


Stable real-energy spectral dynamics with topological transitions and non-Hermitian many-body localization

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In this paper, the interplay between non-Hermiticity, quasidisorder, and repulsive interaction is studied for hard-core bosons confined in a one-dimensional optical lattice, where non-Hermiticity is induced by the nonreciprocal hoppings and the on-site gain and loss breaking the time-reversal symmetry. Although the energy spectra of the static system are fully complex, with the evolution of the initial state, the real part of the expectation value of the Hamiltonian under the time-evolved wave function changes stably. By means of the entanglement entropy and its dynamical evolution, as well as the inverse participation ratio, the many-body localization (MBL) is found to play the key role in the stability of the dynamical behavior of the real part of the expectation value, independent of whether the spectrum of the static Hamiltonian is real or complex. In the delocalization phase, the dynamical evolution of the real part of the expectation value is unstable. Meanwhile, the nearest-neighbor level spacings statistics shows the MBL transition accompanied by the transition from the Ginibre distribution to the complex Poisson distribution, different from the one in the time-reversal invariant system. In addition, the dynamical stability of the real part of the energy and the MBL transition can be characterized by the winding number, indicating that the MBL transition and the topological transition occur simultaneously, and the realization of the Hamiltonian is discussed.

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I. INTRODUCTION

The eigenstate thermalization hypothesis [1–6] is a widely accepted concept that explains the thermalization of the eigenstates. It refers to the fact that the eigenstates of the Hamiltonian of an isolated system will themselves exhibit the properties of heating, resulting in these thermalized eigenstates being ergodic. However, the many-body localization (MBL) violates this hypothesis, making it an active field in condensed physics over the past decade. In the thermalization phase, spectral statistics indicates that the energy spectrum follows the Wigner-Dyson distribution while in the MBL phase it follows the Poisson distribution [6,7]. The difference in statistical distributions before and after the MBL transition attracts wide interest in the study of spectral properties of Gaussian and Poisson ensembles [8–13].

In fact, spectral statistics is one of the tools to understand the thermalization to MBL from the energy perspective. The MBL transition can be characterized by the many-body eigenstate as well. The half-chain entanglement entropy is obtained from the subspace decomposition of eigenstates. It has been shown that for the thermal phase where eigenstates are delocalized the entanglement entropy follows the volume law, while in the MBL phase it follows the area law [14–29]. Additionally, the MBL transition is observed

through diagonal entropy [30–32]; local parameter, which weighs the stability of the eigenstates under local perturbations, and grows with system size in the delocalized phase and decreases in the MBL phase [33]; many-body inverse participation ratio (IPR) [34], or normalized participation ratio [35]; quantum Fisher information [32,36–39]; and particle imbalance [40–53]. Experimentally, the MBL has been realized in various controllable platforms, such as the ultracold atomic systems [40–46,54], trapped ions [36,37], superconducting processors [55–60], nuclear spins [61], and solid material [62].

Similarly to the Hermitian systems, the level statistics can be applied to non-Hermitian systems as well. Differently, the non-Hermiticity leads to a new level statistics distribution, namely the Ginibre distribution [63]. Due to the emergence of non-Hermiticity, the originally Hermitian tenfold topological classifications [64–66] have been generalized to the 38-fold topological classifications [67–69]. Accordingly, the Ginibre distribution is widely studied in the open quantum systems with various symmetries [70–80]. However, it is constrained by the symmetry of non-Hermitian systems. In general, the level statistics in the systems with transposition symmetry (classes $A1^\dagger$ and AII^\dagger) will deviate from the Ginibre distribution [70,71,81].

Recently, non-Hermitian spectral statistics has been employed to analyze the MBL transition. In a class of non-Hermitian many-body disordered or quasidisordered systems with a complex-real transition in energy, level statistics in the delocalized phase (with complex energies) presents the

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Ginibre distribution, while it presents the real Poisson distribution in the MBL phase (with real energies) [70,82–84]. Just as in the Hermitian case, the half-chain entanglement entropy can still be used to extract the transition point and the scaling exponent of the MBL transition. Additionally, for the delocalized states, the entanglement entropy follows the volume law, while for the MBL states it follows the area law [70,82–85]. It is worth noting that, for the delocalized phase, i.e., the complex energy regime, the real part of the energy is unstable during the time evolution, while for the MBL phase, i.e., the real-energy regime, the dynamical process is stable [70,82]. This raises the question of whether MBL or real energy plays the key role in maintaining the stable dynamical behavior, which is not addressed in the previous works [70,82]. In this paper, we attempt to study a non-Hermitian many-body system with a fully complex energy spectrum and with MBL to answer this question and give a clear understanding on the stable dynamical behavior. We note that for the non-Hermitian many-body systems with complex-real-energy transition the winding number [86,87] can be used to characterize the MBL transition (where nonzero winding number corresponds to the delocalized phase whereas the zero winding number corresponds to the MBL phase) [82,83]. However, the topological transition in a complex spectral case is not uncovered [83]. Whether the topological transition occurs in complex spectral systems, and, if the topological transition exists, whether it follows the MBL transition, demand the answer. Additionally, we will discuss the relationship between the topological and MBL transitions. Some research showed that the exceptional point [88–91] leads to nontrivial topology [92–94]. We will go into the topological origin of the complex energy spectrum system and investigate the connection between topological transitions and exceptional points.

The paper is organized as follows. Section II introduces the model and Hamiltonian. Section III analyzes and discusses the properties of the energy spectrum, including the complex energy spectrum and the energy evolution. Section IV analyzes and discusses the MBL transition, including the spectrum statistic, half-chain entanglement entropy, entropy evolution, scaling exponent of entropy, the many-body IPR, and the time-dependent density imbalance. Section V studies the topological transition and introduces how to realize the Hamiltonian. Section VI presents the summary.

II. MODEL AND HAMILTONIAN

Here, we study a non-Hermitian one-dimensional hard-core bosonic system and the Hamiltonian is

$$\hat{H} = \sum_{j=1}^L [-J(e^{-g}\hat{c}_{j+1}^\dagger\hat{c}_j + e^g\hat{c}_j^\dagger\hat{c}_{j+1}) + U\hat{n}_j\hat{n}_{j+1} + V_j\hat{n}_j], \quad (1)$$

where \hat{c}_j (\hat{c}_j^\dagger) is the particle annihilation (creation) operator and $\hat{n}_j = \hat{c}_j^\dagger\hat{c}_j$ is the particle-number operator. Mismatched hopping strengths $-Je^{-g}$ and $-Je^g$ form nonreciprocal hoppings with J the unit of energy and g the dimensionless parameter. U is the repulsive interaction between nearest-neighbor sites. Here, the quasidisordered on-site potential V_j

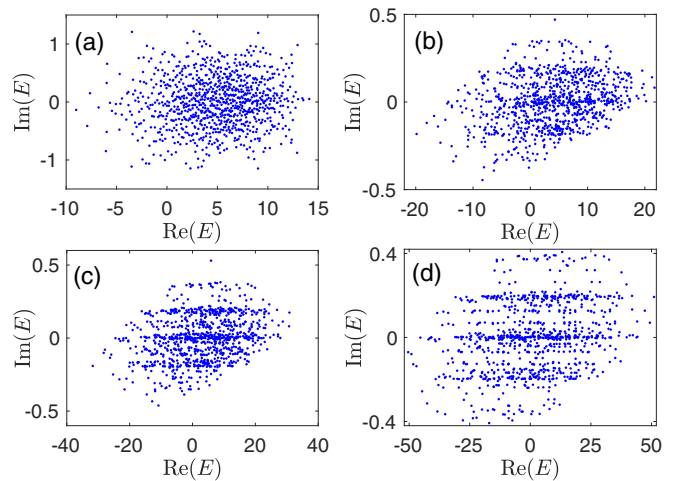


FIG. 1. Energy spectra in the complex plane. (a) $V = 2J$. (b) $V = 5J$. (c) $V = 8J$. (d) $V = 14J$. Other parameters are $L = 12$, $\gamma = 0.1J$, and $\varphi = \pi/4$.

with odd-site gain ($+i\gamma$) and even-site loss ($-i\gamma$) is considered, i.e., $V_j = V \cos(2\pi\alpha j + \varphi) - i\gamma(-1)^j$, where V is the strength of the quasidisorder potential, φ is a random phase, and α is the incommensurate parameter.

The non-Hermiticity of this system is controlled by the parameters g and γ together. When $\gamma = 0$, if we perform a transformation, i.e., $\hat{c}_j \rightarrow e^{-gj}\hat{c}_j$ and $\hat{c}_j^\dagger \rightarrow e^{gj}\hat{c}_j^\dagger$, the Hamiltonian presented in Eq. (1) can go back to a Hermitian many-body interacting model [95], in which the authors give the numerical estimates for the MBL transition point. For nonzero γ , the model in Eq. (1) breaks the time-reversal symmetry. In the following, we take $U = 2J$ and $g = 0.5$ as an example to reveal the MBL property and its relationship with topological transition in the absence of time-reversal symmetry. The strength of the quasidisorder potential V is chosen as the control parameter for the underlying transitions. The incommensurate parameter is chosen at $\alpha = (\sqrt{5} - 1)/2$ and the periodic boundary condition is considered in the following analyses. All the calculations are performed in the subspace with particles $N = L/2$.

III. ENERGY SPECTRUM PROPERTIES

At first, the energy spectrum characteristic is discussed. In the presence of time-reversal symmetry, i.e., $\gamma = 0$, we have known that the energy spectra present a complex-to-real transition [82]. However, in the absence of time-reversal symmetry, i.e., $\gamma \neq 0$, we find that there is no complex-to-real transition. Taking $\gamma = 0.1J$, $\varphi = \pi/4$, and various V , the resulting spectra are plotted in in Figs. 1(a)–1(d). Intuitively, with the increase of the strength of quasidisorder potential, there is always a nonvanishing imaginary part of the energy spectrum in the complex plane. It means that the time-reversal symmetry is essential for the system to have a static real-energy spectrum. Meanwhile, we note that for the systems with time-reversal symmetry the energy spectra characterized by the evolution of the real part of the energy are dynamically stable in the real-energy regime [70,82]. It raises a question whether the real energy

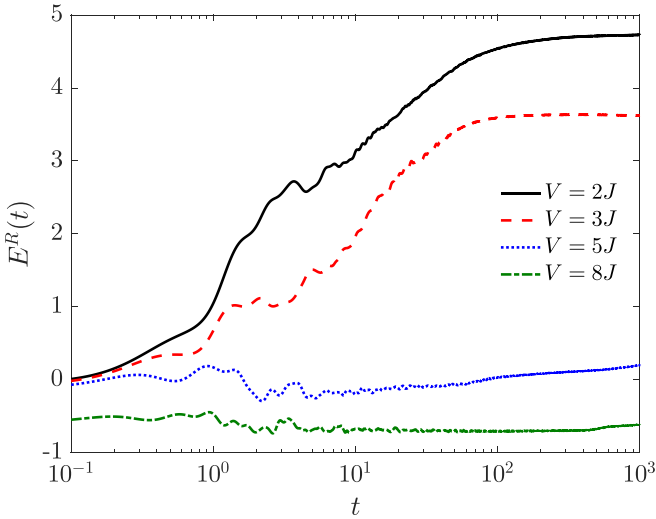


FIG. 2. Time evolution of $E^R(t)$ for $V = 2J, 3J, 5J$, and $8J$. The system size is $L = 12$ and the initial state is taken as $|\psi(t=0)\rangle = |101010\dots\rangle$. We take $\gamma = 0.1J$ and 512 samples in the calculation.

(or the time-reversal symmetry) is essential in maintaining the stably dynamical behaviors. Next, we discuss the dynamical stability of the energy spectra, which is characterized by the evolution of the real part of the energy $E^R(t) = \text{Re}[\langle \psi(t) | \hat{H} | \psi(t) \rangle]$, where $|\psi(t)\rangle$ is the wave function evolving from the initial state to time t [70] and $\bar{\cdot}$ denotes the ensemble average. We emphasize that the disorder configuration of the random phase φ is determined at $t = 0$, and does not change at $t > 0$. $|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(t=0)\rangle / \sqrt{\mathcal{N}}$ describes a quantum trajectory without any quantum jumps, which is microscopically justified in the context of continuously measured systems [70,96]. Here, the initial state $|\psi(t=0)\rangle$ is chosen as $|101010\dots\rangle$ and \mathcal{N} is the normalization coefficient, defined as $\mathcal{N} = \langle \psi(t) | e^{iH^\dagger t} e^{-iHt} | \psi(t) \rangle$. According to the Baker-Campbell-Hausdorff expansion, we know that $e^{iH^\dagger t} e^{-iHt}$ contains the difference term $H^\dagger - H$ and the commutation term $[H^\dagger, H]$. Due to the non-Hermiticity of H , we have $(H^\dagger - H) \neq 0$ and $[H^\dagger, H] \neq 0$. Therefore, $|\psi(t)\rangle$ and \mathcal{N} will change during the time evolution, and may lead to the time dependence of $E^R(t)$.

Taking $L = 12$, $\gamma = 0.1J$, and $\varphi = \pi/4$, we plot the time evolution of $E^R(t)$ t under various V in Fig. 2. We take 256 samples in the calculations. We can see that when the potential strengths are relatively weak, such as $V = 2J$ (black curve) and $V = 3J$ (red curve), $E^R(t)$ is quite unstable and evidently deviates from the initial state during the dynamical process. However, it seems that there exists a phase transition which leads to the different dynamical process of $E^R(t)$. For $V = 5J$ (blue curve) and $V = 8J$ (green curve), the dynamical processes are relatively stable, because during the time evolution, $E^R(t)$ only slightly deviates from the initial state. Even with a longer evolution time, such as $t \in (10^2, 10^3)$, the dynamical evolutions of $E^R(t)$ under $V = 5J$ and $8J$ are still stable. Recall that in the presence of time-reversal symmetry the energy spectra present complex-real transition, and the complex energies lead to an unstably dynamical process of $E^R(t)$, while the real energies result in a stable one [70,82].

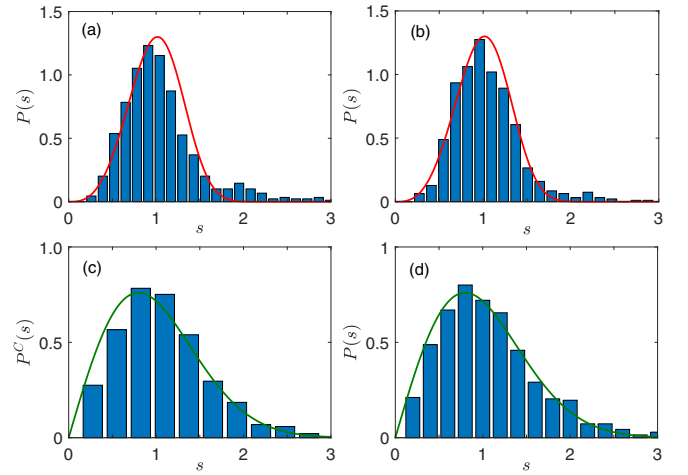


FIG. 3. The nearest-level-spacing distribution of the unfolded energies for (a) $V = 2J$, (b) $V = 3J$, (c) $V = 5J$, and (d) $V = 8J$. The red curve is the Ginibre distribution function $P_{\text{Gin}}^C(s)$ and the green curve is the Poisson distribution function $P_{\text{Po}}^C(s) = \pi s / 2e^{-(\pi/4)s^2}$ on the complex energy plane. The energies used in the statistics are taken from the middle of the energy spectra with a proportion of 20%. Other parameters are $\gamma = 0.1J$, $L = 14$, and $\varphi = \pi/3$.

Here in our model without time-reversal symmetry, the energy spectra are all complex, but $E^R(t)$ presents two types of dynamical behaviors. It indicates that the stable dynamical evolution of $E^R(t)$ is independent of time-reversal symmetry and real-energy spectrum.

IV. MANY-BODY LOCALIZATION TRANSITION

Inspired by an early estimate on the location of the MBL point of a Hermitian system [95] and recent works on non-Hermitian MBL [70,82–85], we are aware that the stable dynamical behavior of $E^R(t)$ in the system without time-reversal symmetry and real-energy spectrum may be caused by the MBL. We first study the nearest-level-spacing statistic to check the mentioned point. If a non-Hermitian many-body system exists MBL transition, there are different spectral statistic laws before and after MBL transition [70,71]. For an energy E_β , the nearest-level spacing is defined by the minimal distance in the complex energy plane, i.e., $\min_{\beta'} |E_\beta - E_{\beta'}|$. For the delocalized phase, it is known that the statistical distribution obeys the Ginibre distribution $P_{\text{Gin}}^C(s) = cp(cs)$ [97,98], in which

$$p(s) = \lim_{M \rightarrow \infty} \left[\prod_{m=1}^{M-1} e_m(s^2) e^{-s^2} \right] \sum_{m=1}^{M-1} \frac{2s^{2m+1}}{m! e_m(s^2)}$$

with $e_m(s^2) = \sum_{\ell=0}^m s^{2\ell} / \ell!$ and $c = \int_0^\infty sp(s) ds = 1.1429\dots$. In our numerical computing, energies used in the statistics are taken from the middle of the energy spectra with a proportion of 20%. For parameters V taken before V_c , such as $V = 2J$ and $3J$, the corresponding statistical distributions are shown in Figs. 3(a) and 3(b), respectively. The distributions match with the Ginibre distribution (the red curve shows). For larger parameters exceeding V_c , such as $V = 5J$ and $8J$, the corresponding

statistical distributions are plotted in Figs. 3(a) and 3(b), respectively. Intuitively, the distributions are matched with the Poisson distribution $P_{\text{Po}}^C(s)$ on the complex plane [70,82,83], satisfying $P_{\text{Po}}^C(s) = \pi s / 2e^{-(\pi/4)s^2}$. According to the symmetry classes of non-Hermitian random matrices, here our model shall belong to the A class [71]. Although the A class and the complex-conjugation symmetry (or say time-reversal symmetry) class [70,82,83,85] are members of the Ginibre symmetry classes [71], the corresponding statistical characteristics are different. To be specific, in our model, the spectral statistics present a transition from the Ginibre distribution to the complex Poisson distribution (which has the same distribution function as the Wigner-Dyson distribution), whereas in the complex-conjugation symmetry class the spectral statistics display a transition from the Ginibre distribution to the real Poisson distribution. In the complex-conjugation class, the statistic distribution transition is accompanied by the MBL transition. The different statistical rules of the energy spectrum suggest that the system has experienced the MBL, and also imply that the MBL plays a leading role in maintaining the stable dynamic evolution of $E^R(t)$.

To further check the presence of MBL, we study the half-chain entanglement entropy and the IPR, which are based on the many-body eigenstate. The half-chain entanglement entropy is defined as

$$S = -\text{Tr}(\rho^r \ln \rho^r), \quad (2)$$

where $\rho^r = \text{Tr}_{L/2}[\rho^r]$ is the reduced density matrix with $|\psi^r\rangle$ the right eigenstate. It is known that for the delocalized phase entanglement entropy obeys the volume law and for the localized phase it obeys the area law [70]. Therefore, the entanglement entropy for the delocalized phase is visibly larger than that of the localized phase [70,82,83,85]. In view of this characteristic, entanglement entropy is usually used to distinguish the delocalized phase from the MBL one.

Under different system sizes and $\gamma = 0.1J$, we take the eigenstates from the middle 4% spectrum to calculate the corresponding entanglement entropy. The ensemble-averaged entropy \bar{S} over the system size L , i.e., \bar{S}/L , is plotted in Fig. 4. Generally speaking, it is a common and effective method to determine the location of MBL transition by the crossing of entanglement entropy curves. But there are exceptions where the crossing shifts with system size, so that the existence of the absolutely stable MBL is under debate [99–101]. For the current studied model, the crossing of entanglement entropy curves does not shift with system size. Therefore, it is feasible to determine the localization of MBL transition from the crossing of the entanglement entropy curves. As can be seen, the entropy exhibits a system size independent crossover at the transition point $V_c^{\text{MBL}} \simeq 4.5 \pm 0.1J$. We note that the MBL transition point of our system is less than that of the system with time-reversal symmetry [82]. It means that the absence of the time-reversal symmetry can suppress the delocalization, and cause the MBL to occur earlier in the system. Additionally, the entropy presents a transition from the volume law to the area law. Before V_c^{MBL} the entropy is larger than that after V_c^{MBL} . Meanwhile, the entropy shows an L dependence. When V is less than V_c^{MBL} , the entropy increases as L increases, and when V is larger than V_c^{MBL} the

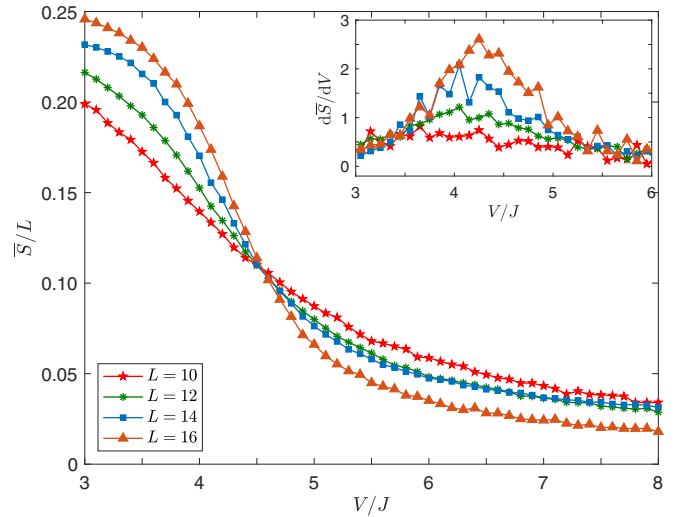


FIG. 4. Ensemble average of the half-chain entanglement entropy over system size \bar{S}/L as a function of the on-site potential strength V . With the increase of V , \bar{S}/L shows crossover at $V_c^{\text{MBL}} = 4.5 \pm 0.1J$, signaling the appearance of the MBL transition. Here, the right eigenstates are taken from the middle of the energy spectra with a proportion of 4%. The inset shows the slope of \bar{S} . We take 500 samples for $L = 10, 12$, and 200 samples for $L = 14$, and 96 samples for $L = 16$. The other parameter is $\gamma = 0.1J$.

entropy decreases as L increases. Additionally, we can employ the slope of \bar{S} to estimate the MBL transition point. Because \bar{S} decays with the increase of V , the definition of the slope of \bar{S} , i.e., $d\bar{S}/dV = [\bar{S}(V) - \bar{S}(V + \delta V)]/\delta V$, is beneficial for visualization. In the calculation, we take $\delta V = 0.1J$ and the slopes for different L are plotted in the inset of Fig. 4. It shows that the slopes peak near the MBL transition point, and the feature is more evident for larger system sizes.

The MBL transition can be characterized by the dynamical behaviors of the entanglement entropy as well. Still taking $|\psi(t=0)\rangle = |101010\dots\rangle$ as the initial state, we calculate the ensemble average of the time-dependent entropy $\bar{S}(t)$ under the system size $L = 14$, and the results are shown in Fig. 5(a). It can be seen that when evolution time τ is short $\bar{S}(t)$ values for different V synchronously grow with τ , soon enter their own evolutionary trajectories, and finally tend to steady values. For parameters $V = 2J$ and $3J$ chosen in the delocalized phase, we can see that $\bar{S}(t)$ is larger than that chosen from the MBL phase ($V = 5J$ and $8J$ cases). Moreover, the entanglement entropies of the steady states gradually decrease with the increase of V , and this variation tendency is consistent with that of the static entanglement entropy. From the entanglement entropy and its dynamical behaviors, we know that there actually exists a delocalization to MBL transition behind the complex energy, two types of different dynamical phenomena of $E^R(t)$, and the transition from Ginibre's distribution to complex Poisson distribution. To reveal the critical exponent of the MBL transition, we will study the scaling behavior of the entanglement entropy near the MBL transition point. For a finite-size system, the entanglement entropy around V_c^{MBL} satisfies the following scaling behavior [70]:

$$S/L = f[(V - V_c^{\text{MBL}})L^{1/\nu}], \quad (3)$$

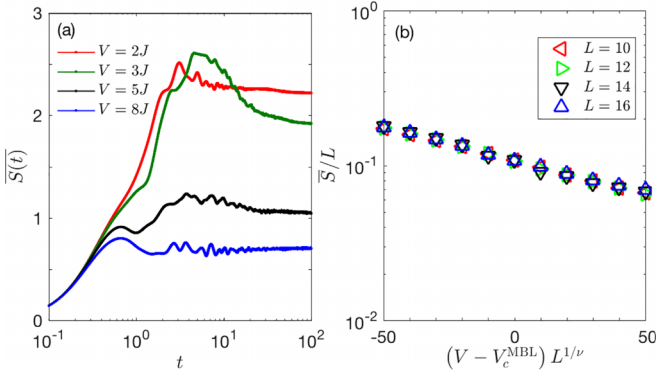


FIG. 5. (a) Ensemble average of the time-dependent half-chain entanglement entropy $\bar{S}(t)$ for various V . We take $|\psi(t=0)\rangle = |101010\dots\rangle$ as the initial state, and use $L = 14$ and 512 samples in the calculations. (b) Finite-size scaling collapse of the entropy as a function of $(V - V_c^{\text{MBL}})L^{1/\nu}$, where we take $V_c^{\text{MBL}} = 4.55J$ and $\nu = 0.6$. The associated eigenstates are taken from the middle 4% spectrum. We take 512 samples for $L = 10, 12$, and 256 samples for $L = 14$, and 32 samples for $L = 16$. The other parameter is $\gamma = 0.1J$.

where ν is the critical exponent and $f(x)$ is the scaling function. With $V_c^{\text{MBL}} = 4.55J$ and different system size, the corresponding ensemble averages of the entropy over the system size \bar{S}/L are plotted in Fig 5(b). It shows that \bar{S}/L values for different L collapse onto a single curve with $\nu \simeq 0.6$, different from the scaling exponent $\nu = 1.3$ in the complex-conjugation symmetry class [70,83], $\nu = 1.8$ in the transposition symmetry class [70], and $\nu = 1$ and 1.5 , corresponding to the cases with nonreciprocal hopping parameters $g = 0.3$ and 0.6 , respectively [83].

Theoretically, the IPR plays the role of indicator to reveal the properties of the eigenstates in the noninteracting systems. The IPR of extended eigenstates scales like $1/L$ and approaches finite constant for the localized states [102]. We try to employ this single-particle feature (i.e., $\text{IPR} \propto 1/L$) to estimate the MBL transition point (marked by V_c^{IPR}). The IPR of a many-body eigenstate reads

$$\text{IPR} = \sum_k^{\text{Dim}} |\psi(k)|^4, \quad (4)$$

where $\psi(k)$ is the amplitude of the eigenstate $|\psi\rangle$ in the Fock basis $\{|k\rangle\}$ with $\psi(k) = \langle k|\psi\rangle$, and ‘‘Dim’’ is the size of the Hilbert space. Under different system sizes, we can obtain the corresponding ensemble averaged IPR, i.e., $\overline{\text{IPR}}$ as a function of V , and the V solution to $\text{IPR} = 1/L$ is the estimated V_c^{IPR} . Taking $L = 10, 12, 14$, and 16 , $\overline{\text{IPR}}$ values as a function of V are plotted in Figs. 6(a)–6(d). For various system sizes, $\overline{\text{IPR}}$ increases from a near-zero constant to a finite constant as V increases, presenting the delocalization to MBL transition. Meanwhile, there is always a solution V_c^{IPR} to $\overline{\text{IPR}} = 1/L$. V_c^{IPR} presents a dependence on the finite-size effect. When L is small, as shown in Figs. 6(a)–6(c), V_c^{IPR} deviates from V_c^{MBL} . Until a larger system size $L = 16$, the estimated MBL transition point $V_c^{\text{IPR}} = 4.4J$ – $4.5J$ is much closer to V_c^{MBL} . The slight deviation can be attributed to the finite-size effect. It

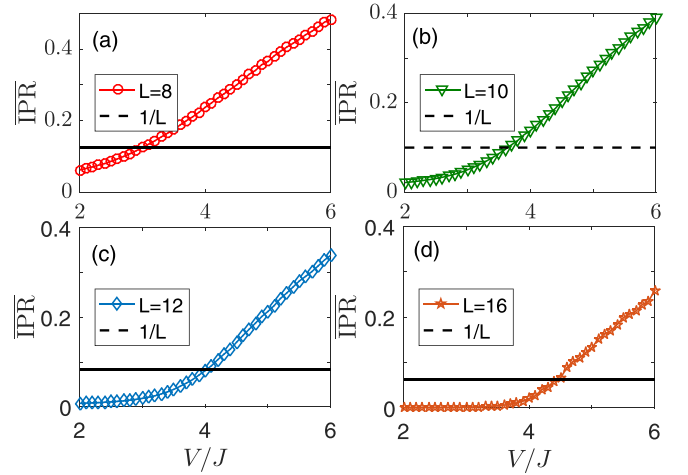


FIG. 6. Ensemble averaged IPR, i.e., $\overline{\text{IPR}}$ as a function of V . We take 1000 samples for (a) $L = 8$, (b) 10 , and (c) 12 , and 100 samples for (d) $L = 16$. The other parameter is $\gamma = 0.1J$.

shows that the solution to $\text{IPR} = 1/L$ can be used to estimate the MBL transition point.

In experiments, the dynamics of the density imbalance [denoted by $I(t)$] is an observable measurement to detect the many-body localization [40–46,58,59], which is defined as

$$I(t) = \frac{n_o(t) - n_e(t)}{N}, \quad (5)$$

where $n_o(t)$ and $n_e(t)$ are the time-dependent densities (particle populations) at odd and even sites, respectively. It was studied that in the long-time evolution limit, for the delocalization phase, $I(t)$ is stable at a finite value, implying that some initial information is preserved, while for the many-body localized phase $I(t)$ approaches zero, implying that the initial formation is lost [40–46,58,59]. Next, we employ the quantity $I(t)$ to detect the non-Hermitian many-body localization.

Taking the initial state $|\psi(t=0)\rangle = |101010\dots\rangle$, $\gamma = 0.1J$, and $L = 12$, the ensemble averaged density imbalance $\overline{I(t)}$ as a function of the evolution time with different V is plotted in Fig. 7. Intuitively, for $V = 2J$, $\overline{I(t)}$ is a finite value in the long-time evolution limit, which implies that the system is in the many-body delocalized phase. In contrast, for $V = 8J$, $\overline{I(t)}$ approaches to zero in the long-time evolution limit, denoting that the system is in the many-body localized phase. Through the above research on the energy spectrum and its statistical law, half-chain entanglement entropy, IPR, and density imbalance, we have clearly understood that the stable dynamic evolution of $E^R(t)$ is attributed to the MBL of the system, and has nothing to do with the real or complex energy spectrum.

V. TOPOLOGICAL TRANSITION

We note that the complex-real transition and the non-Hermitian MBL transition are found to be accompanied by the topological transition in the time-reversal symmetric case [82]. Meanwhile, there is no topological transition uncovered in a many-body complex energy spectrum case [83]. It drives

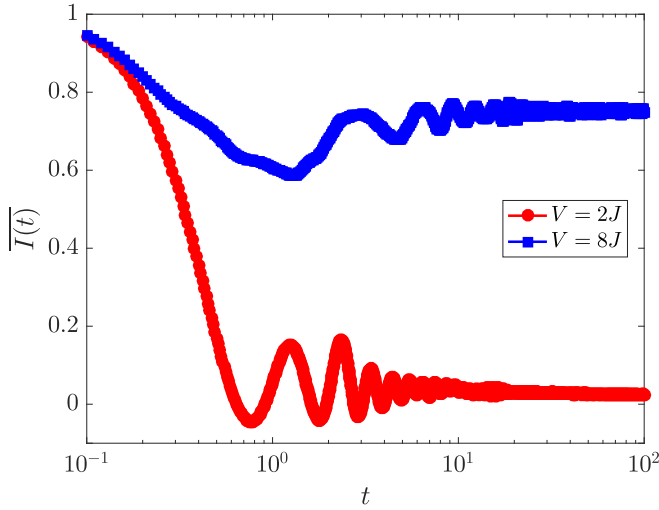


FIG. 7. Ensemble averaged density imbalance $\overline{I(t)}$ with different V . We take 512 samples, $\gamma = 0.1J$, and $L = 12$.

us to investigate whether there is topological transition in the current studied system with a complex energy spectrum, and whether the topological transition is accompanied by the MBL transition. To answer these questions, we calculate the winding number [86,87]. We perform a gauge transformation on the creation and annihilation operators, i.e., $\hat{c}_j \rightarrow e^{i\frac{\phi}{L}j}\hat{c}_j$ and $\hat{c}_j^\dagger \rightarrow e^{-i\frac{\phi}{L}j}\hat{c}_j^\dagger$. Then the ϕ -dependent Hamiltonian $\hat{H}(\phi)$ reads

$$\begin{aligned} \hat{H}(\phi) = & \sum_{j=1}^L [-J(e^{-g}e^{-i\frac{\phi}{L}j}\hat{c}_{j+1}^\dagger\hat{c}_j + e^g e^{i\frac{\phi}{L}j}\hat{c}_j^\dagger\hat{c}_{j+1}) \\ & + U\hat{n}_j\hat{n}_{j+1} + V_j\hat{n}_j], \end{aligned} \quad (6)$$

and accordingly the winding number is defined as

$$W = \int_0^{2\pi} \frac{d\phi}{2\pi i} \partial_\phi \ln \det\{H(\phi) - E_b\}, \quad (7)$$

where E_b is just the base energy. In the calculation, E_b is taken at $E_b = 0$. We emphasize that here the topology is reflected in the trajectory of $\det H(\phi)/\det H(0)$ based on $H(\phi)$, and is quantized by the winding number [86,87]. When the closed trajectory of $\det H(\phi)/\det H(0)$ encircles E_b once, the winding number accumulates by 1. If the closed trajectory does not encircle E_b or there is no closed trajectory, the winding number is zero [82]. With system size $L = 10$ and a single phase $\varphi = \pi/3$, the trajectories of $\det H(\phi)/\det H(0)$ for different V are plotted in Figs. 8(a) and 8(b), respectively. As shown in Fig. 8(a), the trajectory of $\det H(\phi)/\det H(0)$ encircles the base energy seven times, leading to the winding number $W = 7$. In Fig. 8(b), it is seen that although there is a closed trajectory in the complex plane the trajectory does not encircle the base energy, resulting in the winding number $W = 0$. The results show that there actually exists topological transition even if the energy spectra are all complex. Meanwhile, the results also show that the topological transition is not intrinsically related to the real-complex transition of the energy spectrum. It is noted that there is no tight connection between the MBL transition and the real-complex transition of

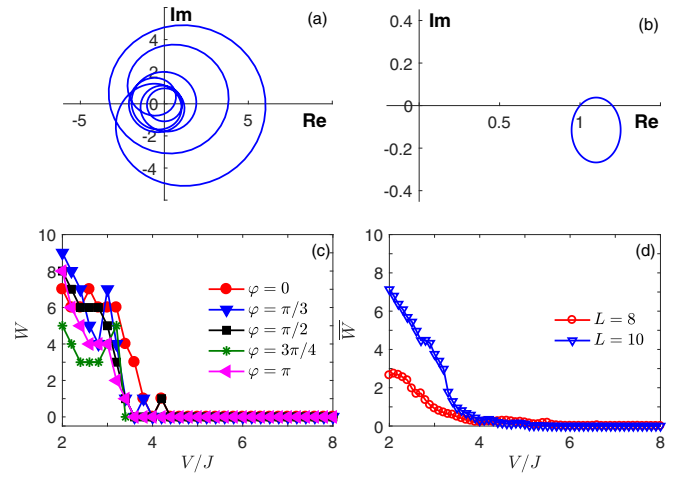


FIG. 8. Trajectory of $\det H(\phi)/\det H(0)$ in the complex plane for (a) $V = 3J$ and (b) $V = 5J$, where the parameters chosen are $L = 10$ and $\varphi = \pi/3$. (c) The φ dependence of W under the system size $L = 10$. (d) The average of W after 1000 ensemble averages. The other parameter is $\gamma = 0.1J$.

the energy spectrum, and we therefore believe that the MBL transition is the main cause of topological transition.

Note that here the frequency of the on-site potential is incommensurate, which leads to a φ -dependent energy spectrum. Therefore, for different φ , the corresponding winding number at a specific V will be different. In Fig. 8(c), W values as a function of V with $\varphi = 0$ (red), $\varphi = \pi/3$ (blue), $\varphi = \pi/2$ (black), $\varphi = 3\pi/4$ (green), and $\varphi = \pi$ (magenta) are plotted. Here, the system size is taken at $L = 10$. From this diagram, we can see that although the winding number W presents a φ dependence there exists a feature that a critical potential strength V_c can divide the system into two parts. Before V_c , the system is nontrivial with nonzero W , and after V_c the system is trivial with $W = 0$. To extract the critical parameter V_c , we perform the finite-size analysis. Figure 8(d) shows the averages of W as the function of V with two different system sizes. Here we have averaged 1000 ensembles. We can see that the two curves intersect at about $V = 4.4 \pm 0.1J$, which implies that the transition point of topological transition is about $V_c = 4.4 \pm 0.1J$. The results show that the topological transition actually exists in this current studied system without real-energy spectrum and time-reversal symmetry. By comparing the transition points of MBL and topological transition, we know that their transition points overlap, indicating that the topological transition occurs synchronously with the MBL transition.

In addition, it was studied that the exceptional point [88–91] could lead to the nontrivial topology [92–94]. Therefore, we will investigate the connection between the uncovered topological phase transition and the exceptional points. In order to determine whether there are exceptional points in the system, we define the minimal energy gap ΔE_{gap} as $\Delta E_{\text{gap}} = \min_{\beta, \beta'} |E_\beta - E_{\beta'}|$, where E_β and $E_{\beta'}$ are two different energies. At first, we study the $\gamma = 0$ case. With different V and φ , the corresponding ΔE_{gap} values (shown with the discrete data points) are plotted in Figs. 9(a), 9(b), 9(c), and 9(d) respectively. The red dashed lines are the fitting

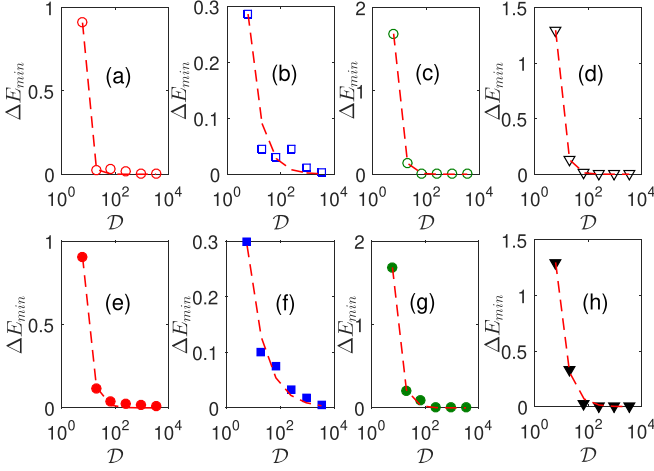


FIG. 9. Minimal energy gap ΔE_{gap} as a function of the size of system subspace \mathcal{D} . (a) $V = 2J$, $\varphi = 0$, and $\gamma = 0$. (b) $V = 2J$, $\varphi = \pi/3$, and $\gamma = 0$. (c) $V = 8J$, $\varphi = 0$, and $\gamma = 0$. (d) $V = 8J$, $\varphi = \pi/3$, and $\gamma = 0$. (e) $V = 2J$, $\varphi = 0$, and $\gamma = 0.1$. (f) $V = 2J$, $\varphi = \pi/3$, and $\gamma = 0.1$. (g) $V = 8J$, $\varphi = 0$, and $\gamma = 0.1$. (h) $V = 8J$, $\varphi = \pi/3$, and $\gamma = 0.1$. The discrete data are the calculated minimal energy gap and the red dashed lines are the corresponding fitting curves.

curves, satisfying $f(\mathcal{D}) = a\mathcal{D}^b$ (where a and b are the fitting parameters with a being a positive number and b being a negative number, presented in Table I, and \mathcal{D} is the size of the system subspace). Here, we choose the inverse power of the system subspace as the fitting function because the system subspace grows exponentially with the system size [95]. The fitting results show that when the system subspace tends to infinite ΔE_{min} values under different V all approach to zero, indicating that there exist exceptional points regardless of whether the potential strength V is small or large. Meanwhile, the decreasing of ΔE_{min} as an inverse power of \mathcal{D} indicates that the system subspace is exponentially dependent of the number of particles. We note that in the $\gamma = 0$ case $V = 2J$ corresponds to nonzero winding number, and $V = 8J$ corresponds to zero winding number [82]. It means that the topological transition followed by complex-real-energy transition in Ref. [82] is not caused by the exceptional points. Next, we investigate whether the topological phase transition in our complex energy model is related to exceptional points. Taking different V , the corresponding ΔE_{gap} values (shown with discrete data points) as a function of the system size are plotted in Figs. 9(e), 9(f), 9(g), and 9(h), respectively. The fitting results (see Table I for details) show that ΔE_{gap} decays with system size in a power-law form at chosen V , and approaches to zero at larger system size. It indicates that

in our complex energy model there are exceptional points as well, regardless of whether the potential strength is small or large. Noting that when V is changed from $V = 2J$ to $8J$ the system undergoes a topological transition from nonzero W to $W = 0$. As a result, similar to that in the system with complex-real-energy transition, the nontrivial topology in our complex energy model is independent of the exceptional point and originates from the nontrivial trajectory of $\det H(\phi)/\det H(0)$ surrounding the base energy.

Up to now, we have known that the topology origin of our complex energy system and that the counterintuitive dynamical behaviors are caused by the MBL and not the real-energy spectrum. Meanwhile, the topological transition is found to be simultaneous with the MBL transition, and the stability of the time evolution of $E^R(t)$ can be predicted by the winding number which is defined in the complex plane after a gauge transformation. In the following, we introduce how to realize the Hamiltonian presented in Eq. (1). The dynamical process of the density matrix ρ for an open system is governed by a Lindblad master equation [103]:

$$\dot{\rho}_t = -i[\mathcal{H}, \rho] + \sum_j \mathcal{D}[L_j]\rho, \quad (8)$$

where \mathcal{H} is a Hermitian Hamiltonian, which is just the H when $g = \gamma = 0$, $\mathcal{D}[L_j] = L_j\rho L_j^\dagger - \{L_j^\dagger L_j, \rho\}/2$, and L_j is the Lindblad dissipator describing the quantum jump between the system and the environment. Under the postselection [86] or no-jump condition [96], the Lindblad dynamical evolution can be governed by a non-Hermitian effective Hamiltonian H_{eff} , which is expressed as

$$H_{\text{eff}} = \mathcal{H} - \frac{i}{2} \sum_j L_j^\dagger L_j. \quad (9)$$

We note that only considering the one-body loss can only achieve the nonreciprocal hoppings, but not the odd (even)-site gain (loss) [86]. To achieve both, we shall consider local (or say, site-dependent) one-body loss L_j^{loss} and gain L_j^{gain} , which are denoted as

$$\begin{aligned} L_j^{\text{loss}} &= \sqrt{\kappa_j} \hat{c}_j \pm i\sqrt{\kappa_{j+1}} \hat{c}_{j+1}, \\ L_j^{\text{gain}} &= \sqrt{\beta_j} \hat{c}_j^\dagger \mp i\sqrt{\beta_{j+1}} \hat{c}_{j+1}^\dagger, \end{aligned} \quad (10)$$

where κ_j and β_j are the strengths of the local one-body loss and gain, respectively. This means that there are both gain and loss of particles at two nearest adjacent lattice sites of the system. Therefore, the summation in H_{eff} shall extend over all lattice sites and the dissipators (i.e., including the loss and gain).

TABLE I. The fitting parameters of the fitting curves presented in Fig. 9.

	Fig. 9(a)	Fig. 9(b)	Fig. 9(c)	Fig. 9(d)	Fig. 9(e)	Fig. 9(f)	Fig. 9(g)	Fig. 9(h)
a	154.20	1.582	75.92	39.64	17.58	1.00	36.03	11.03
b	-2.865	-0.954	-2.125	-1.909	-1.654	-0.687	-1.709	-1.194

Employing the commutation relation of the hard-core bosons $[\hat{c}_j, \hat{c}_k^\dagger] = \delta_{jk}(1 - 2\hat{c}_j^\dagger\hat{c}_j)$, we arrive at H_{eff} as

$$H_{\text{eff}} = \sum_j (J_j^R \hat{c}_{j+1}^\dagger \hat{c}_j + J_j^L \hat{c}_j^\dagger \hat{c}_{j+1}) + \sum_j V_j \hat{n}_j + \sum_j U \hat{n}_j \hat{n}_{j+1} + i \sum_j \gamma_j \hat{n}_j - i \sum_j \beta_j, \quad (11)$$

where

$$J_j^R = -J \mp \frac{\sqrt{\kappa_{j+1}}\sqrt{\kappa_j}}{2} \mp \frac{\sqrt{\beta_{j+1}}\sqrt{\beta_j}}{2},$$

$$J_j^L = -J \pm \frac{\sqrt{\kappa_{j+1}}\sqrt{\kappa_j}}{2} \pm \frac{\sqrt{\beta_{j+1}}\sqrt{\beta_j}}{2},$$

$$\gamma_j = \beta_j - \kappa_j. \quad (12)$$

The last term $-i \sum_j \beta_j$ in H_{eff} denotes a background loss. If the strengths of site-dependent loss and gain are staggered, and satisfy the necessary condition, i.e., $|\beta_j - \kappa_j| = \gamma$, we can realize the Hamiltonian in Eq. (1). Therefore, we believe that with the help of current experimental techniques the non-Hermitian MBL transition and topological transition we have studied can be observed experimentally.

VI. SUMMARY

In this paper, we have investigated a non-Hermitian quasidisordered many-body system that lacks real-energy spectra. Through analysis of spectral statistics, entanglement entropy, inverse participation ratio, and winding number, we have discovered that the many-body localization transition, the spectral statistics transition, and the topological transition driven by quasidisorder occur simultaneously. In the many-body delocalized phase, the energy spectra obey the Ginibre distribution and the entanglement entropy obeys the volume law. Meanwhile, the inverse participation ratio approaches zero, accompanied by a nonzero winding number in the delocalized phase. In contrast, in the many-body localized phase, the energy spectra obey the complex Poisson distribution,

and the entanglement entropy obeys the area law, accompanied by the finite inverse participation ratio and zero winding number. By analyzing the non-Hermitian energy gap, we find that the energy gap decays exponentially with the system size, signaling the existence of the exceptional point. However, the non-Hermitian many-body topology is independent of the exceptional point based on our findings that the exceptional point exists in both the nonzero winding number and zero winding number phase regions, regardless of the presence or absence of real-complex transition. We argue that this non-Hermitian many-body topology originates from the nontrivial trajectory of $\det H(\phi)/\det H(0)$ surrounding the base energy. In addition, the critical exponent of the MBL transition is obtained. Furthermore, this many-body localization transition can be experimentally observed by measuring the density imbalance [40–46,58,59]. We have theoretically calculated the dynamics of density imbalance and find that under the long-time evolution limit the delocalization phase corresponds to near-zero density imbalance, whereas the localized phase corresponds to finite density imbalance. It is noteworthy that despite the absence of real energy in the many-body localization transition phase the real part of the complex energy exhibited relatively stable dynamical behavior, indicating that the many-body localization transition plays a critical role in maintaining this stability. We demonstrate that the Hamiltonian of the studied model can be realized by the Lindblad dynamical evolution under postselection or no-jump condition. Recently, two works have studied non-Hermitian, disorder-free many-body systems that exhibit complex-to-real-energy transitions [104,105]. It would be interesting to investigate whether the many-body localization transition or the real energy is the crucial factor in maintaining the stable dynamical behavior of the real part of the energy in such disorder-free systems.

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