Modular extension of topological orders from congruence representations

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We present an efficient method to compute the modular extension of both fermionic topological orders and $\mathbb{Z}_2$-symmetric bosonic topological orders in two spatial dimensions, from congruence representations of $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups. To demonstrate the validity of our approach, we provide explicit calculations for topological orders with ranks up to 10 for the fermionic cases and up to 6 for the bosonic cases. Along the way, we clarify the relation between fermionic rational conformal field theories, which live on the boundary of the corresponding fermionic topological orders, and modular extensions. We show that the $\mathrm{SL}_2(\mathbb{Z})$ representation of the Ramond-Ramond sector can be determined from the Nuveu-Schwarz-Nuveu-Schwarz sector using the modular extensions.

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I. INTRODUCTION

Topological orders are gapped phases of matter at zero temperature beyond the Landau paradigm, which are characterized by topology-dependent ground-state degeneracy, non-Abelian geometric phases, and long-range entanglement [1–5]. Fractional quantum Hall states [3,6,7] and gapped quantum spin liquid states [1,8,9] are the best known examples. The intriguing connection between topological order and both topological quantum field theory (TQFT) [10] and rational conformal field theory (RCFT) [11] has garnered significant attention, not only from the field of condensed matter physics but also from high-energy physics and mathematics. The impetus driving the investigation of topological order, however, transcends its theoretical profundity. Interestingly, $(2+1)$-dimensional topological orders host exotic pointlike excitations called anyons, which play a pivotal role in the realization of the topological quantum computation [12–15].

The trivial topological order is the universality class of states that can be adiabatically connected to a product state without closing the spectral gap. A topological order is called *invertible* if a state in that order can be adiabatically connected to a state in the trivial topological order after being stacked with a state in another topological order. Universal properties of $(2+1)$-dimensional topological orders up to invertible topological orders are captured and formulated by braided fusion category (BFC) theory [16–19]. Roughly speaking, BFCs are mathematical structures formed by the equivalence classes of anyons with their fusion and braiding data. Specifically, $(2+1)$-dimensional bosonic and fermionic topological orders without symmetry are described by modular tensor categories (MTCs) [17,20] and super-MTCs [18,21,22], respectively. When considering global symmetry, the classification of topological orders becomes more intricate, leading to the concepts of symmetry-protected topological (SPT) orders and symmetry-enriched topological (SET) orders [5]. In such a case, bulk excitations are described by BFCs with the Müger center [23], which is the subcategory of excitations that exhibit trivial braiding with every other excitation. If the symmetry is bosonic, the Müger center is given by a symmetric fusion category (SFC) Rep(G), which is the category of representations of the symmetry group G. We will call such a BFC *G-BFC* for brevity.

Each BFC has a pair of r-dimensional symmetric matrices (S, T) associated with it. Here, the dimension r is called *rank*, which is equal to the number of inequivalent anyon types. The matrices S and T encode the mutual and self statistics of anyons, respectively. While T is always unitary and diagonal, S may be degenerate. For MTCs, S matrices are unitary and, in this case, the pair (S, T) is called *modular data*. In this paper, we use the term modular data for referring to any pair (S, T), regardless of whether it corresponds to an MTC or not. Modular data are gauge invariant and proven to be valuable tools for the analysis and classification of topological orders. Indeed, authors of numerous previous works [17,18,20,22–26] that investigated topological orders heavily relied on modular data. However, it should be emphasized that, in certain cases, a given set of modular data may not uniquely determine a BFC [23,27]. In other words, multiple distinct BFCs can share the same (S, T). For MTCs, such cases are known to occur only when the rank r is sufficiently high, typically $r \geqslant 49$ [27], and it is believed that, in the cases of sufficient row ranks, modular data can uniquely determine an MTC. On the other hand, there is an effort to overcome the limitation of (S, T) by

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introducing higher-genus invariants [28]. For generic BFCs, such ambiguity can be encountered even in low ranks [23].

Recently, it was realized that the modular data are closely related to *congruence representations* of $\mathrm{SL}_2(\mathbb{Z})$ or one of its subgroups [20,22,29]. Specifically, the modular data of an MTC and a super-MTC form a *projective* congruence representation of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_\theta < \mathrm{SL}_2(\mathbb{Z})$, respectively. Moreover, all congruence representations of $\mathrm{SL}_2(\mathbb{Z})$ were classified recently [30], allowing a systematic classification of modular data by constructing candidate data from the representations and then checking consistency conditions. Using this approach, the modular data of MTCs and super-MTCs were classified up to rank 11 (and partially up to rank 12) [26] and 10 [22], respectively. A natural extension of these works would be classifying modular data of G-BFCs, which characterize SPT and SET orders. However, the modular data of G-BFCs do not form group representations by themselves; thus, it is impossible to directly apply a similar approach. Nevertheless, a given G-BFC can be mapped to an MTC in two ways: symmetry breaking and modular extension.

Symmetry breaking is a map from a given G-BFC to its underlying MTC [23]. In this paper, we make the symmetry-breaking procedure explicit for the $G = \mathbb{Z}_2$ case in terms of modular data. Since the modular data of MTCs can be classified by projective congruence representations of $\mathrm{SL}_2(\mathbb{Z})$, a systematic study of modular data of $\mathbb{Z}_2$-BFCs is then possible. In this way, we can classify $\mathbb{Z}_2$-BFCs without imposing any upper bound on fusion coefficients or total quantum dimension.

Modular extension is an MTC that contains a super-MTC or a G-BFC. Physically, it corresponds to a gauged version of a given super-MTC or G-BFC. If a given $(d+1)$-dimensional topological order can be realized on a d-dimensional lattice with on-site symmetry action, then the topological order is said to be *anomaly free*. In terms of category theory, this anomaly-free condition is translated into the condition of existence of a modular extension [18,23,31]. For super-MTCs or $\mathbb{Z}_2$-BFCs, it is known that they always admit modular extensions and thus are anomaly free. However, it is possible that given tentative modular data may turn out to be invalid by disallowing modular extension, which implies that it cannot actually be realized by a super-MTC or a $\mathbb{Z}_2$-BFC. Thus, the existence of modular extensions serves as a necessary condition for confirming the validity of candidate modular data [23]. Consequently, by explicitly computing the modular extensions, we can rule out such invalid modular data. Moreover, modular extensions are closely related to the boundary theory of fermionic topological orders, as we will explain in Sec. II E 3.

However, a systematic approach to compute the modular data of modular extensions has remained elusive. In this paper, we provide a systematic procedure for computing the modular data of fermionic topological orders and $\mathbb{Z}_2$-SETs, by using a hidden structure of modular data. Roughly speaking, the modular data of these topological orders can be decomposed via a basis change to different sectors, each of which transforms in a congruence representation of $\mathrm{SL}_2(\mathbb{Z})$ or one of its index-3 congruence subgroups. We compute the modular data of all 16 modular extensions of each super-MTC found in Ref. [22] and all modular extensions of each $\mathbb{Z}_2$-BFC we found. Importantly, the fact that each super-MTC admits 16 modular extensions is consistent with the previous theorem [21] and serves as strong evidence for the validity of the super-MTCs given in Ref. [22]. We also find that each one of the modular extensions of the new classes of super-MTCs indeed aligns with those previously given in Ref. [32].

In Sec. II, we provide an overview of related concepts, including the mathematical description of $(2+1)$-dimensional topological orders, the relation between modular data and congruence representations, the algebraic consistency conditions that BFCs satisfy, the relation between gauging and modular extensions, and the relation between fermionic topological orders and both TQFT and RCFT. In Sec. III, we introduce our method for classifying modular data of $\mathbb{Z}_2$-BFCs and compute modular extensions of super-MTCs and $\mathbb{Z}_2$-BFCs from congruence representations of $\mathrm{SL}_2(\mathbb{Z})$ and its congruence subgroups. We summarize the result of this paper and make some comments on observations in Sec. IV. The lists of modular extensions of super-MTCs and $\mathbb{Z}_2$-BFCs are presented in the Supplemental Material [33].

II. BACKGROUND

In this section, we introduce some essential background. We focus mainly on the physical intuition and concrete formulas, while details are omitted. For more detailed and mathematical explanation, readers are encouraged to see, for example, Refs. [16,19,34,35].

A. Category-theoretic description of (2 + 1)-dimensional topological orders

Within the framework of category theory, the characterization of a $(2+1)$-dimensional topological order denoted by C is achieved through the association with a BFC $\mathcal{C}$. Roughly speaking, a BFC consists of a set of different anyon types and their fusion/braiding rules. The fusion and braiding rules should satisfy some consistency equations known as the pentagon and hexagon equations. These BFCs may contain the Müger center $\mathcal{E}$ [18]. Here, $\mathcal{E}$ characterizes the global symmetry of C, which can be understood as the subcategory of local pointlike excitations. For example, when C possesses finite on-site bosonic symmetry, denoted by a group G, its local pointlike excitations carry group representations of G, thereby giving rise to $\mathcal{E} = \mathrm{Rep}(G)$, i.e., the category of group representations of G. On the other hand, if the symmetry contains the fermion-number parity, being denoted by G^f, then $\mathcal{E} = \mathrm{sRep}(G^f)$, i.e., the category of group superrepresentations of G^f.

The simplest case, $(2+1)$-dimensional bosonic topological orders without symmetry, have $\mathcal{E} = \mathcal{B}_0$, where $\mathcal{B}_0$ is the category of finite-dimensional vector spaces. The BFCs in this case become MTCs by themselves, i.e., all pointlike excitations are nonlocal, thus braid nontrivially among themselves. In juxtaposition, $(2+1)$-dimensional fermionic topological orders without symmetry have $\mathcal{E} = \mathcal{F}_0 \equiv \mathrm{sRep}(\mathbb{Z}_2^f)$ (also frequently referred to as sVec, the category of supervector spaces, in the literature) where the nontrivial element of $\mathbb{Z}_2^f \simeq \mathbb{Z}_2$ represents the fermion-number parity. The BFCs in this case are called super-MTCs [21]. If a nontrivial

(2 + 1)-dimensional topological order is endowed with a nontrivial bosonic (fermionic) symmetry group G (G^f), then such C is called an SET order [5] and described by a G-BFC (G^f-BFC). In contrast, if the topological order is trivial, then such C is called an SPT order [5].

B. Modular data and congruence representation

Though a BFC is in principle defined by the solutions of the pentagon and hexagon equations, i.e., a set of F and R tensors [34], these tensors are gauge-dependent quantities and difficult to classify. As noted in the introduction, it is more convenient to work with the modular data (S, T) which are gauge invariant.

Let us explain in more detail the meaning of modular data. First, the elements of the first row of the S matrix are real, and none of them are equal to zero. The ath element in the first row divided by S_{11} corresponds to the *quantum dimension* d_a of anyon a, i.e., $d_a = S_{1a}/S_{11}$. The vacuum is labeled by 1 and has $d_1 = 1$. The normalization factor is written as $S_{11} = 1/D$, and $D \equiv \sqrt{\sum_i d_i^2}$ is called the *total quantum dimension*. This total quantum dimension D is an important physical observable captured by the topological entanglement entropy [36,37]. The entries of S_{ab} capture information about the *mutual braiding* of anyons a and b. Moreover, the S matrix is also related to the fusion coefficients N_c^{ab} via the Verlinde formula. On the other hand, the T matrix is diagonal. The ath diagonal element of it is given by $T_{aa} = \exp(2\pi i\theta_a)$, where θ_a is the *topological spin* of anyon a. The topological spin of the vacuum is always given by $\theta_1 = 0$.

The modular data (and the corresponding BFC) are called *unitary* if d_a are all positive. In this case, the quantum dimension d_a of an anyon a equals the *Frobenius-Perron dimension* of a, denoted FPdim(a), defined as the largest eigenvalue of the fusion matrix N^a [here, we interpret the fusion coefficients N_c^{ab} as matrices $(N^a)_c^b$ for each a]. If the total Frobenius-Perron dimension of a fusion category $\mathcal{C}$, $\text{FPdim}(\mathcal{C}) := \sum_a \text{FPdim}(a)^2$, satisfies $\text{FPdim}(\mathcal{C}) = D_{\mathcal{C}}^2$, then $\mathcal{C}$ is called *pseudounitary* [38]. There are pseudounitary fusion categories which are not unitary [39–41].

The modular data have an interesting connection to congruence representations, and this fact has been exploited recently to make progress on the classification problem [20,22,26]. For an MTC, the modular data form a projective congruence representation of $\text{SL}_2(\mathbb{Z})$ [20]. Since congruence representations of $\text{SL}_2(\mathbb{Z})$ are classified [30], the modular data of an MTC can be established through a two-step process. First, we construct candidate S and T matrices from the classification table of congruence representations. Then we test that the pair (S, T) satisfies all the necessary consistency conditions, thereby confirming it as modular data [20].

Things are similar for super-MTCs but with some modification. The modular data of a super-MTC always can be decomposed as [29]

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \hat{S}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{T}. \tag{1}$$

Here, due to sign ambiguity, only $\hat{T}^2$ is well defined. Physically, this corresponds to extracting the local fermion part from the modular data, while leaving only the information of nonlocal anyons that braid nontrivially among themselves in $(\hat{S}, \hat{T}^2)$. Importantly, $\hat{S}$ is unitary and, together with $\hat{T}^2$, generates a projective congruence representation of Γ_θ [29], which is an index-3 congruence subgroup of $\text{SL}_2(\mathbb{Z})$. Since the congruence representations of Γ_θ can be obtained from those of $\text{SL}_2(\mathbb{Z})$, a similar approach can be applied to compute the modular data of super-MTCs [22].

At first glance, it seems that the modular data of a general BFC other than an MTC or a super-MTC does not have such a connection because they do not form a representation of any group. However, by breaking the symmetry of a given BFC [23], we can get the modular data of an MTC. Hence, they are somehow indirectly connected to the congruence representations of $\text{SL}_2(\mathbb{Z})$. Another connection with congruence representations comes through modular extensions [31]. In Secs. III B and III C, we introduce the procedures to obtain the modular data of both symmetry-broken MTCs and modular extensions for the bosonic $G = \mathbb{Z}_2$ case. Using the method, we classify (2 + 1)-dimensional bosonic $\mathbb{Z}_2$-SETs and their modular extensions up to rank 6.

C. Consistency conditions for modular data

The modular data of BFCs should satisfy a set of algebraic consistency conditions [17,18]. The conditions listed below are necessary conditions; a set of sufficient conditions is not known. However, we believe that they are stringent enough to considerably narrow down the candidates for valid topological orders.

1. Verlinde's formula

The S-matrix of a BFC satisfies

$$\frac{S_{il}S_{jl}}{S_{1l}} = \sum_k N_k^{ij} S_{kl}, \tag{2}$$

where N_k^{ij} are nonnegative integers called *fusion coefficients*. The coefficients further satisfy

$$\begin{aligned} N_k^{ij} &= N_k^{ji}, \\ N_j^{1i} &= \delta_{ij}, \\ \sum_k N_1^{ik} N_1^{kj} &= \delta_{ij}, \\ \sum_k N_k^{ij} N_k &= N_i N_j, \end{aligned} \tag{3}$$

where $(N_i)_{jk} \equiv N_k^{ij}$. These coefficients define fusion rules of anyons, e.g., $a \times b = \sum_c N_c^{ab} c$, where a, b, and c denote anyons. If a given BFC is an MTC, then Eq. (2) can be written as

$$N_k^{ij} = \sum_l \frac{S_{il}S_{jl}S_{kl}^*}{S_{1l}}. \tag{4}$$

Hence, for an MTC, the fusion coefficients are uniquely determined by its S matrix.

2. Rational condition

Define

$$V_{ijkl}^{r} = N_{r}^{ij}N_{\bar{r}}^{kl} + N_{r}^{il}N_{\bar{r}}^{jk} + N_{r}^{ik}N_{\bar{r}}^{jl} - (\delta_{ir} + \delta_{jr} + \delta_{kr} + \delta_{lr}) \sum_{m} N_{m}^{ij}N_{\bar{m}}^{kl}, \tag{5}$$

then

$$\sum_{r} V_{ijkl}^{r}\theta_r \in \mathbb{Z}. \tag{6}$$

3. Balancing equation

The elements of the S matrix are given by

$$S_{ij} = \frac{1}{D}\sum_{k} N_{k}^{ij} e^{2\pi i(\theta_i+\theta_j-\theta_k)} d_k, \tag{7}$$

where $D = \sqrt{\sum_i d_i^2}$ is the total quantum dimension.

4. Frobenius-Schur indicator

The quantity

$$\nu_k = \frac{1}{D^2}\sum_{ij} N_{k}^{ij} d_i d_j \cos[4\pi(\theta_i - \theta_j)], \tag{8}$$

called the Frobenius-Schur indicator, satisfies $\nu_k \in \mathbb{Z}$ if $k = \bar{k}$.

5. Weak modularity

Define

$$\Theta = \frac{1}{D}\sum_{i} \exp(2\pi i\theta_i) d_i^2, \tag{9}$$

then

$$S^{\dagger}TS = \Theta T^{\dagger}S^{\dagger}T^{\dagger}. \tag{10}$$

Importantly, $\Theta = |\Theta| \exp(2\pi i\frac{c}{8})$, where c is the chiral central charge mod 8. From this relation, one can compute c of a given topological order. For super-MTCs, however, $|\Theta| = 0$; thus, one cannot rely on this method and should compute their modular extensions. Recently, it was proposed that the chiral central charge of super-MTCs can be obtained without computing modular extensions if one uses congruence representations of Γ_θ [22].

D. Gauging and modular extension

Gauging is the promotion of global symmetry to gauge symmetry. In terms of category theory, the gauging process is understood as computing modular extensions of a given BFC $\mathcal{C}$ with Müger center $\mathcal{E}$. A modular extension $\mathcal{M}$ of a BFC $\mathcal{C}$ with Müger center $\mathcal{E}$ is an MTC $\mathcal{M}$ together with a faithful embedding of $\mathcal{C}$ [18,31]. A modular extension $\mathcal{M}$ of $\mathcal{C}$ is minimal if the total quantum dimension of $\mathcal{M}$ satisfies $D^2_{\mathcal{M}} = D^2_{\mathcal{E}}D^2_{\mathcal{C}}$. When $\mathcal{E} = \text{sVec}$ or $\text{Rep}(\mathbb{Z}_2)$, $D^2_{\mathcal{M}} = 2D^2_{\mathcal{C}}$. In this paper, *modular extension* will always mean a minimal modular extension. We will frequently refer to the MTC $\mathcal{M}$ as a modular extension, without considering the embedding—except when we discuss the results (Sec. IV) and the counting of modular extensions.

It was recently proven that every super-MTC admits a modular extension [42]. Interestingly, it was proven in advance that, if a super-MTC admits a modular extension at all, then there should be 16 different modular extensions [31]. These have distinct c mod 8 and are related to each other by stacking with the $p + ip$ superconductor, the generator of invertible fermionic topological orders [21]. For a topological order with bosonic symmetry described by $\mathcal{E} = \text{Rep}(G)$, the modular extensions all have the same c mod 8 and are related to each other by stacking with G-SPTs [31]. For our purposes, since we work with fermionic or $\mathbb{Z}_2$-symmetric modular data given in terms of a degenerate S matrix, modular extension in practice will simply mean adding anyons to make S nondegenerate.

For explicit description of the modular data of modular extensions for fermionic topological orders, see Eq. (11), which embeds the degenerate modular data Eq. (1) of a super-MTC inside nondegenerate modular data; for $\mathbb{Z}_2$-SETs, see Eq. (31), which is degenerate, and Eq. (34), which extends it.

For fermionic topological orders specified by $(\mathcal{C}, c)$m where $\mathcal{C}$ is a super-MTC and c is the chiral central charge, the corresponding modular extension $\mathcal{M}$ (which is fixed by c mod 8) can be considered a bosonzied description of the same phase, obtained by gauging $\mathbb{Z}_2$-fermion parity symmetry.

Gauging bosonic $\mathbb{Z}_2$ symmetry gives rise to emergent $\mathbb{Z}_2$ one-form symmetry, which is generated by the Wilson line operator corresponding to the SFC $\mathcal{E} = \text{Rep}(\mathbb{Z}_2)$. We will refer to this line operator as the $\mathbb{Z}_2$ charge, or as q. We note that, in the formalism of Ref. [43], a G-SET is described by an MTC together with a permutation action ρ of G on the anyons, plus the data of symmetry fractionalization, which is classified by $H^2_\rho(G, \mathcal{A})$. In our formalism, a G-SET is described instead by a BFC with $\text{Rep}(G)$ as its Müger center. This formalism automatically considers the symmetry fractionalization data, i.e., different symmetry fractionaliztion classes correspond to different BFCs. While it is difficult to see the exact correspondence concretely, in principle, the BFC-based classification considers all possible symmetry fractionalization patterns [44].

We note, however, that these BFCs may not always be distinguishable solely based on the modular data; we may need R and F tensors in addition to the S and T matrices to tell them apart [23]. Sometimes, different symmetry fractionalization classes lead to the same degenerate S but different fusion rules (recall that a degenerate S matrix only partially fixes the fusion rules); sometimes, they may lead to the same S and the same fusion rules. In such cases, it is expected that modular extension will resolve the ambiguity: two BFCs which share the same modular data will nevertheless give rise to distinct modular data after modular extension. At the level of modular data, this manifests itself as the same degenerate modular data leading to more than one class of nondegenerate modular data after extension (here, modular data of different classes cannot be related to each other via stacking with G-SPTs).

For a given topological order described by $\mathcal{C}$ to be anomaly free, i.e., realizable on a lattice model in the same dimension with on-site symmetry, $\mathcal{C}$ must have a modular extension, or in more physical terms, the symmetry must be gaugable. The lack of a modular extension, in the bosonic case, signals an anomaly in $H^4[G, U(1)]$, which means it can only be realized on the boundary of a (3 + 1)-dimensional SPT phase [31,42,43]. Despite their significance in shedding light on the intricate interplay between topological orders and symmetry, a systematic method for computing modular extensions has not yet been developed. In Sec. III, we introduce an algorithmic method for computing modular data of modular extensions. Using this method, we shall compute modular data of modular extensions of super-MTCs up to rank 10 and those of $\mathbb{Z}_2$-BFCs up to rank 6.

E. More on (2 + 1)-dimensional fermionic topological orders

1. Structure of modular extensions

A modular extension of a super-MTC is given by a spin MTC, which is a regular MTC containing a fermionic quasiparticle, i.e., an anyon ψ such that $\psi\times\psi = 1$ and $\theta_\psi = -1$. If there are multiple anyons which have this property, we need to specify a distinguished fermion.

The presence of a fermion ψ gives rise to the following structure in the spin MTC [21,29]:

(1) Each anyon α has mutual braiding ± 1 with ψ. We divide the anyons into two sectors $\mathcal{C}_{\rm NS}\oplus\mathcal{C}_{\rm R}$ based on whether they braid trivially or nontrivially with ψ.

(2) In the trivial-braiding sector $\mathcal{C}_{\rm NS}$, $\psi\alpha := \alpha\times\psi$ is always a distinct anyon from α. We can thus (noncanonically) divide the anyons in $\mathcal{C}_{\rm NS}$ into two sets Π_0 and $\psi\Pi_0$ of equal size.

(3) In the nontrivial-braiding sector $\mathcal{C}_{\rm R}$, $\alpha\times\psi$ can be either distinct from α or equal to α. We refer to the former case as a long orbit (with respect to fusion with ψ), while the latter case is referred to as a short orbit (the anyon absorbs ψ). Long orbits can again be divided into two sets of equal size, Π_v and $\psi\Pi_v$; we call the set of short orbits Π_σ.

The modular data then have the form:

$$S^{\rm spin} = \begin{pmatrix} \frac{1}{2}\hat{S} & \frac{1}{2}\hat{S} & A & A & X \\ \frac{1}{2}\hat{S} & \frac{1}{2}\hat{S} & -A & -A & -X \\ A^T & -A^T & B & -B & 0 \\ A^T & -A^T & -B & B & 0 \\ X^T & -X^T & 0 & 0 & 0 \end{pmatrix},$$

$$T^{\rm spin} = \begin{pmatrix} \hat{T} & 0 & 0 & 0 & 0 \\ 0 & -\hat{T} & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v & 0 & 0 \\ 0 & 0 & 0 & \hat{T}_v & 0 \\ 0 & 0 & 0 & 0 & \hat{T}_\sigma \end{pmatrix}, \quad (11)$$

in the basis $\Pi = \Pi_0\cup\psi\Pi_0\cup\Pi_v\cup\psi\Pi_v\cup\Pi_\sigma$. These matrices are written in a block form [29]:

$$\tilde{S}^{\rm spin} = \begin{pmatrix} \hat{S} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2A & 2X & 0 \\ 0 & 2A^T & 0 & 0 & 0 \\ 0 & 2X^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix},$$

$$\tilde{T}^{\rm spin} = \begin{pmatrix} 0 & \hat{T} & 0 & 0 & 0 \\ \hat{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{T}_v & 0 & 0 \\ 0 & 0 & 0 & \hat{T}_\sigma & 0 \\ 0 & 0 & 0 & 0 & \hat{T}_v \end{pmatrix}, \quad (12)$$

in the basis $\tilde{\Pi} = \Pi_0^+\cup\Pi_0^-\cup\Pi_v^+\cup\Pi_\sigma\cup\Pi_v^-$, where $\Pi_0^\pm = \{X\pm X^\psi | X\in\Pi_0\}$ and $\Pi_v^\pm = \{Y\pm Y^\psi | Y\in\Pi_v\}$.

2. Torus Hilbert space of (2 + 1)-dimensional spin TQFT

It is well known that an MTC defines a (2 + 1)-dimensional TQFT, which assigns a Hilbert space of states to each closed 2-manifold and linear maps between such Hilbert spaces to cobordisms between 2-manifolds. On the torus, given an MTC $\mathcal{C}$, each state can be labeled by an anyon $a\in\mathcal{C}$; we denote this as $|a\rangle\in\mathcal{H}(T^2)$. From this point of view, the S and T matrices then tell us how these states on the torus transform into each other under modular transformations of the torus.

In the fermionic case, there should also be a correspondence between the categorical description of anyons and the transformation properties of the torus Hilbert space of a TQFT. Because of fermions, the TQFT should depend on the spin structure on manifolds; such a TQFT is called a *spin TQFT*. On the torus, we have four spin structures, specifying whether the boundary conditions for fermions are antiperiodic [Neveu-Schwarz (NS)] or periodic [Ramond (R)] along the two cycles of the torus. We shall label the four tori as NS-NS, NS-R, R-NS, and R-R.

Given a spin MTC $\mathcal{M}$ or a modular extension of a super-MTC, we can construct the Hilbert space on the four tori by condensing the distinguished fermion ψ. States in each sector take the form [45]:

(1) NS-NS:

$$\frac{1}{\sqrt{2}}(|a\rangle + |a\times\psi\rangle), \quad (13)$$

where $a\in\Pi_0$.

(2) NS-R:

$$\frac{1}{\sqrt{2}}(|a\rangle - |a\times\psi\rangle), \quad (14)$$

where $a\in\Pi_0$.

(3) R-NS:

$$\frac{1}{\sqrt{2}}(|x\rangle + |x\times\psi\rangle), \quad (15)$$

where $x\in\Pi_v$, and

$$|m\rangle, \quad (16)$$

where $m\in\Pi_\sigma$.

(4) R-R:

$$\frac{1}{\sqrt{2}}(|x\rangle - |x\times\psi\rangle), \quad (17)$$

where $x\in\Pi_v$ [46].

We see that, given an MTC $\mathcal{C}$, the basis change of Eq. (12) corresponds precisely to the sector basis: $\Pi_{\text{NS-NS}} = \Pi_0^+$, $\Pi_{\text{NS-R}} = \Pi_0^-$, $\Pi_{\text{R-NS}} = \Pi_v^+ \cup \Pi_\sigma$, and $\Pi_{\text{R-R}} = \Pi_v^-$. Treating each sector as a block, we can rewrite Eq. (12) as

$$\tilde{S} = \begin{pmatrix} \hat{S} & 0 & 0 \\ 0 & 0 & \hat{S}' \\ 0 & \hat{S}'^T & 0 \end{pmatrix} \oplus S_{\text{R-R}},$$
$$\tilde{T} = \begin{pmatrix} 0 & \hat{T} & 0 \\ \hat{T} & 0 & 0 \\ 0 & 0 & T_{\text{R-NS}} \end{pmatrix} \oplus T_{\text{R-R}}. \tag{18}$$

As expected, S takes the NS-R sector to the R-NS sector and vice versa, while T takes the NS-NS sector to the NS-R sector and vice versa. The first three sectors mix under $\text{SL}_2(\mathbb{Z})$, while the R-R sector transforms into itself under $\text{SL}_2(\mathbb{Z})$.

In Appendix A, we show that the $\text{SL}_2(\mathbb{Z})$ representation of the first three sectors is in fact the induced representation of the Γ_θ representation of the NS-NS sector, which defines the corresponding super-MTC.

3. Relation to two-dimensional fermionic RCFT

Let us begin by recalling some well-known facts about RCFT and their relation to MTCs [11]. In an RCFT, the torus partition function can be expressed as

$$\begin{aligned} Z(\tau,\bar{\tau}) &:= \text{Tr}[q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)}] \\ &= \sum_{i,j} M_{ij}\chi_i(\tau)\bar{\chi}_j(\bar{\tau}), \end{aligned} \tag{19}$$

where $\chi_i(\tau)$ are *characters*; $i = 1, \dots, N$, where N is finite; and M_{ij} is some matrix. The characters $\chi_i(\tau)$ transform covariantly under $\text{SL}_2(\mathbb{Z})$:

$$\begin{aligned} \chi_i(\tau+1) &= \sum_j T_{ij}\chi_j(\tau), \\ \chi_i(-1/\tau) &= \sum_j S_{ij}\chi_j(\tau), \end{aligned} \tag{20}$$

while $Z(\tau,\bar{\tau})$ is invariant.

Bulk S and T form a projective representations of $\text{SL}_2(\mathbb{Z})$, whereas boundary S and T form a linear representation of $\text{SL}_2(\mathbb{Z})$. They are related via

$$\begin{aligned} S_{\text{CFT}} &= S_{\text{MTC}}, \\ T_{\text{CFT}} &= \exp\left(-\frac{2\pi ic}{24}\right)T_{\text{MTC}}, \end{aligned} \tag{21}$$

where c is the chiral central charge. Recall that bulk S and T determine c mod 8 but not mod 24.

In a fermionic conformal field theory [47–49], the partition function depends on the spin structure. We label the four spin structures on the torus as NS-NS, NS-R, R-NS, and R-R (or equivalently, NS, $\widetilde{\text{NS}}$, R, and $\widetilde{\text{R}}$), where NS and R denote antiperiodic and periodic boundary conditions, respectively. Here, NS-R ($\widetilde{\text{NS}}$) denotes antiperiodic along the spatial direction and periodic along the temporal direction, while R-NS (R) denotes periodic along the spatial direction and antiperiodic along the temporal direction. The partition functions are

$$\begin{aligned} Z^{\text{NS}}(\tau,\bar{\tau}) &= \text{Tr}_{\mathcal{H}_{\text{NS}}}[q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)}], \\ Z^{\widetilde{\text{NS}}}(\tau,\bar{\tau}) &= \text{Tr}_{\mathcal{H}_{\text{NS}}}[(-1)^F q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)}], \\ Z^{\text{R}}(\tau,\bar{\tau}) &= \text{Tr}_{\mathcal{H}_{\text{R}}}[q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)}], \\ Z^{\widetilde{\text{R}}}(\tau,\bar{\tau}) &= \text{Tr}_{\mathcal{H}_{\text{R}}}[(-1)^F q^{L_0-(c/24)}\bar{q}^{\bar{L}_0-(c/24)}]. \end{aligned} \tag{22}$$

In a fermionic RCFT, the partition function of each sector can be written in terms of a finite number of characters: $\chi_i^{\text{NS}}(\tau)$ for the NS sector, $\chi_i^{\widetilde{\text{NS}}}(\tau)$ for the $\widetilde{\text{NS}}$ sector, etc. Since we are interested in the interplay between RCFT and MTC (which captures the chiral information about RCFTs), for the rest of the paper, we focus on the characters and not the full partition functions.

The simplest example of a fermionic RCFT is the free Majorana fermion CFT. It has one character per sector (except for the $\widetilde{\text{R}}$ sector which is empty), given by

$$\begin{aligned} \chi^{\text{NS}}(\tau) &= \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}, \\ \chi^{\widetilde{\text{NS}}}(\tau) &= \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \\ \chi^{\text{R}}(\tau) &= \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}, \\ \chi^{\widetilde{\text{R}}}(\tau) &= 0. \end{aligned} \tag{23}$$

It is well known that these can be written in terms of the characters of a bosonic RCFT, in this case, the Ising CFT. The Ising CFT has three chracters χ_h with conformal dimension $h = 0, \frac{1}{2}, \frac{1}{16}$. We can write the above characters as

$$\begin{aligned} \chi^{\text{NS}} &= \tfrac{1}{2}(\chi_0 + \chi_{1/2}), \\ \chi^{\widetilde{\text{NS}}} &= \tfrac{1}{2}(\chi_0 - \chi_{1/2}), \\ \chi^{\widetilde{\text{R}}} &= \frac{\sqrt{2}}{2}\chi_{1/16}. \end{aligned} \tag{24}$$

Such a process is referred to as *fermionization*. Like how each bosonic RCFT character corresponded to a basis state on the torus (labeled by an anyon) of the bulk TQFT, in a fermionic RCFT, a character corresponds to a basis state in a particular sector. The basis states, as we have seen in Sec. II E 2, are written as a linear combination of bosonic basis states.

On the other hand, reversing the above gives us the Ising characters in terms of the fermionic characters. If we know the fermionic characters, we can *bosonize* the theory to obtain bosonic RCFT characters.

A less trivial example is given by the WZW $\text{SU}(2)_6$ model, which has seven characters with conformal weight $h = 0, \frac{3}{32}, \frac{1}{4}, \frac{15}{32}, \frac{3}{4}, \frac{35}{32}, \frac{3}{2}$. The corresponding MTC can be thought of as a modular extension of the super-MTC $\text{PSU}(2)_6$, which contains $h = \{0, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}\}$; $h = \{\frac{3}{32}, \frac{35}{32}, \frac{15}{32}\}$ correspond to anyons which are added in to form the modular extension.

Given the seven characters of $SU(2)_6$, the linear combination:

$$\begin{aligned}\chi_0^{\rm NS} &= \chi_0 + \chi_{3/2},\\ \chi_{1/4}^{\rm NS} &= \chi_{1/4} + \chi_{3/4},\\ \chi_0^{\widetilde{\rm NS}} &= \chi_0 - \chi_{3/2},\\ \chi_{1/4}^{\widetilde{\rm NS}} &= \chi_{1/4} - \chi_{3/4},\\ \chi_{3/32}^{\rm R-NS} &= \chi_{3/32} + \chi_{35/32},\\ \chi_{15/32}^{\rm R-NS} &= \chi_{15/32},\\ \chi_{3/32}^{\widetilde{\rm R}} &= \chi_{3/32} - \chi_{35/32},\end{aligned} \tag{25}$$

gives rise to a fermionic RCFT with two characters in each sector, except for the R-R sector which has a single character [50].

If we begin with a bosonic RCFT and fermionize it, we will automatically obtain the fermionic characters in every sector. Conversely, if we know the fermionic characters in every sector, we can obtain the bosonized theory. In practice, however, we may only have partial information in the fermionic side. See, for example, Refs. [49,51,52], which classify characters in the NS-NS and R-NS sectors. The constraints of fermionic RCFTs allow us to obtain the NS-R and R-NS sector characters immediately via modular transformations from the NS-NS sector. However, the R-R sector cannot be obtained in such a manner. Thus, the question arises: Given the NS-NS sector characters, what can we say about the R-R sector?

The authors of Ref. [53] initiated the investigation of this question. Although they focused on partition functions rather than the characters, the basic idea is one which we shall follow: Given a fermionic RCFT, there should exist a consistent bosonization, and this fact can be used to constrain the R-R sector. As we have seen earlier, on the spin-TQFT level, bosonization essentially corresponds to modular extension. Hence, if we can compute the modular extensions, we could treat this question systematically and give an answer at least at the level of representations.

More specifically, suppose we are given the NS-NS characters of a fermionic RCFT. The NS-NS sector determines the associated Γ_θ representation ρ and the chiral central charge c. Here, ρ determines a super-MTC, and c specifies a modular extension. This modular extension gives the $SL_2(\mathbb{Z})$-representation for all the sectors; the representation of the R-R sector, which was otherwise not reliably available, can be determined in this way.

III. METHOD

In this section, we elaborate on the methods for computing modular data of modular extensions of super-MTCs and $\mathbb{Z}_2$-BFCs and classifying modular data of $\mathbb{Z}_2$-BFCs. Our methods take advantage of congruence representations of $SL_2(\mathbb{Z})$ which have completely been classified recently [30]. To make the connection to representation theory more manifest, we assume that modular data form *linear* congruence representations rather than projective congruence representations, i.e., we assume that the T matrices carry the chiral central charge factor $\exp(-2\pi i\frac{c}{24})$. In other words, we adopt the convention from CFT as shown in Eq. (21). Furthermore, for brevity, all representations henceforth are assumed to be congruence representations.

A. Modular extensions of (2 + 1)-dimensional fermionic topological orders

A modular extension of a super-MTC is called a spin MTC, which is an MTC with a distinguished fermion ψ. For each super-MTC, there always exists a modular extension [42], and there are always 16 different modular extensions [29]. Since a spin MTC is an MTC, it can be captured by the classification of (2 + 1)-dimensional bosonic topological orders without symmetry. However, they are in general of very high ranks: for a super-MTC of rank r, its modular extensions can have ranks between $\frac{3}{2}r$ and $2r$. Furthermore, in principle, if there are more than one anyon with fermionic self-statistics, then one has to choose which should be ψ, and each choice leads to a different super-MTC. Therefore, it would be more efficient if we can compute modular extensions from the data of each super-MTC.

As explained in Sec. II E 1, the modular data of a modular extension can be written in a block form in the $\tilde{\Pi}$ basis. Interestingly, the upper blocks of the matrices in Eq. (12) are equivalent to the generators of a representation induced from Γ_θ to $SL_2(\mathbb{Z})$. Specifically, for given $\hat{S}$ and $\hat{T}^2$, the induced representation is formed by

$$S^{\rm ind} = \begin{pmatrix}\hat{S} & 0 & 0\\ 0 & 0 & \hat{S}^2\\ 0 & \mathbb{1} & 0\end{pmatrix},\quad T^{\rm ind} = \begin{pmatrix}0 & \hat{T}^2 & 0\\ \mathbb{1} & 0 & 0\\ 0 & 0 & (\hat{S}\hat{T}^2)^{-1}\end{pmatrix}. \tag{26}$$

Since the modular data are symmetric matrices, the induced representation shown in Eq. (26) should be symmetrized. We first conjugate $S^{\rm ind}$ and $T^{\rm ind}$ by a unitary matrix $U_1 = {\rm diag}(\mathbb{1}, \mathbb{1}, \hat{S})$:

$$S^{(1)} \equiv U_1 S^{\rm ind} U_1^{-1} = \begin{pmatrix}\hat{S} & 0 & 0\\ 0 & 0 & \hat{S}\\ 0 & \hat{S} & 0\end{pmatrix},\quad T^{(1)} \equiv U_1 T^{\rm ind} U_1^{-1} = \begin{pmatrix}0 & \hat{T}^2 & 0\\ \mathbb{1} & 0 & 0\\ 0 & 0 & \hat{S}^3\hat{T}^{-2}\end{pmatrix}. \tag{27}$$

Note that we have used the fact that $\hat{S}^4 = \mathbb{1}$ and $\hat{S}^2\hat{T}^{-2} = \hat{T}^{-2}\hat{S}^2$, which follow from the fact that $(\hat{S}, \hat{T}^2)$ form a representation of Γ_θ. Similarly, we introduce $U_2 = {\rm diag}(\mathbb{1}, \hat{T}, \mathbb{1})$, which leads to

$$S^{(2)} \equiv U_2 S^{(1)} U_2^{-1} = \begin{pmatrix}\hat{S} & 0 & 0\\ 0 & 0 & \hat{T}\hat{S}\\ 0 & \hat{S}\hat{T}^{-1} & 0\end{pmatrix},\quad T^{(2)} \equiv U_2 T^{(1)} U_2^{-1} = \begin{pmatrix}0 & \hat{T} & 0\\ \hat{T} & 0 & 0\\ 0 & 0 & \hat{S}^3\hat{T}^{-2}\end{pmatrix}. \tag{28}$$

Now we want to find a unitary matrix A such that $A\hat{S}\hat{T}^{-1} = A^*\hat{S}\hat{T}$, or $A^TA = \hat{S}\hat{T}^2\hat{S}^3$. Once we find such A, the unitary matrix $U_3 = \mathrm{diag}(\mathbb{1}, \mathbb{1}, A)$ will symmetrize $S^{(2)}$, i.e., $S^{(3)} \equiv U_3 S^{(2)} U_3^{-1}$ is symmetric. Furthermore, since one can easily check that $A\hat{S}^3\hat{T}^{-2}A^\dagger$ is symmetric from the fact that $A^T A\hat{S}^3\hat{T}^{-2} = \hat{T}^{-2}\hat{S}^3 A^T A$, we can see that $T^{(2)}$ remains symmetric under conjugation by U_3, i.e., $T^{(3)} \equiv U_3 T^{(2)} U_3^{-1}$ is symmetric.

To find A, we first perform the Autonne-Takagi factorization to get a unitary matrix V such that $VA^TAV^T = V\hat{S}\hat{T}^2\hat{S}^3V^T \equiv \Lambda$ is a real diagonal matrix with nonnegative entries. Since A^TA is unitary, we find $\Lambda^2 = \mathbb{1}$. This implies $\Lambda = \mathbb{1}$ because Λ is real nonnegative diagonal. Thus, we get $A^TA = V^\dagger V^*$ and choose $A = V^*$.

As a result, we obtain the symmetrized matrices:

$$S^{(3)} = \begin{pmatrix} \hat{S} & 0 & 0 \\ 0 & 0 & \hat{T}\hat{S}A^\dagger \\ 0 & A^*\hat{S}\hat{T} & 0 \end{pmatrix},$$

$$T^{(3)} = \begin{pmatrix} 0 & \hat{T} & 0 \\ \hat{T} & 0 & 0 \\ 0 & 0 & A^*\hat{S}^3A^\dagger \end{pmatrix}, \tag{29}$$

where we have rewritten the last block of $T^{(3)}$ using the fact that $A^TA = \hat{S}\hat{T}^2\hat{S}^3$. Since $A^*\hat{S}^3A^\dagger$ is a complex symmetric matrix, it can be written as $A^*\hat{S}^3A^\dagger = X + iY$, where X and Y are real symmetric matrices. From the fact that $A^*\hat{S}^3A^\dagger$ is unitary, i.e., $(A^*\hat{S}^3A^\dagger)(A^*\hat{S}^3A^\dagger)^\dagger = X^2 + Y^2 - i[X, Y] = \mathbb{1}$, we can see that $[X, Y] = 0$. This implies that there exists a real orthogonal matrix P that simultaneously diagonalizes X and Y and thus diagonalizes $A^*\hat{S}^3A^\dagger$ [54]. As a result, we get the following symmetric matrices:

$$S^{\mathrm{sym}} \equiv U_4 S^{(3)} U_4^{-1} = \begin{pmatrix} \hat{S} & 0 & 0 \\ 0 & 0 & M \\ 0 & M^T & 0 \end{pmatrix},$$

$$T^{\mathrm{sym}} \equiv U_4 T^{(3)} U_4^{-1} = \begin{pmatrix} 0 & \hat{T} & 0 \\ \hat{T} & 0 & 0 \\ 0 & 0 & W \end{pmatrix}, \tag{30}$$

where $U_4 = \mathrm{diag}(\mathbb{1}, \mathbb{1}, P)$, $M = \hat{T}\hat{S}A^\dagger P^T$, and $W = PA^*\hat{S}^3A^\dagger P^T$. Now the matrices shown in Eq. (30) are orthogonally equivalent to the upper four blocks of $\tilde{S}^{\mathrm{spin}}$ and $\tilde{T}^{\mathrm{spin}}$ in Eq. (12).

In fact, the orthogonal matrices relating the two should act nontrivially only on the last block (the one containing W) because the orthogonal transformation must leave $\hat{S}$ and $\hat{T}$ invariant. If the eigenvalues of W are all distinct, the orthogonal matrices are signed diagonal matrices and there are only a finite number of possibilities. If W has overlapping eigenvalues, however, more general orthogonal matrices are allowed if they do not destroy the diagonal structure, and we need to scan over all those possibilities. This situation is somehow like that of the unresolved cases which appeared in previous literature [20,22]. For the cases where W has overlapping eigenvalues, we scanned over orthogonal matrices in a brute-force way and found a proper one for each case. In this way, after some steps we explain below, we found 16 different unitary modular extensions together with 16 nonunitary ones for each unitary modular data of a super-MTC. Since it is known that a super-MTC has 16 different unitary modular extensions [29], it means we have covered all possibilities, at least in the unitary case—unless the given modular data correspond to multiple inequivalent super-MTCs, which is unlikely to occur in low rank.

Next, we note that $\hat{T}_v$ is contained in W, though the explicit entries are not known *a priori*. This is consistent with the fact that the rank of a modular extension of a given rank-r super-MTC is between $\frac{3r}{2}$ and $2r$. We thus check for all possible choices of $\hat{T}_v$ that are contained in W whether there exists a $\mathrm{SL}_2(\mathbb{Z})$ representation with whose T matrix is equal to the choice $\hat{T}_v$. For given candidates of $\mathrm{SL}_2(\mathbb{Z})$ representations, we construct candidate modular data in the sector basis. Lastly, we go back to the Π basis and check the consistency conditions given in Sec. II C.

One of 16 different modular extensions is determined by the chiral central charge factor $\exp(-2\pi i\frac{c}{12})$ of $\hat{T}^2$ that we begin with. Suppose that we are given modular data $(\hat{S}, \hat{T}^2)$. As we mentioned above, $\hat{T}^2$ is assumed to carry the chiral central charge factor $\exp(-2\pi i\frac{c}{12})$. Note that we have chosen c mod 12 rather than mod $\frac{1}{2}$. Indeed, there are 24 different Γ_θ representations that give rise to the same fermion-quotient modular data, whose phase factors differ by the factor $\exp(2\pi i\frac{1}{24})$. In addition, since the chiral central charges of MTCs are defined mod 8, there are threefold choices for c for a modular extension. The choice of a chiral central charge mod 8 fixes the specific modular extension of the 16 possibilities.

B. Classification of (2 + 1)-dimensional $\mathbb{Z}_2$-enriched bosonic topological orders

Here, (2 + 1)-dimensional $\mathbb{Z}_2$-enriched bosonic topological orders are classified by $\mathbb{Z}_2$-BFCs. For a given bosonic topological order C with an on-site $\mathbb{Z}_2$ symmetry, each indecomposable local excitation is labeled by an irreducible representation of $\mathbb{Z}_2$. Since there is only one nontrivial irreducible representation, there is only one nontrivial label for local excitation, denoted by q. The anyons of C are then categorized by whether they are acted upon trivially or nontrivially by the $\mathbb{Z}_2$ symmetry.

Suppose that an anyon a is acted upon nontrivially by the $\mathbb{Z}_2$ symmetry. In other words, the anyons change the anyon label when they fuse with q, i.e., $a \times q = a^q \neq a$. We say that such anyons like a are in the A sector, while a^q are in the A^q sector. Since $q \times q = 1$, distinction between A and A^q sectors is arbitrary. In contrast, the anyons that remain unchanged under fusion with q are said to be in the B sector, i.e., for b in the B sector, $b \times q = b^q = b$. The modular data $S^{\mathbb{Z}_2}$ and $T^{\mathbb{Z}_2}$ should satisfy $S^{\mathbb{Z}_2}_{ij} = S^{\mathbb{Z}_2}_{i^q j^q}$ and $T^{\mathbb{Z}_2}_{ij} = T^{\mathbb{Z}_2}_{i^q j^q}$; thus, they can be written as the following block form:

$$S^{\mathbb{Z}_2} = \begin{pmatrix} A & A & B \\ A & A & B \\ B^T & B^T & C \end{pmatrix},$$

$$T^{\mathbb{Z}_2} = \begin{pmatrix} T_A & 0 & 0 \\ 0 & T_A & 0 \\ 0 & 0 & T_B \end{pmatrix}. \tag{31}$$

Now we break the $\mathbb{Z}_2$ symmetry to get an MTC. We note that the anyons in the A sector are merely doubled by the $\mathbb{Z}_2$ symmetry action, while those in the B sector can be understood as a composite of the anyons exchanged by the action. For example, let us consider the toric code MTC, with anyon labels $\{1, e, m, f\}$, and $\mathbb{Z}_2$ symmetry that exchanges e and m. Then after enriching the $\mathbb{Z}_2$ symmetry, the anyons in the A and the A^q sectors are $\{1, f\}$ and $\{q, f^q\}$, respectively, while the only anyon in the B sector is $e \oplus m$. Breaking the symmetry is basically undoing this, which leads us to

$$S^{\mathrm{SB}} = \begin{pmatrix} 2A & B & B \\ B^T & \frac{C+K}{2} & \frac{C-K}{2} \\ B^T & \frac{C-K}{2} & \frac{C+K}{2} \end{pmatrix},$$
$$T^{\mathrm{SB}} = \begin{pmatrix} T_A & 0 & 0 \\ 0 & T_B & 0 \\ 0 & 0 & T_B \end{pmatrix}. \tag{32}$$

Here, the square matrix K is extra data which cannot directly be computed from Eq. (31). By an orthogonal transformation, the matrices in Eq. (31) are simultaneously block-diagonalized:

$$\tilde{S}^{\mathrm{SB}} = \begin{pmatrix} 2A & \sqrt{2}B & 0 \\ \sqrt{2}B^T & C & 0 \\ 0 & 0 & K \end{pmatrix},$$
$$\tilde{T}^{\mathrm{SB}} = \begin{pmatrix} T_A & 0 & 0 \\ 0 & T_B & 0 \\ 0 & 0 & T_B \end{pmatrix}, \tag{33}$$

where the upper and the lower blocks form a $\mathrm{SL}_2(\mathbb{Z})$ representation, respectively.

From a known list of $\mathrm{SL}_2(\mathbb{Z})$ representations, we get candidates for the upper blocks in Eq. (33) and reconstruct the matrices in Eq. (31) from them. Then we numerically solve Verlinde's formula shown in Eq. (2) to get nonnegative integer solutions N_k^{ij} that satisfy the other consistency conditions explained in Sec. II C as well. It is important to emphasize that one modular data candidate can yield different sets of solutions. Furthermore, though a solution satisfies all the consistency conditions, it can be invalid by disallowing modular extensions. Thus, we explicitly compute the modular data of modular extensions of $\mathbb{Z}_2$-BFCs.

The $\mathrm{SL}_2(\mathbb{Z})$ representation we start out with may be reducible. We note that our method may miss certain modular data coming from unresolved $\mathrm{SL}_2(\mathbb{Z})$ representations [20]. We also note that, direct sums only involving 2D and 1D irreps lead to the symmetry-broken MTC being integral, and since integral MTCs have been classified up to rank 12 [25], we do not need to construct the representation separately in these cases; we can simply take the modular data of symmetry-broken MTCs of Ref. [25] and obtain the corresponding $\mathbb{Z}_2$-BFCs.

C. Modular extension of (2 + 1)-dimensional $\mathbb{Z}_2$-enriched bosonic topological orders

As we explained above, satisfying all the consistency conditions given in Sec. II C does not guarantee that candidate modular data are valid: if candidate modular data do not admit a modular extension, then the candidate is invalid. Thus, we explicitly compute the modular data of modular extensions of the candidate $\mathbb{Z}_2$-BFCs. Physically, a modular extension corresponds to a gauged SET.

Like the fermionic modular extension case, the starting point is to write the modular data of the modular extension in a block form. To do so, we again decompose the label set of anyons into disjoint subsets: $\Pi = \Pi_{00} \cup \Pi_{10} \cup \Pi_{01} \cup \Pi_{11}$. Here, $\Pi_{00} \cup \Pi_{10}$ and $\Pi_{01} \cup \Pi_{11}$ are the sets for the anyons that braid trivially and nontrivially with the $\mathbb{Z}_2$ charge q, respectively, and $\Pi_{00} \cup \Pi_{01}$ and $\Pi_{10} \cup \Pi_{11}$ are the sets for the anyons that are variant and invariant under the fusion with q, respectively. For example, if $a \in \Pi_{10}$, then $a \times q = a$ and $M_{aq} \equiv \theta_{a\times q}/\theta_a\theta_q = 1$. Overall, we have

$$S^{\mathbb{Z}_2\text{-gauged}} = \begin{pmatrix} A & A & B & X & X \\ A & A & B & -X & -X \\ B^T & B^T & C & 0 & 0 \\ X^T & -X^T & 0 & Y & -Y \\ X^T & -X^T & 0 & -Y & Y \end{pmatrix},$$
$$T^{\mathbb{Z}_2\text{-gauged}} = \begin{pmatrix} T_A & 0 & 0 & 0 & 0 \\ 0 & T_A & 0 & 0 & 0 \\ 0 & 0 & T_B & 0 & 0 \\ 0 & 0 & 0 & T_X & 0 \\ 0 & 0 & 0 & 0 & -T_X \end{pmatrix}. \tag{34}$$

These matrices can be simultaneously block-diagonalized by an orthogonal transformation:

$$\tilde{S}^{\mathbb{Z}_2\text{-gauged}} = \begin{pmatrix} 2A & \sqrt{2}B & 0 & 0 & 0 \\ \sqrt{2}B^T & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2X & 0 \\ 0 & 0 & 2X^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 2Y \end{pmatrix},$$
$$\tilde{T}^{\mathbb{Z}_2\text{-gauged}} = \begin{pmatrix} T_A & 0 & 0 & 0 & 0 \\ 0 & T_B & 0 & 0 & 0 \\ 0 & 0 & T_A & 0 & 0 \\ 0 & 0 & 0 & 0 & T_X \\ 0 & 0 & 0 & T_X & 0 \end{pmatrix}, \tag{35}$$

where the upper blocks of $\tilde{S}^{\mathbb{Z}_2\text{-gauged}}$ and $\tilde{T}^{\mathbb{Z}_2\text{-gauged}}$ are the same as the upper blocks of $\tilde{S}^{\mathrm{SB}}$ and $\tilde{T}^{\mathrm{SB}}$ in Eq. (33). Importantly, the lower blocks of $\tilde{S}^{\mathbb{Z}_2\text{-gauged}}$ and $\tilde{T}^{\mathbb{Z}_2\text{-gauged}}$ form an induced representation from a $\Gamma_0(2)$ representation, whose T matrix is given by T_A.

Here, $\Gamma_0(2)$ is an index-3 subgroup of $\mathrm{SL}_2(\mathbb{Z})$, which is generated by $\mathfrak{s}\mathfrak{t}^2\mathfrak{s}$ and $\mathfrak{t}$. Given a $\Gamma_0(2)$ representation π, its induced representation is generated by

$$S^{\text{ind}} = \begin{pmatrix} 0 & [\pi(\mathfrak{s}\mathfrak{t}^2\mathfrak{s})\pi(\mathfrak{t})]^2 & 0 \\ \pi(\mathbb{1}) & 0 & 0 \\ 0 & 0 & \pi(\mathfrak{s}\mathfrak{t}^2\mathfrak{s})\pi(\mathfrak{t}) \end{pmatrix},$$

$$T^{\text{ind}} = \begin{pmatrix} \pi(\mathfrak{t}) & 0 & 0 \\ 0 & 0 & \pi(\mathbb{1}) \\ 0 & \pi(\mathfrak{s}\mathfrak{t}^2\mathfrak{s})[\pi(\mathfrak{s}\mathfrak{t}^2\mathfrak{s})\pi(\mathfrak{t})]^2 & 0 \end{pmatrix}. \quad (36)$$

Again, these matrices are equivalent to the lower blocks of the matrices in Eq. (35). We could do a similar procedure as the fermionic modular extension case, i.e., symmetrizing the induced representation. That is one possible way, but we proceed in a different way; we compute the eigenvalues of T^{ind} and search $SL_2(\mathbb{Z})$ representations with the same set of T eigenvalues. Since the $SL_2(\mathbb{Z})$ representations are all given in a symmetric form, we do not need to concern ourselves with the symmetrization of the induced representation.

Given the $SL_2(\mathbb{Z})$ representations with the desired T eigenvalues, we construct the candidate modular data in the basis of Eq. (35) and go back to the Π basis. By checking the consistency conditions given in Sec. II C, we get modular data of modular extensions of $\mathbb{Z}_2$-BFCs.

IV. RESULT AND DISCUSSION

We list our results in the Supplemental Material [33]. Here, we make some observations and comments on our results.

A. Fermionic modular extensions

It had been known that each unitary super-MTC always admits 16 unitary modular extensions [31,42]. We find that it also admits 16 additional nonunitary modular extensions, where the only difference is that the quantum dimensions of the anyon in $\mathcal{C}_{\text{R}}$ come with a minus sign—thus, the total quantum dimension is the same as their unitary counterparts, and these are pseudounitary. In terms of modular data, they differ by conjugating with a sign-diagonal matrix whose negative elements are on the rows corresponding to the added anyons. So each unitary super-MTC admits 16 pairs of modular extensions, where each pair contains a unitary and a nonunitary MTC.

We find that nonunitary super-MTCs also admit 16 pairs of modular extensions. Both MTCs in the pair are nonunitary, but the two are related again by a minus sign on the quantum dimensions of the anyons of the $\mathcal{C}_{\text{R}}$ sector. Like the unitary cases, they are connected by conjugating with a sign-diagonal matrix. This leads us to conjecture that, just as for unitary super-MTCs, nonunitary super-MTCs also admit 16 pairs of modular extensions.

Every entry belonging to the new classes of modular data found in Ref. [22], with $D^2 = 472.379$ and 475.151, has modular extensions (on the level of modular data). This is consistent with the explicit realization of these modular data via super-MTCs carried out for specific examples from these classes in Ref. [32].

We find that, for the vast majority of cases, a given super-MTC leads to the modular extensions of two different ranks; in such a case, those modular extensions whose c differ by an integer have the same rank. For example, for the rank 8 super-MTC $PSU(2)_{14}$ (rank 8 #66 in our table), modular extensions with $c = \frac{1}{8} + k$ ($k \in \mathbb{Z}$) have rank 13, such as the $SO(7)_{-3}$ MTC which corresponds to $c = \frac{1}{8}$, but those modular extensions with $c = \frac{1}{8} + k + \frac{1}{2}$ have rank 15, such as the $SU(2)_{14}$ MTC with $c = \frac{21}{8}$. There are exceptions to this, however: For $PSU(2)_4$ at rank 4, every modular extension has rank 7, while for $PSU(2)_6 \boxtimes_{\mathcal{F}_0} PSU(2)_6$ [the fermionic stacking of two $PSU(2)_6$ super-MTCs] of rank 8, every modular extension is of rank 14.

We present the chiral central charge c, quantum dimensions d_i, and topological spins θ_i of modular extensions for each super-MTC in the Supplemental Material [33]. We always choose $D > 0$ when computing Θ. If we had chosen $D < 0$, which is a conventional choice for nonunitary MTCs, then the central charge would have shifted by 4.

For the rank-8 super-MTCs #7–10 [22], we failed to compute the modular data of modular extensions of them within our methodology since their R-NS sector spins are threefold degenerate. This degeneracy allows the freedom of three-dimensional orthogonal transformations keeping the T matrix diagonal. Fortunately, for unitary super-MTCs #7 and 8, each one of their modular extensions are known [22], and we find that each one of modular extensions of #9 and 10 corresponds to $5^B_{\#8} \boxtimes 4^B_3$ and $5^B_{\#7} \boxtimes 4^B_{-3}$, respectively [17,20]. Starting from the known modular extension, we can get another with the chiral central charge shifted by $\frac{1}{2}$, by stacking it with the Ising MTC and condensing a boson. By doing the same procedure successively, we can obtain all 16 modular extensions.

1. The Ramond-Ramond sector of fermionic RCFT

We mentioned in Sec. II E 3 that modular extensions can be used to gain information about the R-R sector from the NS-NS sector. Armed with our data of modular extensions, we can now see how this works in practice.

Let us illustrate this with a pair of examples: the $c = \frac{21}{8}$ and $\frac{225}{8}$ entries from the rank-4 table of Ref. [49]. It has four characters in the NS-NS sector, and we identify the corresponding super-MTC as $PSU(2)_{14}$. The $c = \frac{21}{8}$ theory corresponds to the $SU(2)_{14}$ modular extension, which is of rank 15. Performing the basis change of Eq. 12, we obtain the R-R sector representation as

$$S_{\text{R-R}} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\sqrt{2} \\ -1 & 1 & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} & 0 \end{pmatrix},$$

$$T_{\text{R-R}} = \exp\left(-\frac{2\pi i}{16}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \exp\left(2\pi i \frac{3}{16}\right) \end{bmatrix}. \quad (37)$$

On the other hand, the $c = \frac{225}{8}$ theory has $c = \frac{33}{8}$ mod 8, and the corresponding modular extension is of rank 13. The corresponding R-R sector representation is a trivial one-dimensional representation, so this has a single R-R character which is constant.

We note that Ref. [55] has proved a simple criterion for whether the R-R sector partition function is a constant in terms of the R-NS sector exponents, without making use of modular extension—so the whole modular extension is not necessary if we are interested simply in whether the R-R sector is a constant or not. Our approach, however, allows us to obtain the exact representation of the R-R sector, not just whether it is a constant.

We may also mention that Ref. [56] has given a construction of a class of fermionic RCFTs, which automatically constructs the R-R sector as well as the other sectors, though the MTCs corresponding to this class of theories are all Abelian. Our approach works even for RCFTs whose corresponding MTCs contain non-Abelian anyons.

B. Classification of $\mathbb{Z}_2$-SET orders and modular extensions

Because $(2+1)$-dimensional $\mathbb{Z}_2$-SPTs are classified by $H^3[\mathbb{Z}_2, U(1)] = \mathbb{Z}_2$ [57], each $\mathbb{Z}_2$-BFC will have two modular extensions [31,43]. Recall, however, that a modular extension of a BFC $\mathcal{C}$ is an MTC $\mathcal{M}$ together with an embedding of $\mathcal{C}$ into $\mathcal{M}$. It may happen that a BFC $\mathcal{C}$ has two modular extensions with the same MTC $\mathcal{M}$, distinguished only by the embedding of $\mathcal{C}$ into $\mathcal{M}$.

Such a phenomenon was already observed in Ref. [23], for some rank-5 BFCs which come from rank-4 MTCs and lead to rank-9 modular extension, and referred to as a situation with a single TO-equivalence class. In the terminology of Ref. [58], the SET absorbs an SPT under stacking. In this situation, our algorithm yields only a single gauged MTC in which the BFC $\mathcal{C}$ is embedded, even though there are technically two modular extensions. We observe the same phenomenon for some rank-6 BFCs leading to rank-10 modular extensions: For rank-6 BFC labeled $6_1^{\zeta_2^1}$; self-dual, and similar, we find only a single gauged MTC. In Appendix B, we provide a simple test for when this occurs.

It may also happen that two distinct BFCs are indistinguishable from their modular data—they will have different R and F symbols but the same S, T. This can happen if there are different symmetry fractionalization classes for the same symmetry action on anyons. Sometimes, two BFCs with the same S, T may be distinguished based on the fusion rules, but sometimes even the fusion rules fail to distinguish them, and one needs to take the modular extension to distinguish them, as discussed for certain rank-5 examples in Ref. [23].

We find that, for $6_1^{\zeta_2^1}$ (and similar) from Ref. [23], there are two different BFCs with the same S and T, but they can be distinguished by the fusion rules. The theory labeled $6_1^{\zeta_2^1}$; self-dual has self-dual fusion rules (every anyon is its own antiparticle), which leads to a single modular extension (which absorbs stacking with $\mathbb{Z}_2$-SPTs); the other, non-self-dual fusion rule leads to a different set of modular extensions.

V. CONCLUSIONS

In this paper, we computed modular extensions of super-MTCs and $\mathbb{Z}_2$-BFCs up to rank 10 and 6 using induced representations from the congruence representations of Γ_θ and $\Gamma_0(2)$, respectively. Furthermore, we classified $\mathbb{Z}_2$-BFCs up to rank 6 by their modular data, using congruence representations of $\mathrm{SL}_2(\mathbb{Z})$.

Our method for computing modular data of modular extensions is much more efficient than previous approaches based on finding plausible fusion coefficients and checking the other consistency conditions. High-rank modular extensions were inaccessible in the previous approach. We also classified the modular data of $\mathbb{Z}_2$-BFCs without setting any upper limit of fusion coefficients or total quantum dimensions. Therefore, at least up to unresolved cases, we can argue that the classification is complete up to rank 6.

The fact that the modular data form congruence representations of $\mathrm{SL}_2(\mathbb{Z})$ or its subgroups is advantageous for studying topological orders. First, apart from general representations of $\mathrm{SL}_2(\mathbb{Z})$, the congruence representations are those of finite groups and much restricted, leading to the recent complete classification [30]. Furthermore, congruence representations of the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ can be obtained from those of $\mathrm{SL}_2(\mathbb{Z})$, using the concept of induced representations [22]. These advantages lead to recent progress in classification of modular data [20,22].

While we have worked out the systematic relationship between the R-R sector of fermionic RCFTs and modular extensions of super-MTCs, this still does not determine the R-R sector characters themselves (i.e., the concrete functions of the modular parameter τ) from the NS-NS sector characters. It would be interesting to find out a way to use our representation information about the R-R sector to compute the R-R sector characters explicitly.

Our methods may be generalized to symmetry groups other than $\mathbb{Z}_2$, though the challenge is to find the relevant congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and their representations. For more complicated groups, such as $\mathbb{Z}_2 \times \mathbb{Z}_2$, the 't Hooft anomaly given by $H^4[G, U(1)]$ can be nontrivial, and hence, there may be an obstruction to finding a modular extension even when the modular data define a genuine G-BFC. Since such an anomaly defines a $(3+1)$-dimensional topological order, this may lead to a computationally efficient approach to studying $(3+1)$-dimensional topological orders through BFCs/anomalous $(2+1)$-dimensional SETs, an example of which was Ref. [59].

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APPENDIX A: INDUCED REPRESENTATION INSIDE THE MODULAR EXTENSION

Given the NS-NS sector Γ_θ representation, we can determine the representations of the NS-R and R-NS sectors since these sectors are connected via modular transformations. This has been shown in Appendix C of Ref. [60]. Here, we give an alternative, more physical, argument in terms of RCFT characters.

Given $\chi_i^{\mathrm{NS}}(\tau)$, we can define a basis of characters in the NS-R and R-NS sectors simply by transforming the NS-NS sector basis characters:

$$\chi_i^{\widetilde{\mathrm{NS}}}(\tau) := \chi^{\mathrm{NS}}(t\cdot\tau),$$
$$\chi_i^{\mathrm{R}}(\tau) := \chi^{\mathrm{NS}}(st\cdot\tau). \tag{A1}$$

Note that this basis may not be the basis in which the $\mathrm{SL}_2(\mathbb{Z})$ representation is symmetric; however, once we obtain the $\mathrm{SL}_2(\mathbb{Z})$ representation, we can simply do a basis change via a unitary matrix appropriately.

Now the action of S is

$$\begin{aligned}\chi_i^{\mathrm{NS}}(s\cdot\tau) &= \hat{S}_{ij}\chi_j^{\mathrm{NS}}(\tau),\\ \chi_i^{\widetilde{\mathrm{NS}}}(s\cdot\tau) &= \chi^{\mathrm{NS}}(st\cdot\tau) = \chi_i^{\mathrm{R}}(\tau),\\ \chi_i^{\mathrm{R}}(s\cdot\tau) &= \chi_i^{\mathrm{NS}}(s^2t\cdot\tau) = \chi_i^{\mathrm{NS}}(ts^2\cdot\tau)\\ &= (\hat{S}^2)_{ij}\chi_i^{\mathrm{NS}}(t\cdot\tau)\\ &= (\hat{S}^2)_{ij}\chi_j^{\widetilde{\mathrm{NS}}}(\tau).\end{aligned} \tag{A2}$$

In other words, acting on the basis $\begin{bmatrix}\chi_i^{\mathrm{NS}}(\tau)\\ \chi_i^{\widetilde{\mathrm{NS}}}(\tau)\\ \chi_i^{\mathrm{R}}(\tau)\end{bmatrix}$, we have

$$S = \begin{pmatrix}\hat{S} & 0 & 0\\ 0 & 0 & \hat{S}^2\\ 0 & \mathbb{1} & 0\end{pmatrix}. \tag{A3}$$

The action of T is

$$\begin{aligned}\chi_i^{\mathrm{NS}}(t\cdot\tau) &= \chi_i^{\widetilde{\mathrm{NS}}},\\ \chi_i^{\widetilde{\mathrm{NS}}}(t\cdot\tau) &= \chi^{\mathrm{NS}}(t^2\cdot\tau) = \hat{T}_{ij}\chi_j^{\mathrm{NS}}(\tau),\\ \chi_i^{\mathrm{R}}(t\cdot\tau) &= \chi_i^{\mathrm{NS}}(tst\cdot\tau) = \chi_i^{\mathrm{NS}}[st(st^2)^{-1}\cdot\tau]\\ &= (\hat{S}\hat{T}^2)^{-1}_{ij}\chi_i^{\mathrm{NS}}(st\cdot\tau)\\ &= (\hat{S}\hat{T}^2)^{-1}_{ij}\chi_i^{\mathrm{R}}(\tau).\end{aligned} \tag{A4}$$

This leads to

$$T = \begin{bmatrix}0 & \hat{T}^2 & 0\\ \mathbb{1} & 0 & 0\\ 0 & 0 & (\hat{S}\hat{T}^2)^{-1}\end{bmatrix}. \tag{A5}$$

This is nothing but the $\mathrm{SL}_2(\mathbb{Z})$ representation induced from the Γ_θ representation $(\hat{S},\hat{T}^2)$. Since this induced representation is the $\mathrm{SL}_2(\mathbb{Z})$ representation acting upon the NS-NS, NS-R, and R-NS sectors, it is nothing but the first block of Eq. (18), i.e., a part of the $\mathrm{SL}_2(\mathbb{Z})$ representation of the modular extension, simply in a different basis.

In fact, $T_{\text{R-NS}} = \begin{pmatrix}T_v & 0\\ 0 & T_\sigma\end{pmatrix}$ can be easily computed from the above: It is simply the eigenvalues of $(\hat{S}\hat{T}^2)^{-1}$. The form of Eq. (18) corresponds to a basis of R-NS characters which are diagonal under T.

For $\mathbb{Z}_2$-symmetry-enriched phases, the same principle applies. If we are given an MTC which contains a $\mathbb{Z}_2$ charge (i.e., an anyon g with $\theta_g = 1$ and $g\times g = 1$), we can condense this and obtain states on the different sectors of the torus [45]. The torus sectors are given by the $\mathbb{Z}_2$ holonomies along the two cycles. In this case, the $(0, 0)$ sector of the torus transforms into itself under $\mathrm{SL}_2(\mathbb{Z})$, while the $(0, 1)$, $(1, 0)$, and $(1, 1)$ sectors transform into each other under $\mathrm{SL}_2(\mathbb{Z})$. Hence, if we take the $(0, 1)$ sector, which carries a representation of $\Gamma_0(2)$, and form the induced representation Eq. (36), this will be the lower block of the $\mathrm{SL}_2(\mathbb{Z})$ representation Eq. (35) of the modular extension.

APPENDIX B: TEST FOR THE ABSORPTION OF $\mathbb{Z}_2$-SPT ORDER

Gauged $\mathbb{Z}_2$-SPT corresponds to the double semion (DS) topological order, which has anyons $\{1, s\bar{s}, s, \bar{s}\}$ with spins $\{0, 0, \frac{1}{4}, -\frac{1}{4}\}$. This is a modular extension of $\mathrm{Rep}(\mathbb{Z}_2)$, with the first two spins $\{0, 0\}$ belonging to $\mathrm{Rep}(\mathbb{Z}_2)$. Note that $s\bar{s}$ plays the role of the $\mathbb{Z}_2$ charge.

Hence, once we have a modular extension $\mathcal{C}$ (which has the $\mathbb{Z}_2$ charge q) of a $\mathbb{Z}_2$-BFC $\mathcal{B}$, to obtain another modular extension $\mathcal{C}'$, we stack with DS and condense the composite boson $(q, s\bar{s})$ in $\mathcal{C}\boxtimes\mathrm{DS}$ [23,31].

An anyon (α, x) of the stacked theory $\mathcal{C}\boxtimes\mathrm{DS}$ survives the condensation (i.e., is not confined) if it has trivial braiding with $(q, s\bar{s})$. This can only be that case if one of the following hold:

(1) α has trivial braiding with q, and x has trivial braiding with $s\bar{s}$ (i.e., $x = 1$ or $s\bar{s}$), or

(2) α has -1 braiding with q, and x is s or $\bar{s}$.

The first type of anyons is exactly those which come from the $\mathbb{Z}_2$-BFC $\mathcal{B}$. Both types of anyons come in pairs α and $\alpha^q := q\times\alpha$, which have spins $\theta_{\alpha^q} = \theta_\alpha$ for the first type and $\theta_{\alpha^q} = -\theta_\alpha$ for the second type. In the stacked theory,

$$\begin{aligned}(\alpha, 1)\times(\alpha, s\bar{s}) &= (\alpha, 1),\\ (\alpha, s)\times(q, s\bar{s}) &= (\alpha^q, \bar{s}).\end{aligned} \tag{B1}$$

Thus, under the condensation of $(q, s\bar{s})$, $(\alpha, 0)$ and $(\alpha, s\bar{s})$ are identified, leading to exactly the anyons α of the new modular extension $\mathcal{C}'$. The anyons (α, s) and $(\alpha^q, \bar{s})$ are also identified. Note that these indeed have the same topological spin, as required, since $\theta_s = \theta_\alpha + \frac{1}{4} = \theta_{\alpha^q} - \frac{1}{4}$.

Hence, given a modular extension $\mathcal{C}$ of $\mathcal{B}$, the other modular extension $\mathcal{C}'$ will have anyons given by the following procedure: The anyons of $\mathcal{C}$ in the original $\mathbb{Z}_2$-BFC $\mathcal{B}$ will remain the same, while the anyons of $\mathcal{C}$ in the sector with nontrivial braiding with q will have their spins shifted by $\frac{1}{4}$. If this shift leads to the same set of spins, then the modular extension absorbs the stacking with the $\mathbb{Z}_2$-SPT.

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