Foliated field theories and multipole symmetries

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Due to the recent studies of the fracton topological phases, the host of which deconfined quasiparticle excitations with mobility restrictions, the concept of symmetries has been updated. Focusing on one of these new symmetries, multipole symmetries, including global, dipole, and quadruple symmetries, and gauge fields associated with them, we construct new sets of $\mathbb{Z}_N$ $(2+1)$-dimensional foliated background-field (BF) theories, where BF theories of conventional topological phases are stacked in layers with couplings between them. By investigating gauge-invariant nonlocal operators, we show that our foliated BF theories exhibit unusual ground-state degeneracy depending on the system size; it depends on the greatest common divisor between N and the system size. Our result provides an important connection between UV lattice models of the fracton topological phases and other unconventional ones in view of foliated field theories.

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I. INTRODUCTION

Topologically ordered phases are unconventional phases of matter and have been one of the central subjects in the condensed matter physics community [1–6]. The prominent feature of these phases is that they host exotic fractionalized excitations, called anyons [2,7,8]. These excitations are not only theoretically intriguing but also may find potential application in quantum information science as exchanging these excitations can be utilized for quantum computers [9,10].

Recently, new types of topologically ordered phases have been introduced, referred to as the fracton topological phases [11–16]. The distinctive feature of these phases is that a mobility constraint is imposed on quasiparticle excitations, leading to the subextensive ground state degeneracy (GSD). Due to this feature, conventional effective field theory description of the topologically ordered phases [17–20] cannot be applied to the fracton topological phases. Fractons have attracted a lot of interest in the field of high energy physics. Indeed, fractons have been recently studied in the context of the gravity theory [21–24], the branes [25], and holography [26–28]. Given the novelty of these phases, one of the challenges is to establish a consistent framework for continuum field theories.

One of the attempts to tackle this problem is to introduce new types of symmetries—*subsystem symmetries* and *multipole symmetries* [29–35]. Here, subsystem symmetry means that a theory is invariant under a symmetry action which acts on a submanifold. The multipole symmetry, especially the U(1) multipole symmetry, is the generalization of the global U(1) symmetry in the sense that a theory is invariant under the global phase rotation which depends on the spatial coordinate in polynomial form. For instance, in the case of a scalar theory which respects the global and dipole U(1) symmetries, the Lagrangian is invariant under $\Phi \rightarrow e^{i\alpha+i\beta x}\Phi$, where α and β are constants and x denotes the spatial coordinate [32]. Recent work has investigated fracton topological phases in view of such a new type of symmetries [36–59].

Another strategy to construct effective field theories of the fracton topological phases is to introduce the so-called foliated background-field (BF) theories [60–65]. Such theories are introduced so that (2+1)-dimensional (2+1D) BF theories are stacked in layers. An important aspect of these theories is that the gauge transformation is modified due to the couplings between the layers so that a constraint is imposed on the form of the gauge invariant operators, such as the Wilson loops, contributing to the subextensive GSD. To our knowledge, there are a handful of foliated BF theories that are known to describe the fracton topological phases, such as the X-cube model. Complete understanding of the foliated BF theories remains elusive.

In this work, we explore new sets of 2+1D foliated BF theories by taking one of the new types of symmetries, multipole symmetries, into account. Introducing gauge fields associated with the multipole symmetries, one can systematically construct new foliated BF theories. While the previous foliated BF theories consist of a subextensive number of layers of toric codes [60–62], our BF theories are made of a *finite* number of layers of the toric codes with couplings. We further show that, due to these couplings which modify the gauge transformations, such theories exhibit unusual GSD dependence on

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the system size; the GSD depends on the greatest common divisor between a quantum number N characterizing the fractional charges and the system size, which is in contrast with the previous foliated BF theories which show subextensive GSD [60,61]. We also show that our foliated BF theories have the UV lattice model counterparts which were studied in the literature [66–68], sometimes referred to as the higher rank topological phases. Consideration given in this work provides an important connection between the fracton topological phases and the higher rank topological phases [66–70] in view of foliated field theories.

The rest of this work is organized as follows. In Sec. II, to see clearly how our strategy of building up the foliated BF theories works, we discuss a simple example of construction of a BF theory of a conventional topological phase, starting with a theory with a global U(1) symmetry. In Sec. III, we study a foliated BF theory with global and dipole symmetry. We also discuss the GSD of the theory on a torus geometry and the UV lattice model counterpart. In Sec. IV, we further explore a foliated BF theory with global, dipole, and quadrupole symmetry. Finally, in Sec. V, we conclude our work with a few future directions. Technical issues are relegated into the Appendixes.

II. WARM-UP: CONSTRUCTION OF THE $\mathbb{Z}_N$ TORIC CODE

Before going into detailed discussion of the foliated BF theories, we demonstrate a way to construct the BF theory of the $\mathbb{Z}_N$ toric code [10] starting with a theory with global U(1) symmetry and the corresponding U(1) gauge field for clearer illustrations. Throughout this paper, we study theories in $2+1$ dimensions. Also, we employ the differential form for notational simplicity.

We consider a theory with global U(1) zero-form symmetry (i.e., symmetry operation acts on an entire space) and its charge Q. The conserved charge is described by

$$Q(V) = \int_V *j, \tag{2.1}$$

where j and V represents the conserved one-form current and two dimensional spatial volume and $*$ represents the Hodge dual. This charge is global in the sense that it commutes with the translation operation, P_I $(I = x, y)$, i.e.,

$$[iP_I, Q] = 0. \tag{2.2}$$

We introduce a one-form U(1) gauge field a, which couples with the current j with the coupling term being described by

$$S_c = \int_V a \wedge *j. \tag{2.3}$$

With the gauge transformation (χ: gauge parameter) $a \to a + d\chi$ and the condition that the coupling term (2.3) is gauge invariant, we have the conservation law of the current $d * j = 0$. Defining a gauge invariant flux as $f := da$, we introduce the following Lagrangian with discarding the coupling term S_c:

$$\mathcal{L}_{TC} = \frac{N}{2\pi} b \wedge f = \frac{N}{2\pi} b \wedge da, \tag{2.4}$$

where b represents a one-form U(1) gauge field. The theory (2.4) is nothing but the BF description of the $\mathbb{Z}_N$ toric code [10,71]. The equation of motion of the theory (2.4) implies that the following gauge invariant field strengths vanish [72]:

$$\begin{aligned} B^a &= \partial_x a_y - \partial_y a_x,\ E^a_x = \partial_\tau a_x - \partial_x a_0,\ E^a_y = \partial_\tau a_y - \partial_y a_0, \\ B^b &= \partial_x b_y - \partial_y b_x,\ E^b_x = \partial_\tau b_x - \partial_x b_0,\ E^b_y = \partial_\tau b_y - \partial_y b_0. \end{aligned} \tag{2.5}$$

It is known that the BF theory (2.4) has nontrivial ground state degeneracy (GSD) when we put the theory on a Riemann surface with nonzero genus [10,18,71]. Indeed, on the torus geometry, the number of distinct noncontractible loops of the gauge fields, a_x, and a_y amounts to the GSD. To see this, we consider the following Wilson loop:

$$W_{0x}(y) = \exp\left[i\oint dx\, a_x(x,y)\right],$$

which can be intuitively understood as the trajectory of the $\mathbb{Z}_N$ fractional charge going around the torus in the x direction. Since the magnetic flux vanishes, this loop does not depend on y. Likewise, we also think of the following Wilson loop:

$$W_{0y}(x) = \exp\left[i\oint dy\, a_y(x,y)\right],$$

which is associated with the trajectory of the $\mathbb{Z}_N$ fractional charge in the y direction. Using the fluxless condition, one can verify that it does not depend on x. From $W_{0x}^N(y) = W_{0y}^N(x) = 1$, it follows that there are N^2 distinct Wilson loops of the gauge field a, implying that the GSD is given by N^2. The way we evaluate the GSD on the torus geometry by counting a distinct number of the Wilson loops will be frequently used in the case of the foliated BF theories discussed in the subsequent sections.

One can construct the UV Hamiltonian corresponding to the BF theory (2.4). Focusing on the field strength of the gauge field a, that is, B^a, E^a_x, and E^a_y in (2.5), the Hamiltonian of the lattice can be constructed in a way that the ground state does not have the electric and magnetic charges. Such a task can be accomplished by defining the Hamiltonian so that the ground state is the trivial eigenstate of the operators of the Gauss law $G = \partial_x E^a_x + \partial_y E^a_y$ and the magnetic flux B^a. To implement this construction, we think of a 2D discrete lattice and on each link we define a local Hilbert space spanned by

$$|\eta\rangle_{(\hat{x}+1/2,\hat{y})}, \quad |\eta\rangle_{(\hat{x},\hat{y}+1/2)}, \tag{2.6}$$

where the subscripts $(\hat{x}+1/2,\hat{y})$ and $(\hat{x},\hat{y}+1/2)$ denote the coordinate of the horizontal and vertical links, respectively, while η labels $\mathbb{Z}_N$ i.e., $\eta = 0, 1, \dots, N-1$ (mod N). We also introduce the $\mathbb{Z}_N$ Pauli operators $X_{(\hat{x}+1/2,\hat{y})}$ and $Z_{(\hat{x}+1/2,\hat{y})}$ acting on the state on the horizontal link $|\eta\rangle_{(\hat{x}+1/2,\hat{y})}$ as

$$\begin{aligned} X_{(\hat{x}+1/2,\hat{y})}|\eta\rangle_{(\hat{x}+1/2,\hat{y})} &= |\eta+1\rangle_{(\hat{x}+1/2,\hat{y})}, \\ Z_{(\hat{x}+1/2,\hat{y})}|\eta\rangle_{(\hat{x}+1/2,\hat{y})} &= \omega^\eta|\eta\rangle_{(\hat{x}+1/2,\hat{y})}, \end{aligned} \tag{2.7}$$

with $\omega := e^{2\pi i/N}$. We also define the $\mathbb{Z}_N$ Pauli operators $X_{(\hat{x},\hat{y}+1/2)}$ and $Z_{(\hat{x},\hat{y}+1/2)}$ acting on the state on the vertical link in a similar way. Using these Pauli operators, we define the

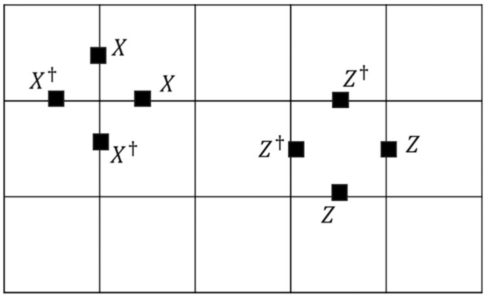

FIG. 1. Two types of the terms defined in (2.8). The black dot represents the $\mathbb{Z}_N$ Pauli operator.

following two types of operators (Fig. 1):

$$V_{(\hat{x},\hat{y})}=X^\dagger_{(\hat{x}-1/2,\hat{y})}X_{(\hat{x}+1/2,\hat{y})}X^\dagger_{(\hat{x},\hat{y}-1/2)}X_{(\hat{x},\hat{y}+1/2)},$$
$$P_{(\hat{x},\hat{y})}=Z^\dagger_{(\hat{x},\hat{y}+1/2)}Z_{(\hat{x}+1,\hat{y}+1/2)}Z^\dagger_{(\hat{x}+1/2,\hat{y})}Z_{(\hat{x}+1/2,\hat{y}+1)}, \tag{2.8}$$

which correspond to the Gauss law and the magnetic flux, respectively. Then we introduce the Hamiltonian as [10]

$$H_{TC}=-\sum_{\hat{x},\hat{y}}[V_{(\hat{x},\hat{y})}+P_{(\hat{x},\hat{y})}]+(\text{H.c.}). \tag{2.9}$$

The ground state $|\Omega\rangle$ of this Hamiltonian does not have electric or magnetic excitations:

$$V_{(\hat{x},\hat{y})}|\Omega\rangle=P_{(\hat{x},\hat{y})}|\Omega\rangle=|\Omega\rangle \quad {}^\forall\,\hat{x},\hat{y} \tag{2.10}$$

and the number of the ground states on the torus agrees with the one of the BF theory (2.4).

In what follows, we will discuss a way to construct foliated BF theories, starting with a theory with *multipole* symmetries as well as a global one [73,74]. The spirit of obtaining such phases is essentially the same as what we have discussed here. Namely, starting with conserved charges associated with multipole symmetries, and introducing gauge fields related to them, we define gauge invariant fluxes to introduce the BF theory. We also discuss the UV lattice Hamiltonian in the same way as we describe the Hamiltonian (2.9) from the BF theory. The crucial difference between the present argument and what we will see in the following is that, due to the multipole symmetries, we obtain richer foliated BF theories.

III. DIPOLE SYMMETRIES

We turn to the case with global and dipole symmetries [73,74]. We also discuss the relation between the foliated BF theory and the UV lattice model. Technically, we gauge global U(1) and dipole symmetries (to be defined soon) and Higgs the gauge fields by introducing a BF theory. Suppose that we have a theory with conserved charges associated with global U(1) and dipole symmetries, described by Q, Q_x, and Q_y. These charges are subject to the following relations:

$$[iP_I,Q]=0,\quad [iP_I,Q_J]=\delta_{I,J}Q\quad (I=x,y), \tag{3.1}$$

where the first relation is the same as (2.2). We write the charges Q, Q_x, and Q_y via the integral expression using the conserved currents as

$$Q=\int_V *j,\quad Q_I=\int_V *K_I.$$

To reproduce the relation (3.1), we demand that

$$*K_I=*k_I-x_I*j, \tag{3.2}$$

with k_I being a local (not necessarily conserved) current and $(x_1,x_2)=(x,y)$. A straightforward calculation verifies the relation (3.1). This can be seen as follows [74]. Noting that the translation operator P_I acts as a derivative on quantum fields and does not act on explicit coordinates, we obtain

$$[iP_I,Q_J]=\int_V[\partial_I(*k_J)-x_J(\partial_I*j)]$$
$$=\int_V\partial_I(*k_J-x_J*j)+\int_V(\partial_Ix_J)*j=\delta_{I,J}Q,$$

where, in the second equation, we have performed the partial integration and, in the last equation, we have dropped the total derivative term.

Corresponding to (2.3), we introduce the U(1) one-form gauge fields a and A^I with the coupling term defined by [75]

$$S_{\text{dip}}=\int_V\left(a\wedge *j+\sum_I A^I\wedge *k_I\right). \tag{3.3}$$

We need to have a proper gauge transformation in such a way that the condition of the coupling term being gauge invariant yields the conservation law of the currents. The following gauge transformation does the job [76,77]:

$$a\to a+d\Lambda+\sum_I\sigma_I dx_I,\quad A^I\to A^I+d\sigma_I, \tag{3.4}$$

where Λ,σ_I are the gauge parameters. Indeed, one can verify that the gauge invariance of the coupling term S_{dip} under the gauge transformation (3.4) yields

$$d*j=0,\quad d(*k_I-x_I*j)=d*K_I=0.$$

Now we are in a good place to introduce the foliated BF theory. We define gauge invariant fluxes as

$$f:=da-\sum_I A^I\wedge dx_I,\quad F^I:=dA^I. \tag{3.5}$$

We put these fluxes in the BF theory format, namely, we introduce the following BF theory:

$$\mathcal{L}_{\text{dip}}=\frac{N}{2\pi}b\wedge f+\sum_{I=x,y}\frac{N}{2\pi}c^I\wedge F^I, \tag{3.6}$$

where b and $c^I (I=x,y)$ represent the U(1) one-form gauge fields. To proceed, we also introduce the foliation fields e^I $(I=x,y)$ [60,61]. Generally, foliation is defined to be codimension one submanifold, which is orthogonal to the one-form foliation field e^I. Setting the foliation field by $e^x:=dx$, $e^y:=dy$, and rewriting (3.6), we arrive at the following foliated BF theory:

$$\boxed{\mathcal{L}_{\text{dip}}=\frac{N}{2\pi}a\wedge db+\sum_{I=x,y}\frac{N}{2\pi}A^I\wedge dc^I+\frac{N}{2\pi}A^I\wedge b\wedge e^I.} \tag{3.7}$$

Compared with the foliated BF theories that were previously studied [60,61], where a subextensive number of 2 + 1D BF theories are placed in layers, in our theory (3.7), there are only three layers of the $2+1$D BF theories, corresponding to the first two terms, with coupling between the layers being described by the last term. Note that the similar form of the coupling term appears in other foliated BF theories in fracton topological phases.

We study the GSD of our foliated BF theory on a torus. In addition to the gauge symmetry (3.4), the foliated BF theory (3.7) admits the following gauge symmetry with respect to b and c^I:

$$b \to b + d\lambda, \quad c^I \to c^I + d\gamma^I + \lambda e^I. \tag{3.8}$$

Similar to the foliated BF theory of the X-cube model [60], the unusual gauge symmetries (3.4) and (3.8) put constraints on the form of the gauge invariant operators, which contributes to the unconventional GSD dependence on the system size. To see this point more explicitly, we integrate out some of the fields in (3.7) to transform the Lagrangian to a simpler form with a fewer number of the gauge fields.

Integrating out b_0 gives the following condition $(i, j = x, y)$:

$$\partial_i a_j \epsilon^{ij} - A^I_i \delta^I_j \epsilon^{ij} = 0. \tag{3.9}$$

We also integrate out A^I_0 and obtain the following condition:

$$\partial_i c^I_j \epsilon^{ij} + b_i \delta^I_j \epsilon^{ij} = 0. \tag{3.10}$$

One can eliminate the gauge fields b and A^I by substituting the relations (3.9) and (3.10) into (3.7). In doing so, we introduce gauge fields

$$A_{(ij)} := \partial_i a_j - A^j_i \quad (i, j = x, y), \tag{3.11}$$

whose gauge transformation coming from (3.4) reads

$$A_{(ij)} \;\to\; A_{(ij)} + \partial_i \partial_j \Lambda, \tag{3.12}$$

with the property that $A_{(xy)} = A_{(yx)}$ due to (3.9). After the substitution, the Lagrangian (3.7) becomes

$$\begin{aligned}\mathcal{L}_{\text{dip}} = \frac{N}{2\pi}\big[&-c^x_0(\partial_x A_{(yx)} - \partial_y A_{(xx)}) - c^y_0(\partial_x A_{(yy)} - \partial_y A_{(xy)})\\ &+ \tilde{c}(\partial_\tau A_{(xy)} - \partial_x\partial_y A_0) - c^x_y\big(\partial_\tau A_{(xx)} - \partial_x^2 A_0\big)\\ &+ c^y_x\big(\partial_\tau A_{(yy)} - \partial_y^2 A_0\big)\big],\end{aligned} \tag{3.13}$$

where we have defined $A_0 := a_0$ and $\tilde{c} := c^x_x - c^y_y$ with the gauge transformation $\tilde{c} \to \tilde{c} + \partial_x \gamma^x - \partial_y \gamma^y$.

Similar to the other fracton models, investigating the BF theory (3.13), such as evaluation of the form of the Wilson loops and the GSD, is more challenging than the conventional BF theory due to the presence of the higher order spatial derivatives, giving rise to UV/IR mixing. To circumvent this issue, we follow an approach proposed in [45] and implement a mapping (3.13) to what is called "integer BF theory" [45,67,73], where the gauge fields take integer values defined on a discrete lattice. Relegating the details to Appendix. A, we map the theory (3.13) to the following integer BF theory:

$$\begin{aligned}\mathcal{L}_{\text{dip}} = \frac{2\pi}{N}\big[&-\hat{c}^x_0(\Delta_x \hat{A}_{(yx)} - \Delta_y \hat{A}_{(xx)}) - \hat{c}^y_0(\Delta_x \hat{A}_{(yy)} - \Delta_y \hat{A}_{(xy)})\\ &+ \hat{\tilde{c}}\big(\Delta_\tau \hat{A}_{(xy)} - \Delta_x \Delta_y \hat{A}_0\big) - \hat{c}^x_y\big(\Delta_\tau \hat{A}_{(xx)} - \Delta_x^2 \hat{A}_0\big)\\ &+ \hat{c}^y_x\big(\Delta_\tau \hat{A}_{(yy)} - \Delta_y^2 \hat{A}_0\big)\big],\end{aligned} \tag{3.14}$$

where gauge fields with hat $(\hat{\cdot})$ take $\mathbb{Z}_N$ values, which are defined on a discrete lattice. Also, we set the spatial coordinate of the lattice to be $(\hat{x}, \hat{y})$, which take integer number in the unit of the lattice spacing, and define a derivative operator Δ_x which acts on a function f defined on a site via $\Delta_x f(\tau, \hat{x}, \hat{y}) = f(\tau, \hat{x}+1, \hat{y}) - f(\tau, \hat{x}, \hat{y})$ (Δ_y, Δ_τ is similarly defined). Note that we put the hat on top of the gauge fields in (3.14) to emphasize that they are integer gauge fields. The gauge fields A_0, c^x_0, and c^y_0 reside on the τ links and $\hat{c}^y_x$ and $\hat{c}^x_y$ on the x and y links, respectively. Also, $\hat{A}_{(xx)}$ $\hat{A}_{(yy)}$ are defined on sites, whereas $\hat{A}_{xy}$ and $\hat{\tilde{c}}$ are on the plaquettes in the xy plane.

The integer BF theory (3.14) admits the following gauge symmetry:

$$\begin{aligned}&\hat{c}^x_0 \to \hat{c}^x_0 + \Delta_\tau \xi^x,\ \hat{c}^x_y \to \hat{c}^x_y + \Delta_y \xi^x,\ \hat{c}^y_0 \to \hat{c}^y_0 + \Delta_\tau \xi^y,\\ &\hat{c}^y_x \to \hat{c}^y_x + \Delta_x \xi^y,\ \hat{\tilde{c}} \to \hat{\tilde{c}} + \Delta_x \xi^x - \Delta_y \xi^y,\\ &\hat{A}_0 \to \hat{A}_0 + \Delta_\tau \eta,\ \hat{A}_{(ij)} \to \hat{A}_{(ij)} + \Delta_i \Delta_j \eta,\end{aligned} \tag{3.15}$$

where ξ^x ξ^y and η denote gauge parameters with integer values. Also, the BF theory (3.14) consists of the gauge field $(\hat{c}^x_0, \hat{c}^y_0, \hat{\tilde{c}}, \hat{c}^x_y, \hat{c}^y_x)$ and the field strength for $(\hat{A}_0, \hat{A}_{(ij)})$. The equations of motion of the BF theory imply that the following gauge invariant field strengths vanish:

$$\begin{aligned}&B_x = \Delta_x \hat{A}_{(yx)} - \Delta_y \hat{A}_{(xx)}, \quad B_y = \Delta_x \hat{A}_{(yy)} - \Delta_y \hat{A}_{(xy)},\\ &E_{(ij)} = \Delta_\tau \hat{A}_{(ij)} - \Delta_i \Delta_j \hat{A}_0,\\ &\tilde{B} = \Delta_x^2 \hat{c}^x_y - \Delta_y^2 \hat{c}^y_x - \Delta_x \Delta_y \hat{\tilde{c}}, \quad E^x_y = \Delta_\tau \hat{c}^x_y - \Delta_y \hat{c}^x_0,\\ &E^y_x = \Delta_\tau \hat{c}^y_x - \Delta_x \hat{c}^y_0, \quad \tilde{E} = \Delta_\tau \hat{\tilde{c}} - \Delta_x \hat{c}_0{}^x + \Delta_y \hat{c}^y_0.\end{aligned} \tag{3.16}$$

It is worth mentioning that the field strength B_x, B_y, $E_{(xx)}, E_{(yy)}, E_{(xy)}$ has the same form as the one which was found in the symmetric tensor gauge theory, unconventional Maxwell theory with the two-rank spatial-symmetric-tensor space components, admitting dipole charges [78–80].

The equation of motions ensure that there are no nontrivial local gauge invariant operators. However, analogous to the other BF theories, the theory has nonlocal gauge-invariant operators, which can be constructed from the gauge fields either $\hat{A}_{(ij)}$ or $(\hat{\tilde{c}}, \hat{c}^x_y, \hat{c}^y_x)$. Especially, when placing on a torus, the theory admits the noncontractible Wilson loops of the gauge fields which contribute to the nontrivial GSD. In the following, we evaluate the GSD of our theory on the torus with system size $L_{x/y}$ in the x/y direction, by counting the number of distinct noncontractible Wilson loops of the gauge fields $\hat{A}_{(ij)}$ [81]. We set the periodic boundary condition of the lattice via $(\hat{x}, \hat{y}) \sim (\hat{x}+L_x, \hat{y}) \sim (\hat{x}, \hat{y}+L_y)$.

Let us first focus on the form of the noncontractible Wilson loops of $\hat{A}_{(xx)}$ in the x direction. Due to the gauge symmetry (3.11), there are two types of the loops, described by

$$\begin{aligned}W_x(\hat{y}) &= \exp\left[i\frac{2\pi}{N}\sum_{\hat{x}=1}^{L_x} \hat{A}_{(xx)}(\hat{x}, \hat{y})\right],\\ W_{\text{dip}:x}(\hat{y}) &= \exp\left[i\frac{2\pi}{N}\alpha_x \sum_{\hat{x}=1}^{L_x} \hat{x}\hat{A}_{(xx)}(\hat{x}, \hat{y})\right],\end{aligned} \tag{3.17}$$

where $\alpha_x = \frac{N}{\gcd(N,L_x)}$ and gcd stands for the greatest common divisor. The first loop in (3.17) describes the noncontractible loops of the $\mathbb{Z}_N$ fractional charge with $W_x^N(\hat{y}) = 1$, which can also be found in the other topological phases such as the toric code. The second loop can be interpreted as the "dipole of the Wilson loop," i.e., the loops are formed by trajectory of the fractional charge with its intensity increasing linearly as going around the torus in the x direction, which is reminiscent of the integration of a dipole moment, $x\rho$. These two loops, $W_x(\hat{y})$ and $W_{\text{dip}:x}(\hat{y})$, are characterized by quantum number $\mathbb{Z}_N$ and $\mathbb{Z}_{\gcd(N,L_x)}$, respectively. We also derive the same Wilson loops from a different perspective; see Appendix C.

Likewise, there are two types of the noncontractible loops of $\hat{A}_{(yy)}$ in the y direction, which have the form

$$W_y(\hat{x}) = \exp\left[i\frac{2\pi}{N}\sum_{\hat{y}=1}^{L_y}\hat{A}_{(yy)}(\hat{x},\hat{y})\right],$$
$$W_{\text{dip}:y}(\hat{y}) = \exp\left[i\frac{2\pi}{N}\alpha_y\sum_{\hat{y}=1}^{L_y}\hat{y}\hat{A}_{(yy)}(\hat{x},\hat{y})\right], \tag{3.18}$$

with $\alpha_y = \frac{N}{\gcd(N,L_y)}$, implying that these are labeled by $\mathbb{Z}_N \times \mathbb{Z}_{\gcd(N,L_y)}$.

We turn to introducing noncontractible loops of the gauge field $\hat{A}_{(xy)}$ in either the x or y direction. To this end, we consider the following gauge invariant loops:

$$W_{xy:x}(\hat{y}) = \exp\left[i\frac{2\pi}{N}\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xy)}(\hat{x},\hat{y})\right],$$
$$W_{xy:y}(\hat{x}) = \exp\left[i\frac{2\pi}{N}\sum_{\hat{y}=1}^{L_y}\hat{A}_{(xy)}(\hat{x},\hat{y})\right]. \tag{3.19}$$

Naively, there are N^2 distinct loops; however, a set of constraints reduce this number, as we discuss below.

After having defined gauge invariant noncontractible loops (3.17), (3.18), and (3.19), we need to check the coordinate dependence of the loops to count the distinct number of the loops. As we will see, there are several constraints imposed on the loops of the gauge field $\hat{A}_{(xy)}$. We first check the $\hat{y}$ dependence of the loops, $W_x(\hat{y})$ and $W_{\text{dip}:x}(\hat{y})$ (3.17). From the equation of motion of the BF theory $B_y = 0$, where B_y is given in (3.16), and summing the field along the x direction, we have

$$\sum_{\hat{x}=1}^{L_x}(\Delta_x\hat{A}_{(yx)} - \Delta_y\hat{A}_{(xx)}) = 0 \leftrightarrow \Delta_y\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)} = 0, \tag{3.20}$$

implying that the loop $W_x(\hat{y})$ (3.17) does not depend on $\hat{y}$. Furthermore, multiplying $\hat{x}$ on the equation of motion $B_x = 0$ and summing along the x direction, one finds

$$\Delta_y\sum_{\hat{x}=1}^{L_x}\hat{x}\hat{A}_{(xx)} = -\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xy)}, \tag{3.21}$$

from which we have

$$W_{\text{dip}:x}(\hat{y}) = W_{\text{dip}:x}(\hat{y}-1)\big(W_{xy:x}^{\alpha_x}(\hat{y})\big)^\dagger. \tag{3.22}$$

The iterative use of (3.22) gives

$$W_{\text{dip}:x}(\hat{y}) = W_{\text{dip}:x}(\hat{y}_0)\big(W_{xy:x}^{\alpha_x(\hat{y}-\hat{y}_0)}(\hat{y}_0)\big)^\dagger, \tag{3.23}$$

implying that, in order to deform the loop $W_{\text{dip}:x}(\hat{y})$ from $\hat{y}$ to $\hat{y}_0$, we need to multiply additional loops. Due to the periodic boundary condition, $W_{\text{dip}:x}(\hat{y}) = W_{\text{dip}:x}(\hat{y}+L_y)$, we must have

$$W_{xy:x}^{\alpha_x L_y}(\hat{y}) = 1. \tag{3.24}$$

Therefore, there are $N \times \gcd(N, L_x)$ distinct loops (3.17) with the condition of (3.24).

We turn to checking the $\hat{x}$ dependence of the loops $W_y(\hat{x})$ and $W_{\text{dip}:y}(\hat{x})$ (3.18). A similar line of thought shows that $W_y(\hat{x})$ does not depend on $\hat{x}$ and that

$$W_{\text{dip}:y}(\hat{y}) = W_{\text{dip}:y}(\hat{x}_0)\big(W_{xy:y}^{\alpha_y(\hat{x}-\hat{x}_0)}(\hat{x}_0)\big)^\dagger, \tag{3.25}$$

from which one finds

$$W_{xy:y}^{\alpha_y L_x}(\hat{x}) = 1. \tag{3.26}$$

Hence there are $N \times \gcd(N, L_y)$ distinct loops (3.18) with the condition of (3.26).

We move onto investigating the coordinate dependence of the loops $W_{xy:x}(\hat{y})$ and $W_{xy:y}(\hat{x})$ given in (3.19). From the equation of motion $B_x = 0$, and summing over the field along the x direction, we have

$$\sum_{\hat{x}=1}^{L_x}(\Delta_x\hat{A}_{(yy)} - \Delta_y\hat{A}_{(xy)}) = 0 \leftrightarrow \Delta_y\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xy)} = 0, \tag{3.27}$$

from which it can be shown that the loop $W_{xy:x}(\hat{y})$ can be deformed so that it can go up or down in the y direction, namely, $W_{xy:x}(\hat{y}) = W_{xy:x}(\hat{y}-1) = \cdots = W_{xy:x}(\hat{y}_0)$, where $\hat{y}_0$ denotes an arbitrary y coordinate [i.e., the loop $W_{xy:x}(\hat{y})$ does not depend on $\hat{y}$]. Likewise, one can show that the loop $W_{xy:y}(\hat{x})$ can go left or right, i.e., $W_{xy:y}(\hat{x}) = W_{xy:y}(\hat{x}-1) = \cdots = W_{xy:y}(\hat{x}_0)$, with $\hat{x}_0$ being an arbitrary x coordinate. Using this property, it follows that

$$W_{xy:x}^{L_y}(\hat{y}) = W_{xy:y}^{L_x}(\hat{x}). \tag{3.28}$$

With the conditions (3.24), (3.26), and (3.28), one can count the distinct number of the Wilson loops involving $\hat{A}_{(xy)}$ (3.19). Let us consider a set of loops (3.19) with the following form:

$$W_{xy:x}^{k_x}(\hat{y})W_{xy:y}^{k_y}(\hat{x}), \tag{3.29}$$

with (k_x, k_y) being integers. We need to count the distinct combination of (k_x, k_y). Relegating the details to Appendix B, we find that there are $N \times \gcd(N, L_x, L_y)$ distinct loops of $\hat{A}_{(xy)}$.

To recap the argument, we have counted the distinct number of the noncontractible loops of the gauge fields $\hat{A}_{(ij)}$. For the loops involving $\hat{A}_{(xx)}$ and $\hat{A}_{(yy)}$, there are $[N \times \gcd(N, L_x)] \times [N \times \gcd(N, L_y)]$ distinct loops, whereas there are $N \times \gcd(N, L_x, L_y)$ distinct loops of $\hat{A}_{(xy)}$. The total number of the distinct loops amounts to be the GSD. Hence we arrive at

$$\boxed{\text{GSD} = N^3 \times \gcd(N, L_x) \times \gcd(N, L_y) \times \gcd(N, L_x, L_y).} \tag{3.30}$$

Compared with previous foliated BF theories [60–62], where coupling between the layers of the toric codes yields the subextensive GSD, in our foliated BF theory, the coupling between the three layers of the toric codes gives rise to unusual GSD dependence on the system size.

A. UV stabilizer model

Analogous to the fact that the BF theory description of the $\mathbb{Z}_N$ toric code can be mapped to the stabilizer model in the UV lattice, one can implement a similar mapping from the BF theory to the lattice model. Focusing on gauge-invariant field strengths, B_x, B_y, and $E_{(ij)}$ (3.16), we introduce the UV lattice Hamiltonian in such a way that the ground state does not admit the $\mathbb{Z}_N$ electric and magnetic charge excitations.

To this end, we envisage a 2D square lattice and introduce two types of local Hilbert space of the $\mathbb{Z}_N$ spin, denoted by $|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})}$ with $a, b = 0, 1, \dots, N-1$, as well as the $\mathbb{Z}_N$ Pauli operators $X_{i,(\hat{x},\hat{y})}$, $Z_{i,(\hat{x},\hat{y})}$ $(i = 1, 2)$ that act on the Hilbert space with

$$\begin{aligned} Z_{1,(\hat{x},\hat{y})}\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})} &= \omega^a\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})}\,,\\ Z_{2,(\hat{x},\hat{y})}\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})} &= \omega^b\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})}\,,\\ X_{1,(\hat{x},\hat{y})}\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})} &= |a+1\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})}\,,\\ X_{2,(\hat{x},\hat{y})}\,|a\rangle_{(\hat{x},\hat{y})}\,|b\rangle_{(\hat{x},\hat{y})} &= |a\rangle_{(\hat{x},\hat{y})}\,|b+1\rangle_{(\hat{x},\hat{y})}\,. \end{aligned} \tag{3.31}$$

At each plaquette, labeled by the coordinate $(\hat{x}+1/2, \hat{y}+1/2)$, we also introduce a local Hilbert space represented by $|c\rangle_{(\hat{x}+1/2,\hat{y}+1/2)}$ with $c = 0, \dots, N-1 \mod N$ and the $\mathbb{Z}_N$ Pauli operators, $X_{0,(\hat{x}+1/2,\hat{y}+1/2)}$, $Z_{0,(\hat{x}+1/2,\hat{y}+1/2)}$, which act on the state as

$$\begin{aligned} Z_{0,(\hat{x}+1/2,\hat{y}+1/2)}\,|c\rangle_{(\hat{x}+1/2,\hat{y}+1/2)} &= \omega^c\,|c\rangle_{(\hat{x}+1/2,\hat{y}+1/2)}\,,\\ X_{0,(\hat{x}+1/2,\hat{y}+1/2)}\,|c\rangle_{(\hat{x}+1/2,\hat{y}+1/2)} &= |c+1\rangle_{(\hat{x}+1/2,\hat{y}+1/2)}\,. \end{aligned} \tag{3.32}$$

With these preparations, we define the following mutually commuting operators:

$$\begin{aligned} V_{(\hat{x},\hat{y})} &:= X_{1,(\hat{x}+1,\hat{y})}X_{1,(\hat{x}-1,\hat{y})}(X^\dagger_{1,(\hat{x},\hat{y})})^2X_{2,(\hat{x},\hat{y}+1)}X_{2,(\hat{x},\hat{y}-1)}(X^\dagger_{2,(\hat{x},\hat{y})})^2\\ &\quad\times X_{0,(\hat{x}+1/2,\hat{y}+1/2)}X^\dagger_{0,(\hat{x}-1/2,\hat{y}+1/2)}X_{0,(\hat{x}-1/2,\hat{y}-1/2)}X^\dagger_{0,(\hat{x}-1/2,\hat{y}-1/2)},\\ P_{(\hat{x},\hat{y}+1/2)} &:= Z^\dagger_{1,(\hat{x},\hat{y}+1)}Z_{1,(\hat{x},\hat{y})}Z_{0,(\hat{x}+1/2,\hat{y}+1/2)}Z^\dagger_{0,(\hat{x}-1/2,\hat{y}+1/2)},\\ Q_{(\hat{x}+1/2,\hat{y})} &:= Z_{2,(\hat{x}+1,\hat{y})}Z^\dagger_{2,(\hat{x},\hat{y})}Z_{0,(\hat{x}+1/2,\hat{y}+1/2)}Z^\dagger_{0,(\hat{x}+1/2,\hat{y}-1/2)}, \end{aligned} \tag{3.33}$$

which are portrayed in Figs. 2(a) and 2(b). The operator $V_{(\hat{x},\hat{y})}$ corresponds to the Gauss law term involving the eclectic fields, $E_{(ij)}$, $G = \Delta_x^2E_{(xx)} + \Delta_y^2E_{(yy)} + \Delta_x\Delta_yE_{(xy)}$, whereas the operators $P_{(\hat{x},\hat{y}+1/2)}$ and $Q_{(\hat{x}+1/2,\hat{y})}$ are associated with the magnetic flux B_x and B_y, respectively [78]. The Hamiltonian is defined in such a way that the ground state does not contain the electric and magnetic excitations, namely,

$$H_{\text{dip}} := -\sum_{\hat{x},\hat{y}}[V_{(\hat{x},\hat{y})} + P_{(\hat{x},\hat{y}+1/2)} + Q_{(\hat{x}+1/2,\hat{y})}] + (\text{H.c.}). \tag{3.34}$$

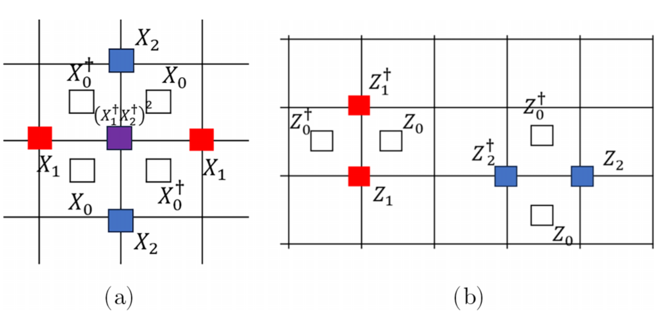

FIG. 2. Two types of the terms that constitute Hamiltonian (3.34). The red (blue) square represents the Pauli operator acting on the local Hilbert space $|a\rangle_{(\hat{x},\hat{y})}$ ($|b\rangle_{(\hat{x},\hat{y})}$), whereas the small square with black line denotes the Pauli operator that acts on $|c\rangle_{(\hat{x}+1/2,\hat{y}+1/2)}$. A purple square indicates the composite of the Pauli operators which act on both $|a\rangle_{(\hat{x},\hat{y})}$ and $|b\rangle_{(\hat{x},\hat{y})}$.

It is interesting to note that, in [66,70], the same Hamiltonian was obtained by placing the tensor gauge theory [78] on the lattice and Higgs it, which is in contrast with the present case where we Higgs the continuum gauge theory with the dipole symmetries and put it on the lattice.

IV. QUADRUPOLE

After having seen the example of the BF theories with dipole symmetries, in this section, we extend the previous argument to the case with quadrupole symmetry.

A. Construction of the foliated BF theory

We envisage a theory with quadrupole U(1) symmetry in addition to the global charge and dipole ones whose charges are denoted as Q, Q_x, Q_y, and Q_{xy}. These charges are subject to the following commutation relation with the translation operators:

$$[iP_I, Q] = 0, \quad [iP_I, Q_J] = \delta_{I,J}Q, \quad [iP_I, Q_{xy}] = Q_{\bar{I}}, \tag{4.1}$$

where

$$\bar{I} = \begin{cases} x & \text{for } I = y,\\ y & \text{for } I = x. \end{cases}$$

We also write the charges via integral expression of one-form currents as

$$Q = \int_V *j, \quad Q_I = \int_V *K_I, \quad Q_{xy} = \int_V *\ell. \tag{4.2}$$

If we set

$$*K_I = *k_I - x_I * j, \quad *\ell := *J + xy * j - x_I * k_I, \tag{4.3}$$

where k_I and J represent nonconserved local current, then a simple calculation shows that the form (4.2) jointly with (4.3) yields the commutation relations (4.1). We introduce one-form U(1) gauge fields as a, A^I, and a' that are coupled with the currents with coupling term

$$S_{qp} = \int_V a \wedge *j + \sum_I A^I \wedge *k_I + a' \wedge *J. \tag{4.4}$$

With the following gauge transformation (Λ, σ_I, Λ': gauge parameters):

$$a \to a + d\Lambda + \sum_I \sigma_I dx_I, \quad A^I \to A^I + d\sigma_I + \Lambda' dx_{\bar{I}},$$
$$a' \to a' + d\Lambda', \tag{4.5}$$

jointly with the condition S_{qp} is invariant under the gauge transformation, one can verify the three kinds of currents are conserved, i.e., $d * j = d * K_I = d * \ell = 0$.

We further introduce gauge invariant fluxes by

$$f := da - \sum_I A^I \wedge dx^I, \quad F^I := dA^I - a' \wedge dx^{\bar{I}},$$
$$f' := da'. \tag{4.6}$$

Using these fluxes, we defined a BF theory as

$$\mathcal{L}_{qp} = \frac{N}{2\pi}\left[b \wedge f + \sum_{I=x,y} c^I \wedge F^I + D \wedge f'\right], \tag{4.7}$$

where c^I and D denote U(1) one-form gauge fields. Writing the foliated field as $e^I = dx^I$, Eq. (4.7) is transformed into the following foliated BF theory:

$$\boxed{\begin{aligned}\mathcal{L}_{qp} = {}& \frac{N}{2\pi} a \wedge db + \frac{N}{2\pi} a' \wedge dD + \sum_{I=x,y} \frac{N}{2\pi} A^I \wedge dc^I \\ & + \frac{N}{2\pi} A^I \wedge b \wedge e^I + \frac{N}{2\pi} a' \wedge c^I \wedge e^{\bar{I}}.\end{aligned}} \tag{4.8}$$

Compared with the previous foliated BF theory (3.7), we now have four layers of the toric codes, corresponding to the first three terms in (4.8), with the coupling terms between the layers being given by the last two terms. In the next subsection, we show that the foliated BF theory exhibits unusual GSD dependence on the system size on torus geometry by studying the noncontractible Wilson loops.

B. Ground state degeneracy

In addition to (4.5), the theory respects the following gauge symmetry:

$$D \to D + d\lambda + \sum_I \xi^I e^{\bar{I}}, \quad c^I \to c^I - d\xi^I. \tag{4.9}$$

Analogous to the previous foliated BF theory (3.7), the theory admits unusual gauge symmetries due to the presence of the couplings between the layers. This puts constraints on the form of the gauge invariant operators, contributing to the lattice dependence of the GSD. To see this point, we simplify the Lagrangian (4.8) by integrating out some of the fields.

By integrating out b_0 and a'_0, we have

$$\partial_i a_j \epsilon^{ij} - A^I_i \delta^I_j \epsilon^{ij} = 0 \;\leftrightarrow\; \partial_x a_y - A^y_x = \partial_y a_x - A^x_y \tag{4.10}$$

and

$$\partial_i D_j \epsilon^{ij} + \sum_{I \neq J} c^I_i \delta^J_j \epsilon^{ij} = 0 \;\leftrightarrow\; c^x_x - c^y_y = -\partial_i D_j \epsilon^{ij}. \tag{4.11}$$

Furthermore, integrating out the gauge field A^I_0 and c^I_0 gives rise to the following two conditions:

$$\partial_i c^I_j \epsilon^{ij} + b_i \delta^I_j \epsilon^{ij} = 0, \quad \partial_i A^I_j \epsilon^{ij} + a'_i \delta^{\bar{I}}_j \epsilon^{ij} = 0, \tag{4.12}$$

which can be rewritten as

$$\begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} -\partial_i c^y_j \epsilon^{ij} \\ \partial_i c^x_j \epsilon^{ij} \end{pmatrix}, \quad \begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = \begin{pmatrix} \partial_i A^x_j \epsilon^{ij} \\ -\partial_i A^y_j \epsilon^{ij} \end{pmatrix}. \tag{4.13}$$

To proceed, we introduce gauge fields as

$$A_{(xx)} := \partial_x a_x - A^x_x, \quad A_{(yy)} := \partial_y a_y - A^y_y,$$

and eliminate the gauge fields a' and b by use of the relations (4.10)–(4.13); the Lagrangian is rewritten as

$$\boxed{\begin{aligned}\mathcal{L}_{qp} = \frac{N}{2\pi}\big[& A_0\big(\partial_x^2 B_{(yy)} - \partial_y^2 B_{(xx)}\big) + A_{(xx)}\big(\partial_\tau B_{(yy)} - \partial_y^2 B_0\big) \\ & - A_{(yy)}\big(\partial_\tau B_{(xx)} - \partial_x^2 B_0\big)\big],\end{aligned}} \tag{4.14}$$

where

$$A_0 := a_0, \quad B_0 := D_0, \quad B_{(xx)} := \partial_x D_x + c^y_x,$$
$$B_{(yy)} := \partial_y D_y + c^x_y. \tag{4.15}$$

Analogous to the previous section, in order to study the BF theory, we implement a mapping from (4.14) to the following integer BF theory defined on a discrete lattice (see Appendix. A):

$$\mathcal{L}_{qp} = \frac{2\pi}{N}\big[\hat{A}_0\big(\Delta_x^2 \hat{B}_{(yy)} - \Delta_y^2 \hat{B}_{(xx)}\big) + \hat{A}_{(xx)}\big(\Delta_\tau \hat{B}_{(yy)} - \Delta_y^2 \hat{B}_0\big)$$
$$- \hat{A}_{(yy)}\big(\Delta_\tau \hat{B}_{(xx)} - \Delta_x^2 \hat{B}_0\big)\big], \tag{4.16}$$

where $\hat{B}_0$, $\hat{B}_{(xx)}$, $\hat{B}_{(yy)}$, $\hat{A}_0$, $\hat{A}_{(xx)}$, and $\hat{A}_{(yy)}$ denote the gauge fields which take integer values. The gauge fields $\hat{A}_0$ and $\hat{B}_0$ are defined on the τ links, whereas $\hat{A}_{(ii)}$ and $B_{(ii)}$ $(i = x, y)$ are on sites. The gauge fields respect the following gauge symmetry (ξ, ξ': integer gauge parameters):

$$\hat{B}_0 \to \hat{B}_0 + \Delta_\tau \xi, \quad \hat{B}_{(xx)} \to \hat{B}_{(xx)} + \Delta_x^2 \xi,$$
$$\hat{B}_{(yy)} \to \hat{B}_{(yy)} + \Delta_y^2 \xi, \quad \hat{A}_0 \to \hat{A}_0 + \Delta_\tau \xi',$$
$$\hat{A}_{(xx)} \to \hat{A}_{(xx)} + \Delta_x^2 \xi', \quad \hat{A}_{(yy)} \to \hat{A}_{(yy)} + \Delta_y^2 \xi'. \tag{4.17}$$

The theory (4.16) reminds us of the BF theory description of the $\mathbb{Z}_N$ toric code (2.4) with the difference being that the spatial derivatives are replaced with the second order. The equations of motion in the Lagrangian imply that the following gauge invariant field strengths vanish:

$$F_A = \Delta_x^2 \hat{A}_{(yy)} - \Delta_y^2 \hat{A}_{(yy)}, \quad E_{xA} = \Delta_\tau \hat{A}_{(xx)} - \Delta_x^2 \hat{A}_0,$$
$$E_{yA} = \Delta_\tau \hat{A}_{(yy)} - \Delta_y^2 \hat{A}_0, \quad F_B = \Delta_x^2 \hat{B}_{(yy)} - \Delta_y^2 \hat{B}_{(yy)},$$
$$E_{xB} = \Delta_\tau \hat{B}_{(xx)} - \Delta_x^2 \hat{B}_0, \quad E_{yB} = \Delta_\tau \hat{B}_{(yy)} - \Delta_y^2 \hat{B}_0. \tag{4.18}$$

As in the case with the other BF theories, the equations of motion ensure that there are no nontrivial local gauge invariant operators, yet the theory admits nonlocal gauge-invariant operators, contributing to the GSD. With the simplified Lagrangian (4.16), we evaluate the GSD on the torus geometry by counting the distinct number of the noncontractible Wilson loops of the gauge fields $\hat{A}_{(xx)}$ and $\hat{A}_{(yy)}$. Below, similar to the previous section, we think of the theory (4.16) on a discrete 2D lattice with periodic boundary condition and evaluate the GSD [82].

As for the noncontractible loops of the gauge field, $\hat{A}_{(xx)}$, referring to the gauge transformation (4.17), we have two types of such loops in the form of

$$W_x(\hat{y}) = \exp\left[i\frac{2\pi}{N}\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y})\right],$$
$$W_{\mathrm{dip}:x}(\hat{y}) = \exp\left[i\frac{2\pi}{N}\alpha_x\sum_{\hat{x}=1}^{L_x}\hat{x}\hat{A}_{(xx)}(\hat{x},\hat{y})\right], \qquad (4.19)$$

with $\alpha_x = \frac{N}{\gcd(N,L_x)}$. Justification of the existence of these loops is given in Appendix C, where we discuss thoroughly the Wilson loops in the UV lattice model. In order to count the number of distinct such loops, we need to check whether these loops depend on $\hat{y}$. Let us first focus on the loop $W_x(\hat{y})$. From the equation of motion, $F_A = \Delta_x^2\hat{A}_{(yy)} - \Delta_y^2\hat{A}_{(yy)} = 0$, and summing over the field along the x direction, we have

$$\Delta_y^2\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}) = 0 \;\leftrightarrow\; \Delta_y\left(\sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y})\right) = 0, \qquad (4.20)$$

from which it follows that $\sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y})$ is independent of $\hat{y}$. For an arbitrary $\hat{y}_0$, we set

$$\sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y}) = \sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y}_0), \qquad (4.21)$$

which can be rewritten as

$$\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}+1) - \sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}) = \sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y}_0). \qquad (4.22)$$

Iterative use of (4.22) gives

$$\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}) = \sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}_0) + (\hat{y}-\hat{y}_0)\sum_{\hat{x}=1}^{L_x}\Delta_y\hat{A}_{(xx)}(\hat{x},\hat{y}_0), \qquad (4.23)$$

from which we have

$$W_x(\hat{y}) = W_x(\hat{y}_0)[\tilde{W}_x(\hat{y}_0)]^{(\hat{y}_0-\hat{y})},$$
$$\tilde{W}_x(\hat{y}_0) := \exp\left[i\frac{2\pi}{N}\sum_{\hat{x}=1}^{L_x}\hat{A}_{(xx)}(\hat{x},\hat{y}_0+1) - \hat{A}_{(xx)}(\hat{x},\hat{y}_0)\right], \qquad (4.24)$$

indicating that, in order to deform the loop $W_x(\hat{y})$ from $\hat{y}$ to $\hat{y}_0$, we need to multiply other loops. The relation (4.24) is corroborated by a close investigation of the stabilizer model presented in Appendix C.

The distinct loops of $W_x(\hat{y})$ are labeled by two quantum numbers, corresponding to two loops, $W_x(\hat{y}_0)$ and $\tilde{W}_x(\hat{y}_0)$. The loop $W_x(\hat{y}_0)$ is labeled by $\mathbb{Z}_N$ as we have $W_x^N(\hat{y}_0) = 1$. Due to the periodic boundary condition, $W_x(\hat{y}+L_y) = W_x(\hat{y})$ and (4.24), the loop $\tilde{W}_x(\hat{y}_0)$ is subject to $\tilde{W}_x^{L_y}(\hat{y}_0) = \tilde{W}_x^N(\hat{y}_0) = 1$, from which it follows that the loop $\tilde{W}_x(\hat{y}_0)$ is labeled by $\mathbb{Z}_{\gcd(N,L_y)}$. In total, the distinct number of the loop $W_x(\hat{y})$ is given by $N \times \gcd(N, L_y)$.

The similar consideration shows that there are $\gcd(N, L_x) \times \gcd(N, L_x, L_y)$ distinct loops of $W_{\mathrm{dip}:x}(\hat{y})$. Overall, there are $N \times \gcd(N, L_x) \times \gcd(N, L_x) \times \gcd(N, L_x, L_y)$ distinct noncontractible loops of the gauge field $\hat{A}_{(xx)}$ in the x direction. One can analogously count the number of the distinct noncontractible loops of $\hat{A}_{(yy)}$ in the y direction, arriving at the same number. Therefore, the GSD, which is equivalent to the distinct number of loops of the gauge fields $\hat{A}_{(xx)}$ and $\hat{A}_{(yy)}$, is given by

$$\boxed{\mathrm{GSD} = [N \times \gcd(N, L_x) \times \gcd(N, L_x) \times \gcd(N, L_x, L_y)]^2.} \qquad (4.25)$$

Analogous to the previous case, the foliated BF theory shows the GSD dependence on the UV lattice, involving the greatest common divisor between N and the system size, which is in sharp contrast with the subextensive GSD dependence found in the preexisting foliated BF theories.

C. UV stabilizer model

Similar to the previous section, we can map from the BF theory to the UV lattice stabilizer model. To start, we introduce a 2D square lattice where we place two types of N-qubit states ($\mathbb{Z}_N$ clock states) on each vertex. We represent the basis of the two types of the clock states as $|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})}$, with $a, b \in \mathbb{Z}_N$, and the $\mathbb{Z}_N$ Pauli operators acting on the state as

$$Z_{1,(\hat{x},\hat{y})}|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})} = \omega^a|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})},$$
$$Z_{2,(\hat{x},\hat{y})}|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})} = \omega^b|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})},$$
$$X_{1,(\hat{x},\hat{y})}|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})} = |a+1\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})},$$
$$X_{2,(\hat{x},\hat{y})}|a\rangle_{(\hat{x},\hat{y})}|b\rangle_{(\hat{x},\hat{y})} = |a\rangle_{(\hat{x},\hat{y})}|b+1\rangle_{(\hat{x},\hat{y})}. \qquad (4.26)$$

With this preparation, we introduce the following terms (see Fig. 3):

$$V_{(\hat{x},\hat{y})} := X_{1,(\hat{x}+1,\hat{y})}X_{1,(\hat{x}-1,\hat{y})}(X^\dagger_{1,(\hat{x},\hat{y})})^2X_{2,(\hat{x},\hat{y}+1)}X_{2,(\hat{x},\hat{y}-1)}(X^\dagger_{2,(\hat{x},\hat{y})})^2,$$
$$P_{(\hat{x},\hat{y})} := Z^\dagger_{1,(\hat{x},\hat{y}+1)}Z^\dagger_{1,(\hat{x},\hat{y}-1)}Z^2_{1,(\hat{x},\hat{y})}Z_{2,(\hat{x}+1,\hat{y})}Z_{2,(\hat{x}-1,\hat{y})}(Z^\dagger_{2,(\hat{x},\hat{y})})^2. \qquad (4.27)$$

Note that $V_{(\hat{x},\hat{y})}$ and $P_{(\hat{x},\hat{y})}$ is what corresponds to the Gauss law and flux operator in the BF theory (4.8), where it reads as $G = \Delta_x^2E_{xA} + \Delta_y^2E_{yA}$ and $F_A = \Delta_x^2\hat{A}_{(yy)} - \Delta_y^2\hat{A}_{(yy)}$, respectively. It is straightforward to check that these operators mutually commute. The Hamiltonian is defined by

$$H_{qp} = -\sum_{\hat{x},\hat{y}}[V_{(\hat{x},\hat{y})} + P_{(\hat{x},\hat{y})}] + (\mathrm{H.c.}). \qquad (4.28)$$

This model is exactly solvable as individual terms in the Hamiltonian commute with one another. Further, the model exhibits unusual behavior of the ground state degeneracy on

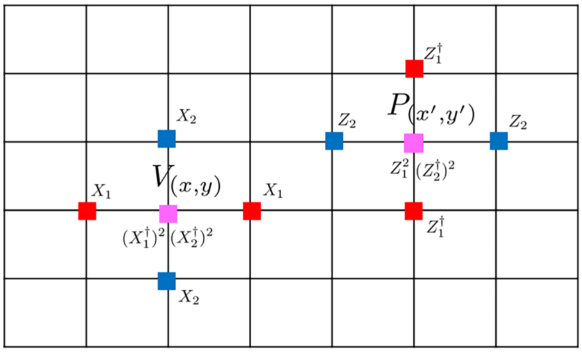

FIG. 3. Two types of the terms defined in (4.27). The red and blue squares distinguish between the Pauli operators that act on the state $|a\rangle$ and those on $|b\rangle$, respectively, whereas the pink square represents the composite of the Pauli operators that act on both $|a\rangle$ and $|b\rangle$.

torus geometry, depending on the system size [68]. Differing the details to Appendix C, we derive the GSD of the stabilizer model, arriving at the same value as (4.25).

D. Multipole one-form symmetries

It is known that in the $2+1$D toric code, there are one-form symmetries [83], which are prototype examples of the generalized global symmetries [84]. In condensed matter physics language, the one-form symmetries in the toric code are nothing but the closed loops of the fractionalized charges on the lattice. Since our UV lattice model (4.28) has the simple form, one can explicitly see that there are such one-form symmetries, corresponding to closed loops. In addition, such symmetries follow the similar relation of the multipole symmetries that we started with (4.1), which is not seen in the regular toric code.

To this end, we focus on the case with $N=2$, while other N cases can be argued in a similar way. In this case, two types of the terms $V_{\hat{x},\hat{y}}$ and $P_{(\hat{x},\hat{y})}$ constituting the Hamiltonian (4.28) become [Fig. 4(a)]

$$\begin{aligned} V_{(\hat{x},\hat{y})} &= X_{1,(\hat{x}+1,\hat{y})}X_{1,(\hat{x}-1,\hat{y})}X_{2,(\hat{x},\hat{y}+1)}X_{2,(\hat{x},\hat{y}-1)},\\ P_{(\hat{x},\hat{y})} &= Z_{1,(\hat{x},\hat{y}+1)}Z_{1,(\hat{x},\hat{y}-1)}Z_{2,(\hat{x}+1,\hat{y})}Z_{2,(\hat{x}-1,\hat{y})}. \end{aligned} \tag{4.29}$$

There are four types of closed loops of the operators Z_1 and Z_2 that commute with the Hamiltonian, as portrayed in Figs. 4(b)–4(e). The first loop [Fig. 4(b)], denoted by $\tilde{Q}_{xy}$, consists of two horizontal lines of Z_1 and two vertical lines of Z_2, where each Z_1 (Z_2) is separated by a single site in the horizontal (vertical) direction. The second loop [Fig. 4(c)], labeled by $\tilde{Q}_x$, is formed in such a way that there are four horizontal lines of Z_1, each of which is separated by a single site, and there are two vertical lines of Z_2. The role of Z_1 and Z_2 is switched in the third loop, $\tilde{Q}_y$ [Fig. 4(d)]. The fourth loop, $\tilde{Q}$, consists of four horizontal lines of Z_1 and four vertical lines of Z_2. We consider translating these operators by one lattice site in either the x or y direction and see how these loops change. Denoting a translation operator in the I direction by one lattice site by T_I, we investigate the change

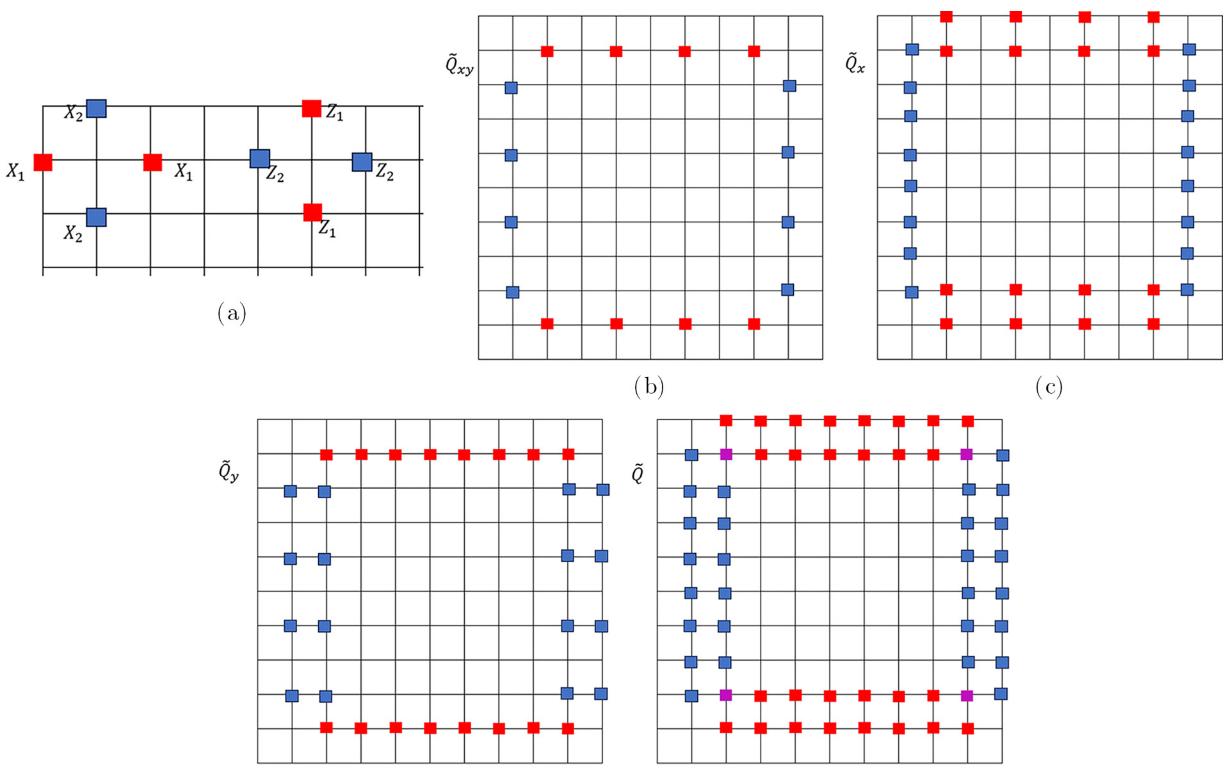

FIG. 4. (a) Two types of the terms $V_{(\hat{x},\hat{y})}$ and $P_{(\hat{x},\hat{y})}$ constituting Hamiltonian (4.28) with $N=2$. [(b)–(d)] Four closed loops of the operators Z_1 and Z_1 that commute with the Hamiltonian. The red and blue squares distinguish between the Pauli operators that act on the state $|a\rangle$ and those on $|b\rangle$.

of the translation by taking the ratio between a loop after the translation and the one before the translation. We have

$$\frac{T_x^{-1}\tilde{Q}_{xy}T_x}{\tilde{Q}_{xy}} = \tilde{Q}_y, \qquad \frac{T_y^{-1}\tilde{Q}_{xy}T_y}{\tilde{Q}_{xy}} = \tilde{Q}_x,$$
$$\frac{T_x^{-1}\tilde{Q}_x T_x}{\tilde{Q}_x} = \tilde{Q}, \qquad \frac{T_y^{-1}\tilde{Q}_y T_y}{\tilde{Q}_y} = \tilde{Q}. \tag{4.30}$$

This relation is inherited from our original consideration of a theory with multipole symmetries (4.1) in the sense that, if we translate a loop, it gives rise to another loop, similar to the fact that translating a quadruple gives a dipole (or translating a dipole gives a charge) in (4.1). It would be interesting to study these one-form symmetries from the perspective higher-form gauging and anomaly. Also, since these one-form symmetries are qualitatively different from those found in the regular toric code due to (4.30), it could be intriguing to address whether the lattice model (4.28) has different features in view of quantum error corrections [9]. We will leave these issues for future works.

V. CONCLUSION AND DISCUSSIONS

Symmetry has been one of the fundamental laws of physics, and is a guiding principle, allowing us to analyze various physical properties. In this work, we have demonstrated that symmetry plays a pivotal role to construct new field theories in the context of the fracton topological phases. We have demonstrated a way to construct the $2+1$D foliated BF theories based on the argument of the multipole symmetries. By introducing gauge fields associated with these symmetries, one can systematically construct new foliated BF theories. These theories exhibit unusual GSD dependence on the system size, involving the greatest common divisor between N and the length of the lattice. We have also shown that these foliated BF theories are an effective field theory description of unconventional topological phases, referred to as the higher rank topological phases. Our consideration provides an important connection between various unconventional topological phases such as fracton and higher rank topological phases in terms of the foliated topological field theories.

One obtains the preexisting foliated BF theories from ours by imposing additional constraints on the gauge fields. For instance, if we replace the gauge field c^I with $\phi^I e^I$ (I is not summed over) in (3.7), where ϕ^I is a zero-form field, we have

$$\mathcal{L}_{EX} = \frac{N}{2\pi} a \wedge db + \sum_{I=x,y} \frac{N}{2\pi} A^I \wedge d\phi^I \wedge e^I + \frac{N}{2\pi} A^I \wedge b \wedge e^I, \tag{5.1}$$

which is known as the foliated field theory description of the exotic $\mathbb{Z}_N$ gauge theory [39]. Compared with (3.7), which has the dipole symmetries, the theory (5.1) respects an additional symmetry–subsystem symmetry. The theory is invariant under $A^I \to A^I + e^I g$, with g being an arbitrary function. Manifestation of the subsystem symmetry in the theory (5.1) is that a gauge invariant operator constructed by the gauge field A^I is mobile only along a submanifold, which forbids moving in the direction parallel to e^I.

Other foliated BF theories of fracton phases, such as the X-cube model, can also be obtained by similar steps [65,74], i.e., starting with the $3+1$D analog of (3.7), and replacement of the gauge fields with the one which depends on the foliation field. Such a procedure might be related to discussion in [85], where one places $3+1$D tensor gauge theory on the lattice and Higgs it, which yields the X-cube model. Conversely, it could be interesting to see whether our theories (3.7) and (4.8) can be obtained by the preexisting foliated BF theories [60]. These theories might be related with one another via dimension reduction. We leave elaborating on these issues, including investigating whether our theories describe other higher rank topological phases such as the one in [86], for future studies.

There are several research directions regarding the present work that we would like to pursue for future studies. In this paper, we focus on the construction of the $2+1$D foliated BF theories, which can be generalized in several ways. For instance, one could study the construction of the $3+1$D or even higher dimensional foliated BF theories, with various multipole symmetries higher than dipole or quadrupole, such as ocutopoles. Also, one could think of higher-form symmetries analog of the multipole symmetries in the same way as we did in this paper and see what the resulting foliated BF theories are. Studying a boundary theory and topological defects of our foliated BF theories to see how multipole degrees of freedom are incorporated could be an interesting direction.

It would be also interesting to study whether our model can be useful in the context of quantum computing. The UV stabilizer lattice models that we have discussed here are distinct from the conventional toric code due to the sensitivity to the local geometry. Also, as seen from Sec. IV D, the model admits one-form symmetries which have the similar commutation relation with the translation operators as the ones of the multipole symmetries that we started with. In the case of the toric code, it is known that the capability of the quantum error correction is characterized by the classical Ising universal class [9]. Since the UV lattice model that we discuss respects the multipole symmetries, one naively would expect that robustness of the model against errors is qualitatively different (see [87] for relevant discussion.). It would be intriguing to investigate whether the UV lattice models discussed in the present paper show different universality classes that characterize the capability of the error correction.

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APPENDIX A: MAPPING BETWEEN REAL BF AND INTEGER BF THEORY

Generally, the BF theory with multipole symmetry studied in the present paper contains higher order spatial derivatives, making analysis much more challenging compared with conventional BF theories of topologically ordered phases. To circumvent this problem, we make use of a mapping from the BF theory to what is called integer BF theory where gauge fields take integer values defined on a discrete lattice, proposed in [45]. The latter BF theory is especially useful to study the properties of the model with multipole symmetries, such as the Wilson loops. For clearer illustration purposes, we first review how such a mapping works in the case of the toric code and then we apply this mapping to our BF theory.

Let us introduce the following BF theory defined on a discrete infinite Euclidean lattice:

$$L = \frac{2\pi}{N}[\hat{a}_\tau(\Delta_x\hat{b}_x - \Delta_y\hat{b}_x) + \hat{a}_x(\Delta_y\hat{b}_\tau - \Delta_\tau\hat{b}_y) + \hat{a}_y(\Delta_\tau\hat{b}_x - \Delta_x\hat{b}_\tau)]. \tag{A1}$$

Here, gauge fields $\hat{a}_\mu$ $\hat{b}_\mu$ $(\mu = \tau, x, y)$ take integer values defined on a μ link of the infinite Euclidean lattice. Following the terminology in [45,67,73], we term such a theory integer BF theory. This integer BF theory corresponds to the BF theory of the toric code (2.4). To see why, we first think of the equivalent description of (A1) which is given by

$$\begin{aligned} L = {}& \frac{N}{2\pi}[a_\tau(\Delta_x b_x - \Delta_y b_x - 2\pi m_{xy}) + a_x(\Delta_y b_\tau - \Delta_\tau b_y - 2\pi m_{y\tau}) + a_y(\Delta_\tau b_x - \Delta_x b_\tau - 2\pi m_{\tau x})] \\ & - N n_{xy} b_\tau - N n_{\tau y} b_x - N n_{\tau x} b_y + n_{xy}\Delta_\tau\tilde{\phi} + n_{y\tau}\Delta_x\tilde{\phi} + n_{\tau x}\Delta_y\tilde{\phi} + m_{xy}\Delta_\tau\phi + m_{y\tau}\Delta_x\phi + m_{\tau x}\Delta_y\phi. \end{aligned} \tag{A2}$$

Here, a_μ b_μ $(\mu = \tau, x, y)$ represent gauge fields with real values. Note the distinction between the fields with or without hat. We put a hat on top of the fields to emphasize that they are the fields with integer values. Further, we have intruded the Stueckelberg fields, ϕ, $\tilde{\phi}$ to ensure that a_μ b_μ respects the gauge symmetry and $m_{\mu\nu}$, $n_{\mu\nu}$ as the integer fields defined on a plaquette in the $\mu\nu$ plane, which take the role of the Lagrangian multiplier. Indeed, summing over the integer fields $m_{\mu\nu}$, $n_{\mu\nu}$ gives the following constraints:

$$a_\mu = \frac{2\pi}{N}\hat{a}_u + \frac{1}{N}\Delta_\mu\phi, \quad b_\mu = \frac{2\pi}{N}\hat{b}_u + \frac{1}{N}\Delta_\mu\tilde{\phi}, \tag{A3}$$

where $\hat{a}_\mu$ $\hat{b}_\mu$ $(\mu = \tau, x, y)$ are integer fields. Substituting (A3) to (A2), we come back to (A1). The theory (A2) admits the following gauge symmetry:

$$\begin{aligned} & a_\mu \to a_\mu + \Delta_\mu\alpha + 2\pi\hat{k}_\mu, \quad b_\mu \to b_\mu + \Delta_\mu\beta + 2\pi\hat{q}_\mu, \quad \phi \to \phi + N\alpha + 2\pi\hat{k}_\phi, \quad \tilde{\phi} \to \tilde{\phi} + N\beta + 2\pi\hat{q}_{\tilde{\phi}}, \\ & m_{xy} \to m_{xy} + \Delta_x\hat{k}_y - \Delta_y\hat{k}_x, \quad m_{y\tau} \to m_{y\tau} + \Delta_y\hat{k}_\tau - \Delta_\tau\hat{k}_y, \quad m_{\tau x} \to m_{\tau x} + \Delta_\tau\hat{k}_x - \Delta_x\hat{k}_\tau, \\ & n_{xy} \to n_{xy} + \Delta_x\hat{q}_y - \Delta_y\hat{q}_x, \quad n_{y\tau} \to n_{y\tau} + \Delta_y\hat{q}_\tau - \Delta_\tau\hat{q}_y, \quad n_{\tau x} \to n_{\tau x} + \Delta_\tau\hat{q}_x - \Delta_x\hat{q}_\tau. \end{aligned} \tag{A4}$$

Here, α, β represent the real gauge parameters, whereas $\hat{k}_\mu$, $\hat{q}$, $\hat{k}_\phi$, $\hat{q}_{\tilde{\phi}}$ denote the integer gauge parameters. To see the relation to the original BF theory of the toric code (2.4), we sum over the Stueckelberg fields, ϕ $\tilde{\phi}$, in (A2), which gives the following fluxless condition:

$$\Delta_\tau m_{xy} + \Delta_x m_{y\tau} + \Delta_y m_{\tau x} = 0 \tag{A5}$$

and similarly for $n_{\mu\nu}$. Since we think of the infinite Euclidean lattice, using the gauge symmetry (A4), jointly with the fluxless condition (A5) allows us to set the fields $m_{\mu\nu}$ $n_{\mu\nu}$ to be zero. (In the case of the lattice with periodic boundary condition, $m_{\mu\nu}$ $n_{\mu\nu}$ can be set to be zero except for a few cells, which capture the holonomy. Accordingly, one needs to take into account the transition function of the gauge fields, a_μ, b_ν; see [45] for more discussion on this point.) What remains in (A2) is

$$L = \frac{N}{2\pi}[a_\tau(\Delta_x b_x - \Delta_y b_x) + a_x(\Delta_y b_\tau - \Delta_\tau b_y) + a_y(\Delta_\tau b_x - \Delta_x b_\tau)]. \tag{A6}$$

After taking the continuum limit appropriately, Eq. (A6) becomes the original BF theory of the toric code (2.4).

After having reviewed the mapping between real BF theory and the integer BF theory, now we apply this technique to our theories. In Sec. III, we considered the BF theory with dipole symmetry given in (3.13). Since the way we carry out the mapping between theories closely parallels the one in the case of the toric code, we succinctly describe how the mapping works. The corresponding integer BF theory defined on the infinite discrete Euclidean lattice is given by

$$\begin{aligned} \mathcal{L}_{\text{dip}} = \frac{2\pi}{N}\big[& -\hat{c}_0^x(\Delta_x\hat{A}_{(yx)} - \Delta_y\hat{A}_{(xx)}) - \hat{c}_0^y(\Delta_x\hat{A}_{(yy)} - \Delta_y\hat{A}_{(xy)}) + \hat{\hat{c}}(\Delta_\tau\hat{A}_{(xy)} - \Delta_x\Delta_y\hat{A}_0) \\ & - \hat{c}_y^x\big(\Delta_\tau\hat{A}_{(xx)} - \Delta_x^2\hat{A}_0\big) + \hat{c}_x^y\big(\Delta_\tau\hat{A}_{(yy)} - \Delta_y^2\hat{A}_0\big)\big], \end{aligned} \tag{A7}$$

where the gauge fields with hat ($\hat{\cdot}$) take integer values. To see such a mapping work, let us first rewrite (A7) as

$$\begin{aligned}\mathcal{L}_{\text{dip}} = \frac{N}{2\pi}\big[&-c_0^x\big(\Delta_x A_{(yx)} - \Delta_y A_{(xx)} - 2\pi m_{xy}^x\big) - c_0^y\big(\Delta_x A_{(yy)} - \Delta_y A_{(xy)} - 2\pi m_{xy}^y\big)\\ &+ \tilde{c}\big(\Delta_\tau A_{(xy)} - \Delta_x\Delta_y A_0 - 2\pi\tilde{m}\big) - c_y^x\big(\Delta_\tau A_{(xx)} - \Delta_x^2 A_0 - 2\pi m_{\tau x}^x\big) + c_x^y\big(\Delta_\tau A_{(yy)} - \Delta_y^2 A_0 - 2\pi m_{y\tau}^y\big)\big]\\ &- NA_0 n_{xy} - NA_{(xy)}\tilde{n} - NA_{(xx)}n_{y\tau} - NA_{(yy)}n_{\tau x} - m_{xy}^x\Delta_\tau\phi^x - m_{xy}^y\Delta_\tau\phi^y\\ &+ \tilde{m}(\Delta_x\phi^x - \Delta_y\phi^y) + m_{\tau x}\Delta_y\phi^x + n_{xy}\Delta_\tau\tilde{\phi} + \tilde{\Delta}_x\Delta_y\tilde{\phi} + n_{y\tau}\Delta_x^2\tilde{\phi} + n_{\tau x}\Delta_y^2\tilde{\phi}.\end{aligned} \quad \text{(A8)}$$

Here, $(c_0^x, c_0^y, c_y^x, c_x^y, \tilde{c})$, $(A_0, A_{(xx)}, A_{yy}, A_{(xy)})$ denote the gauge fields which take *real* values with the Stueckelberg fields $\phi^x, \phi^y, \tilde{\phi}$ to ensure the gauge symmetry. Also, integer fields $m_{\mu\nu}^i, \tilde{m}, n_{\mu\nu}^i, \tilde{n}$ $(\mu, \nu = x, y, \tau, i = x, y)$ are Lagrangian multipliers; summing over these fields yields

$$\begin{aligned}&c_0^i = \frac{2\pi}{N}\hat{c}_0^i + \frac{1}{N}\Delta_i\phi^i\ (i = x, y), \quad c_x^y = \frac{2\pi}{N}\hat{c}_x^y + \frac{1}{N}\Delta_x\phi^y,\\ &c_y^x = \frac{2\pi}{N}\hat{c}_y^x + \frac{1}{N}\Delta_y\phi^x, \quad \tilde{c} = \frac{2\pi}{N}\hat{\tilde{c}} + \frac{1}{N}(\Delta_x\phi^x - \Delta_y\phi^y),\\ &A_0 = \frac{2\pi}{N}\hat{A}_0 + \frac{1}{N}\Delta_\tau\tilde{\phi}, \quad A_{(ij)} = \frac{2\pi}{N}\hat{A}_{(ij)} + \frac{1}{N}\Delta_i\Delta_j\tilde{\phi},\end{aligned} \quad \text{(A9)}$$

where the fields with a hat represent integer fields. One can easily check that substituting this into (A8) gives (A7). Similar to the previous case of the toric code, summing over the Stueckelberg fields $\phi^x, \phi^y, \tilde{\phi}$ gives the fluxless condition of the fields $m_{\mu,\nu}^i, \tilde{m}, n_{\mu\nu}^i, \tilde{n}$. Using this condition jointly with the gauge symmetries of these fields allows us to suppress the fields $m_{\mu,\nu}^i, \tilde{m}, n_{\mu\nu}^i, \tilde{n}$. What remains in (A8) is given by

$$\begin{aligned}\mathcal{L}_{\text{dip}} = \frac{N}{2\pi}\big[&- c_0^x(\Delta_x A_{(yx)} - \Delta_y A_{(xx)}) - c_0^y(\Delta_x A_{(yy)} - \Delta_y A_{(xy)})\\ &+ \tilde{c}(\Delta_\tau A_{(xy)} - \Delta_x\Delta_y A_0) - c_y^x\big(\Delta_\tau A_{(xx)} - \Delta_x^2 A_0\big)\\ &+ c_x^y\big(\Delta_\tau A_{(yy)} - \Delta_y^2 A_0\big)\big].\end{aligned} \quad \text{(A10)}$$

Taking the continuum limit, we obtain the original BF theory (3.13). Based on this argument, to study the original BF theory (3.13), we instead study the integer BF theory defined on the discrete lattice (A7), allowing us to study its physical properties.

By the analogous lines of thoughts, we study the integer BF theory (4.16) with quadrupole instead of the BF theory introduced in (4.14) in Sec. IV.

APPENDIX B: DERIVATION OF EQ. (3.30)

In this Appendix, we provide an argument to derive (3.30). The composite of loops (3.29) is subject to the conditions

$$(k_x, k_y) \sim (k_x + N, k_y) \sim (k_x + \alpha_y L_y, k_y) \quad \text{(B1)}$$

and

$$(k_x, k_y) \sim (k_x, k_y + N) \sim (k_x, k_y + \alpha_x L_y), \quad \text{(B2)}$$

with $\alpha_{x/y} = \frac{N}{\gcd(N, L_{x/y})}$. Also, (k_x, k_y) has to satisfy

$$(k_x, k_y) \sim (k_x + L_x, k_y - L_y). \quad \text{(B3)}$$

From (B1) and (B3), there are $\gcd(N, \alpha_y L_x, L_x)$ distinct number of k_x and, for these distinct values of k_x, there are $\gcd(N, \alpha_x L_y)$ distinct number of k_y. In total, there are $\gcd(N, \alpha_y L_x, L_x) \times \gcd(N, \alpha_x L_y)$ distinct number of the composite loops (3.29). Since $\gcd(N, \alpha_y L_x, L_x) = \gcd(N, L_x)$, it follows that

$$\begin{aligned}&\gcd(N, \alpha_y L_x, L_x) \times \gcd(N, \alpha_x L_y)\\ &\quad = \gcd(N, L_x) \times \gcd\left(N, \frac{N}{\gcd(N, L_x)}L_y\right)\\ &\quad = N\gcd[\gcd(L_x, N), L_y] = N\gcd(N, L_x, L_y).\end{aligned} \quad \text{(B4)}$$

Hence there are $N\gcd(N, L_x, L_y)$ distinct loops of the gauge field $\hat{A}_{(xy)}$. Taking the other noncontractible loops into account, we arrive at the fact that the GSD, which is the number of the distinct noncontractible loops of the gauge fields $\hat{A}_{(ij)}$, is given by (3.30).

APPENDIX C: INVESTIGATION OF THE LATTICE MODEL BY THE LAPLACIAN

In this Appendix, we analyze the stabilizer model (4.28) obtained from the foliated BF theory, especially investigating the Wilson loops that contribute to the nontrivial GSD on a torus geometry from a different perspective. As we discussed in the main text, the model contains the second order spatial derivatives, involving the nearest neighboring states. Due to this property, one can make use of the *Laplacian*, which is the graph theoretical analog of the second order derivatives [89], allowing us to systematically investigate the lattice model, such as the Wilson loops. To see how, we first recall the Hamiltonian has the following form:

$$H_{qp} = -\sum_{\hat{x},\hat{y}}[V_{(\hat{x},\hat{y})} + P_{(\hat{x},\hat{y})}] + (\text{H.c.}), \quad \text{(C1)}$$

and that the ground state $|\Omega\rangle$ satisfies $V_{(\hat{x},\hat{y})}|\Omega\rangle = P_{(\hat{x},\hat{y})}|\Omega\rangle = |\Omega\rangle$, i.e., the ground state does not admit an electric and magnetic excitation. To discuss the Wilson loops, we also investigate the excitations. When acting on an operator $Z_{1,(\hat{x},\hat{y})}$ on the ground state, it violates the condition $V_{(\hat{x},\hat{y})} = 1$; that is,

$$\begin{aligned}V_{(\hat{x},\hat{y})}(Z_{1,(\hat{x},\hat{y})}|\Omega\rangle) &= \omega^{-2}(Z_{1,(\hat{x},\hat{y})}|\Omega\rangle),\\ V_{(\hat{x}\pm1,\hat{y})}(Z_{1,(\hat{x},\hat{y})}|\Omega\rangle) &= \omega(Z_{1,(\hat{x},\hat{y})}|\Omega\rangle),\end{aligned} \quad \text{(C2)}$$

yielding electric charges. This implies that, by acting on an operator $Z_{1,(\hat{x},\hat{y})}$, an electric charge is induced at the coordinate $(\hat{x} \pm 1, \hat{y})$ and two conjugate of the electric charges are obtained at $(\hat{x}, \hat{y})$. Denoting the $\mathbb{Z}_N$ eclectic charge as $e_{(\hat{x},\hat{y})}$ (and its conjugate as $\overline{e}_{(\hat{x},\hat{y})}$), Eq. (C2) can be described by the

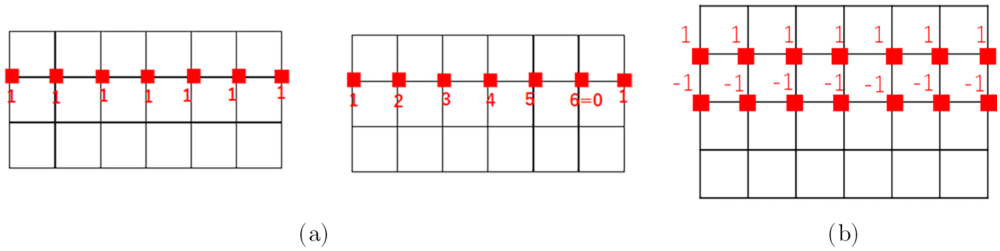

FIG. 5. (a) Examples of the two types of the loops given in (C8) [left: $W_x(\hat{y})$; right: $W_{\rm dip}(\hat{y})$] in the case of $N = L_x = 6$. The red dots denote the operators $Z_{1,(\hat{x},\hat{y})}$. The integer numbers correspond to the power entering in the form of the loops (C8). The periodic boundary condition is imposed so that the left and right edges are identified. (b) Configuration corresponding to the second term of (C14), which is the composite of the loops with opposite charge located adjacent in the y direction.

fusion rule, reading

$$I \rightarrow e_{(\hat{x}-1,\hat{y})}\bar{e}^{2}_{(\hat{x},\hat{y})}e_{(\hat{x}+1,\hat{y})}. \tag{C3}$$

The fusion rule of the magnetic charges can be similarly discussed.

Using this property, one can study the GSD of the model on a torus geometry by counting the distinct number of Wilson loops. To this end, the Laplacian comes into play. We consider the model (C1) on a 2D lattice with periodic boundary condition and the system size being $L_x \times L_y$. In this lattice, we think of a closed loop of the electric charge in the horizontal direction at $\hat{y}$, described by $\prod_{\hat{x}=1}^{L_x} Z^{a_{\hat{x}}}_{1,(\hat{x},\hat{y})}$ with $a_{\hat{x}} \in \mathbb{Z}_N$. From (C3), the electric charges induced by acting this operator on the ground state are described by the fusion rule

$$I \rightarrow \prod_{\hat{x}=1}^{L_x} \otimes e^{r_{\hat{x}}}_{(\hat{x},\hat{y})} \quad (r_{\hat{x}} \in \mathbb{Z}_N), \tag{C4}$$

with

$$\mathbf{r} = -L_{L_x\times L_x}\mathbf{a}. \tag{C5}$$

Here, $\mathbf{a} := (a_1, \dots, a_{L_x})^T$, $\mathbf{r} := (r_1, \dots, r_{L_x})^T$, and $L_x \times L_x$ matrix, $L_{L_x\times L_x}$, is the Laplacian, the graph theoretical analog of the second order spatial derivative, which has the following form:

$$L_{L_x\times L_x} = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}. \tag{C6}$$

Since the closed noncontractible loop $\prod_{\hat{x}=1}^{L_x} Z^{a_{\hat{x}}}_{1,(\hat{x},\hat{y})}$ has to commute with the Hamiltonian, the fusion rules (C4) and (C5) have to be trivial, i.e., $\mathbf{r} = \mathbf{0} \mod N$. Thus the closed loops of the electric charges are characterized by the *kernel* of the Laplacian.

By evaluating the Laplacian (C6), the solution of $L_{L_x\times L_x}\mathbf{a} = \mathbf{0}$ is given by [68]

$$\mathbf{a} = \alpha_0(1, 1, \dots, 1)^T + \alpha_1 \times \frac{N}{\gcd(N, L_x)}(1, 2, 3, \dots, L_x)^T. \tag{C7}$$

Here, $\alpha_0 \in \mathbb{Z}_N$ and $\alpha_1 \in \mathbb{Z}_{\gcd(N,L_x)}$. Hence there are two kinds of closed loops in the horizontal direction ($\alpha_x = \frac{N}{\gcd(N,L_x)}$):

$$W_x(\hat{y}) = \prod_{\hat{x}=1}^{L_x} Z_{1,(\hat{x},\hat{y})}, \quad W_{{\rm dip}:x}(\hat{y}) = \prod_{\hat{x}=1}^{L_x} \left(Z^{\hat{x}}_{1,(\hat{x},\hat{y})}\right)^{\alpha_x}. \tag{C8}$$

The first loop is interpreted as the conventional Wilson loop [Fig. 5(a), left], which is formed by the trajectory of the $\mathbb{Z}_N$ electric charge, traveling around the torus in the x direction. This corresponds to the first term of (4.19) in the BF theory. The second loop can be interpreted as the "dipole of the Wilson loop" [Fig. 5(a), right] in the sense that the loop is formed by the $\mathbb{Z}_N$ electric charge going around the torus with its intensity increasing linearly. This corresponds to the second term in (4.19). The first loop $W_x(\hat{y})$ is labeled by $\mathbb{Z}_N$, whereas the second one $W_{{\rm dip}:x}(\hat{y})$ is by $\mathbb{Z}_{\gcd(N,L_x)}$. By evaluating the fusion rules, one can similarly discuss the configuration of the (part of) closed loops in the lattice model (3.34) obtained from the foliated BF theory with dipole symmetry (3.7). For instance, the form of the loops given in (3.17) can be similarly derived.

Having identified loops of the electric charges in the x direction (C8), one needs to check whether these loops are $\hat{y}$ dependent or not to count the distinct number of configurations of the loops. Let us focus on the loop $W_x(\hat{y})$. By multiplying the following operator:

$$M(\hat{y}) = \prod_{\hat{x}=1}^{L_x} P_{(\hat{x},\hat{y})}, \tag{C9}$$

where $P_{(\hat{x},\hat{y})}$ is given in (4.27), the loop $W_x(\hat{y})$ is deformed as

$$M(\hat{y})W_x(\hat{y}) = W_x(\hat{y}-1)W_x^{\dagger}(\hat{y})W_x(\hat{y}+1). \tag{C10}$$

We need to count the distinct configuration of the loop $W_x(\hat{y})$ up to the deformation (C10). To this end, we think of deforming the composite of the loops, $\prod_{\hat{y}=1}^{L_y} W_x^{b_{\hat{y}}}(\hat{y})$, with $b_{\hat{y}} \in \mathbb{Z}_N$. Multiplying it with the following operator:

$$\prod_{\hat{y}=1}^{L_y} M^{c_{\hat{y}}}(\hat{y}) \quad (c_{\hat{y}} \in \mathbb{Z}_N) \tag{C11}$$

and referring to (C10), it follows that the composite of the loops after multiplying (C11) becomes

$$\prod_{\hat{y}=1}^{L_y} W^{\tilde{b}_{\hat{y}}}(\hat{y}) \quad (\tilde{b}_{\hat{y}} \in \mathbb{Z}_N), \tag{C12}$$

with

$$\tilde{\mathbf{b}} = \mathbf{b} - L_{L_y\times L_y}\mathbf{c}. \tag{C13}$$

Here, $\mathbf{b} = (b_1, \ldots, b_{L_y})^T$ with other vectors $\tilde{\mathbf{b}}, \mathbf{c}$ being similarly defined and the $L_y \times L_y$ matrix $L_{L_y\times L_y}$ denotes the Laplacian, which has the same form as (C6), where the matrix size is replaced with $L_y \times L_y$. It follows that the composite of loops, $\prod_{\hat{y}=1}^{L_y} W_x^{b_{\hat{y}}}(\hat{y})$, labeled by the vector $\mathbf{b}$ and the ones, $\prod_{\hat{y}=1}^{L_y} W_x^{\tilde{b}_{\hat{y}}}(\hat{y})$, labeled by $\tilde{\mathbf{b}}$ which are related via (C13) with $^\exists\mathbf{c}$ are identified. Therefore, the number of distinct configurations of the loop $W_x(\hat{y})$ is found to be $\mathbf{s} := \mathbb{Z}_N^{L_y}/\mathrm{Im}(L_{L_y\times L_y})$, which is the *cokernel* of the Laplacian. Note that this corresponds to the discussion around Eq. (4.20) in the main text, where we count the distinct number of the loops $W(\hat{y})$ subject to the condition (4.20). Namely, in (4.20), we discuss the distinct configuration of the loops up to the operation of the second order derivative ∂_y^2, which corresponds to the present consideration where we need to find distinct configurations of the loops up to the Laplacian, $L_{L_y\times L_y}$.

The cokernel is given by [68]

$$\mathbf{s} = \beta_0(0, \ldots, 0, 1)^T + \beta_1(0 \ldots, 0, 1, -1)^T, \tag{C14}$$

with $\beta_0 \in \mathbb{Z}_N$, $\beta_1 \in \mathbb{Z}_{\gcd(N,L_y)}$. Equation (C14) indicates that the distinct configurations of the loops are characterized by a single loop and the composite of loops [Fig. 5(b)], which have the opposite charge, located adjacent to each other in the y direction. These two types of loops are labeled by $\mathbb{Z}_N$ and $\mathbb{Z}_{\gcd(N,L_y)}$. This corresponds to the discussion around (4.24).

Overall, we have counted the distinct number of loops $W_x(\hat{y})$, which is given by $N \times \gcd(N, L_y)$. An analogous line of thought shows that there are $\gcd(N, L_x) \times \gcd(N, L_x, L_y)$ distinct configurations of the loops $W_{\mathrm{dip}:x}(\hat{y})$. Thus, in total, there are $N \times \gcd(N, L_y) \times \gcd(N, L_x) \times \gcd(N, L_x, L_y)$ different configurations of the loops of the electric charge. So far, we have considered closed loops of the electric charges. Regarding the closed loops of the magnetic charges, a similar argument follows as the electric charges; thus they are labeled by the same quantum numbers. Taking this into consideration, we finally arrive at the fact that the GSD is the same value as (4.25).

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$$A^{(1)} \to A^{(1)} + d\lambda_A^{(0)}, \quad B^{(2)} \to B^{(2)} + d\Lambda_B^{(1)} + \frac{\kappa}{2\pi}\lambda_A^{(0)} dA^{(1)},$$

while $\lambda_A^{(0)}$ and $\Lambda_B^{(1)}$ are the zero-form and one-form gauge parameters.
[77] As discussed in [74], one can regard the gauge group as U(1), taking into account that the quantization condition of the dipole gauge field depends on the length of the dipole. If we define the theory on a discrete lattice, it can be understood by the fact that the dipole charge is quantized as the lattice spacing becomes a minimal unit of distance. Throughout this paper, we set such a length to be 1.
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[82] Similar to the previous case, we put the theory on the discrete 2D lattice with coordinate $(\hat{x}, \hat{y})$, and, due to the periodic boundary condition, $(\hat{x} + L_x, \hat{y}) \sim (\hat{x}, \hat{y} + L_y) \sim (\hat{x}, \hat{y})$.
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