + \text{H.c.}\big]\eta_1\eta_2\eta_3\eta_0, \tag{144}$$

$$Q^{31}_{-+} \sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_4\Psi_L + \text{H.c.}\big]\eta_1\eta_2\eta_3\eta_0, \tag{145}$$

$$\begin{aligned} Q^{11}_{-+} - Q^{22}_{-+} &\sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_3\Psi_L + \text{H.c.}\big]\eta_1\eta_2\eta_3\eta_0,\\ Q^{11}_{-+} + Q^{22}_{-+} - 2Q^{33}_{-+} &\sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_8\Psi_L + \text{H.c.}\big]\\ &\quad\times\eta_1\eta_2\eta_3\eta_0, \end{aligned} \tag{146}$$

$$\text{Tr}(Q_{-+}) \sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}(C + \Psi_R\Psi_L) + \text{H.c.}\big]\eta_1\eta_2\eta_3\eta_0, \tag{147}$$

showing (see Appendix C) that Q^{ab}_{-+} is a linear combination of operators transforming in the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representations of SU(3). In contrast with the symmetric case, all operators have the same prefactor λ^2 and there are no differences in expectation values. As a result, all the operators have the same autocorrelation function except $\text{Tr}(Q_{-+})$, which has an extra contribution from the correlator of $e^{-i\sqrt{3}\theta_c}$. That contribution does not produce cross correlation with $e^{-i\sqrt{3}\theta_c}\Psi_R\Psi_L$ thanks to the unbroken U(1) symmetry $\vartheta_c \to \vartheta_c + \alpha$. Moreover, $e^{-i\sqrt{3}\theta_c}$ creates an excitation with larger charge Q than the operators transforming in the $\mathbf{6}, \bar{\mathbf{6}}$ representations. In turn, this implies the creation of a larger number of solitons and a faster exponential decay for the correlator of $e^{-i\sqrt{3}\theta_c}$. We have found this time that an operator transforming in the singlet representation of SU(2), Tr(Q_{-+}), and five operators $Q^{a\neq b}_{-+}$, $Q^{xx}_{-+} - Q^{yy}_{-+}$, and $Q^{xx}_{-+} + Q^{yy}_{-+} - 2Q^{zz}_{-+}$ transforming in the spin-2 representation turn out to transform in the same representation of SU(3). Asymptotically, these six operators show the same correlation function, revealing SU(3) symmetry. If we turn to $(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)$,

$$[(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)]^1 = \cos\vartheta_1\sin(\varphi_2 + \varphi_3)\sin\vartheta_0 i\eta_1\eta_0, \tag{148}$$

$$[(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)]^2 = \cos\vartheta_2\sin(\varphi_1 + \varphi_3)\sin\vartheta_0 i\eta_2\eta_0, \tag{149}$$

$$[(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)]^3 = \cos\vartheta_3\sin(\varphi_1 + \varphi_2)\sin\vartheta_0 i\eta_3\eta_0, \tag{150}$$

we can rewrite

$$\begin{aligned}&[(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)]^j\\ &\quad\sim \tfrac{1}{4}\sin\vartheta_0\big[e^{i\sqrt{3}\varphi_c}(\psi_{Rj} + \psi^\dagger_{Lj}) + \text{H.c.}\big],\end{aligned} \tag{151}$$

showing that $(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)$ is a linear combination of operators transforming in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations of SU(3). We note that the operator carries the same quantum numbers as the solitons or antisolitons, implying that they will give rise to sharp peaks in its dynamical structure factor. If we turn our attention to $(\epsilon_1 - \epsilon_3)(\mathbf{n}_2 + \mathbf{n}_4) - (\mathbf{n}_1 - \mathbf{n}_3)(\epsilon_2 + \epsilon_4)$, we find

$$\begin{aligned}&(\epsilon_1 - \epsilon_3)(n_2 + n_4)^1 - (n_1 - n_3)(\epsilon_2 + \epsilon_4)^1\\ &\quad\sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_7\Psi_L + \text{H.c.}\big],\end{aligned} \tag{152}$$

$$\begin{aligned}&(\epsilon_1 - \epsilon_3)(n_2 + n_4)^2 - (n_1 - n_3)(\epsilon_2 + \epsilon_4)^2\\ &\quad\sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_5\Psi_L + \text{H.c.}\big],\end{aligned} \tag{153}$$

$$\begin{aligned}&(\epsilon_1 - \epsilon_3)(n_2 + n_4)^3 - (n_1 - n_3)(\epsilon_2 + \epsilon_4)^3\\ &\quad\sim \sin\vartheta_0\big[e^{-i\sqrt{3}\vartheta_c}\Psi_R\Lambda_2\Psi_L + \text{H.c.}\big],\end{aligned} \tag{154}$$

showing that $(\epsilon_1 - \epsilon_3)(\mathbf{n}_2 + \mathbf{n}_4) - (\mathbf{n}_1 - \mathbf{n}_3)(\epsilon_2 + \epsilon_4)$ is a linear combination of operators transforming in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representation. On the lattice, the expressions to consider are

$$\begin{aligned}Q^{ab}_{-+}(n) \sim \tfrac{1}{2}[&(S_{n,1} - S_{n,3} - S_{n-1,1} + S_{n-1,3})^a\\ &\times (S_{n,2} + S_{n,4} - S_{n-1,2} - S_{n-1,4})^b\\ &+ (S_{n,2} + S_{n,4} - S_{n-1,2} - S_{n-1,4})^a\\ &\times (S_{n,1} - S_{n,3} - S_{n-1,1} + S_{n-1,3})^b],\end{aligned} \tag{155}$$

$$\begin{aligned}&(\epsilon_1 - \epsilon_3)(\mathbf{n}_2 + \mathbf{n}_4) + (\epsilon_2 + \epsilon_4)(\mathbf{n}_1 - \mathbf{n}_3)\\ &\quad\sim [\mathbf{S}_{n,1}\cdot(\mathbf{S}_{n+1,1} - \mathbf{S}_{n-1,1}) - \mathbf{S}_{n,3}\cdot(\mathbf{S}_{n+1,3} - \mathbf{S}_{n-1,3})]\\ &\qquad\times(\mathbf{S}_{n,2} + \mathbf{S}_{n,4} - \mathbf{S}_{n-1,2} - \mathbf{S}_{n-1,4})\\ &\qquad+ [\mathbf{S}_{n,2}\cdot(\mathbf{S}_{n+1,2} - \mathbf{S}_{n-1,2}) + \mathbf{S}_{n,4}\cdot(\mathbf{S}_{n+1,4} - \mathbf{S}_{n-1,4})]\\ &\qquad\times(\mathbf{S}_{n,1} - \mathbf{S}_{n,3} - \mathbf{S}_{n-1,1} + \mathbf{S}_{n-1,3}),\end{aligned} \tag{156}$$

$$\begin{aligned}(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4) &\sim (\mathbf{S}_{n+1,1} - \mathbf{S}_{n+1,3} - \mathbf{S}_{n,1} + \mathbf{S}_{n,3})\\ &\quad\times(\mathbf{S}_{n+1,2} + \mathbf{S}_{n+1,4} - \mathbf{S}_{n,2} - \mathbf{S}_{n,4}).\end{aligned} \tag{157}$$

We can also consider $Q^{ab}_{+-} = (n_1 + n_3)^a(n_2 - n_4)^b + (n_1 + n_3)^a(n_2 - n_4)^b$ and the vector product $(\mathbf{n}_1 + \mathbf{n}_3) \times (\mathbf{n}_2 - \mathbf{n}_4)$. As before, their bosonized expressions are obtained from those of $\overleftrightarrow{Q}_{+-}$ and $(\mathbf{n}_1 - \mathbf{n}_3) \times (\mathbf{n}_2 + \mathbf{n}_4)$ by the duality transformation $\vartheta_j \to \frac{\pi}{2} - \vartheta_j$ and $\varphi_j \to \frac{\pi}{2} - \varphi_j$. In the end, the operators Q^{ab}_{-+} are also linear combination of operators in the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representation, while the components $(\mathbf{n}_1 + \mathbf{n}_3) \times (\mathbf{n}_2 - \mathbf{n}_4)$ are also linear combination of operators in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ representation. The corresponding expressions on the lattice are obtained by swapping the odd and the even indices in Eqs. (155)–(157).

V. CONCLUSION

We have found that the field theory describing the low-energy excitations of the four-leg spin tube in the limit of weak rung exchange has an enlarged SU(3) symmetry, broken down to SU(2) only by marginal perturbations. By adding diagonal interactions, the marginal perturbations can be canceled, enhancing the SU(3) symmetry of the spectrum at low energy. The spectrum of the low-energy theory organizes in multiplets of SU(3) classified by isospin and hypercharge [72]. While the isospin is directly related with the total spin, the hypercharge is nonlocal in the spin operators of the tube. The SU(3) symmetry is thus revealed by apparently accidental degeneracies of the spectrum when it is decomposed into the expected SU(2) spin multiplets. In particular, we have identified two degenerate SU(2) triplets that correspond to the fundamental and conjugate representations of SU(3). Such degeneracies should be detectable in exact diagonalization studies [52,86]. We have shown that the triplet excitations would give rise to coherent peaks in the dynamical spin structure factor near zero momentum. Moreover, ferroquadrupolar (or nematic) correlations can reveal the enlarged SU(3) symmetry. Such correlations functions are accessible with density matrix renormalization group [87–90] or quantum Monte Carlo [86,91]. The same dynamical symmetry enlargement should be observed in a two-leg spin-1 ladder with biquadratic interactions [74] along the legs when the biquadratic interactions are tuned to the Takhtajan-Babujian [49,50] point. Although this is a less realistic model, it is less computationally expensive for numerical simulations. Concerning the spectrum of the model, open questions remain concerning first the presence of spin singlet gapped excitations resulting from the nonmagnetic modes, and second the existence of soliton-antisoliton bound states. Since the low-energy theory does not seem to be integrable, these questions will have to be addressed by other nonperturbative methods such as the truncated conformal space approximation [92]. Another open issue is the nature of edge states in a semi-infinite four-leg spin tube [93]. If the rung interactions are ferromagnetic, one would expect spin-1 edge states [94] similar to those of the spin-2 chain. If the open boundary conditions are compatible with the bulk SU(3) symmetry, those spin-1 edge states could turn out to be in a **3** or a $\bar{\mathbf{3}}$ representation of SU(3). Beyond the case of spin systems, an emergent SU(3) symmetry should also be present in the four-leg Hubbard tube at half-filling as a consequence of spin-charge separation. Upon doping, the spin gap is robust, and the SU(3) emergent symmetry should be observable in the four-leg Hubbard tube [95–97] and the four-leg t-J tube [98].

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APPENDIX A: EXPRESSION OF STAGGERED SPIN COMPONENTS OF A PAIR OF SPIN-1/2 CHAINS IN TERMS OF ISING ORDER AND DISORDER OPERATORS

Here we summarize the derivation in Refs. [6,45]. For the sake of definiteness, we treat the case of the chains with an odd index. Analogous relations are obtained for the chains of an even index. Considering the addition of two $SU(2)_1$ currents, we find

$$J^+_{R,1} + J^+_{R,3} = \frac{1}{2\pi a}\left[e^{-i\sqrt{2}(\theta_1-\phi_1)(x)} + e^{-i\sqrt{2}(\theta_3-\phi_3)(x)}\right]. \quad \text{(A1)}$$

Introducing

$$\phi_{o,+} = \frac{1}{\sqrt{2}}(\phi_1+\phi_3), \quad \text{(A2)}$$

$$\phi_{o,-} = \frac{1}{\sqrt{2}}(\phi_1-\phi_3), \quad \text{(A3)}$$

and the corresponding dual variables, we rewrite

$$J^+_{R,1} + J^+_{R,3} = \frac{1}{2\pi a}\left[e^{-i(\theta_{o,+}-\phi_{o,+})-i(\theta_{o,-}-\phi_{o,-})} + e^{-i(\theta_{o,+}-\phi_{o,+})+i(\theta_{o,-}-\phi_{o,-})}\right], \quad \text{(A4)}$$

and we introduce fermion operators

$$\psi_{\nu,o,+} = \frac{1}{\sqrt{2\pi a}}e^{i(\theta_{o,+}-r_\nu\phi_{o,+})}\eta_{o,+}, \quad \text{(A5)}$$

$$\psi_{\nu,o,-} = \frac{1}{\sqrt{2\pi a}}e^{i(\theta_{o,-}-r_\nu\phi_{o,-})}\eta_{o,-}. \quad \text{(A6)}$$

We can rewrite the currents

$$J^+_{R,1} + J^+_{R,3} = \psi^\dagger_{R,o,+}\eta_{o,+}\eta_{o,-}(\psi^\dagger_{R,o,-}+\psi_{R,o,-}). \quad \text{(A7)}$$

The operator $i\eta_{o,+}\eta_{o,-}$ is Hermitian and satisfies $(i\eta_{o,+}\eta_{o,-})^2 = 1$. It can be diagonalized [99] with eigenvalues ± 1, allowing us to replace $\eta_{o,+}\eta_{o,-}$ with $\pm i$ in Eq. (A7). Picking $\eta_{o,+}\eta_{o,-} = -i$, and introducing Majorana fermion operators

$$\psi_{R,o,+} = \frac{1}{\sqrt{2}}(\zeta_{R,o,2} + i\zeta_{R,o,1}), \quad \text{(A8)}$$

$$\psi_{R,o,-} = \frac{1}{\sqrt{2}}(\zeta_{R,o,3} + i\zeta_{R,o,0}), \quad \text{(A9)}$$

the currents are finally rewritten in the form

$$J^a_{R,1} + J^a_{R,3} = -\frac{i}{2}\epsilon_{abc}\zeta_{R,o,b}\zeta_{R,o,c}. \quad \text{(A10)}$$

We also have

$$n^+_1 + n^+_3 = 2e^{i\theta_{o,+}}\cos\theta_{o,-}, \quad \text{(A11)}$$

$$n^3_1 + n^3_3 = 2\sin\phi_{o,+}\cos\phi_{o,-}, \quad \text{(A12)}$$

$$\epsilon_1 + \epsilon_3 = 2\cos\phi_{0,+}\cos\phi_{o,-} \quad \text{(A13)}$$

and using [70] we can write

$$\cos\phi_{o,+} = \mu_{o,1}\mu_{o,2}\cos\phi_{o,-} = \mu_{o,3}\mu_{o,0}, \quad \text{(A14)}$$

$$\sin\phi_{o,+} = \sigma_{o,1}\sigma_{o,2}i\eta_{o,1}\eta_{o,2}\sin\phi_{o,-} = \sigma_{o,3}\sigma_{o,0}i\eta_{o,3}\eta_{o,0}, \quad \text{(A15)}$$

$$\cos\theta_{o,+} = \sigma_{o,1}\mu_{o,2}\eta_{o,1}\cos\theta_{o,-} = \sigma_{o,3}\mu_{o,0}\eta_{o,3}, \quad \text{(A16)}$$

$$\sin\theta_{o,+} = \mu_{o,1}\sigma_{o,2}\eta_{o,2}\sin\theta_{o,-} = \mu_{o,3}\sigma_{o,0}\eta_{o,0}, \quad \text{(A17)}$$

where $\sigma_{o,j}$ and $\mu_{o,j}$ are Ising order and disorder operators, and $\eta_{o,j}$ $(j = 0, 1, 2, 3)$ are Majorana fermion operators with $\eta^2_{o,j} = 1$. The $SU(2)_1$ primary operators are rewritten

$$n^1_1 + n^1_3 = \mu_{o,1}\sigma_{o,2}\sigma_{o,3}\mu_{o,0}(i\eta_{o,2}\eta_{o,3}), \quad \text{(A18)}$$

$$n^2_1 + n^2_3 = \sigma_{o,1}\mu_{o,2}\sigma_{o,3}\mu_{o,0}(i\eta_{o,3}\eta_{o,1}), \quad \text{(A19)}$$

$$n^3_1 + n^3_3 = \sigma_{o,1}\sigma_{o,2}\mu_{o,3}\mu_{o,0}(i\eta_{o,1}\eta_{o,2}), \quad \text{(A20)}$$

$$\epsilon_1 + \epsilon_3 = \mu_{o,1}\mu_{o,2}\mu_{o,3}\mu_{o,0}. \quad \text{(A21)}$$

For differences of $SU(2)_1$ primaries, we have

$$n^+_1 - n^+_3 = 2ie^{i\theta_{o,+}}\sin\theta_{o,-}, \quad \text{(A22)}$$

$$n^3_1 - n^3_3 = 2\cos\phi_{o,+}\sin\phi_{o,-}, \quad \text{(A23)}$$

$$\epsilon_1 - \epsilon_3 = 2\sin\phi_{0,+}\sin\phi_{o,-}, \quad \text{(A24)}$$

leading to

$$n_1^1 - n_3^1 = \sigma_{o,1}\mu_{o,2}\mu_{o,3}\sigma_{o,0}(i\eta_{o,1}\eta_{o,0}), \tag{A25}$$

$$n_1^2 - n_3^2 = \mu_{o,1}\sigma_{o,2}\mu_{o,3}\sigma_{o,0}(i\eta_{o,2}\eta_{o,0}), \tag{A26}$$

$$n_1^3 - n_3^3 = \mu_{o,1}\mu_{o,2}\sigma_{o,3}\sigma_{o,0}(i\eta_{o,3}\eta_{o,0}), \tag{A27}$$

$$\epsilon_1 - \epsilon_3 = \sigma_{o,1}\sigma_{o,2}\sigma_{o,3}\sigma_{o,0}\eta_{o,1}\eta_{o,2}\eta_{o,3}\eta_{o,0}. \tag{A28}$$

The differences are obtained from the sums by the duality transformation $\mu \leftrightarrow \sigma$.

With the help of Eq. (A18), the most relevant operator reads

$$\begin{aligned}\mathcal{H}_{\text{int},b} = \frac{J_\perp\lambda^2}{a}\mu_{e,0}\mu_{o,0}[&\mu_{o,1}\mu_{e,1}\sigma_{o,2}\sigma_{e,2}\sigma_{o,3}\sigma_{e,3}\eta_{o,2}\eta_{e,2}\eta_{o,3}\eta_{e,3}\\ &+\mu_{o,2}\mu_{e,2}\sigma_{o,3}\sigma_{e,3}\sigma_{o,1}\sigma_{e,1}\eta_{o,3}\eta_{e,3}\eta_{o,1}\eta_{e,1}\\ &+\mu_{o,3}\mu_{e,3}\sigma_{o,1}\sigma_{e,1}\sigma_{o,2}\sigma_{e,2}\eta_{o,1}\eta_{e,1}\eta_{o,2}\eta_{e,2}].\end{aligned} \tag{A29}$$

The products of Majorana fermion operators $\eta_{o,j}\eta_{e,j}$ commute among themselves, making them simultaneously diagonalizable. Since

$$\eta_{o,2}\eta_{e,2}\eta_{o,3}\eta_{e,3}\eta_{o,3}\eta_{e,3}\eta_{o,1}\eta_{e,1}\eta_{o,1}\eta_{e,1}\eta_{o,2}\eta_{e,2} = -1, \tag{A30}$$

the product of eigenvalues has to be -1. The most symmetrical choice is to take the eigenvalue -1 for all products of four Majorana fermions. This gives Eq. (38).

APPENDIX B: ALTERNATIVE DERIVATION OF THE INTERCHAIN INTERACTION

Using Eqs. (27) and (28), and the fields defined in Eq. (30), we first write the interaction

$$\begin{aligned}(\mathbf{n}_1+\mathbf{n}_3)\cdot(\mathbf{n}_2+\mathbf{n}_3) &= 2\cos(\theta_{e+}-\theta_{o+})\\ &\times[\cos(\theta_{e-}+\theta_{o-})+\cos(\theta_{e-}-\theta_{o-})]\\ &+[\cos(\phi_{e+}-\phi_{o+})-\cos(\phi_{e+}+\phi_{o+})]\\ &\times[\cos(\phi_{e-}-\phi_{o-})+\cos(\phi_{e-}+\phi_{o-})],\end{aligned} \tag{B1}$$

and then we introduce the fields

$$\begin{pmatrix}\phi_c\\ \phi_s\\ \phi_f\\ \phi_{sf}\end{pmatrix} = \begin{pmatrix}\frac12 & \frac12 & \frac12 & \frac12\\ \frac12 & \frac12 & -\frac12 & -\frac12\\ \frac12 & -\frac12 & \frac12 & -\frac12\\ \frac12 & -\frac12 & -\frac12 & \frac12\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}, \tag{B2}$$

and we rewrite

$$\begin{aligned}&(\mathbf{n}_1+\mathbf{n}_3)\cdot(\mathbf{n}_2+\mathbf{n}_3)\\ &= 2\cos(\sqrt{2}\theta_f)[\cos(\sqrt{2}\theta_s)+\cos(\sqrt{2}\theta_{sf})]\\ &\quad+[\cos(\sqrt{2}\phi_f)-\cos(\sqrt{2}\phi_c)]\\ &\quad\times[\cos(\sqrt{2}\phi_s)+\cos(\sqrt{2}\phi_{sf})].\end{aligned} \tag{B3}$$

Noting that the operators $\cos\sqrt{2}\theta_\nu$, $\sin\sqrt{2}\theta_\nu$, and $\sin\sqrt{2}\phi_\nu$ transform as the components of a vector under SU(2) rotation, we make a unitary transformation such that

$$\mathcal{U}^\dagger\cos\sqrt{2}\theta_\nu\mathcal{U} = \sin\sqrt{2}\phi_\nu, \tag{B4}$$

$$\mathcal{U}^\dagger\sin\sqrt{2}\theta_\nu\mathcal{U} = \sin\sqrt{2}\theta_\nu, \tag{B5}$$

$$\mathcal{U}^\dagger\sin\sqrt{2}\phi_\nu\mathcal{U} = -\cos\sqrt{2}\theta_\nu \tag{B6}$$

for $\nu = s, f, sf$ to obtain

$$\begin{aligned}&\mathcal{U}^\dagger(\mathbf{n}_1+\mathbf{n}_3)\cdot(\mathbf{n}_2+\mathbf{n}_3)\mathcal{U}\\ &= 2\sin(\sqrt{2}\phi_f)[\sin(\sqrt{2}\phi_s)+\sin(\sqrt{2}\phi_{sf})]\\ &\quad+[\cos(\sqrt{2}\phi_f)-\cos(\sqrt{2}\phi_c)][\cos(\sqrt{2}\phi_s)\\ &\quad+\cos(\sqrt{2}\phi_{sf})].\end{aligned} \tag{B7}$$

Now, with the fields

$$\varphi_0 = \frac{\phi_s-\phi_{sf}}{\sqrt{2}}\ \varphi_1 = -\frac{\phi_s+\phi_{sf}}{\sqrt{2}}, \tag{B8}$$

$$\varphi_2 = \frac{\phi_f+\phi_c}{\sqrt{2}}\ \varphi_3 = \frac{\phi_f-\phi_c}{\sqrt{2}}, \tag{B9}$$

we recover the form

$$\begin{aligned}&\mathcal{U}^\dagger(\mathbf{n}_1+\mathbf{n}_3)\cdot(\mathbf{n}_2+\mathbf{n}_4)\mathcal{U}\\ &= \cos\varphi_0[\cos(\varphi_1+\varphi_2-\varphi_3)+\cos(\varphi_3+\varphi_1-\varphi_2)\\ &\quad+\cos(\varphi_2+\varphi_3-\varphi_1)-3\cos(\varphi_1+\varphi_2+\varphi_3)],\end{aligned} \tag{B10}$$

and with a change of variables analogous to Eq. (44),

$$\begin{pmatrix}\varphi_c\\ \varphi_a\\ \varphi_b\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3}\\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & -\frac{2}{\sqrt6}\end{pmatrix}\begin{pmatrix}\varphi_2\\ \varphi_3\\ \varphi_1\end{pmatrix}, \tag{B11}$$

we recover the form Eq. (45) for the interaction. Note that in Eq. (B11), we have made a circular permutation of $\varphi_{1,2,3}$ compared with Eq. (44). The reason is that such a choice of variables gives us

$$-\frac{\sqrt{2}}{\pi}\partial_x\varphi_a = -\frac{\sqrt{2}}{\pi}\partial_x\phi_c, \tag{B12}$$

and this implies that $\partial_x\varphi_a$ is proportional to the magnetization density. While the approach in this Appendix is convenient to establish the SU(3) symmetry in the low-energy Hamiltonian, it is impractical to derive bosonized expressions of the SU(2)$_4$ currents $\sum_n J^{x,y}_{R/L,n}$. The reason is that the transformation under the SU(2) rotation of the operators $e^{i\pm(\theta_\nu\pm\phi_\nu)/\sqrt{2}}$ is ambiguous. Indeed, the expression $e^{i(\theta_\sigma-\phi_\sigma)/\sqrt{2}}$ appears both in the bosonized representation of the spin-up annihilation operator and in the bosonized representation of the spin-down creation operator [38]. However, those operators transform differently under SU(2) rotation. To have a well-defined transformation under SU(2) rotation, we must specify if we are considering $e^{i(\theta_\sigma-\phi_\sigma)/\sqrt{2}}\eta_\uparrow$ or $e^{i(\theta_\sigma-\phi_\sigma)/\sqrt{2}}\eta_\downarrow$.

APPENDIX C: TRANSFORMATION OF FERMION BILINEARS

If we consider a fermion bilinear

$$\sum_{\alpha,\beta}\psi_{R,\alpha}M_{\alpha\beta}\psi_{L,\beta}, \tag{C1}$$

since we can always write $M = (M+{}^tM)/2 + (M-{}^tM)/2$, we can without loss of generality consider the cases $M = {}^tM$

and $M = -{}^tM$ separately. In the first case, the fermion bilinear can be written

$$\frac{1}{2}\sum_{\alpha,\beta} M_{\alpha\beta}(\psi_{R,\alpha}\psi_{L,\beta} + \psi_{R,\beta}\psi_{L,\alpha}), \tag{C2}$$

and under a SU(3) rotation $\psi_{\nu,\alpha} = \sum_{\alpha'} \tilde{\psi}_{\nu\alpha'}$, it becomes

$$\frac{1}{2}\sum_{\alpha,\beta,\alpha',\beta'} M_{\alpha\beta}U_{\alpha\alpha'}U_{\beta\beta'}(\tilde{\psi}_{R,\alpha'}\tilde{\psi}_{L,\beta'} + \tilde{\psi}_{R,\beta'}\tilde{\psi}_{L,\alpha'}),$$
$$= \frac{1}{2}\sum_{\alpha',\beta'}(tUMU)_{\alpha'\beta'}(\tilde{\psi}_{R,\alpha'}\tilde{\psi}_{L,\beta'} + \tilde{\psi}_{R,\beta'}\tilde{\psi}_{L,\alpha'}). \tag{C3}$$

The matrix M is transformed in the new symmetric matrix ${}^tUMU$. Symmetric matrices in $M_3(\mathbb{C})$ are a six-dimensional vector space, showing that the fermion bilinear is in the **6** representation of SU(3). The Gell-Mann matrices $\Lambda_1, \Lambda_3, \Lambda_4, \Lambda_6, \Lambda_8$ and the identity matrix span the space of symmetric matrices. With M antisymmetric, the fermion bilinear is now written

$$\frac{1}{2}\sum_{\alpha,\beta} M_{\alpha\beta}(\psi_{R,\alpha}\psi_{L,\beta} - \psi_{R,\beta}\psi_{L,\alpha}), \tag{C4}$$

and M now transforms in the antisymmetric matrix ${}^tUMU$. Antisymmetric matrices in $M_3(\mathbb{C})$ are a three-dimensional vector space, spanned by $\Lambda_2, \Lambda_5, \Lambda_7$. Moreover, since $\det U = 1$, antisymmetric combinations of $U_{\alpha\alpha'}U_{\beta\beta'}$ are combining into U^{-1}. The three-dimensional representation corresponds to $\bar{\mathbf{3}}$.

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