Spectral form factors of unconventional superconductors

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We show that spectral form factors of unconventional gapped superconductors have singularities occurring periodically in time. These are the superconductors whose gap function vanishes somewhere in momentum space (Brillouin zone) but whose fermionic excitation spectrum is fully gapped. Many, although not all, of these superconductors are topologically nontrivial. In contrast, conventional fully gapped superconductors have featureless spectral form factors which are analytic in time. Some gapless superconductors may also have singularities in their spectral form factors, but they are not as ubiquitous and their appearance may depend on the details of the interactions among fermionic particles which form the superconductor and on the underlying lattice where the particles move. This work builds on the prior publication [S. Gaur, V. Gurarie, and E. A. Yuzbashyan, Phys. Rev. B **106**, L220506 (2022)] where Loschmidt echo of topological superconductors, related but not identical to spectral form factors, was studied. It follows that spectral form factors could be used as a test of the structure of the superconducting gap functions.

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Spectral form factor is a way to characterize the energy spectrum of quantum systems. It is defined as a trace of the evolution operator

$$\mathcal{Z} = \operatorname{tr} e^{-i\hat{H}t} = \sum_n e^{-iE_n t}, \tag{1}$$

and can be written as a sum over all the energy levels E_n of a quantum system. It is closely related to the thermal partition function of a quantum system, coinciding with its formal analytic continuation to the complex values of temperature.

Fourier transform of the absolute value square of the spectral form factor produces the correlation between energy levels of the system

$$\int dt\, e^{i\omega t} |\mathcal{Z}|^2 = 2\pi \sum_{nm} \delta(\omega - E_n + E_m). \tag{2}$$

Behavior of the spectral form factors in quantum many-body systems attracted some attention recently [1,2].

Spectral form factors are related, although not identical, to the Loschmidt echo, which can be defined as the expectation of the evolution operator with respect to a state which is not an eigenstate of the Hamiltonian $Z = \langle \Psi | e^{-i\hat{H}t} | \Psi \rangle$. It was proposed some time ago that the singularities in the Loschmidt echo as a function of t may reflect the nature of the Hamiltonian $\hat{H}$ as well as of the state $|\Psi\rangle$, in particular serving as a probe of whether $|\Psi\rangle$ and the ground state of $\hat{H}$ correspond to different quantum phases [3–5]. However, this has not been unambiguously demonstrated. In the recent publication [6] it was shown that the Loschmidt echo of topological superconductors may also have singularities, dependent on the initial state $|\Psi\rangle$.

In this work we propose to look instead at spectral form factors of topological superconductors. In contrast to the Loschmidt echo, those depend only on the Hamiltonian itself and reflect its properties. We show that spectral form factors in unconventional gapped superconductors have singularities which occur periodically in time, while their conventional counterparts have featureless spectral form factors. It follows that spectral form factors could be used as a test of the structure of the superconducting gap functions. Combined with the new proposals which make it possible to measure spectral form factors in some atomic systems [1], this makes spectral form factors interesting observables to study.

More precisely, consider the gap function $\Delta(\mathbf{p})$, where $\mathbf{p}$ is the (quasi)momentum. In unconventional and especially in topological superconductors it cannot be nonzero everywhere. Rather it vanishes at points, lines, or surfaces in the momentum space or in the Brillouin zone. Consider the spectrum of Bogoliubov quasiparticles $E(\mathbf{p})$ for those values $\mathbf{p}$ where $\Delta(\mathbf{p}) = 0$. Suppose E_- is the minimum of $E(\mathbf{p})$ for all such $\mathbf{p}$. If $\Delta(\mathbf{p})$ vanishes at a single point, then we obviously cannot minimize $E(\mathbf{p})$ and instead take E_- to be equal to $E(\mathbf{p})$ calculated at this point. Then the singularities occur at times $t_n = \pi(1+2n)/(2E_-)$ where n is an arbitrary integer. We show that the nature of the singularities depends on the dimensionality of space and is given by

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln |t - t_n| \tag{3}$$

in two-dimensional superconductors and

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{|t - t_n|} \tag{4}$$

in three-dimensional superconductors.

For superconductors whose underlying fermionic particles move on a lattice, the singularities could also occur at $t_n = \pi(1+2n)/(2E_+)$, where E_+ is the maximum of the excitation spectrum computed where $\Delta(\mathbf{p}) = 0$ (if E_+ is different from E_-). The existence of these singularities is not as ubiquitous as that of the ones associated with E_-.

We would like to emphasize that to see this behavior it is important to look at the Hamiltonians of interacting fermions, as opposed to the Bogoliubov–de Gennes Hamiltonians with a given gap function. While we end up using mean field theory to calculate spectral form factors, the gap function in the resulting effective Bogoliubov–de Gennes Hamiltonian turns out to depend on the time interval t over which the spectral form factor is defined. This is reminiscent of the gap function dependent on the temperature when calculating thermal partition functions of superconductors.

Let us now demonstrate this by first studying the example of the two dimensional $p_x + ip_y$ (often abbreviated as $p + ip$) superconductor. Specifically, we have in mind attractively interacting identical fermions with the Hamiltonian [7,8]

$$H = \sum_{\mathbf{p}} \epsilon(p)\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \frac{\lambda}{V}\sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \mathbf{k}\cdot\mathbf{q}\, \hat{a}^\dagger_{\frac{\mathbf{p}}{2}+\mathbf{k}}\hat{a}^\dagger_{\frac{\mathbf{p}}{2}-\mathbf{k}}\hat{a}_{\frac{\mathbf{p}}{2}-\mathbf{q}}\hat{a}_{\frac{\mathbf{p}}{2}+\mathbf{q}}. \quad (5)$$

Here,

$$\epsilon(p) = \frac{p^2}{2m} - \mu \quad (6)$$

is the kinetic energy of these interacting spinless fermions, λ is the interaction constant, and V is the volume of the system. These fermions are known to form a $p_x + ip_y$ paired fermionic superfluid, which for brevity we will refer to as a p-wave superconductor. It is a class D superconductor [9] which is topological if $\mu > 0$ and has a gap as long as $\mu \neq 0$. The Bogoliubov–de Gennes (BdG) Hamiltonian of this superconductor takes the following standard form

$$\hat{H} = \sum_{\mathbf{p},\ p_y>0} \begin{pmatrix}\hat{a}^\dagger_{\mathbf{p}} & \hat{a}_{-\mathbf{p}}\end{pmatrix}\begin{pmatrix}\epsilon(p) & \Delta(\mathbf{p}) \\ \bar{\Delta}(\mathbf{p}) & -\epsilon(p)\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{p}} \\ \hat{a}^\dagger_{-\mathbf{p}}\end{pmatrix}. \quad (7)$$

Here $\Delta(\mathbf{p}) = (p_x + ip_y)\Delta_p$ and $\bar{\Delta}(\mathbf{p}) = (p_x - ip_y)\bar{\Delta}_p$ are the gap functions. Δ_p and $\bar{\Delta}_p$ are the magnitudes of the gap functions (the subscript p emphasizes that these are p-wave gap functions). To avoid double counting, the summation over $\mathbf{p}$ is restricted to $p_y > 0$. Below, all the sums over $\mathbf{p}$ for p-wave superconductors will be restricted in this way.

Let us use the BdG Hamiltonian to calculate the spectral form factor. To do that, we diagonalize the BdG Hamiltonian for each $\mathbf{p}$. Its eigenvalues $\omega_\pm(p)$ are

$$\omega_\pm(p) = \pm E(p), \quad (8)$$

where

$$E(p) = \sqrt{\epsilon(p)^2 + p^2\bar{\Delta}_p\Delta_p}. \quad (9)$$

Therefore the trace of its evolution operator is

$$\mathcal{Z} = \prod_{\mathbf{p}} S_{\mathbf{p}},\ S_{\mathbf{p}} = e^{-itE(p)} + e^{itE(p)} = 2\cos\left(tE(p)\right). \quad (10)$$

Before proceeding to study $\mathcal{Z}$, let us briefly discuss its analytic properties. Each factor $S_{\mathbf{p}}$ is obviously an analytic function of time t. However, if $S_{\mathbf{p}}$ vanishes for some values of $\mathbf{p}$ at some critical time $t = t_c$ with all $S_{\mathbf{p}}$ remaining nonzero if t deviates from t_c, this could make $\mathcal{Z}$ nonanalytic at t_c (we postpone the discussion whether $S_{\mathbf{p}}$ can indeed behave in this way until later). Indeed, suppose $S_{\mathbf{p}}$ vanishes at $t = t_c$ if $\mathbf{p} = \mathbf{p}_c$. Quite generally we should expect that in the vicinity of $\mathbf{p} = \mathbf{p}_c$ and $t = t_c$, $S_{\mathbf{p}}$ has the following expansion

$$S_{\mathbf{p}} \approx C(t - t_c + \alpha|\mathbf{p} - \mathbf{p}_c|^2), \quad (11)$$

where α and C are some complex constants (we will see later that, even though it may not be obvious right now, the factors $S_{\mathbf{p}}$ are generally complex valued). This immediately leads to

$$\frac{\partial \ln \mathcal{Z}}{\partial t} = \sum_{\mathbf{p}} \frac{\partial \ln S_{\mathbf{p}}}{\partial t} \approx \sum_{\mathbf{p}} \frac{1}{t - t_c + \alpha|\mathbf{p} - \mathbf{p}_c|^2}. \quad (12)$$

On the right-hand side above, the approximate expression for $S_{\mathbf{p}}$ valid with $\mathbf{p}$ in the vicinity of $\mathbf{p}_c$ and $t - t_c$ small is substituted. The sum above is obviously a singular function of time at $t = t_c$, with the details of the singularity dependent on the dimensionality of space and on whether $\mathbf{p}_c$ is zero or nonzero. This makes $\ln\mathcal{Z}$ as well as $\mathcal{Z}$ itself a nonanalytic function of time at $t = t_c$ (note an obvious similarity between the thermal free energy and $\ln\mathcal{Z}$ introduced above).

Let us now go back to Eq. (10). For a Hamiltonian Eq. (7) with given Δ_p, $\bar{\Delta}_p$, and $\epsilon(p)$, Eq. (10) gives the answer for its spectral form factor. However, in a superconductor, Δ_p and $\bar{\Delta}_p$ are not fixed beforehand but must be determined self-consistently, by matching the Hamiltonian Eq. (5) with the BdG Hamiltonian Eq. (7). To understand how to do it, let us recall that to calculate thermal partition function tr $\exp(-\hat{H}/(k_BT))$, we must determine Δ_p and $\bar{\Delta}_p$ by solving the gap equation. In a p-wave superconductor, it takes the form

$$\frac{1}{V}\sum_{\mathbf{p}} \frac{p^2\, \tanh\left[\frac{E(p)}{2k_BT}\right]}{E(p)} = \frac{1}{\lambda}, \quad (13)$$

where T is the temperature and k_B is the Boltzmann constant. This equation is solved for the product $\bar{\Delta}_p\Delta_p$ which enters $E(p)$. The solution to this equation can be used, for example, to calculate the thermal partition function of the superconductor. In order to adapt this to calculating the spectral form factor, we replace $1/(k_BT) \to it$, with the result

$$\frac{i}{V}\sum_{\mathbf{p}} \frac{p^2\, \tan\left[\frac{tE(p)}{2}\right]}{E(p)} = \frac{1}{\lambda}. \quad (14)$$

This should be understood as an equation to determine $\bar{\Delta}_p\Delta_p$, which should then be substituted into Eq. (10). See Appendix A for the steps necessary for a formal derivation of Eq. (14) from the Hamiltonian Eq. (7).

In principle, there could be many solutions of the Eq. (14). To find the one we should use, we should identify the solution which gives the largest contribution to the spectral form factor. One strategy to do it could consist of first finding the solution of Eq. (13) for the temperatures T where the solution $\bar{\Delta}_p\Delta_p$ is nonzero, and then analytically continuing to the imaginary values of T. We will leave the detailed study of the solutions of Eq. (13) for future work.

Now observe that this equation predicts that the product $\bar{\Delta}_p\Delta_p$ must not be real. Indeed, if it is real, the left-hand side of this equation is necessarily imaginary, while the right-hand side is real. Similar situation occurs in evaluation of the Loschmidt echo where one also finds [6] that $\bar{\Delta}$ and Δ are not complex conjugates of each other. With $\bar{\Delta}_p\Delta_p$ being complex, $E(p)$ is also generally complex.

As a result, the factors $\cos(tE(p))$ generally do not vanish at any t. The exception to that is $p = 0$ where $E(0) = |\epsilon(0)| = |\mu|$, and is independent of $\bar{\Delta}_p\Delta_p$. Quite remarkably, this takes us to the previously discussed scenario given by Eq. (11) with $\mathbf{p}_c = 0$. Specifically, for t close to any of the values t_n given by

$$t_n = \frac{\pi}{2|\mu|}(1+2n), \tag{15}$$

with an arbitrary integer n, we can write

$$S_{\mathbf{p}} = 2\cos(tE(p)) \approx C(t - t_n + \alpha p^2). \tag{16}$$

Here

$$C = 2(-1)^{n+1}|\mu|, \tag{17}$$

and

$$\alpha = \pi\left(\frac{1}{2}+n\right)\frac{\bar{\Delta}_p\Delta_p m - \mu}{2\mu^2|\mu|m}. \tag{18}$$

Importantly, α is complex due to $\bar{\Delta}_p\Delta_p$ being complex. Note that this matches the conjectured form Eq. (11).

Working in the large V limit and replacing summation over $\mathbf{p}$ with integration we find

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{4\pi}\int \frac{p\,dp}{t - t_n + \alpha p^2}. \tag{19}$$

The integral above is taken over p varying from 0 to infinity, although we must remember that only the approximate value for the expression being integrated is written above valid for small p only. In particular, that means that the integral above can be cut off at some momentum scale, avoiding any divergencies at large p. It is then straightforward to see that the leading singularity is

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} \approx -\frac{1}{8\pi\alpha}\ln|t - t_n|. \tag{20}$$

The expression here is approximate, valid when t is in the vicinity of t_n. Therefore we arrive at a conclusion advertised earlier. The spectral form factor for the two-dimensional (2D) p-wave chiral superconductor has periodic logarithmic singularities which occur at times t_n, defined above in Eq. (15).

It is important for this argument that α is complex and is not real, which in turn is related to $\bar{\Delta}_p\Delta_p$ being complex.

Let us contrast this behavior with that of attractively interacting spin-1/2 fermions obeying the Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}}\sum_{\sigma=\uparrow,\downarrow}\xi_p\,\hat{a}^\dagger_{\mathbf{p},\sigma}\hat{a}_{\mathbf{p},\sigma} - \frac{\lambda}{V}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}}\hat{a}^\dagger_{\frac{\mathbf{k}}{2}+\mathbf{p},\uparrow}\hat{a}^\dagger_{\frac{\mathbf{k}}{2}-\mathbf{p},\downarrow}\hat{a}_{\frac{\mathbf{k}}{2}-\mathbf{q},\downarrow}\hat{a}_{\frac{\mathbf{k}}{2}+\mathbf{q},\uparrow}, \tag{21}$$

and forming a conventional s-wave superconductor. Its Bogoliubov–de Gennes Hamiltonian takes the form

$$\hat{H} = \sum_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}\uparrow} \quad \hat{a}_{-\mathbf{p}\downarrow})\begin{pmatrix}\epsilon(p) & \Delta_s\\ \bar{\Delta}_s & -\epsilon(p)\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{p}\uparrow}\\ \hat{a}^\dagger_{-\mathbf{p}\downarrow}\end{pmatrix}. \tag{22}$$

Here Δ_s, $\bar{\Delta}_s$ are momentum-independent s-wave gap functions. The spectral form factor takes the same form Eq. (10) but with the spectrum

$$E_s(p) = \sqrt{\epsilon(p)^2 + \bar{\Delta}_s\Delta_s}. \tag{23}$$

Here $\bar{\Delta}_s\Delta_s$ is controlled by the gap equation almost identical to the one for the p-wave superconductor, given by

$$\frac{i}{2V}\sum_{\mathbf{p}}\frac{\tan\left[\frac{tE_s(p)}{2}\right]}{E_s(p)} = \frac{1}{\lambda}. \tag{24}$$

The main point is that, just like in case of Eq. (14), the solution of this equation necessarily corresponds to $\bar{\Delta}_s\Delta_s$ being complex. As a result, $E_s(p)$ is complex. Unlike in case of the p-wave superconductor, $E_s(p)$ is complex for all p without exceptions. As a result, none of the factors $S_{\mathbf{p}}$ defined in Eq. (10) vanish for any time t, and the spectral form factor $\mathcal{Z}$ is analytic at all times.

We see that the key distinction between s-wave and 2D p-wave superconductors was the presence, in case of the latter, of a point $\mathbf{p} = 0$ in the gap function $\Delta(\mathbf{p}) = (p_x + ip_y)\Delta_p$ where it vanishes. Furthermore, despite having to analytically continue the solution of the gap Eq. (13) to imaginary temperature $1/T \to it$, we expect that the analytically continued gap function must also vanish as $\mathbf{p} \to 0$. Indeed, from the structure of the Hamiltonian Eq. (5) the p-wave gap function must satisfy

$$\Delta(\mathbf{p}) = -\Delta(-\mathbf{p}). \tag{25}$$

This enforces that the gap function must always vanish at $\mathbf{p} = 0$ even if it is a solution of the analytically continued gap Eq. (14). More generally, the key necessary condition for a nonanalytic spectral form factor is that the gap function $\Delta(\mathbf{p})$ vanishes at certain values of $\mathbf{p}$, not only at finite temperature, but also when analytically continued to imaginary values of temperature.

A good second example of a p-wave superconductor is class DIII three-dimensional (3D) topological superconductor [9] (Helium III B phase) with the Bogoliubov–de Gennes Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}} \quad \hat{a}_{-\mathbf{p}})\begin{pmatrix}\epsilon(p) & ip_\mu\sigma^y\sigma^\mu\Delta_p\\ -ip_\mu\sigma^\mu\sigma^y\bar{\Delta}_p & -\epsilon(p)\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{p}}\\ \hat{a}^\dagger_{-\mathbf{p}}\end{pmatrix}, \tag{26}$$

where σ^y and σ^μ are Pauli matrices acting on the spin indices of the operators $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$. Its spectrum is also given by Eq. (9), but with $\mathbf{p}$ being the 3D vector. By analogy with the previous analysis leading to Eq. (19), we immediately find

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{2\pi^2}\int \frac{p^2\,dp}{t - t_n + \alpha p^2} \sim \sqrt{|t - t_n|}. \tag{27}$$

On the other hand, let us examine 2D spin-singlet chiral d-wave superconductor, which belongs to the symmetry class C. The corresponding Bogoliubov-de-Gennes Hamiltonian is

$$\hat{H} = \sum_{\mathbf{p}}(\hat{a}^\dagger_{\mathbf{p}\uparrow} \quad \hat{a}_{-\mathbf{p}\downarrow})\begin{pmatrix}\epsilon(p) & \Delta(\mathbf{p})\\ \bar{\Delta}(\mathbf{p}) & -\epsilon(p)\end{pmatrix}\begin{pmatrix}\hat{a}_{\mathbf{p}\uparrow}\\ \hat{a}^\dagger_{-\mathbf{p}\downarrow}\end{pmatrix}, \tag{28}$$

with $\Delta(\mathbf{p}) = (p_x + ip_y)^2\Delta_d$, $\bar{\Delta}(\mathbf{p}) = (p_x - ip_y)^2\bar{\Delta}_d$. What sets this example apart from others is that while the gap function vanishes at $\mathbf{p} = 0$, it is not automatically obvious that the gap function analytically continued to imaginary temperature would still vanish in this limit. To elucidate this further, we suppose that the gap function consists of both d-wave and s-wave pieces, $\Delta(\mathbf{p}) = \Delta_s + (p_x + ip_y)^2\Delta_d$, $\bar{\Delta}(\mathbf{p}) = \bar{\Delta}_s + (p_x - ip_y)^2\bar{\Delta}_d$. With rotationally invariant interactions, the gap equation should decouple into two separate equations for Δ_s, $\bar{\Delta}_s$ and for Δ_d, $\bar{\Delta}_d$. If Δ_s is equal to zero for any temperature T, its analytic continuation to imaginary values of T should also be zero. At the same time, just as earlier, $\bar{\Delta}_d\Delta_d$ becomes a complex number, with the spectrum given by $E(p) = \sqrt{\epsilon^2(p) + p^4\bar{\Delta}_d\Delta_d}/2$. leading to the following singularity in the spectral form factor (below β is real, while α is complex)

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{2\pi}\int \frac{p\,dp}{t - t_n + \beta p^2 + \alpha p^4} \sim \ln|t - t_n|. \quad (29)$$

However, if Δ_s is nonzero at some range of temperature, then it may still be nonzero after the analytic continuation $1/(k_BT) \to it$. Then the superconductor will have a nonsingular spectral form factor. To decide whether a particular superconductor of this form will have singularities in its spectral form factor we need to examine the original Hamiltonian of the interacting fermions which led to this superconductor and see if any s-wave pairing is possible in addition to the d-wave pairing. Therefore, the singularities in this case are not as ubiquitous as in the p-wave case. All superconductors that we looked at so far were gapped to fermionic excitations. Let us now look at an example of a gapless superconductor. As an example, consider a 3D p-wave spin-triplet superconductor which has the Bogoliubov-de-Gennes Eq. (7) with the gap function which behaves as

$$\Delta(\mathbf{p}) = (p_x + ip_y)\Delta_p. \quad (30)$$

This gap function vanishes if $p_x = p_y = 0$, for all p_z. Furthermore, given $\epsilon(p) = p^2/(2m) - \mu$ with $\mu > 0$, the excitation spectrum

$$E(\mathbf{p}) = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \left(p_x^2 + p_y^2\right)\bar{\Delta}_p\Delta_p} \quad (31)$$

vanishes at $p_x = p_y = 0$, $p_z = \sqrt{2m\mu}$. Suppose just as in the previous examples, once the temperature is made imaginary, $\bar{\Delta}_p\Delta_p$ becomes complex, but otherwise no other terms appear in the gap function. However, unlike the previous examples of gapful superconductors, setting $p_x = p_y = 0$, we find that $E(p_z)$ now ranges from zero to infinity. As a result, the spectral form factor $\mathcal{Z}(t)$ is now an analytic function of time t.

Now it is further possible to imagine that the fermions which formed this superconductor move on a lattice, as opposed to a continuous space. If so, then $E(p_z)$ at $p_x = p_y = 0$ now has a maximum somewhere as p_z is varied. Denoting the maximum E_+ it is straightforward to see that this would lead to singularities in the spectral form factor occurring at times $t_n = \pi(2n+1)/(2E_+)$. These arguments show that singularities are possible even in gapless superconductors, but they are not as ubiquitous and their existence requires some assumptions.

Note, however, that if $\mu < 0$, then the resulting superconductor is gapful, although not topological. It will still have singularities controlled by $E_- = |\mu|$.

Coming back to the gapful (topological) superconductors, we can rely on the classification of the topological superconductors [9] to see that there are five distinct classes of topological superconductors of interest, three in the two-dimensional space and two more in the three-dimensional space. We can summarize the behavior of their spectral form factors in the following table.

Class	Gap function	Spectral form factor
D, 2D	$(p_x + ip_y)\Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln|t - t_n|$
C, 2D	$(p_x + ip_y)^2\Delta_d$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln|t - t_n|$
DIII, 2D	$(\sigma^z p_x + ip_y)\Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln|t - t_n|$
DIII, 3D	$ip_\mu\sigma^y\sigma^\mu\Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{|t - t_n|}$
CI, 3D	vanishes on surfaces	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{|t - t_n|}$

The first two entries as well as the fourth entry in the table above were already worked out above. In particular, class D and class DIII superconductors are p-wave and the singularities in their spectral form factor are ubiquitous. The class C superconductor may have singularities in their spectral form factor if its gap equation excludes the possibility of an additional s-wave gap function. The last entry refers to the class CI topological spin-singlet superconductor in three dimensions [10]. It is in the same class as the conventional s-wave spin-singlet superconductor and therefore will have singularities in the spectral form factor only if its gap equation excludes the possibility of an additional s-wave gap function. If this is excluded, then working out its singularities relies on the understanding that its gap function vanishes on 2D surfaces in its 3D Brillouin zone. Starting from the point on the surface where $E(p)$ has its minimum, following the arguments given here it is easy to see that

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \int \frac{d^2q_1 dq_2}{t - t_n - \alpha q_1^2 - \beta q_2^2}. \quad (32)$$

Here q_1 is the coordinate parametrizing the surface and q_2 is the direction perpendicular to the surface, α is real while β is complex. By analogy with Eq.(28) we find

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{|t - t_n|}, \quad (33)$$

just as stated in the table above. Therefore, we see that the type of the singularity in the spectral form factor which occurs in topological superconductors depends only on the dimensionality of space. Finally, as was already mentioned at the beginning, we would like to remark that spectral form factors nowadays are accessible to measurement in experiments, using the techniques of atomic physics. For example, if the superconductor is realized by means of cold ions [11], its spectral form factor could, in principle, be measured by directly evolving a random initial product state up to some time t and measuring the distribution of Cooper pairs in the resulting state via a protocol proposed and developed in Ref. [1] (see Appendix B for the further exposition of this method). Therefore, spectral form factors can be used as a probe of the structure of the superconducting order parameter.

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APPENDIX A: REAL-TIME GAP EQUATION FOR S-WAVE SUPERCONDUCTORS

Here we present the derivation of the real-time gap equation for the s-wave spin-singlet superconductor. The gap equations for other types of superconductors can be derived similarly. We begin with the Hamiltonian Eq. (21) for the spin-1/2 attractively interacting fermions. We are interested in calculating the spectral form factor, that is the quantity

$$\mathcal{Z} = \operatorname{tr} e^{-i\hat{H}t}. \tag{A1}$$

Let us set up the coherent path integral for the purpose of this calculation.

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \exp\Bigg(i\int_0^t d\tau \int d^3x \Bigg\{\sum_{\sigma=\uparrow,\downarrow}\Bigg(i\bar{\psi}_\sigma\dot{\psi}_\sigma - \frac{\nabla\bar{\psi}_\sigma\nabla\psi_\sigma}{2m} + \mu\bar{\psi}_\sigma\psi_\sigma\Bigg) + \lambda\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow\Bigg\}\Bigg). \tag{A2}$$

It is well known that in order to represent the spectral form factor, the fermionic fields ψ and $\bar{\psi}$ must satisfy the boundary conditions

$$\psi_\sigma(t) = -\psi_\sigma(0),\ \bar{\psi}_\sigma(t) = -\bar{\psi}_\sigma(0). \tag{A3}$$

As standard in the theory of superconductivity we introduce the Hubbard-Stratonovich field Δ_s, which results in

$$\mathcal{Z} = \int \mathcal{D}\Delta_s\mathcal{D}\bar{\Delta}_s\, e^{iW}, \tag{A4}$$

where

$$e^{iW} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iS},$$
$$S = \int_0^t d\tau \int d^3x \Bigg\{\sum_{\sigma=\uparrow,\downarrow}\Bigg(i\bar{\psi}_\sigma\dot{\psi}_\sigma - \frac{\nabla\bar{\psi}_\sigma\nabla\psi_\sigma}{2m} + \mu\bar{\psi}_\sigma\psi_\sigma\Bigg) - \bar{\Delta}_s\psi_\downarrow\psi_\uparrow - \Delta_s\bar{\psi}_\uparrow\bar{\psi}_\downarrow - \frac{\bar{\Delta}_s\Delta_s}{\lambda}\Bigg\}. \tag{A5}$$

We calculate the integral over Δ_s and $\bar{\Delta}_s$ in the saddle-point approximation. Varying W over $\bar{\Delta}_s(\mathbf{r},\tau)$ at some time τ and at some position $\mathbf{r}$, we find

$$\frac{1}{\mathcal{Z}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left(\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau) + \frac{1}{\lambda}\Delta_s(\mathbf{r},\tau)\right)e^{iS} = 0. \tag{A6}$$

We will look for the solution of this equation in terms of $\Delta_s(\mathbf{r},\tau)$ and $\bar{\Delta}_s(\mathbf{r},\tau)$, which are constant in space and time and so, from now on, denote them simply as Δ_s and $\bar{\Delta}_s$. This gives

$$\Delta_s = -\frac{\lambda}{\mathcal{Z}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)e^{iS}. \tag{A7}$$

We note that recasting this equation, and the corresponding equation for $\bar{\Delta}_s$, in the operator formalism gives

$$\Delta_s = -\lambda \operatorname{tr}\left(e^{-i\hat{H}_{\rm BdG}(t-\tau)}\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})e^{-i\hat{H}_{\rm BdG}\tau}\right),$$
$$\bar{\Delta}_s = -\lambda \operatorname{tr}\left(e^{-i\hat{H}_{\rm BdG}(t-\tau)}\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r})e^{-i\hat{H}_{\rm BdG}\tau}\right). \tag{A8}$$

Here $\hat{H}_{\rm BdG}$ is the Bogoliubov–de Gennes Hamiltonian which follows from the action S in Eq. (A5). Note that, perhaps unexpectedly, the equation for $\bar{\Delta}_s$ is not the complex conjugate of the equation for Δ_s, therefore as pointed out in the main text of this article, $\bar{\Delta}_s\Delta_s$ does not have to be real. To proceed further, we need to calculate the anomalous Green's function which appears on the right-hand side of Eq. (A7). This is computed by taking advantage of the functional integral over ψ, $\bar{\psi}$ being Gaussian. We rewrite the action S by using the Nambu notations in the frequency and momentum space. The frequencies as always are discrete and have the fermionic Matsubara form, to ensure the antiperiodic boundary conditions Eq. (A3):

$$\omega_n = \frac{\pi}{t}(1+2n). \tag{A9}$$

We define

$$\psi_{\mathbf{p},\omega_n} = \frac{1}{\sqrt{V}}\int_0^t d\tau \int d^3x\, \psi(\mathbf{r},\tau)\, e^{i\omega_n\tau - i\mathbf{p}\cdot\mathbf{r}}, \tag{A10}$$

$$\psi(\mathbf{r},\tau) = \frac{1}{t\sqrt{V}}\sum_{\omega_n,\mathbf{p}}\psi_{\mathbf{p},\omega_n}\, e^{-i\omega_n\tau + i\mathbf{p}\cdot\mathbf{r}}.$$

We find

$$S = \frac{1}{t}\sum_{\mathbf{p},\omega_n}\begin{pmatrix}\bar{\psi}_{\mathbf{p},\omega_n,\uparrow} & \psi_{-\mathbf{p},-\omega_n,\downarrow}\end{pmatrix} \times \begin{pmatrix}\omega_n - \frac{p^2}{2m} + \mu & -\Delta_s \\ -\bar{\Delta}_s & \omega_n + \frac{p^2}{2m} - \mu\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{p},\omega_n,\uparrow} \\ \bar{\psi}_{-\mathbf{p},-\omega_n,\downarrow}\end{pmatrix} - \frac{tV}{\lambda}\bar{\Delta}_s\Delta_s. \tag{A11}$$

To calculate the anomalous Green's function we invert the matrix

$$\begin{pmatrix}\omega_n - \frac{p^2}{2m} + \mu & -\Delta_s \\ -\bar{\Delta}_s & \omega_n + \frac{p^2}{2m} - \mu\end{pmatrix}^{-1} = \frac{1}{\omega_n^2 - \left(\frac{p^2}{2m} - \mu\right)^2 - \bar{\Delta}_s\Delta_s} \times \begin{pmatrix}\omega_n + \frac{p^2}{2m} - \mu & \Delta_s \\ \bar{\Delta}_s & \omega_n - \frac{p^2}{2m} + \mu\end{pmatrix} \tag{A12}$$

and read off the anomalous Green's function from the upper right corner of this matrix. We find

$$\frac{1}{\mathcal{Z}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)e^{iS} = -\frac{i}{tV}\sum_n\sum_{\mathbf{p}}\frac{\Delta_s}{\omega_n^2 - \left(\frac{p^2}{2m} - \mu\right)^2 - \bar{\Delta}_s\Delta_s}. \tag{A13}$$

Therefore, the saddle-point Eq. (A6) becomes

$$\frac{i}{tV}\sum_n\sum_{\mathbf{p}}\frac{1}{\omega_n^2-\left(\frac{p^2}{2m}-\mu\right)^2-\bar{\Delta}_s\Delta_s}=\frac{1}{\lambda}. \tag{A14}$$

Summation over the Matsubara frequencies can now be carried out explicitly, with the result

$$\frac{i}{2V}\sum_{\mathbf{p}}\frac{\tan\left[\frac{tE_s(p)}{2}\right]}{E_s(p)}=\frac{1}{\lambda}, \tag{A15}$$

where

$$E_s(p)=\sqrt{\left(\frac{p^2}{2m}-\mu\right)^2+\bar{\Delta}_s\Delta_s}. \tag{A16}$$

This is the real-time gap equation which appears as Eq. (25) in our paper. Given the solution $\bar{\Delta}_s\Delta_s$ of this equation, we can calculate the spectral form factor by following Eq. (10). Following a very similar blueprint we can also derive Eq. (14) from the Hamiltonian Eq. (7).

Note that if instead we had been interested in computing the thermal partition function $\operatorname{tr} e^{-\hat{H}/(k_BT)}$, we would have employed the imaginary-time formalism. It largely coincides with the formalism described here, and differs only by the replacement $it\to 1/(k_BT)$ where T is temperature and k_B Boltzmann constant, and by the analytic continuation of the time τ used above to imaginary values. It would have resulted in

$$\frac{k_BT}{V}\sum_n\sum_{\mathbf{p}}\frac{1}{\omega_n^2+\left(\frac{p^2}{2m}-\mu\right)^2+\bar{\Delta}_s\Delta_s}=\frac{1}{\lambda}, \tag{A17}$$

where

$$\omega_n=\pi k_BT(1+2n), \tag{A18}$$

instead of Eqs. (A9) and (A14). Carrying out the summation over the Matsubara frequencies results in

$$\frac{1}{2V}\sum_{\mathbf{p}}\frac{\tanh\left[\frac{E_s(p)}{2k_BT}\right]}{E_s(p)}=\frac{1}{\lambda}, \tag{A19}$$

which is the standard thermal gap equation [8,12].

APPENDIX B: MEASURING THE SPECTRAL FORM FACTOR

Spectral form factors are new types of observables which only recently came within reach of experiment. It may not be obvious that they can be measured experimentally. We would like to present here a brief overview of the measurement techniques which were recently suggested in the literature.

The simplest object to measure would be the Loschmidt echo. That could be defined as

$$\mathcal{E}=|\langle\psi|e^{-i\hat{H}t}|\psi\rangle|^2. \tag{B1}$$

If the system under study is equivalent to a number of interacting spins, or qubits, and if there is experimental control over each of these spins, one could prepare the initial state $|\psi\rangle$ (assuming it is a product state), evolve it in time, and find the probability that after that evolution it is still the same state $|\psi\rangle$ as initially. This program was carried out in a system of cold ions [5], where the state of each ion can be addressed independently.

Spectral form factor cannot be measured using this approach as it is given by

$$|\mathcal{Z}|^2=\left|\sum_n\langle n|e^{-i\hat{H}t}|n\rangle\right|^2. \tag{B2}$$

Here $|n\rangle$ could be the eigenstates of $\hat{H}$ or any other complete set of orthonormal states. Instead, an alternative approach was proposed in Ref. [1] which allows to measure it. Just as in the example above, this approach still requires that the system under study consists of interacting spins or qubits.

In this work we study interacting fermions. However, all the relevant Hamiltonians presented here can be mapped into a system of interacting spin. The mapping, which has extensively been discussed in the literature previously, consists of defining the Anderson pseudospin operators. For the s-wave Hamiltonian Eq. (21) the Anderson pseudospin operators are defined by

$$\hat{S}^+_{\mathbf{p}}=\hat{a}^\dagger_{\mathbf{p},\uparrow}\hat{a}^\dagger_{-\mathbf{p},\downarrow},\ \hat{S}^-_{\mathbf{p}}=\hat{a}_{-\mathbf{p},\downarrow}\hat{a}_{\mathbf{p},\uparrow}, \tag{B3}$$

$$\hat{S}^z_{\mathbf{p}}=\tfrac{1}{2}(\hat{a}^\dagger_{\mathbf{p},\uparrow}\hat{a}_{\mathbf{p},\uparrow}+\hat{a}^\dagger_{-\mathbf{p},\downarrow}\hat{a}_{-\mathbf{p},\downarrow}-1). \tag{B4}$$

It is straightforward to check that they satisfy SU(2) algebra, as required for spins. In terms of these, the Hamiltonian becomes

$$\hat{H}=2\sum_{\mathbf{p}}\epsilon(p)\hat{S}^z_{\mathbf{p}}-\frac{\lambda}{V}\sum_{\mathbf{k},\mathbf{q}}\hat{S}^+_{\mathbf{k}}\hat{S}^-_{\mathbf{q}}. \tag{B5}$$

Within mean-field theory, this Hamiltonian reduces to

$$\hat{H}=2\sum_{\mathbf{p}}\epsilon(p)\hat{S}^z_{\mathbf{p}}-\Delta\sum_{\mathbf{k}}\hat{S}^+_{\mathbf{k}}-\bar{\Delta}\sum_{\mathbf{k}}\hat{S}^-_{\mathbf{k}}, \tag{B6}$$

where Δ satisfies a gap equation almost identical to Eq. (25):

$$\frac{i}{2V}\sum_{\mathbf{p}}\frac{\tan\left[tE_s(p)\right]}{E_s(p)}=\frac{1}{\lambda}. \tag{B7}$$

The absence of a factor of 2 in Eq. (B7) when compared to Eq. (25) is due to a small difference between the spin system and the original interacting fermions. As can be readily seen, a spin-flip excitation corresponds to exciting two Bogoliubov quasiparticles (with the opposite spin and the same excitation energy) in the superconductor. This does not affect the qualitative features of the spectral form factor.

We can now aim at creating a spin system obeying Eq. (B5). A version of the spin system equivalent to a p-wave superconductor Eq. (5) obeys the Hamiltonian

$$\hat{H}=\sum_{\mathbf{p}}\epsilon(p)\hat{S}^z_{\mathbf{p}}-\frac{\lambda}{V}\sum_{\mathbf{k},\mathbf{q}}\mathbf{k}\cdot\mathbf{q}\,\hat{S}^+_{\mathbf{k}}\hat{S}^-_{\mathbf{q}} \tag{B8}$$

with the identification

$$\hat{S}^{+}_{\mathbf{p}} = \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{-\mathbf{p}},\ \hat{S}^{-}_{\mathbf{p}} = \hat{a}_{-\mathbf{p}}\hat{a}_{\mathbf{p}}, \tag{B9}$$

$$\hat{S}^{z}_{\mathbf{p}} = \tfrac{1}{2}(\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^{\dagger}_{-\mathbf{p}}\hat{a}_{-\mathbf{p}} - 1). \tag{B10}$$

Ref. [11] addressed the question of how the Hamiltonian Eq. (B8) can be created in a cold ion system where each spin (qubit) can be independently controlled and measured, at least in principle. The question remains how this addressability can be used to measure spectral form factors.

This question was resolved in another work [1]. That work proposed a protocol towards measuring the spectral form factor. The protocol itself is not elementary. It consists of the following steps. If the Hilbert space of a quantum many-body system can be represented by a collection of N qubits, remarkably the square of the absolute value of the spectral form factor can be measured in terms of the probabilities $|\langle \mathbf{s}|U^{\dagger}e^{-i\hat{H}t}U|0\rangle|^2$. Here $|0\rangle$ is the state where all qubits are initialized in the "all spin-up" state, $\langle \mathbf{s}|$ is the state where the jth qubit points up if $s_j = 0$ or down if $s_j = 1$. $U = \prod_{j=1}^{N} u_j$ and u_j is a unitary SU(2) rotation of the jth qubit. It can be shown that

$$|\mathrm{tr}\, e^{-i\hat{H}t}|^2 = \int \left[\prod_j du_j\right] \sum_{s_i=0,1} (-2)^{-\sum_{j=1}^{N} s_j} |\langle \mathbf{s}|U^{\dagger}e^{-i\hat{H}t}U|0\rangle|^2. \tag{B11}$$

Here the integrals du_j are over the SU(2) group's Haar measure.

To implement this proposal, it is envisioned that a system of spins is initialized in the "all spin-up" state. Subsequently it is rotated by a random rotation U, evolved in time, rotated again by $U^{\dagger}$, and the spins are measured producing the data of s_j. This is repeated many times and $(-2)^{-\sum_j s_j}$ is averaged over many realizations of U as well as many repetitions of the same experiment. Since the quantum mechanical probability of observing an outcome of a set of s_j is given by $|\langle \mathbf{s}|U^{\dagger}e^{-i\hat{H}t}U|0\rangle|^2$, it should be clear that this procedure will, upon averaging over many measurements, produce the spectral form factor as long as Eq. (B11) is correct.

The derivation of Eq. (B11) is given in Ref. [1].

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