Editors' Suggestion

Superfluid stiffness within Eliashberg theory: The role of vertex corrections

Zachary M. Raines, Shang-Shun Zhang, and Andrey V. Chubukov
School of Physics and Astronomy and William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA

(Received 28 December 2023; revised 7 March 2024; accepted 11 March 2024; published 5 April 2024)

In this work we consider the superfluid stiffness of a generically non-Galilean-invariant interacting system and investigate under what conditions the stiffness may nonetheless approach the Galilean-invariant value n/m. Within Eliashberg theory we find that the renormalized stiffness is approximately given by n/m in the case when the $l = 0$ and 1 components of the effective Fermi-surface projected interaction are approximately equal over a range of frequencies. This holds, in particular, when the interaction is peaked at zero momentum transfer. We examine this result through three complementary lenses: the $\delta(\omega)$ term in the conductivity, the phase dependence of the Luttinger-Ward free energy, and the coupling of the amplitude and phase sectors in the Hubbard-Stratonovich collective mode action. From these considerations we show that the value of the stiffness is determined by the strength of renormalization of the current vertex and that the latter can be interpreted as the shift of the self-consistent solution due to flow of the condensate, or alternatively as coupling of the phase mode to $l = 1$ fluctuations of the order parameter. We highlight that even though the superfluid stiffness in some non-Galilean systems approaches the Galilean value, this is not enforced by symmetry, and in general the stiffness may be strongly suppressed from its BCS value. As a corollary we obtain the generic form of the phase action within Eliashberg theory and charge and spin Ward identities for a superconductor with frequency-dependent gap function.

DOI: 10.1103/PhysRevB.109.144505

I. INTRODUCTION

The superfluid stiffness $D_s(T)$ is one of the key characteristics of a superconductor: it determines the strength of the δ-functional contribution to the optical conductivity and the energy cost of phase fluctuations. In two dimensions (2D), D_s has the dimension of energy and we explicitly define it via $\sigma(\omega \to 0) = e^2\pi D_s(T)\delta(\omega) + \cdots$ or, equivalently, via $E_{\rm cond} = (1/8)D_s[\nabla\phi(r)]^2$, where $E_{\rm cond}$ is the condensation energy per unit volume, and $\phi(r)$ is the phase of a superconducting order parameter $\Delta(r) = \Delta e^{i\phi(r)}$ [1,2].

In a clean Bardeen-Cooper-Schrieffer (BCS) superconductor, $D_s(T = 0) = E_F/\pi$, where E_F is the Fermi energy [3,4]. For a parabolic dispersion this reduces to $D_s(T = 0) = n/m$, where n is the total electron density and m the bare electron mass. In a dirty BCS superconductor, D_s is reduced and can be substantially smaller than in the clean case [2,5,6]. At a small E_F (the low-density limit), D_s can become smaller than the bound-state energy of two fermions in a vacuum E_0. In this situation the system displays, even at weak coupling, Bose-Einstein condensation (BEC) behavior where bound pairs of fermions are formed at $T_p \sim E_0$, while actual superconductivity with a macroscopic phase coherence sets in at smaller $T_c \approx (\pi/8)D_s(T_c) \sim E_F$ [4,7].[1]

The subject of this paper is the analysis of the superfluid stiffness at $T = 0$ in strongly coupled clean superconductors, with special attention to systems in the vicinity of a quantum critical point (QCP), where superconductivity emerges out of a non-Fermi liquid (NFL). We will not discuss here disorder effects [8] nor the behavior at small E_F. We assume that E_F is larger than the fermion-boson interaction strength and analyze the behavior of the stiffness within Eliashberg theory. To shorten notations, below we label $D_s(T = 0)$ simply as D_s.

Our primary goal is to understand the interplay between the contributions to D_s from the quasiparticle residue $1/Z$ and from the renormalization of the current vertex. Without vertex renormalization, D_s is renormalized down from the BCS value to $D_s \sim E_F/Z$ and is substantially reduced at strong coupling, when the quasiparticle residue is small. It was argued, however [9], that in a Galilean-invariant system, $1/Z$ is exactly canceled out by vertex renormalization due to a special Ward identity, which states that the renormalization factor for the current vertex is exactly Z. As a result, $D_s = D_s^{\rm Gal}$ is unaffected by interactions and remains the same as for a BCS superconductor ($D_s^{\rm Gal} = n/m$ at $T = 0$).

Our goal is to understand the interplay between $1/Z$ and vertex renormalizations in systems near a QCP. A frequently used low-energy model for such systems is one of fermions near the Fermi-surface Yukawa coupled to soft dynamical bosonic collective fluctuations in the corresponding spin or charge channel. The bosonic dynamics plays a crucial role for the pairing and non-Fermi-liquid behavior in the normal state [10,11]. This dynamical model is, however, non-Galilean invariant, even if a fermionic dispersion can be approximated as parabolic, because the dynamical term in the bosonic propagator is not invariant under a Galilean boost in which the momentum q of a boson remains unchanged while the frequency ω shifts to $\omega + vq$, where v is the velocity of the boost. Meanwhile, a QCP towards spin or charge order and

[1] A more accurate expression is $T_c \sim E_F/\log\log E_0/E_F$.

superconductivity near it can develop already in a Galilean-invariant system. For the latter, one then has to add additional four-fermion interactions, e.g., an effective interaction mediated by two dynamical bosons (the Aslamazov-Larkin–type terms). It is *a priori* unclear how these additional interactions, which are generally less singular near a QCP than the direct Yukawa coupling with a soft boson, account for the cancellation of the $1/Z$ factor in the stiffness.

We argue that near cancellation happens already without the additional terms. The key here is the observation that for a Galilean-invariant system, a spin or charge order emerges with $q = 0$ [12], hence, soft bosonic excitations carry small momenta q. We argue that in this situation, the leading term in the renormalization of the current vertex is the same as in the renormalization of one of the components of the spin vertex ($\sigma^i_{\alpha\beta} c^\dagger_{k,\alpha} c_{k,\beta}$ at the bare level). The fully renormalized spin vertex is related to the self-energy by the Ward identity, associated with the global spin conservation, and cancels out $1/Z$. This holds for both Galilean-invariant and Galilean-non-invariant systems. The subleading terms, which distinguish between the renormalizations of the current and spin vertices, are small in q and remain nonsingular at a QCP. For a Galilean-invariant system, these subleading terms cancel out by additional, less singular interaction terms in the fermion-boson Hamiltonian.

In this communication we discuss the interplay between Z and the renormalization of the current vertex near a $q = 0$ QCP in some detail. We obtain a generic expression for D_s for interacting fermions near a QCP and show under which condition it reduces to the BCS result $D_s = n/m$. This condition (the equivalence between two functions of Matsubara frequency) is satisfied for a Galilean-invariant system, but also approximately holds for a non-Galilean-invariant system. We call these systems *effectively Galilean*. We obtain how D_s changes once the condition is violated and illustrate this for the case of fermions interacting with a boson with propagator $\chi(q, \Omega_m) \propto 1/(\omega_D^2 + \omega_m^2 + (cq)^2)$. We use the boson velocity c as a control parameter and show within Eliashberg theory how the renormalization of D_s evolves between the limits of large c, when the scattering is predominantly in forward direction, and small c, when scattering by any q is equally probable. We show that in the first case $D_s \approx n/m$, while in the second case D_s is reduced to $\sim(n/mZ)$. This last result holds for the interaction mediated by an Einstein phonon.

These results appear naturally when the superfluid stiffness is identified with the prefactor for the $\delta(\omega)$ term in the optical conductivity. We also show how the fully dressed D_s emerges in the Luttinger-Ward (LW) description of a superconductor with coordinate-dependent phase $\phi = \mathbf{q} \cdot \mathbf{r}$ or, equivalently, of a superconductor with a nonzero total momentum q of a pair. The key issue we discuss here is how the corrections to the current vertex emerge in this approach. We show that they originate from the change of the fermionic self-energy due to the phase twist $\exp(i\mathbf{q} \cdot \mathbf{r})$ and that the existence of such corrections is a general feature of linear response in the LW formalism. We also show how the fully renormalized superfluid stiffness can be derived within the Hubbard-Stratonovich (HS) formalism in the context of the phase action.

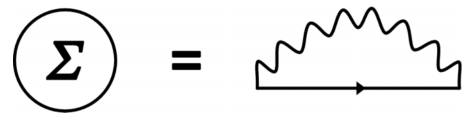

FIG. 1. Self-energy diagram within the Eliashberg framework. The solid line is the full Nambu Green's function, and the wavy line is the interaction V. Vertex corrections to the self-energy are neglected as they are small for the typical frequency and momentum scales contributing to $\hat{\Sigma}$ [10,12–14].

We restrict our analysis primarily to the cases when superconductivity emerges near a $q = 0$ QCP and soft bosonic excitations are peaked at $q = 0$. For a QCP with a finite Q [e.g., towards (π, π) antiferromagnetic order], the renormalization of the current vertex is unrelated to that of the spin vertex, and in general the fully dressed superfluid stiffness scales with $1/Z$.

The structure of the paper is as follows. In Sec. II, we outline the model and provide a brief review of the Eliashberg theory of superconductivity. In Sec. III, we explicitly calculate the superfluid stiffness within Eliashberg theory as the weight of the δ function in the DC conductivity, with a particular focus on the role of vertex corrections and the notion of effective Galilean invariance. In Sec. IV, we recontextualize Eliashberg theory in terms of the LW functional and employ this description to naturally obtain the superfluid stiffness, including vertex corrections. Finally, in Sec. V, we make contact with the HS description of Eliashberg theory and present the associated phase action, including the important role played by coupling of phase and amplitude gap fluctuations. Some technical aspects are discussed in Appendixes A–E. In particular, in Appendix B we explicitly derive Ward identities for charge, spin, and momentum for a superconductor with a frequency-dependent gap.

II. MODEL

We consider a model of fermions described by a Matsubara action

$$S = -\sum_k \bar{\psi}_{k\sigma}(i\epsilon_n - \xi_{\mathbf{k}})\psi_{k\sigma} - \frac{1}{2}\int dx\,dx' V(x - x')\bar{\psi}_\sigma(x)\psi_\sigma(x)\bar{\psi}_{\sigma'}(x')\psi_{\sigma'}(x'), \quad (1)$$

where the effective interaction $V(x - x')$ is mediated by a soft dynamical boson. The notations are $k = (\epsilon_n, \mathbf{k})$ and $x = (\tau, \mathbf{x})$. We assume the fermion dispersion ξ and interaction V to be rotationally invariant. We define Nambu spinors $\Psi(x) = (\psi_\uparrow(x), \bar{\psi}_\downarrow(x))$ with Green's function $\hat{\mathcal{G}}(x, x') \equiv -\langle \Psi(x)\bar{\Psi}^T(x')\rangle$. Eliashberg theory approximates the matrix self-energy by the one-loop self-consistent expression (Fig. 1)

$$\hat{\Sigma}(x, x') = V(x - x')\hat{\tau}_3\hat{\mathcal{G}}(x, x')\hat{\tau}_3 = \begin{pmatrix} -i\Sigma(x, x') + \chi(x, x') & \phi(x, x') \\ \phi^*(x', x) & -i\Sigma(x, x') - \chi(x, x') \end{pmatrix}, \quad (2)$$

where ϕ represents the pairing vertex while Σ and χ are, respectively, the odd and even parts of the normal-state self-energy.[2] For a particle-hole symmetric system, χ can be neglected (see Appendix C), and we do so henceforth. The self-energy can be compactly expressed as

$$\hat{\Sigma}(x,x') = -i\Sigma(x,x')\hat{\tau}_0 + \phi(x,x')\hat{\tau}_1. \tag{3}$$

For a translationally invariant system, the equations for Σ and real pairing amplitude ϕ can be written in momentum space

$$\hat{\Sigma}(k) = T\sum_{k'} V(k-k')\hat{\tau}_3\hat{\mathcal{G}}(k)\hat{\tau}_3. \tag{4}$$

The dependence of the self-energy on the magnitude of momentum is weak within Eliashberg theory, due to the separation of momentum scales between fermions and bosons (see below), allowing us to approximate

$$\hat{\Sigma}(k) \to \hat{\Sigma}_n(\mathbf{k}_F), \quad V(k-k') \to V_{n-n'}(\mathbf{k}_F - \mathbf{k}'_F). \tag{5}$$

Then

$$\hat{\Sigma}_n(\mathbf{k}_F) = -i\Sigma_n\hat{\tau}_0 + \phi_n\hat{\tau}_1. \tag{6}$$

Within the same approximation, the Nambu Green's function takes the form

$$\mathcal{G}(k) = \frac{-i(\epsilon_n + \Sigma_n)\hat{\tau}_0 - \xi_{\mathbf{k}}\hat{\tau}_3 - \phi_n\hat{\tau}_1}{(\epsilon_n+\Sigma_n)^2 + \xi_{\mathbf{k}}^2 + \phi_n^2}. \tag{7}$$

We can now directly perform the integral over $\xi_{\mathbf{k}}$ and obtain the Fermi-surface projected Eliashberg equations:

$$i\hat{\Sigma}_n(\mathbf{k}_F)\hat{\tau}_3 = \pi\nu T\sum_{n'}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_{n-n'}(|\mathbf{k}_F - \mathbf{k}'_F|)\hat{g}_{n'}(\mathbf{k}'_F) \tag{8}$$

with S_n the surface area of the n sphere and ν the density of states per spin at the Fermi surface (we keep dimension d arbitrary, but will later apply the results to $d = 2$). Here, $\hat{g}_n(\mathbf{k}_F)$ is the ξ-integrated Green's function weighted with $\hat{\tau}_3$ [15–17]:

$$\hat{g}_n(\mathbf{k}_F) \equiv \frac{i}{\pi}\int d\xi\ \hat{\tau}_3\hat{\mathcal{G}}_n(\xi,\mathbf{k}_F) \equiv g_n(\mathbf{k}_F)\hat{\tau}_3 + f_n(\mathbf{k}_F)\hat{\tau}_2. \tag{9}$$

For simplicity of presentation we consider s-wave superconductivity, in which case we obtain the isotropic Eliashberg equations

$$\tilde{\Sigma}_n = \epsilon_n + \pi\nu T\sum_{n'}V^{l=0}_{n-n'}\underbrace{\frac{\tilde{\Sigma}_{n'}}{\sqrt{\tilde{\Sigma}^2_{n'} + \phi^2_{n'}}}}_{g_n},$$

$$\phi_n = \pi\nu T\sum_{n'}V^{l=0}_{n-n'}\underbrace{\frac{\phi_{n'}}{\sqrt{\tilde{\Sigma}^2_{n'} + \phi^2_{n'}}}}_{f_n}, \tag{10}$$

where we have defined $\tilde{\Sigma}_n = \epsilon_n + \Sigma_n$ and the Fermi-surface average of the interaction

$$V^{l=0}_m \equiv \oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_m(|\mathbf{k}_F - \mathbf{k}'_F|). \tag{11}$$

It will also be convenient to define the related quantities[3]

$$Z_n \equiv 1 + \frac{\Sigma_n}{\epsilon_n} \equiv \frac{\tilde{\Sigma}_n}{\epsilon_n}, \quad \Delta_n \equiv \frac{\phi_n}{Z_n}, \tag{12}$$

which obey equations

$$\Delta_n = \pi\nu T\sum_{n'}V^{l=0}_{n-n'}\frac{\Delta_{n'} - \frac{\epsilon_{n'}}{\epsilon_n}\Delta_n}{\sqrt{\epsilon^2_{n'} + \Delta^2_{n'}}},$$

$$Z_n = 1 + \frac{\pi\nu T}{\epsilon_n}\sum_{n'}V^{l=0}_{n-n'}\frac{\epsilon_{n'}}{\sqrt{\epsilon^2_{n'} + \Delta^2_{n'}}}. \tag{13}$$

We see that there is only one self-consistent equation for Δ_n, while Z_n is a functional of Δ_n [11]. Equations (10) and (13) are the central equations that define the equilibrium theory.

There are two approximations used in derivation of the Eliashberg equations. First, Eq. (5) is valid when bosons are slow modes compared to fermions, i.e., for the same frequency, a typical bosonic momentum is much larger than a typical fermionic momentum. This approximation is justified when the fermion-boson coupling is much smaller than the Fermi energy. Second, vertex corrections are neglected. For fermions interacting by exchanging soft collective bosons, these corrections are, in most cases, $O(1)$ parameterwise, but are small numerically [10,12–14]. We emphasize in this regard that a typical frequency and a typical momentum of a soft boson in the self-energy diagram are such that $v_F q \gg \omega$. In this limit, a correction to the boson-fermion vertex in the self-energy diagram is related to the derivative of the self-energy over the momentum, which is at most logarithmic at a QCP, and is [10] weaker than the derivative over frequency, which has a power-law divergence at a QCP. This reasoning, however, does not hold for the corrections to the external current and density vertices, as for them the incoming momentum $q = 0$, while ω is finite. In this situation, vertex corrections are generally of order of the frequency derivative of the self-energy at $k = k_F$ and are large and singular near a QCP.

III. SUPERFLUID WEIGHT IN THE CONDUCTIVITY

Given a solution to Eqs. (10) and (13) we may calculate the conductivity in the superconducting state, from which the superfluid stiffness can be extracted. In this section, we outline the calculation of the superfluid stiffness from the conductivity within the Eliashberg paradigm, comparing the generic result with that for an exactly Galilean-invariant system.

The optical conductivity can be expressed in terms of the retarded velocity-velocity correlator J [18]. In the superconducting state $\sigma'(\omega)$ has a delta-function piece

$$\sigma'(\omega) = e^2\pi\delta(\omega)\,\text{Re}\,J^R(\omega,\mathbf{q}=0) + \cdots. \tag{14}$$

We then identify the superfluid stiffness via $D_s \equiv -\text{Re}\,J^R(\omega\to 0,\mathbf{q}=0)$. The low-energy velocity-velocity

[2] We have assumed time-reversal symmetry so that the normal-state self-energy of the two spin species are equal.

[3] Note that, strictly speaking, Z_n^{-1} is *not* the quasiparticle residue $Z_{\text{res}}^{-1} \equiv 1 + (\partial\Sigma/\partial\epsilon_n)_{\epsilon_n\to 0}$.

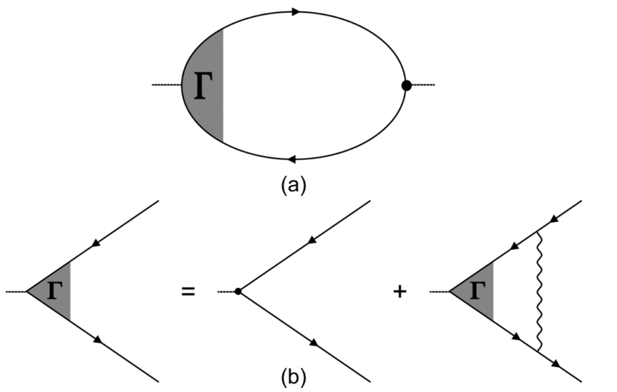

FIG. 2. Diagrams contributing to the optical conductivity: (a) Paramagnetic velocity-velocity bubble determining the weight of the δ function in the optical conductivity. The dot represents the bare current vertex, while the shaded vertex represents the renormalized current vertex. (b) Bethe-Salpeter equation for the renormalized current vertex. The dot is the bare current vertex $\mathbf{v}_F$, and the shaded vertex is the renormalized vertex $\mathbf{v}_F\hat{\Gamma}_n$. The thick lines are the full Green's functions of the theory, whereas the wavy line is the interaction $V(k-k')$.

correlator is expressed in terms of the Nambu Green's functions $\hat{\mathcal{G}}$ by the diagram in Fig. 2(a) as[4]

$$\hat{J}(Q) = -T\sum_k \mathrm{tr}[\boldsymbol{\gamma}_{\mathbf{k}}\hat{\mathcal{G}}_{K+Q}\hat{\boldsymbol{\Gamma}}_{K+Q,K}\hat{\mathcal{G}}_K], \tag{15}$$

where $Q = (i\Omega_m, 0)$ and $K = (i\epsilon_n, \mathbf{k})$. Here $\boldsymbol{\gamma}_{\mathbf{k}}$ is the bare velocity vertex and $\boldsymbol{\Gamma}_{K+Q,K}$ the renormalized velocity vertex within the ladder approximation [Fig. 2(b)] satisfying the Bethe-Salpeter equation

$$\hat{\boldsymbol{\Gamma}}_{K+Q,K} = \hat{\boldsymbol{\gamma}}_{\mathbf{k}} + T\sum_{K'} V_{K-K'}\hat{\tau}_3\hat{\mathcal{G}}_{K'+Q}\hat{\boldsymbol{\Gamma}}_{K'+Q,K'}\hat{\mathcal{G}}_{K'}\hat{\tau}_3. \tag{16}$$

The vertex correction is evaluated within the ladder approximation, consistent with the Eliashberg scheme for calculation of the self-energy.[5]Nonladder vertex correction diagrams, e.g., crossed diagrams, are suppressed to the same degree as vertex corrections to the self-energy. The bare velocity vertex is the conventional $\hat{\gamma}_{\mathbf{k}} = \nabla_{\mathbf{k}}\xi_{\mathbf{k}}\hat{\tau}_0$. Near the Fermi surface this is simply $\hat{\boldsymbol{\gamma}}_{\mathbf{k}} = \mathbf{v}_F\hat{\tau}_0$. For a rotationally symmetric interaction, in the $\mathbf{q}\to 0$ limit, the renormalized current vertex must also be proportional to $\mathbf{v}_F$, allowing us to split the renormalized vertex into a product of $\mathbf{v}_F$ and a rotational scalar, which only depends on frequency: $\hat{\boldsymbol{\Gamma}}_{K+Q,K} = \mathbf{v}_F\hat{\Gamma}_{n+m,n}$, where n and m stand for ϵ_n and Ω_m. The matrix $\hat{\Gamma}_{n+m,n}$ obeys the Bethe-Salpeter equation in the form

$$\hat{\Gamma}_{n+m,n} = \hat{\tau}_0 + \nu T\sum_{n'} V^{l=1}_{n-n'} \times \int d\xi_k \hat{\tau}_3\hat{\mathcal{G}}_{n'+m}(\xi_k)\hat{\Gamma}_{n'+m,n'}\hat{\mathcal{G}}_{n'}(\xi_k)\hat{\tau}_3, \tag{17}$$

where we have defined the generalized $l=1$ harmonic of the interaction [cf. Eq. (11)]

$$\left(\frac{v_F^2}{d}\delta_{ij}\right)V^{l=1}_m \equiv \oint_{\mathrm{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\mathrm{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}\mathbf{v}_{Fj}\mathbf{v}'_{Fj}V_m(|\mathbf{k}_F-\mathbf{k}'_F|). \tag{18}$$

Within the Eliashberg theory, particle-hole symmetry restricts solutions of Eq. (17) to be of the form

$$\Gamma = \Gamma^{(0)}\hat{\tau}_0 + \Gamma^{(1)}\hat{\tau}_1 \tag{19}$$

(see Appendix A for details). There is no coupling to the phase sector, and we are able to safely take the limit $\Omega_m\to 0$ (at $T=0$) without encountering any nonanalyticity. In terms of the renormalized vertex $\hat{\Gamma}_n = \hat{\Gamma}_{n,n}$ the general expression for the superfluid stiffness is

$$D_s = \frac{v_F^2}{d}\nu T\sum_n\left(\Pi^{00}_n\Gamma^{(0)}_n + \Pi^{01}_n\Gamma^{(1)}_n\right), \tag{20}$$

where

$$\Pi^{\mu\nu}_n \equiv \int d\xi\,\mathrm{tr}[\hat{\tau}^\mu\hat{\mathcal{G}}_n(\xi)\hat{\tau}^\nu\hat{\mathcal{G}}_n(\xi)]. \tag{21}$$

Explicitly evaluating the fermionic bubbles one finds

$$\Pi^{00}_n = \frac{2\pi\Delta_n^2}{Z_n\left(\epsilon_n^2+\Delta_n^2\right)^{3/2}}, \quad \Pi^{01}_n = \frac{i2\pi\Delta_n\epsilon_n}{Z_n\left(\epsilon_n^2+\Delta_n^2\right)^{3/2}} \tag{22}$$

so that

$$D_s = \frac{v_F^2}{d}2\pi\nu T\sum_n\frac{\Delta_n}{Z_n\left(\epsilon_n^2+\Delta_n^2\right)^{3/2}}\left(\Delta_n\Gamma^{(0)}_n + i\epsilon_n\Gamma^{(1)}_n\right). \tag{23}$$

Equation 23 is a general result for the superfluid stiffness within Eliashberg theory. At $T\to 0$, $T\sum_n \to \int d\epsilon_n/(2\pi)$.

In the Galilean-invariant case, there is a special Ward identity relating the fully renormalized current vertex to the self-energy as

$$\hat{\Gamma}_{n+m,n} \equiv 1 + i\frac{\hat{\Sigma}_{n+m}-\hat{\Sigma}_n}{\Omega_m}. \tag{24}$$

This relation is obtained from a combination of the Ward identity for conservation of momentum, and the identity $\mathbf{j} = e(\mathbf{k}/m)$ allowing the renormalized current vertex to be expressed in terms of the renormalized momentum vertex (see Appendix B 2 d). At $\Omega_m\to 0$, this reduces to

$$\hat{\Gamma}_{n+m,n} = \hat{\Gamma}_n \equiv 1 + i\frac{\partial\hat{\Sigma}_n}{\partial\epsilon_n}. \tag{25}$$

[4]Below we employ the computational scheme in which we first integrate over the dispersion ξ_k and then over frequency. In this scheme, the diamagnetic term is canceled by the high-energy contribution from the fermion bubble. For this reason we focus only on the low-energy paramagnetic velocity-velocity correlator.

[5]Within our treatment we do not consider the backaction of superconductivity on the bosonic action.

In components, $\Gamma^{(0)}_n = 1 + \frac{\partial \Sigma_n}{\partial \epsilon_n}$ and $\Gamma^{(1)}_n = i\frac{\partial \hat{\phi}_n}{\partial \epsilon_n}$ Using these formulas, we obtain

$$D_s = D_s^{\rm Gal} = \frac{v_F^2}{d} 2\pi \nu T \sum_n \frac{\Delta_n}{Z_n\left(\epsilon_n^2 + \Delta_n^2\right)^{3/2}} \times \left(\Delta_n\left[1 + \frac{\partial \Sigma_n}{\partial \epsilon_n}\right] - \epsilon_n \frac{\partial \phi_n}{\partial \epsilon_n}\right). \tag{26}$$

We now use Eq. (12) and rewrite

$$\frac{\partial \phi_n}{\partial \epsilon_n} = \frac{\Delta_n}{\epsilon_n}\left(1 + \frac{\partial \Sigma_n}{\partial \epsilon_n}\right) - \frac{\Delta_n}{\epsilon_n} Z_n + Z_n \frac{\partial \Delta_n}{\partial \epsilon_n}. \tag{27}$$

Inserting this into Eq. (26), we obtain

$$D_s^{\rm Gal} = \frac{2\pi \nu v_F^2}{d} T \sum_n \frac{\Delta_n}{\left(\epsilon_n^2 + \Delta_n^2\right)^{3/2}}\left(\Delta_n - \epsilon_n \frac{\partial \Delta_n}{\partial \epsilon_n}\right). \tag{28}$$

At $T = 0$, replacing $2\pi T \sum_n$ by $\int d\epsilon_n$, we obtain

$$D_s^{\rm Gal} = \frac{\nu v_F^2}{d} \int d\epsilon_n \frac{\Delta_n}{\left(\epsilon_n^2 + \Delta_n^2\right)^{3/2}}\left(\Delta_n - \epsilon_n \frac{\partial \Delta_n}{\partial \epsilon_n}\right). \tag{29}$$

The integrand is a total derivative:

$$D_s^{\rm Gal} = \frac{v_F^2}{d} \nu \int d\epsilon_n \frac{d}{d\epsilon_n}\left(\frac{\epsilon_n}{\sqrt{\epsilon_n^2 + \Delta_n^2}}\right). \tag{30}$$

Evaluating the integral we then obtain

$$D_s^{\rm Gal} = \frac{2\nu v_F^2}{d} = \frac{n}{m}. \tag{31}$$

This result implies that in an interacting Galilean-invariant system, the superfluid stiffness retains its bare value [19].

To understand how and when Eq. (23) differs from the Galilean-invariant result in a generic case, when there is no Ward identity relating $\hat{\Gamma}_{n+m,n}$ to the self-energy, we recall that there are Ward identities for a generic system of interacting fermions associated with global charge conservation and global spin conservation. The latter, for the vector of matrix spin vertex $\sigma^i_{\alpha\beta}\hat{\Gamma}^{(sp)}_{n+m,n}$, is of interest to us. Specifically, the matrix $\hat{\Gamma}^{(sp)}_{n+m,n}$ obeys the Bethe-Salpeter equation

$$\hat{\Gamma}^{(sp)}_{n+m,n} = 1 + \nu T \sum_{n'} V^{l=0}_{n-n'} \times \int d\xi_k \hat{\tau}_3 \hat{\mathcal{G}}_{n'+m}(\xi_k) \hat{\Gamma}^{(sp)}_{n'+m,n'} \hat{\mathcal{G}}_{n'}(\xi_k) \hat{\tau}_3 \tag{32}$$

whose solution is the same as for $\hat{\Gamma}_{n+m,n}$ in the Galilean-invariant case:

$$\hat{\Gamma}^{(sp)}_{n+m,n} \equiv 1 + i\frac{\hat{\Sigma}_{n+m} - \hat{\Sigma}_n}{\Omega_m} \tag{33}$$

(see Appendix B for details). In the limit $\Omega_m \to 0$, when $\hat{\Gamma}^{(sp)}_{n+m,n} = \hat{\Gamma}^{(sp)}_n$, this reduces to $\Gamma^{(sp),(0)}_n = 1 + \frac{\partial \Sigma_n}{\partial \epsilon_n}$ and $\Gamma^{(sp),(1)}_n = i\frac{\partial \phi_n}{\partial \epsilon_n}$. We emphasize that these relations hold for both the Galilean-invariant case and non-Galilean-invariant case. We also note that Eq. (33) holds only for the spin vertex. For the charge vertex, the equation is somewhat different (see Appendix B).

Comparing with Eq. (17) for the current vertex $\hat{\Gamma}_{n+m,n}$, we see that the only difference in these equations is that the equation for $\hat{\Gamma}^{(sp)}$ involves the $l = 0$ harmonic while the one for $\hat{\Gamma}$ involves the $l = 1$ harmonic. As a consequence, the superfluid stiffness in a non-Galilean system retains its free-fermion value n/m when the harmonics $V_n^{l=0,1}$ are *equal for all frequencies*. We call such a system *effectively Galilean invariant*.

There is one fundamental difference between an *effectively Galilean-invariant* and a truly Galilean-invariant system. In the first, the cancellation between fermionic Z and vertex correction occurs between terms involving only quasiparticles in the vicinity of the Fermi surface. In a generic Galilean-invariant system, the special Ward identity establishes the relation between properties of the system near and far away from the surface. So while $D_s = n/m$ in an *effectively Galilean-invariant* system, the reason why interaction-driven corrections cancel out is in general quite different from that in a truly Galilean-invariant system.

We now investigate in more detail how the vertex corrections restore the Galilean value of D_s when $V_n^{l=0} = V_n^{l=1}$. The role of the vertex corrections in the effectively Galilean-invariant case can be elucidated by the following three cases:

(a) A frequency-independent self-energy (such as from an instantaneous interaction) and $\Delta \sim$ const. This is the BCS case.
(b) A matrix self-energy of the form $\Sigma_n \sim \epsilon_n$, $\Delta \sim$ const. This is the case of superconductivity out of a Fermi liquid away from a QCP.
(c) A matrix self-energy in which both Δ_n and Z_n are strongly frequency dependent. This is the case of superconductivity out of a NFL at a QCP.

For case (a), $Z = 1$ and both vertex corrections $\Gamma^{(0)} - 1$ and $\Gamma^{(1)}$ vanish, giving

$$D_s = D_s^{(0)} = \frac{v_F^2}{d} 2\pi \nu T \sum_n \frac{\Delta^2}{\left(\epsilon_n^2 + \Delta^2\right)^{3/2}} \xrightarrow{T=0} \frac{n}{m}. \tag{34}$$

Indeed, one can easily verify that $Z = \Gamma^{(0)} = 1$ and $\Gamma^{(1)} = 0$ is the solution of Eqs. (13) and (17) for any instantaneous interaction, and thus all BCS-like local interactions are effectively Galilean invariant. For the vertex correction, this follows from the fact that all components of $\int d\epsilon_{n'} (\hat{G}_{n',n'})^2$ vanish, either because the integrand is odd in $\epsilon_{n'}$ or because it can be reexpressed such that the both fermionic poles lie in the same half-plane, and the integral vanishes after closing the integration contour in the other half-plane.

For case (b), the expression for D_s is

$$D_s = \frac{v_F^2}{d} 2\pi \nu T \sum_n \frac{\Delta^2}{\left(\epsilon_n^2 + \Delta^2\right)^{3/2}}\left(\frac{1 + \partial\Sigma_n/\partial\epsilon_n}{Z}\right). \tag{35}$$

The constant factor in the last bracket cancels out because for $\Sigma_n \propto \epsilon_n$, $Z = 1 + \Sigma_n/\epsilon_n = 1 + \partial\Sigma_n/\partial\epsilon_n$.

For case (c), $\Sigma_n/\epsilon_n \neq (\partial\Sigma_n/\partial\epsilon_n)$ and thus Z and $1 + \partial\Sigma_n/\partial\epsilon_n$ no longer cancel. One needs to include the frequency derivative of the pairing vertex on *equal footing* with $\partial\Sigma_n/\partial\epsilon_n$ to get the cancellation of Z and reproduce $D_s = n/m$.

We also note that for all cases (a)–(c) the relation $D_s = n/m$ holds independent of the fermionic dispersion. This is, of course, only approximately true as in linearizing about the Fermi surface we have neglected corrections of order Δ/E_F to D_s. These corrections cancel out only in the truly Galilean-invariant case, where the relation $D_s = n/m$ is exact.

We now consider how the stiffness gets modified when $V^{l=0}_{n-n'} \neq V^{l=1}_{n-n'}$. We define $V^{l=1}_{n-n'} = V^{l=0}_{n-n'} + \delta V_m$ and $\hat\Gamma_n = \hat\Gamma^{(sp)}_n + \delta\hat\Gamma_n$ (in the limit when external bosonic frequency $\Omega_m \to 0$). The vertex $\delta\hat\Gamma_n$ obeys the modified Bethe-Salpeter equation

$$\delta\hat\Gamma_n = \nu T\sum_{n'}\left(V^{l=0}_{n-n'}\int d\xi\ \hat\tau_3\hat{\mathcal{G}}_{k'}\delta\hat\Gamma_{n'}\hat{\mathcal{G}}_{k'}\hat\tau_3 + \delta V_{n-n'}\int d\xi\ \hat\tau_3\hat{\mathcal{G}}_{k'}\hat\Gamma^{(sp)}_{n'}\hat{\mathcal{G}}_{k'}\hat\tau_3\right). \tag{36}$$

Splitting $\delta\hat\Gamma_n$ into components we obtain for the stiffness

$$D_s = D^{\rm Gal}_s + \frac{v_F^2}{d}2\pi\nu T\sum_n \frac{\Delta_n}{Z_n(\epsilon_n^2+\Delta_n^2)^{3/2}} \times \left(\Delta_n\delta\Gamma^{(0)}_n + i\epsilon_n\delta\Gamma^{(1)}_n\right). \tag{37}$$

If the difference between $V^{l=1}_{n-n'}$ and $V^{l=0}_{n-n'}$ is small for all relevant frequencies, this will be a small correction of order δV. This is the case for interactions which are dominated by small-angle scattering, as then scattered particles do not distinguish between different harmonics. That small-angle scattering leads to approximate relations between the renormalized current and spin vertices has previously been appreciated in the normal state [20,21].

As an example, consider an interaction mediated by a propagating boson with mass ω_D and dispersion cq:

$$V_m(q) = \frac{g^2\chi_0}{\Omega_m^2+\omega_D^2+c^2q^2}. \tag{38}$$

For fermions on the Fermi surface $q^2 = |\mathbf{k}_F - \mathbf{k}'_F|^2 = 2k_F^2(1-\cos\theta)$, where θ is the angle between $\mathbf{k}_F$ and $\mathbf{k}'_F$. We can then write

$$V_m(q) = V_m(\theta) = \frac{g^2\chi_0}{2c^2k_F^2}\frac{1}{a_m-\cos\theta} \tag{39}$$

with $a_m = 1 + (\Omega_m^2+\omega_D^2)/(2c^2k_F^2)$ and express

$$V^{l=0}_m = \frac{g^2\chi_0}{2c^2k_F^2}\oint_{\rm FS}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\rm FS}\frac{d\mathbf{k}'_F}{S_{d-1}}\frac{1}{a_m-\cos(\theta-\theta')} \tag{40}$$

and

$$V^{l=1}_m = d\frac{g^2\chi_0}{2c^2k_F^2}\oint_{\rm FS}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\rm FS}\frac{d\mathbf{k}'_F}{S_{d-1}}\frac{\cos\theta\ \cos\theta'}{a_m-\cos(\theta-\theta')}. \tag{41}$$

In $d=2$ we have

$$V^{l=0}_m = \frac{g^2\chi_0}{2c^2k_F^2}\frac{1}{\sqrt{a_m^2-1}},\quad V^{l=1}_m = \frac{g^2\chi_0}{2c^2k_F^2}\left(\frac{a_m}{\sqrt{a_m^2-1}}-1\right) \tag{42}$$

so

$$\delta V_m = \frac{g^2\chi_0}{2c^2k_F^2}\left(\sqrt{\frac{a_m-1}{a_m+1}}-1\right) = -\frac{g^2\chi_0}{2c^2k_F^2}\left(1-\frac{1}{\sqrt{1+\frac{4c^2k_F^2}{\Omega_m^2+\omega_D^2}}}\right) \tag{43}$$

and

$$\delta V_m/V^{l=0}_m = \frac{-2}{1+\sqrt{\frac{a_m-1}{a_m+1}}} = \frac{-1}{1+\sqrt{1+\frac{4c^2k_F^2}{\Omega_m^2+\omega_D^2}}}. \tag{44}$$

The relevant frequencies Ω_m are of order Δ_m. The characteristic scale for the latter is the gap function at zero frequency at $T=0$, which we label simply by Δ. We see that $\delta V_m/V^{l=0}_m$ is small when the velocity c is large enough such that $ck_F \gg (\Delta^2+\omega_D^2)^{1/2}$. This is the limit of small-angle scattering. We furthermore note that in this limit, δV_m is determined by scattering to large angles and remains nonsingular at a QCP even if we set $\Omega_m \to 0$. As a consequence, $\delta D_s = D_s - D^{\rm Gal}_s$ also remains nonsingular. For a Galilean-invariant system, this nonsingular δD_s cancels exactly with contributions coming from interactions with noncritical bosons.

A near cancellation between Γ and Z factors in D_s for small-angle scattering θ is similar to the near cancellation between self-energy and Maki-Thompson contributions to optical conductivity in the normal state, in a similar situation of small momentum scattering (these are the insertions of self-energy and vertex corrections into the conductivity bubble) [22–24]. Like there, in our case the net result for the difference between D_s and $D^{\rm Gal}_s$ contains the additional factor $1-\cos\theta \approx \theta^2/2$ compared to what one would get by including only Z or only Γ. Furthermore, for the truly Galilean-invariant case, the already reduced contribution to the optical conductivity cancels out by additional, Aslamazov-Larkin–type diagrams [24–26]. The same happens in our case: for a Galilean-invariant system the already reduced $D_s - D^{\rm Gal}_s$ is canceled out by other contributions to $\delta(\omega)$, term in the conductivity, likely also Aslamazov-Larkin–type contributions. It is also possible that for a convex Fermi surface there is an additional reduction of $D_s - D^{\rm Gal}_s$ when all contributions to the $\delta(\omega)$ term in the conductivity are added together [27,28]. We do not dwell on this issue here.

At $ck_F \sim \Delta$, $\delta V_m/V^{l=0}_m = O(1)$, i.e., D_s differs from $D^{\rm Gal}_s$. A particularly extreme example where cancellation is absent is the case of $c=0$, when $V^{l=1}=0$. This case describes, in particular, the pairing mediated by a soft Einstein phonon. At $T=0$, we have

$$D_s = D^{\rm Gal}_s\int d\epsilon\frac{\Delta^2(\epsilon)}{Z(\epsilon)[\epsilon^2+\Delta(\epsilon)^2]^{3/2}} \sim D^{\rm Gal}_s/Z(\Delta). \tag{45}$$

Near a QCP, $Z(\Delta)$ is large [29] and D_s is substantially smaller than $D^{\rm Gal}$. For the phonon pairing, $D^{\rm Gal}_s \sim E_F$, while $Z(\Delta) \sim \bar g^2/(\omega_D\Delta)$, where $\bar g = (g^2\chi_0 m)^{1/2}$. At small ω_D, $Z(\Delta) \gg 1$. The actual stiffness is $D_s \sim E_F\Delta\omega_D/\bar g^2$. Eliashberg theory for electron-phonon interactions is valid as long as Eliashberg parameter $\lambda_E = \bar{g}^2/\omega_D E_F$ remains small. Using that at small ω_D, T_c and Δ are both of order $\bar{g}$ [30,31], the stiffness can be reexpressed as $D_s \sim T_c/\lambda_E$. We see that, as long as Eliashberg theory is under control, the dressed stiffness remains larger than T_c. In this situation, phase fluctuations are weak and Eliashberg T_c nearly coincides with the actual T_c. However, at the boundary of applicability of Eliashberg theory, D_s becomes comparable to T_c and phase fluctuations cannot be neglected.

IV. SUPERFLUID STIFFNESS IN THE ELIASHBERG-LUTTINGER-WARD DESCRIPTION

While the superfluid stiffness appears naturally as a transport property in the conductivity, it can also be obtained directly from the thermodynamic properties of the system. In particular, it parametrizes the free-energy cost associated with twisting the phase boundary conditions of the superconducting state [2,32,33]. In this section, we obtain the superfluid stiffness directly from the LW variational free energy for the Green's function in the Nambu representation. Our particular interest here is to understand how the renormalization of the current vertex appears in this approach. We show that it emerges naturally already within the one-loop approximation because of the change of the self-energy due to the phase twist. We argue that the emergence of corrections to the current vertex is a general feature of the linear response in the LW formalism, reflecting the conserving nature of the approach.

Luttinger and Ward showed that a many-body system can be described by the variational free energy [34]

$$\beta\Omega[\hat{\mathcal{G}}] = -\mathrm{Tr}\,\ln(-\hat{\mathcal{G}}^{-1}) - \mathrm{Tr}\big[\hat{G}_0^{-1}\hat{\mathcal{G}}\big] + \Phi[\hat{\mathcal{G}}], \tag{46}$$

where $\mathcal{G}$, the fully dressed Green's function, is to be minimized over, and $\Phi[\hat{\mathcal{G}}]$ is the LW functional, which can be obtained diagrammatically as the sum of all two-particle irreducible vacuum skeleton diagrams. This description has the following properties:

(i) The equilibrium Green's function $\mathcal{G}_{\mathrm{eq}}$ minimizes Ω.

(ii) The self-energy is the functional derivative of the LW functional Φ, $\hat{\Sigma} = \delta\Phi/\delta\hat{\mathcal{G}}$.

(iii) The minimal value of Ω is the equilibrium free energy, $F_{\mathrm{eq}} = \Omega[\mathcal{G}_{\mathrm{eq}}]$.

The variational free energy Ω is also known as the Baym-Kadanoff functional [35,36] and is very closely related to the two-particle irreducible effective action, in that $\Gamma^{(2\mathrm{PI})} = \beta\Omega_{\mathrm{LW}}$ on the Matsubara axis [37].

Eliashberg theory corresponds to the one-loop approximation for the diagrammatic series for the LW functional $\Phi[\hat{\mathcal{G}}]$ [38–40]. Within the one-loop approximation,

$$\Phi[\hat{\mathcal{G}}] = \frac{1}{2}\int dx\,dy\,V(x-y)\mathrm{Tr}[\hat{\tau}_3\hat{\mathcal{G}}(x,y)\hat{\tau}_3\hat{\mathcal{G}}(y,x)]. \tag{47}$$

Minimizing the free energy leads to the Eliashberg equations (4) for the matrix self-energy.

The superfluid stiffness of a superconductor can be obtained by considering the energy cost associated with phase twists of the ground state. Since the generator of the broken U(1) symmetry is simply $\hat{\tau}_3$ in the Nambu basis we consider the free energy of the superconducting state as a function of a phase twist

$$\Psi(x) \to e^{i\mathbf{Q}\cdot\mathbf{r}\hat{\tau}_3}\Psi(x) \tag{48}$$

imposed on the Nambu spinors. In terms of the LW variational free energy we define a modified functional

$$\Omega_Q[\hat{\mathcal{G}}(x-x')] \equiv \Omega[e^{i\mathbf{Q}\cdot\mathbf{r}\hat{\tau}_3}\hat{\mathcal{G}}(x-x')e^{-i\mathbf{Q}\cdot\mathbf{r}'\hat{\tau}_3}] \tag{49}$$

to be minimized over Green's functions with self-energies of the form[6]

$$\hat{\Sigma}_n(\mathbf{k}_F) = -i\Sigma_n(\mathbf{k}_F)\hat{\tau}_0 + \phi_n(\mathbf{k}_F)\hat{\tau}_1. \tag{50}$$

The superfluid stiffness, twice the coefficient of the Q^2 term in $F_Q \equiv \Omega_Q[\hat{\mathcal{G}}_{\mathrm{eq}}(x-x')]$, can then be obtained as

$$D_s \equiv \left.\frac{d^2}{dQ^2}F_Q\right|_{Q\to 0}. \tag{51}$$

Functionally, the relation between Ω and Ω_Q is that we replace

$$\hat{G}_0^{-1} \to \hat{G}_0^{-1} - \mathbf{v}_F\cdot\mathbf{Q} - \frac{Q^2}{2m}\hat{\tau}_3 + \cdots \tag{52}$$

in the LW variational free energy. Within our evaluation scheme (performing the integration over ξ_k first) the diamagnetic term $\propto Q^2\hat{\tau}_3$ can be neglected (see Appendix D). The Q^2 in the action is then entirely due to the source term $\mathbf{v}_F\cdot\mathbf{Q}$. Noting this, we can straightforwardly evaluate the derivatives in Eq. (51) using the saddle-point equation and obtain

$$\begin{aligned} D_s &= -T\sum_k \mathrm{tr}\left(\frac{d\hat{G}_0^{-1}}{dQ}\frac{d\hat{\mathcal{G}}}{dQ}\right) \\ &= -i\pi\nu T\sum_n \oint_{\mathrm{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\mathbf{v}_F\mathrm{tr}\left(\hat{\tau}_3\frac{d\hat{g}}{dQ}\right), \end{aligned} \tag{53}$$

where in the second equality we have used the definition of the ξ_k-integrated Green's function (9). We now define the first-order variation of $\hat{g}$ due to $\mathbf{Q}$ via

$$\hat{g}_n(\mathbf{k}_F) = \hat{\bar{g}}_n + i\mathbf{v}_F\cdot\mathbf{Q}\delta\hat{g}_n + \cdots \tag{54}$$

in terms of the $Q=0$ solution $\hat{\bar{g}}_n$. This allows us to compactly express the superfluid stiffness as

$$D_s = \frac{v_F^2}{d}\pi\nu T\sum_n \mathrm{tr}(\hat{\tau}_3\delta\hat{g}_n) = \frac{v_F^2}{d}2\pi\nu T\sum_n \delta g_n. \tag{55}$$

What remains is to calculate $\delta\hat{g}_n$. We start by noting that the integration over the dispersion ξ_k can be performed for arbitrary $\hat{\Sigma}_n(\mathbf{k}_F)$ and yields (see Appendix C)

$$\hat{g}_n(\mathbf{k}_F) = \frac{\Upsilon_n(\mathbf{k}_F)\hat{\tau}_3 + \phi_n(\mathbf{k}_F)\hat{\tau}_2}{\sqrt{\Upsilon_n(\mathbf{k}_F)^2 + \phi_n(\mathbf{k}_F)^2}}, \tag{56}$$

where we have defined $\Upsilon = \varpi + \Sigma_n$ and $\varpi = \epsilon_n + i\mathbf{v}_F\cdot\mathbf{Q}$. We now introduce, by analogy with Eq. (13),

$$Z_n(\mathbf{k}_F) \equiv \frac{\Upsilon_n(\mathbf{k}_F)}{\varpi_n(\mathbf{k}_F)}, \quad \Delta_n(\mathbf{k}_F) \equiv \frac{\phi_n(\mathbf{k}_F)}{Z_n(\mathbf{k}_F)}. \tag{57}$$

[6]This is what makes this functional correspond to twisted boundary conditions.

Using these notations, we express $\hat{g}_n$ in a form independent of Z as

$$\hat{g}_n(\mathbf{k}_F) = \frac{\varpi_n(\mathbf{k}_F)\hat{\tau}_3 + \Delta_n(\mathbf{k}_F)\hat{\tau}_2}{\sqrt{\varpi_n(\mathbf{k}_F)^2 + \Delta_n(\mathbf{k}_F)^2}}. \tag{58}$$

We see that the gap equation separates into a self-consistency condition for Δ and functional definition of Z in terms of Δ, as in the isotropic case. It is now straightforward to obtain the first-order correction to the ξ_k-integrated Green's function δg_n by defining $\Delta_n(\mathbf{k}_F) = \bar{\Delta}_n + i\mathbf{v}_F \cdot \mathbf{Q}\delta\Delta_n + \cdots$, with $\bar{\Delta}_n$ the equilibrium solution. Expanding Eq. (58) to first order in $\mathbf{v}_F \cdot \mathbf{Q}$, we obtain

$$\delta g_n = \frac{\bar{\Delta}_n}{\left(\epsilon_n^2 + \bar{\Delta}_n^2\right)^{3/2}}(\bar{\Delta}_n - \epsilon_n \delta\Delta_n) \tag{59}$$

and therefore

$$D_s = \frac{v_F^2}{d} 2\pi\nu T \sum_n \frac{\bar{\Delta}_n}{\left(\epsilon_n^2 + \bar{\Delta}_n^2\right)^{3/2}}(\bar{\Delta}_n - \epsilon_n\delta\Delta_n). \tag{60}$$

Note the similarity to Eq. (23).

We now make explicit the relation between the variation of the self-energy due to $\mathbf{Q}$ and the renormalized current vertex $\hat{\Gamma}_n$ which appears in Eq. (23) in the previous section. Similar to Eq. (27), we can use Eq. (57) to reexpress $\delta\Delta$ in terms of $\delta\Sigma$ and $\delta\phi$ via

$$\delta\phi_n = \frac{\bar{\Delta}_n}{\epsilon_n}(1 + \delta\Sigma_n) - \frac{\bar{\Delta}_n}{\epsilon_n}\bar{Z}_n + \bar{Z}_n\delta\Delta_n. \tag{61}$$

Using Eq. (61) we rewrite the superfluid stiffness as

$$D_s = \frac{v_F^2}{d} 2\pi\nu T \sum_n \frac{\bar{\Delta}_n}{\bar{Z}_n\left(\epsilon_n^2 + \bar{\Delta}_n^2\right)^{3/2}}(\bar{\Delta}_n + \bar{\Delta}_n\delta\Sigma_n - \epsilon_n\delta\phi_n). \tag{62}$$

We now expand the Nambu self-energy, Eq. (8), to first order in $\mathbf{v}_F \cdot \mathbf{Q}$ as

$$\hat{\Sigma}_n(\mathbf{k}_F) = \hat{\bar{\Sigma}}_n + i\mathbf{v}_F \cdot \mathbf{Q}\delta\hat{\Sigma}_n + \cdots . \tag{63}$$

Equating the first-order terms using Eqs. (54) and (56) and splitting $\delta\hat{\Sigma}_n$ in components as

$$\delta\hat{\Sigma}_n = -i\delta\Sigma_n\hat{\tau}_0 + \delta\phi_n\hat{\tau}_1, \tag{64}$$

we find

$$i\delta\hat{\Sigma}_n\hat{\tau}_3 = \pi\nu T \sum_{n'} V_{n-n'}^{l=1}\left(\frac{\partial\hat{g}_{n'}}{\partial\Upsilon_{n'}}(1 + \delta\Sigma_{n'}) + \frac{\partial\hat{g}_{n'}}{\partial\phi_{n'}}\delta\phi_{n'}\right). \tag{65}$$

The definition of the ξ_k-integrated Green's function (9) implies the identities[7]

$$\frac{\partial\hat{g}}{\partial\Upsilon} = \frac{1}{\pi}\int d\xi\ \hat{\tau}_3\hat{\mathcal{G}}(\xi)\hat{\mathcal{G}}(\xi),$$
$$\frac{\partial\hat{g}}{\partial\phi} = \frac{i}{\pi}\int d\xi\ \hat{\tau}_3\hat{\mathcal{G}}_n(\xi)\hat{\tau}_1\hat{\mathcal{G}}_n(\xi). \tag{66}$$

[7]Order of limits does not matter here as this is a gapped state.

This allows us to rewrite Eq. (65) as

$$1 + i\delta\hat{\Sigma}_n = 1 + \nu T \sum_{n'} V_{n-n'}^{l=1} \times \int d\xi\ \hat{\tau}_3\hat{\mathcal{G}}_{n'}(\xi)[1 + i\delta\hat{\Sigma}_{n'}]\hat{\mathcal{G}}_{n'}(\xi)\hat{\tau}_3. \tag{67}$$

Equation 67 is identical to Eq. (17) with the identification $\hat{\Gamma}_n \equiv 1 + i\delta\hat{\Sigma}_n$, and we may rewrite Eq. (62) as

$$D_s = \frac{v_F^2}{d} 2\pi\nu T \sum_n \frac{\bar{\Delta}_n}{\bar{Z}_n\left(\epsilon_n^2 + \bar{\Delta}_n^2\right)^{3/2}} \times \left(\bar{\Delta}_n\Gamma_n^0 + i\epsilon_n\Gamma_n^1\right) \tag{68}$$

in agreement with Eq. (23). All the results of Sec. III then follow.

We see from the above analysis that in the LW formalism the correction to the current vertex is equivalent to the first-order change in the self-energy due to the phase twist $\exp(i\mathbf{Q}\cdot\mathbf{r})$ (up to a constant factor). This is a general feature of linear response in the LW formalism, and it reflects the conserving nature of the LW (and Baym-Kadanoff) approach.

V. HUBBARD-STRATONOVICH DESCRIPTION AND THE PHASE ACTION

As a final perspective, we now employ the HS formulation of Eliashberg theory [41,42] to derive the superfluid stiffness in the context of the phase action. We start by presenting and commenting on the final result and then provide its derivation. The action for the phase mode $\theta(i\Omega_m, \mathbf{q})$ to order q^2 is given by

$$S_\theta = \frac{1}{2}\sum_q \theta_{-q}\left(2\nu\Omega_m^2 + D_s q^2\right)\theta_q. \tag{69}$$

This form is identical to the BCS phase action, however, D_s is the fully renormalized stiffness. Note that the prefactor of the Ω_m^2 term remains the same as in BCS theory. We argue below that this term comes from high energies, where fermions are free quasiparticles.

We now derive Eq. (69) starting with the HS decoupling of Eq. (1) in the Nambu basis

$$S_{\text{bos}}[\hat{\Sigma}] = -\frac{1}{2}\int d\tau\, d\tau' d\mathbf{x}\frac{1}{V(x - x')}\text{tr}[\hat{\Sigma}(x, x')\hat{\tau}_3\hat{\Sigma}(x', x)\hat{\tau}_3] - \text{Tr}\ \ln[-\beta(\hat{G}^{-1} - \hat{\Sigma})]. \tag{70}$$

One can verify that the saddle-point equations of Eq. (70) are the Eliashberg equations (4). The phase mode θ enters as parametrization of the HS field

$$\hat{\Sigma}_{\tau\tau'}(\mathbf{x}) = e^{i\theta_\tau(\mathbf{x})\hat{\tau}_3}\hat{\Sigma}_L(x - x')e^{-i\theta_{\tau'}(\mathbf{x})\hat{\tau}_3}, \tag{71}$$

where $\hat{\Sigma}_L$ contains only longitudinal fluctuations around the saddle-point solution:

$$\hat{\Sigma}_L = \hat{\Sigma}_{\text{sp}} - i\delta\Sigma\hat{\tau}_0 + \delta\phi\hat{\tau}_1 \equiv \hat{\Sigma}_{\text{sp}} + \delta\hat{\Sigma}_L. \tag{72}$$

We compute the phase action by making use of the gauge invariance of the theory [18]. Let us define $\hat{U} = e^{i\theta\hat{\tau}_3}$ such that $\hat{\Sigma} = \hat{U}\hat{\Sigma}_L\hat{U}^\dagger$. The first term of the bosonic action (70) is invariant under application of $\hat{U}$. For the trace-logarithm term, we may use the cyclic property of the trace to rewrite it in terms of $\hat{\Sigma}_L$ and the gauge-transformed quantity $\hat{U}^\dagger \mathcal{G}^{-1}\hat{U}$. One can verify that this amounts to replacing the partial derivatives in the inverse Green's function with covariant derivatives

$$\partial_\tau \to D_\tau = \partial_\tau + i\partial_\tau\theta\hat{\tau}_3, \quad \nabla \to \mathbf{D} = \nabla + i\nabla\theta\hat{\tau}_3. \tag{73}$$

The action is then compactly written as

$$S_{\text{bos}} = S_{\text{sp}} + S_{\text{HS}}[\delta\hat{\Sigma}_L] - \text{Tr}\,\ln\{-\beta(\hat{\mathcal{G}}^{-1}[D_\tau, \mathbf{D}] - \delta\hat{\Sigma}_L)\}, \tag{74}$$

where[8]

$$\hat{\mathcal{G}}^{-1}[D_\tau, \mathbf{D}] \approx \hat{\mathcal{G}}_{\text{sp}}^{-1} - (i\partial_\tau\theta\hat{\tau}_3 + \mathbf{v}_F\cdot\nabla\theta), \tag{75}$$

and $\hat{\mathcal{G}}_{\text{sp}}^{-1}$ is the inverse of the saddle-point Green's function (the solution of the Eliashberg equations). Performing a second-order expansion in derivatives and in longitudinal fluctuations leads to the Gaussian action

$$S_{\text{bos}} = S_\theta^{(b)} + S_L^{(b)} + S_c, \tag{76}$$

where $S_\theta^{(b)}$ is the bare phase action, $S_L^{(b)}$ the bare action for the longitudinal mode, and S_c the coupling term. The bare phase action is

$$S_\theta^{(b)} = \frac{1}{2}\sum_q \left(\kappa^{(b)}\Omega_m^2 + D_s^{(b)}q^2\right)|\theta_q|^2, \tag{77}$$

where the constants $\kappa^{(b)}$ and $D_s^{(b)}$ are given by

$$\kappa^{(b)} = -T\sum_k \text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k)\hat{\tau}_3]^2, \;\; D_s^{(b)} = \frac{v_F^2}{d}\nu\sum_k \text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k)]^2, \tag{78}$$

and we recall that $k = (\xi_k, \epsilon_n)$ and $\sum_k = T\sum_n\int d\xi_k$.

In explicit form, we have for $\kappa^{(b)}$

$$\begin{aligned}\kappa^{(b)} &= -\nu\int\frac{d\epsilon_n}{2\pi}\int_{-\Lambda}^{\Lambda} d\xi_k \frac{\text{tr}[(-i\tilde{\Sigma}_n - \xi_k\hat{\tau}_3 - \phi_n\hat{\tau}_1)\hat{\tau}_3]^2}{\left(\tilde{\Sigma}_n^2 + \phi_n^2 + \xi_k^2\right)^2} \\ &= 2\nu\int\frac{d\epsilon_n}{2\pi}\int d\xi_k\frac{\tilde{\Sigma}_n^2 - \xi_k^2 + \phi_n^2}{\left(\tilde{\Sigma}_n^2 + \phi_n^2 + \xi_m^2\right)^2}.\end{aligned} \tag{79}$$

It is natural to do the integration over ξ_k first as this integral can be evaluated exactly. This integration, however, should be done with care as the full integral over ξ_k and ϵ_n is not uniformly convergent. To regulate the integral, we introduce a UV cutoff Λ, and then take it to infinity at the end of the calculation (see Ref. [43] and Appendixes B and E). The integration is dominated by energies $|\xi_k| \sim \Lambda$, for which fermions are essentially free particles, and yields $\kappa^{(b)} = 2\nu$.

For $D_s^{(b)}$ we have

$$\begin{aligned}D_s^{(b)} &= \frac{v_F^2}{d}\nu\int\frac{d\epsilon_n}{2\pi}\int d\xi_k\frac{\text{tr}[(-i\tilde{\Sigma}_n - \xi_k\hat{\tau}_3 - \phi_n\hat{\tau}_1)]^2}{\left(\tilde{\Sigma}_n{}^2 + \phi_n^2 + \xi_k^2\right)^2} \\ &= 2\frac{v_F^2}{d}\nu\int\frac{d\epsilon_n}{2\pi}\int d\xi_k\frac{\xi_k^2 + \phi_n^2 - \tilde{\Sigma}_n^2}{\left(\tilde{\Sigma}_n^2 + \phi_n^2 + \xi_m^2\right)^2}.\end{aligned} \tag{80}$$

Here we replaced $\int_{-\Lambda}^{\Lambda} d\xi_k$ by $\int_{-\infty}^{\infty} d\xi_k$ as the integral over $|\xi_k| > \Lambda$ cancels out with the diamagnetic contribution. Evaluating the integral over ξ, we obtain

$$D_s^{(b)} = \frac{v_F^2}{d}\nu\int d\epsilon_n\frac{\Delta_n^2}{Z_n\left(\Delta_n^2 + \epsilon_n^2\right)^{3/2}}. \tag{81}$$

This is the expression for the stiffness without vertex corrections.

Next, the longitudinal action is

$$\begin{aligned}S_L &= -\frac{1}{2}T^3\sum_{kk'q} V_{k-k'}^{-1}\text{tr}[\delta\hat{\Sigma}_{k,k+q}\hat{\tau}_3\delta\hat{\Sigma}_{k'+q,k'}\hat{\tau}_3] \\ &\quad + \frac{1}{2}T^2\sum_{kk'}\text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k)\delta\hat{\Sigma}_{kk'}\hat{\mathcal{G}}_{\text{sp}}(k')\delta\hat{\Sigma}_{k'k}] \\ &\equiv \frac{1}{2}T^3\sum_{kk'q}\delta\Sigma_{k,k+q}^{\mu}\left[T_{kk'q}^{-1}\right]^{\mu\nu}\delta\Sigma_{k'+q,k'}^{\nu},\end{aligned} \tag{82}$$

and the coupling term is

$$\begin{aligned}S_c &= T\sum_{kq}\theta_q(\Omega_m\text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k+q)\hat{\tau}^3\hat{\mathcal{G}}_{\text{sp}}(k)\delta\hat{\Sigma}_{k,k+q}] + i\mathbf{v}_F \\ &\quad\cdot\mathbf{q}\text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k+q)\hat{\mathcal{G}}_{\text{sp}}(k)\delta\hat{\Sigma}_{k,k+q}]) \\ &\equiv T\sum_{kq}\theta_q\delta\Sigma_{k,k+q}^{\mu}\left(\Omega_m(C^\omega)_{k,k+q}^{\mu} + i\mathbf{v}_F\cdot\mathbf{q}\cdot(C^q)_{k,k+q}^{\mu}\right),\end{aligned} \tag{83}$$

where we have expanded $\delta\hat{\Sigma}$ in Pauli matrices $\delta\hat{\Sigma} = \sum_\mu\delta\Sigma^\mu\hat{\tau}^\mu$ and introduced the couplings

$$\begin{aligned}(C^\omega)_{k,k+q}^{\mu} &= \frac{1}{2}\text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k+q)\hat{\tau}^3\hat{\mathcal{G}}_{\text{sp}}(k)\hat{\tau}^\mu], \\ (C^q)_{k,k+q}^{\mu} &= \frac{1}{2}\text{tr}[\hat{\mathcal{G}}_{\text{sp}}(k+q)\hat{\mathcal{G}}_{\text{sp}}(k)\hat{\tau}^\mu].\end{aligned} \tag{84}$$

Upon integrating out the longitudinal modes $\delta\hat{\Sigma}_L$ the effective phase action can be written in terms of bare constants $\kappa^{(b)}$, $D_s^{(b)}$ and vertex corrections $\delta\kappa$, δD_s:

$$\begin{aligned}S_\theta &= \frac{1}{2}\sum_q\left([\kappa^{(b)} + \delta\kappa]\Omega_m^2 + \left[D_s^{(b)}\delta^{ij} + \delta D_s^{ij}\right](\mathbf{q})_i(\mathbf{q})_j\right)|\theta_q|^2 \\ &\quad + O(q^4),\end{aligned} \tag{85}$$

where

$$\begin{aligned}\delta\kappa &= \lim_{q\to 0} T^2\sum_{kk'}(C^\omega)_{k,k+q}^{\mu}T_{k,k',q}^{\mu\nu}(C^\omega)_{k'+q,k}^{\nu}, \\ \delta D_s^{ij} &= \lim_{q\to 0} T^2\sum_{kk'}\mathbf{v}_F\mathbf{v}_F'\left[i(C^q)_{k,k+q}^{\mu}\right]T_{k,k',q}^{\mu\nu}\left[i(C^q)_{k'+q,k}^{\nu}\right].\end{aligned} \tag{86}$$

Since $T_{k,k',q}^{\mu\nu}$ is nonsingular at $q \to 0$, we can safely set $q = 0$ in the integrands. One can verify that $\lim_{q\to 0}(C^\omega)_{k+q,k}^{\mu} = 0$ (see Appendix A), hence, $\delta\kappa = 0$. For δD_s^{ij} we obtain

$$\delta D_s^{ij} = -\frac{v_F^2\nu^2}{d}\delta^{ij}\sum_{nn'}\Pi_n^{0\mu}T_{\mu\nu}^{l=1}(\epsilon_n, \epsilon_n')\Pi_{n'}^{\nu 0}, \tag{87}$$

[8] As discussed before, the diamagnetic term can safely be dropped when performing integration over the momentum first.

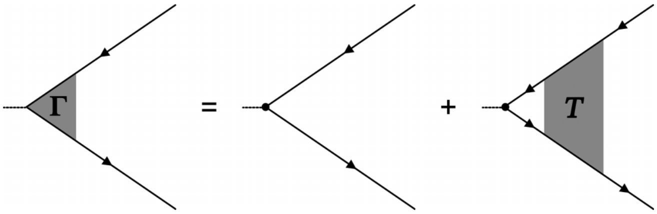

FIG. 3. Relation between the Bethe-Salpeter equation for the renormalized current vertex and the T matrix in the ladder approximation. Explicitly, the renormalized vertex is a quasiparticle contribution containing the bare vertex, and a vertex correction coming from the collective modes.

where $\Pi_n^{\mu\nu}$ is the same as in Eq. (36), and we have defined, in analogy with $V^{l=1}$,

$$\left(\frac{v_F^2}{d}\delta_{ij}\right)T_{\mu\nu}^{l=1}(\epsilon_n, \epsilon_n') \equiv \oint_{\mathrm{FS}} \frac{d\mathbf{k}_F}{S_{d-1}} \oint_{\mathrm{FS}} \frac{d\mathbf{k}_F'}{S_{d-1}} \mathbf{v}_{Fj}\mathbf{v}_{Fj}' T_{\mu\nu}(\epsilon_n, \mathbf{k}_F; \epsilon_n', \mathbf{k}_F'). \tag{88}$$

The relation between the T and V is shown diagrammatically in Fig. 3: V is the interaction and T is the full T matrix in the longitudinal channel, the propagator of longitudinal fluctuations. One can see by inserting the Bethe-Salpeter equation for the T matrix, Fig. 4, into this relation that the Bethe-Salpeter equation for the vertex (16) is obtained. We thus find that Eq. (87) is the contribution to stiffness from corrections to the current vertex, and then combining Eqs. (87) and (80) we reproduce Eq. (23) for the full D_s. From this perspective, vertex corrections to superfluid stiffness involve fluctuations in the $l = 1$ longitudinal sector, although in the far off-shell region (i.e., far from the pole in the T matrix).

To recapitulate, we have shown that within the Eliashberg theory, the phase action is generically of the BCS-type form (69); the only difference is in the value of the superfluid stiffness D_s. This superfluid stiffness contains the renormalization of the effective mass (the Z factor) and the renormalization from the corrections to the current vertex. The mass renormalization factor is present already in the bare stiffness computed using HS decoupling. Vertex corrections arise when we include the coupling to longitudinal gap fluctuations.

VI. CONCLUSION

In this work, we have calculated the superfluid stiffness for a family of 2D non-Galilean-invariant models within the Eliashberg approximation. We showed explicitly, by calculating the delta functional contribution to the conductivity, that in some cases the stiffness approaches its Galilean-invariant value $D_s^{\mathrm{Gal}} = n/m$, despite the absence of Galilean invariance in the model. In particular, when the $l = 0$ and 1 harmonics of the interaction on the Fermi surface are identical, the renormalization of the current vertex is fully determined by the Ward identity for the spin vertex, up to corrections of order $O(\Delta/E_F)$. In this situation, the frequency-dependent renormalization of the current vertex cancels out the frequency-dependent renormalization of the quasiparticle mass, and the stiffness remains the same as in the Galilean-invariant case. We labeled such systems as having *effectively Galilean-invariant* superfluid response.

As an example, we considered a set of models with boson-mediated interaction in the density-density channel, strongly peaked at zero momentum transfer, and isotropic but otherwise arbitrary fermionic dispersion. For such systems, the $l =$ 0 and 1 harmonics of the interaction are nearly identical and differ by $O(\theta_{\mathrm{sc}}^2)$, where θ_{sc} is a characteristic scattering angle. We showed that these systems are effectively Galilean invariant with $D_s \approx n/m + O(\theta_{\mathrm{sc}}^2)$. For a truly Galilean-invariant system, the relation $D_s^{\mathrm{Gal}} = n/m$ is restored by going beyond the single-boson exchange and including Aslamazov-Larkin–type diagrams. We also argued that for arbitrary dispersion the $O(\theta_{\mathrm{sc}}^2)$ term in D_s vanishes only when the boson velocity is taken to infinity, corresponding to an instantaneous action.

We discussed one qualitative difference between an effectively Galilean-invariant system and a truly Galilean-invariant one. In a Galilean-invariant system, the relation $D_s = n/m$ is due to the existence of a special Ward identity relating the renormalized current vertex and spin vertices *exactly*. This Ward identity results from the combination of the Ward identity for momentum conservation and the precise relation $\mathbf{j} = e\mathbf{k}/m$ and thus depends on the behavior of particles both near and far from the Fermi surface. In contrast, the stiffness of an *effectively Galilean-invariant* system approaches n/m by fine tuning of the low-energy interaction parameters of the model so that the relation between current and spin is approximately satisfied. Thus, while the value of the stiffness is approximately the same, the underlying physics is generally quite different.

We also argued that for both Galilean-invariant and non-Galilean-invariant systems with a frequency-dependent gap function, one must include the contribution to the stiffness from the anomalous component of the renormalized current vertex, which is given by the frequency derivative of the pairing vertex. This contribution must be included on an equal footing with the usual renormalizations to the normal current vertex. The presence of the anomalous contribution to D_s reflects the fact that in the superconducting state, in addition to the usual diagram renormalizing the normal current vertex, one must take into account the Doppler shift of the pairing vertex due to the flow of the condensate.

To further elucidate the nature of the vertex corrections we presented complementary perspectives on the stiffness by obtaining the above results from the LW functional and the HS decoupling of our model.

In the LW description of Eliashberg theory, corresponding to keeping only the lowest-order diagram in the LW functional, the correct prescription for calculating linear response

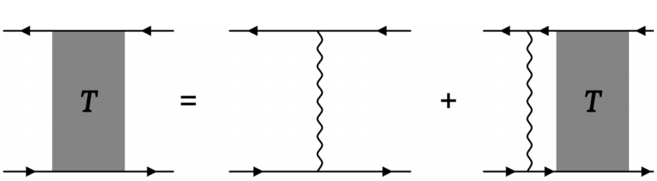

FIG. 4. Bethe-Salpeter equation for the T matrix in the ladder approximation. Solid lines are the full Nambu Green's functions. When restricted to the longitudinal sector T is the collective mode propagator for Gaussian longitudinal fluctuations.

is to minimize the free energy in presence of the external fields and take derivatives of the minimal free energy over the fields to get the associated susceptibilities. Then at the end of the calculation one may set the external fields to zero. This is the sense in which the LW formalism produces "conserving approximations" when the LW functional is truncated at any order. Performing the calculation in this way, we showed that the required vertex corrections to the external current vertex appear naturally as the shift of the self-energy due to the phase winding $\hat{\Gamma} \sim (\partial \hat{\mathcal{G}}^{-1}/dQ)$, exactly reproducing the results of the diagrammatic calculation.

In the HS description, we extracted the stiffness from the phase action of an Eliashberg superconductor. Using the gauge invariance of the action, we showed that the Gaussian action for the phase sector includes the bare phase action as well as a coupling to the $l = 1$ longitudinal modes. Upon integrating out the longitudinal modes we showed that the phase action within Eliashberg theory takes the generic form $S = \frac{1}{2}\int d\tau\, d\mathbf{r}(2\nu|\partial_\tau\theta|^2 + D_s|\nabla\theta|^2)$. Here, D_s is the same stiffness as obtained in the previous sections, with the vertex corrections arising from the longitudinal mode propagators evaluated at $\mathbf{q} = 0$, $i\Omega_m \to 0$. On the other hand, the coefficient of the $(\partial_t\theta)^2$ is unrenormalized from its bare value, reflecting its origin as coming from fermions away from the Fermi surface which are agnostic to emergence of a pairing vertex at low energy.

For clarity and simplicity of presentation, this work focused on s-wave superconductivity in rotationally symmetric systems. The general considerations still apply when either of these constraints are relaxed, but the calculations become more involved as one needs to evaluate products of velocities and form factors of a non-s-wave gap along the Fermi surface. The formalism may also be extended to the case of multiband superconductors in which case one must calculate additional susceptibilities, vertex corrections, and interaction channels due to the presence of band indices. In general, effects which break Galilean invariance in the vicinity of the Fermi surface will suppress the stiffness unless the interaction channels obey a particular relation. This includes effects which are known to strongly affect transport such as Fermi-surface anisotropy [28] or umklapp scattering [44]. Nonetheless, where the dominant interactions only cause small-angle scattering on the Fermi surface, an "effective Galilean-invariance" condition may still be satisfied as the interactions are almost local on the Fermi surface and thus cannot resolve the global shape of the Fermi surface or the umklapp nature of interactions. The situation is likely even more involved in $4e$ superconductors [45].

The key result of our work is that an effectively Galilean-invariant value of the stiffness in a non-Galilean-invariant system requires a specific relationship between the low-energy interaction channels of the system, which is not guaranteed by the symmetries of the system. Indeed, interaction via Einstein phonons is an example of a system which strongly violates these conditions. Therefore, we expect that generically the superfluid stiffness of a quantum critical superconductor, where the pairing vertex is strongly frequency dependent and the Z factor is large, may be strongly reduced from the Galilean-invariant value $D_s^{\text{Gal}} \approx n/m$. We discuss specific examples in a separate paper [46], where we analyze the stiffness for underlying quantum-critical models.

ACKNOWLEDGMENTS

The authors would like to thank P. A. Lee, J. Schmalian, and Y. Wang for stimulating discussions. The research was supported by the U.S. Department of Energy, Office of Science, Basic Energy Sciences, under Award No. DE-SC0014402. This work was completed while the coauthors attended a workshop at KITP in Santa Barbara, CA. KITP is supported in part by Grant No. NSF PHY-1748958.

APPENDIX A: FERMION BUBBLES IN THE LIMIT OF ZERO EXTERNAL MOMENTUM

Consider the bare fermionic bubble in Eliashberg theory

$$\Pi^{\mu\nu}_{n+m,n} \equiv \sum_{\mathbf{k}} \text{tr}[\hat{\tau}^\mu \hat{\mathcal{G}}_{n+m}(\mathbf{k})\hat{\tau}^\mu \hat{\mathcal{G}}_n(\mathbf{k})], \tag{A1}$$

where the Nambu Green's function is

$$\hat{\mathcal{G}}_k = \frac{-i\tilde{\Sigma}_n - \xi\hat{\tau}_3 - \phi_n\hat{\tau}_1}{D_k}, \quad D_k \equiv \tilde{\Sigma}_n^2 + \xi^2 + \phi_n^2. \tag{A2}$$

In the quasiclassical approximation this can be expressed via the integrals

$$\begin{aligned} I^{(1)}_{n+m,n} &\equiv \nu \int d\xi \frac{1}{D_{n+m}D_n} = \frac{\pi\nu}{S_{n+m}S_n(S_{n+m}+S_n)}, \\ I^{(2)}_{n+m,n} &\equiv \nu \int d\xi \frac{\xi^2}{D_{n+m}D_n} = \frac{\pi\nu}{S_{n+m}+S_n}, \end{aligned} \tag{A3}$$

where $S_n^2 = \tilde{\Sigma}_n^2 + \phi_n^2$. Explicitly,

$$\begin{aligned} \Pi^{\mu\nu}_{n+m,n} &= c^{(1);\mu\nu}_{n+m,n} I^{(1)}_{n+m,n} + c^{(2);\mu\nu}_{n+m,n} I^{(2)}_{n+m,n}, \\ c^{(1);\mu\nu}_{n+m,n} &\equiv \text{tr}[\hat{\tau}^\mu(i\tilde{\Sigma}_{n+m} + \phi_{n+m}\hat{\tau}_1)\hat{\tau}^\nu(i\tilde{\Sigma}_n + \phi_n\hat{\tau}_1)], \\ c^{(2);\mu\nu}_{n+m,n} &= \text{tr}[\hat{\tau}^\mu\hat{\tau}_3\hat{\tau}^\nu\hat{\tau}_3] = 2\,\text{diag}(1,-1,-1,1)^{\mu\nu}. \end{aligned} \tag{A4}$$

Explicitly evaluating the traces we find for $c^{(1)}$

$$c^{(1);00}_{n+m,n} = c^{(1);11}_{n+m,n} = 2(\phi_{n+m}\phi_n - \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n), \tag{A5}$$

$$c^{(1);01}_{n+m,n} = c^{(1);10}_{n+m,n} = 2i(\phi_{n+m}\tilde{\Sigma}_n + \phi_n\tilde{\Sigma}_{n+m}), \tag{A6}$$

$$c^{(1);22}_{n+m,n} = c^{(1);33}_{n+m,n} = -2(\phi_{n+m}\phi_n + \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n), \tag{A7}$$

$$c^{(1);23}_{n+m,n} = -c^{(1);32}_{n+m,n} = 2(\phi_{n+m}\tilde{\Sigma}_n - \phi_n\tilde{\Sigma}_{n+m}), \tag{A8}$$

and the remaining elements are zero. Thus, Π is block diagonal

$$\hat{\Pi}_{n+m,n} = \begin{pmatrix} \hat{\Pi}^L_{n+m,n} & 0 \\ 0 & \hat{\Pi}^T_{n+m,n} \end{pmatrix}$$

with longitudinal block

$$\hat{\Pi}^L_{n+m,n} = \frac{2\pi\nu}{S_{n+m}S_n(S_{n+m}+S_n)} \begin{pmatrix} \phi_{n+m}\phi_n - \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n + S_{n+m}S_n & i(\phi_{n+m}\tilde{\Sigma}_n + \phi_{n+m}\tilde{\Sigma}_n) \\ i(\phi_{n+m}\tilde{\Sigma}_n + \phi_{n+m}\tilde{\Sigma}_n) & \phi_{n+m}\phi_n - \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n - S_{n+m}S_n \end{pmatrix} \tag{A9}$$

and transverse block

$$\hat{\Pi}^T_{n+m,n} = \frac{2\pi\nu}{S_{n+m}S_n(S_{n+m}+S_n)}\begin{pmatrix} -\phi_{n+m}\phi_n - \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n - S_{n+m}S_n & (\phi_{n+m}\tilde{\Sigma}_n - \phi_n\tilde{\Sigma}_{n+m}) \\ -(\phi_{n+m}\tilde{\Sigma}_n - \phi_n\tilde{\Sigma}_{n+m}) & -\phi_{n+m}\phi_n - \tilde{\Sigma}_{n+m}\tilde{\Sigma}_n + S_{n+m}S_n \end{pmatrix}. \quad \text{(A10)}$$

In the $m \to 0$ limit these reduce to

$$\hat{\Pi}^L_n = \frac{2\pi\nu}{Z_n\zeta_n^3}\begin{pmatrix} \Delta_n^2 & i\Delta_n\epsilon_n \\ i\Delta_n\epsilon_n & -\epsilon_n^2 \end{pmatrix}, \quad \hat{\Pi}^T_{n+m,n} = -\frac{2\pi\nu}{Z_n\zeta_n}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{(A11)}$$

where $\zeta_n^2 = \epsilon_n^2 + \Delta_n^2$.

APPENDIX B: WARD IDENTITIES IN A SUPERCONDUCTOR WITH FREQUENCY-DEPENDENT GAP FUNCTION

In this Appendix, we derive Ward identities associated with charge and spin conservation in a superconductor with frequency-dependent gap function.

Ward identities are special relations between vertices and self-energies, imposed by the conservation laws. For a system of fermions with U(1) charge (gauge) symmetry and SU(2) spin symmetry, they ensure that the total charge of the system (and, hence, the total number of fermions) and each component of the total spin do not change with time. In practical terms, we focus on Ward identities which relate two-fermion spin and charge density vertices at zero transferred momentum and a finite transferred frequency to the fermionic self-energy. The relations are particularly simple in Eliashberg-type theories, in which the self-energy $\Sigma(k, \epsilon_n)$ has much stronger dependence on frequency than on fermionic momentum, and the latter can be neglected. On the Matsubara axis we then approximate $\Sigma(k, \epsilon_n) \approx \Sigma_n$. Within the same approximation, spin and charge vertices Γ^{ch} and $\boldsymbol{\Gamma}^{sp}$ can also be treated as functions of frequency only. Each vertex depends on Matsubara frequencies and spin projections on the incoming and outgoing fermions, $\Gamma^{ch} = \Gamma^{ch}_{n+m\alpha,n\beta}$ and $\boldsymbol{\Gamma}^{sp} = \boldsymbol{\Gamma}^{sp}_{n+m\alpha,n\beta}$.

For completeness, we also derive the Ward identity associated with the conservation of momentum.

1. Normal state

We define Σ_n in the normal state via $G^{-1}(k, \epsilon_n) = i\epsilon_n - \Sigma_n - \xi_k$, where ξ_k is the fermionic dispersion. The relations between Γ^{ch}, Γ^{sp}, and Σ_n are [22,23,47,48]

$$\Gamma^{ch}_{n+m\alpha,n\beta} = \delta_{\alpha\beta}\Gamma, \quad \boldsymbol{\Gamma}^{sp}_{m+n,\alpha,m\beta} = \boldsymbol{\sigma}_{\alpha\beta}\Gamma, \quad \text{(B1)}$$

where

$$\Gamma = 1 + i\frac{\Sigma_{n+m} - \Sigma_n}{\Omega_m}. \quad \text{(B2)}$$

The bosonic Ω_m is the difference between outgoing and incoming fermionic frequencies.

To set the stage for our analysis in the superconducting state, we present the diagrammatic proof of this relation. For this we note that within Eliashberg theory the fermionic self-energy is obtained within the one-loop approximation, as a convolution of the fermionic propagator and the effective frequency-dependent "local" interaction $V^{l=0}_{n-n'}$, which is $V_{n-n'}(\mathbf{k}_F - \mathbf{k}'_F)$ integrated over the Fermi surface. Within the same computational scheme, the vertex Γ^{ch} is obtained by summing up ladder series of vertex corrections, with the same $V^{l=0}_{n-n'}$. For Γ^{sp}, the analysis is more nuanced: ladder series hold when $V_{n-n'}(\mathbf{k}_F - \mathbf{k}'_F)$ is of density-density form, i.e., when spin projection (up or down) is conserved along the interaction line. If $V_{n-n'}(\mathbf{k}_F - \mathbf{k}'_F)$ is a spin-spin interaction with spin σ matrices in the vertices, one has to add additional Aslamazov-Larkin–type terms to get the proper series for $\Gamma^{(sp)}$ [49]. For simplicity, below we assume that the effective interaction is of the density-density type. The ladder series in Fig. 5 yields the following integral equation for $\Gamma_{n+m,n}$:

$$\Gamma_{n+m,n} = 1 + \nu T\sum_{n'} V^{l=0}_{n-n'}\Gamma_{n'+m,n'}\int d\xi_k G_{n'+m}(\xi_k)G_{n'}(\xi_k), \quad \text{(B3)}$$

where ν is the density of states at the Fermi level. The self-energy is given by

$$\Sigma_n = \nu T\sum_{n'} V^{l=0}_{n-n'}\int d\xi_k G_{n'}(\xi_k) = -\frac{i}{2}\nu T\sum_{n'} V^{l=0}_{n-n'}\text{sgn}n'. \quad \text{(B4)}$$

The product of the two Green's functions in Eq. (B3) can be decoupled as

$$G_{n'+m}(\xi_k)G_{n'}(\xi_k) = [G_{n'+m}(\xi_k) - G_{n'}(\xi_k)] \times \frac{i}{\Omega_m + i(\Sigma_{n+m} - \Sigma_n)}. \quad \text{(B5)}$$

Substituting into Eq. (B3), we obtain

$$\Gamma_{n+m,n} = 1 + i\nu T\sum_{n'} V^{l=0}_{n-n'}\int d\xi_k \frac{\Gamma_{n'+m,n}}{\Omega_m + i(\Sigma_{n'+m} - \Sigma_{n'})} \times [G_{n'+m}(\xi_k) - G_{n'}(\xi_k))]. \quad \text{(B6)}$$

One can straightforwardly verify that $\Gamma_{n+m,n}$ from Eq. (B2) is the solution of this equation. Indeed, substituting this $\Gamma_{n'+m,n'}$ into the right-hand side of Eq. (B6) we find that it reduces to

$$1 + \frac{i}{\Omega_m}\nu T\sum_{n'} V^{l=0}_{n-n'}\int d\xi_k[G_{n'+m}(\xi_k) - G_{n'}(\xi_k)]. \quad \text{(B7)}$$

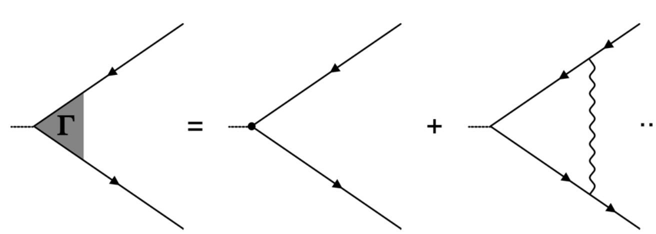

FIG. 5. Ladder series for the renormalized vertex Γ, solid lines are the full Nambu Green's functions, and the wavy line is the interaction.

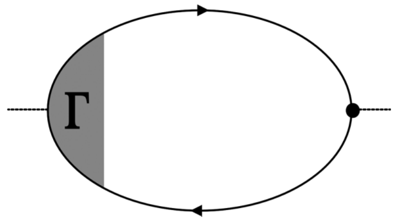

FIG. 6. Dressed polarization bubble including the renormalized vertex Γ.

Using Eq. (B4), we reexpress this as

$$1 + i\frac{\Sigma_{n+m} - \Sigma_n}{\Omega_m}, \tag{B8}$$

which is exactly $\Gamma_{n+m,n}$.

Using the Ward identities, one can straightforwardly demonstrate that charge and spin correlators (the polarization bubbles) vanish at a zero incoming momentum and a finite incoming frequency, as should be the case for a conserved quantity X. (The choice of zero momentum and a finite frequency implies that one probes a variation of the total X in the sample between different times. For a conserved X, there is no such variation.) The fully dressed polarization bubble is shown in Fig. 6. In explicit form,

$$\Pi(q=0,\Omega_m) = \nu T\sum_n \int d\xi_k \Gamma_{n+m,n} G_{n+m}(\xi_k) G_n(\xi_k). \tag{B9}$$

It is natural to integrate over ξ_k first as this integration is straightforward. One cannot, however, integrate over ξ_k in infinite limits as at large frequencies, when $\epsilon_n > \Sigma_n$ and $T\sum_n \to (1/2\pi)\int d\epsilon_n$, the Green's function approaches the unrenormalized form $G_n(\xi_k) = 1/(i\epsilon_n - \xi_k)$ and the double integral $\int d\epsilon_n d\xi_k/(i\epsilon_n - \xi_k)^2$ diverges logarithmically. The physically sound way to regularize the divergence is to restrict the ξ_k integration to $|\xi_k| < \Lambda$ and set $\Lambda \to \infty$ only at the end of the calculation. Carrying out the integration over ξ_k this way, we obtain

$$\Pi^{\rm ch}(q=0,\Omega_m) = \Pi^{\rm sp}(q=0,\Omega_m) = \nu\left(1 - \frac{\Omega_m \Gamma_{n+m,n}}{\Omega_m + i\Sigma_{n+m} - \Sigma_n}\right) = 0 \tag{B10}$$

as it should be.

2. Superconducting state

As in the main text, for definiteness we consider s-wave superconductivity, in which case the pairing vertex ϕ_n and the gap function Δ_n are independent on the angle along the Fermi surface. We also set $T = 0$ to avoid complications due to discreteness of Matsubara frequencies. We keep the notations Σ_n, etc., with the understanding that $\Sigma_n = \Sigma(\epsilon_n)$, where ϵ_n is a continuous variable along the Matsubara axis.

a. Distinction between charge and spin correlations

We argue below that in a superconductor charge and spin correlators have to be treated differently as the first one acquires an additional contribution from coupling to phase fluctuations. The distinction between spin and charge correlators can be seen already for a BCS superconductor. Both spin and charge polarization bubbles have to vanish at zero incoming momentum and a nonzero incoming frequency Ω_m, and we show below that this is indeed the case. However, to prove this for the charge case, extra care is needed.

Specifically, for a BCS superconductor, it is tempting to neglect the interaction and express spin and charge correlators as bubbles made of free-fermion Nambu Green's functions. For the charge bubble this gives

$$\Pi^{\rm ch}_{\rm free}(q=0,\Omega_m) = \frac{\nu}{2\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k {\rm tr}[\hat{\mathcal{G}}_{n+m,\alpha}(\xi_k)\hat{G}_{n,\alpha}(\xi_k)] \tag{B11}$$

and for the spin case we have

$$\Pi^{{\rm sp},ii}_{\rm free}(q=0,\Omega_m) = \frac{\nu}{2\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k {\rm tr} \times [\sigma^i_{\alpha\beta}\hat{\mathcal{G}}_{n+m,\alpha}(\xi_k)\hat{G}_{n,\beta}\sigma^i_{\beta\alpha}(\xi_k)], \tag{B12}$$

where $i = x, y, z$. For definiteness, we set $i = z$ below.

Let us set continuous Ω_m to be finite but infinitesimally small. One can easily verify that for a nonzero Δ, the limit $\Omega_m \to 0$ is entirely regular, and one can just set $\Omega_m = 0$ in the calculations. Using Eq. (7) from the main text for the Green's function in Nambu representation and identifying ϕ_n in a BCS superconductor with Δ_n, we obtain

$$\Pi^{\rm ch}_{\rm free}(q=0,\Omega_m\to 0) = \frac{\nu}{\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k \frac{\left(\xi_k^2 - \Delta^2 - \epsilon_n^2\right)}{\left(\xi_k^2 + \Delta^2 + \epsilon_n^2\right)^2} \tag{B13}$$

and

$$\Pi^{{\rm sp},zz}_{\rm free}(q=0,\Omega_m\to 0) = \frac{\nu}{\pi}\int d\epsilon_n \times \int_{-\Lambda}^{\Lambda} d\xi_k \frac{\left(\xi_k^2 + \Delta^2 - \epsilon_n^2\right)}{\left(\xi_k^2 + \Delta^2 + \epsilon_n^2\right)^2}. \tag{B14}$$

In Gorkov's notations of normal and anomalous Green's functions $G_n(\xi_k) = -(i\epsilon_n + \xi_k)/(\xi_k^2 + \epsilon_k^2 + \Delta^2)$ and $F_n(\xi_k) = \Delta/(\xi_k^2 + \epsilon_k^2 + \Delta^2)$, the polarization bubbles are

$$\Pi^{\rm ch}_{\rm free}(q=0,\Omega_m\to 0) = \frac{\nu}{\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k ([G_n(\xi_k)]^2 - [F_n(\xi_k)]^2), \tag{B15}$$

and

$$\Pi^{{\rm sp},zz}_{\rm free}(q=0,\Omega_m\to 0) = \frac{\nu}{\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k ([G_n(\xi_k)]^2 + [F_n(\xi_k)]^2). \tag{B16}$$

The two polarization bubbles differ in the sign of the F^2 term.

The integration over ϵ_n and ξ_k in Eqs. (B13) and (B14) can be carried out in any order, and the results are

$$\Pi^{{\rm sp},zz}_{\rm free}(q=0,\Omega_m\to 0) = 0,\quad \Pi^{\rm ch}_{\rm free}(q=0,\Omega_m\to 0) = -2\nu. \tag{B17}$$

We see that $\Pi^{{\rm sp},zz}_{\rm free}(q=0\Omega_m\to 0)$ vanishes, as expected, but $\Pi^{\rm ch}_{\rm free}(q=0\Omega_m\to 0)$ remains finite.

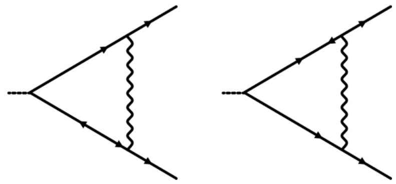

FIG. 7. Diagrams formed from products of normal (G) and anomalous (F) Green's functions diagrams contributing to the coupling of the spin and charge vertices to the particle-particle susceptibility.

Because charge conservation must be satisfied, there must be another contribution to charge polarization, which cancels the free-fermion contribution, as was pointed out by Nambu [50]. Such a contribution has been identified in other contexts as well, e.g., in the analysis of A_{1g} Raman scattering in a BCS superconductor [51–53]. The argument is that charge fluctuations are linearly coupled to phase fluctuations of the superconducting order parameter and this gives rise to the extra contribution $\Pi^{\text{ch}}_{ex}(q,\Omega_m) = S^2(q,\Omega_m)\chi^{\text{pp}}(q,\Omega_m)$, where $S(q,\Omega_m)$ is the coupling and $\chi^{\text{pp}}(q,\Omega_m)$ is the propagator of phase fluctuations. The coupling $S(\Omega_m)$ is generated by the triangular diagram, consisting of the original charge vertex, one normal and one anomalous Green's function, and the four-fermion interaction V (see Fig. 7). This coupling vanishes at $\Omega_m = 0$, but at a finite Ω_m, $S(\Omega_m) \propto V\Omega_m/\Delta$. Naively, this would imply that the extra contribution is irrelevant at $\Omega_m \to 0$. However, phase fluctuations are massless, and their propagator $\chi^{\text{pp}}(0,\Omega_m) \propto \nu(\Delta/V\Omega_m)^2$. As a result, $\Pi^{\text{ch}}_{ex}(q,\Omega_m)$ is independent of Ω_m and is of order ν, like the free-fermion $\Pi^{\text{ch}}_{\text{free}}(q=0,\Omega_m\to 0)$. We now compute explicitly the prefactor in $\Pi^{\text{ch}}_{ex}(q=0,\Omega_m\to 0)\sim\nu$. We first compute the particle-particle propagator. Within the ladder approximation (the same in which the BCS gap equation has been obtained)

$$\chi^{\text{pp}}(q,\Omega_m) = 2\frac{\Pi^{\text{pp}}(q,\Omega_m)}{1 - V^{l=0}\Pi^{\text{pp}}(q,\Omega_m)}, \tag{B18}$$

where the overall factor 2 is due to spin summation and

$$\Pi^{\text{pp}}(q,\Omega_m) = \frac{\nu}{\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k \left(G_n(\xi_k)G_{-n}(\xi_k) + [F_n(\xi_k)]^2\right), \tag{B19}$$

where $V^{l=0} > 0$ is an attractive interaction in the s-wave channel. Using $V^{l=0}\Pi^{\text{pp}}(0,0) = 1$ and expanding in Ω_m, we obtain

$$\chi^{\text{pp}}(0,\Omega_m) = 2\nu\left(\frac{2\Delta}{V^{l=0}\nu\Omega_m}\right)^2. \tag{B20}$$

We next compute the coupling $S(\Omega_m)$. There are two topologically different triangular diagrams involving products of G and F Fig. 7. For the charge side vertex, they add up and yield

$$S(\Omega_m) = V^{l=0}\frac{\nu}{2\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k \frac{\Delta\Omega_m}{\left(\xi_k^2+\epsilon_n^2+\Delta^2\right)^2} = \nu V^{l=0}\frac{\Omega_m}{2\Delta}. \tag{B21}$$

We then obtain

$$\Pi^{\text{ch}}_{ex}(q=0,\Omega_m\to 0) = 2\nu\left(\frac{\nu V^{l=0}\Omega_m}{2\Delta}\right)^2\left(\frac{2\Delta}{\nu V\Omega_m}\right)^2 = 2\nu. \tag{B22}$$

Combining with Eq. (B17), we find that $\Pi^{ch}_{\text{free}}(q=0,\Omega_m\to 0) + \Pi^{\text{ch}}_{ex}(q=0,\Omega_m\to 0) = 0$, as it should be because the total charge is the conserved quantity.

For the spin correlator, the two contributions to the coupling $S(\Omega_m)$ cancel out at order Ω_m. Then there is no additional Ω_m-independent contribution to the spin propagator, consistent with the vanishing of $\Pi^{\text{sp},zz}_{\text{free}}(q=0,\Omega_m\to 0)$. From a physics perspective, this is a consequence of the fact that spin fluctuations do no couple linearly to phase fluctuations.

Below we extend the analysis of a BCS superconductor to the case when the effective four-fermion interaction is a dynamical $V_{n-n'}$. A dynamical interaction gives rise to fermionic self-energy Σ_n, and also the pairing vertex ϕ_n and the gap function Δ_n become functions of frequency. The proof of the Ward identities in this situation is more involved, and for the charge correlator it is further involved by the necessity to include the coupling to phase fluctuations. For this reason, we consider Ward identities associated with spin and charge conservation separately.

b. Ward identity for $\Gamma^{(sp)}_{n+m,n}$

As before, we use matrix Nambu notations and write the self-consistent one-loop equation for the matrix $\hat{\Sigma}_n = -i\Sigma_n\hat{\tau}_0 + \phi_n\hat{\tau}_1$ and the ladder equation for the matrix $\hat{\Gamma}^{\text{sp},ii}_{n+m\alpha,n\beta} = \sigma^i_{\alpha\beta}\hat{\Gamma}_{n+m,n}$, where $\hat{\Gamma}_{n+m,n} = \Gamma^{(0)}_{n+m,n}\hat{\tau}_0 + \Gamma^{(1)}_{n+m,n}\hat{\tau}_1$ [see Eqs. (6) and (19) in the main text]. The equations for $\hat{\Sigma}_n$ and $\hat{\Gamma}_{n+m,n}$ are formally the same as Eqs. (B3) and (B4), but now have matrix form

$$\hat{\Sigma}_n = \nu T\sum_{n'} V^{l=0}_{n-n'}\int d\xi\, \hat{\tau}_3\hat{\mathcal{G}}_{k'}\hat{\tau}_3 \tag{B23}$$

and

$$\hat{\Gamma}_{n+m,n} = 1 + \nu T\sum_{n'} V^{l=0}_{n-n'} \times \int d\xi_k \hat{\tau}_3\hat{\mathcal{G}}_{n'+m}(\xi_k)\hat{\Gamma}_{n'+m,n'}\hat{\mathcal{G}}_{n'}(\xi_k)\hat{\tau}_3. \tag{B24}$$

Splitting $\hat{\Gamma}$ into components and taking the limit of $\Omega_m\to 0$, we obtain the set of two coupled equations, as schematically depicted in Fig. 8,

$$\Gamma^{(0)}_{n+m,n} = 1 + \frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(0)}_{n'+m,n'}\frac{\xi_k^2+\phi_{n'}^2-\tilde{\Sigma}_{n'}^2}{\left(\xi_k^2+\phi_{n'}^2+\tilde{\Sigma}_{n'}^2\right)^2}V^{l=0}_{n-n'} + 2i\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(1)}_{n'+m,n'}\frac{\tilde{\Sigma}_{n'}\phi_{n'}}{\left(\xi_k^2+\phi_{n'}^2+\tilde{\Sigma}_{n'}^2\right)^2}V^{l=0}_{n-n'},\ \Gamma^{(1)}_{n+m,n},$$

$$\Gamma^{(1)}_{n+m,n} = \frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(1)}_{n'+m,n'}\frac{\xi_k^2-\phi_{n'}^2+\tilde{\Sigma}_{n'}^2}{\left(\xi_k^2+\phi_{n'}^2+\tilde{\Sigma}_{n'}^2\right)^2}V^{l=0}_{n-n'} - 2i\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(0)}_{n'+m,n'}\frac{\tilde{\Sigma}_{n'}\phi_{n'}}{\left(\xi_k^2+\phi_{n'}^2+\tilde{\Sigma}_{n'}^2\right)^2}V^{l=0}_{n-n'}. \tag{B25}$$

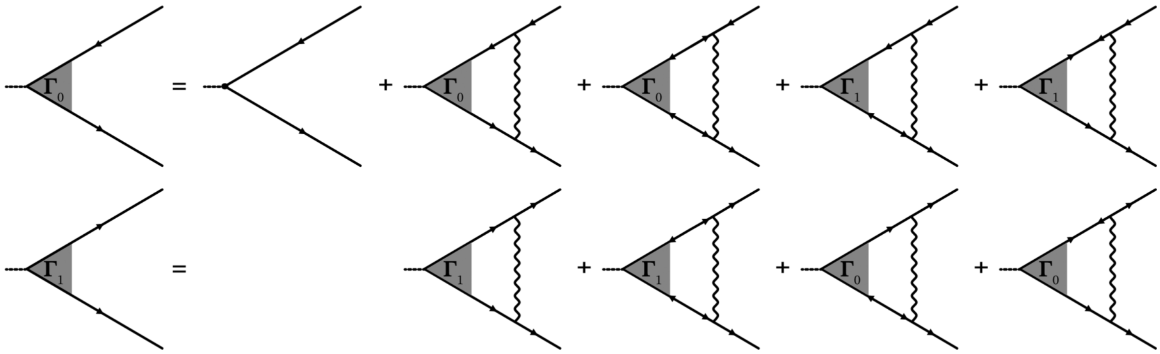

FIG. 8. Bethe-Salpeter equation in the ladder approximation for the renormalized normal Γ^0 and anomalous Γ^1 vertices. The spin and charge diagrams differ in their spin structure and thus the symmetry of vertices under reversal of the direction of the legs. In particular, the relative sign of between two anomalous diagrams in each right-hand side differs for the spin and charge channels.

We assume that the dynamical interaction vanishes in the limit of large frequency transfer. The double integral over ξ_k and $\epsilon_{n'}$ is then ultraviolet convergent, and the integration over ξ_k can be extended to infinite limits. Integrating over ξ_k in each term in Eq. (B25), we reduce it to

$$\begin{aligned}\Gamma^{(0)}_{n+m,n} &= 1 + \frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(0)}_{n'+m,n'}\frac{\Delta^2_{n'}}{Z_{n'}\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'} + i\frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{\epsilon_{n'}\Delta_{n'}}{Z_{n'}\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'},\\ \Gamma^{(1)}_{n+m,n} &= -\frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(0)}_{n'+m,n'}\frac{\epsilon^2_{n'}}{Z_{n'}\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'} - i\frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{\epsilon_{n'}\Delta_{n'}}{Z_{n'}\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'}.\end{aligned} \tag{B26}$$

We assume and verify that the solution of these equations is the matrix extension of Eq. (B2):

$$\hat{\Gamma}_{n+m,n} = \hat{\tau}_0 + i\frac{\hat{\Sigma}_{n+m}-\hat{\Sigma}_n}{\Omega_m} \xrightarrow{\Omega_m\to 0} \hat{\tau}_0 + i\frac{d\hat{\Sigma}_n}{d\epsilon_n}. \tag{B27}$$

Using $\hat{\Sigma}_n = -i\Sigma_n\tau_0 + i\phi_n\tau_1$, we rewrite Eq. (B27) in components

$$\Gamma^{(0)}_{n+m,n} = 1 + \frac{d\Sigma_n}{d\epsilon_n}, \quad \Gamma^{(1)}_{n+m,n} = i\frac{d\phi_n}{d\epsilon_n}. \tag{B28}$$

Substituting these forms into the right-hand side of Eq. (B26) and using Eq. (27) from the main text,

$$\frac{d\phi_n}{d\epsilon_n} = \frac{\Delta_n}{\epsilon_n}\left(1+\frac{d\Sigma_n}{d\epsilon_n}\right) - \frac{\Delta_n}{\epsilon_n}Z_n + Z_n\frac{d\Delta_n}{d\epsilon_n}, \tag{B29}$$

we obtain after simple algebra

$$\Gamma^{(0)}_{n+m,n} = 1 + \frac{\nu}{2}\int d\epsilon_{n'}\frac{\Delta_{n'}\left(\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}\right)}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'}, \quad \Gamma^{(1)}_{n+m,n} = -i\frac{\nu}{2}\int d\epsilon_{n'}\frac{\epsilon_{n'}\left(\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}\right)}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'}. \tag{B30}$$

Note that the quasiparticle residue Z_n cancels out between Eqs. (B26) and (B30). Using

$$\frac{\Delta_{n'}\left(\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}\right)}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{3/2}} = \frac{d}{d\epsilon_{n'}}\left(\frac{\epsilon_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}\right), \quad \frac{\epsilon_{n'}\left(\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}\right)}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{3/2}} = -\frac{d}{d\epsilon_{n'}}\left(\frac{\Delta_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}\right), \tag{B31}$$

integrating by parts, and replacing $dV_{n-n'}/d\epsilon_{n'}$ by $-dV_{n-n'}/d\epsilon_n$, we obtain

$$\Gamma^{(0)}_{n+m,n} = 1 + \frac{d}{d\epsilon_n}\left[\frac{\nu}{2}\int d\epsilon_{n'}\frac{\epsilon_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}\right], \quad \Gamma^{(1)}_{n+m,n} = \frac{d}{d\epsilon_n}\left[\frac{\nu}{2}\int d\epsilon_{n'}\frac{\Delta_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}\right]. \tag{B32}$$

The self-energy and the pairing vertex are given by Eqs. (10) and (12) from the main text:

$$\Sigma_n = \frac{\nu}{2}\int d\epsilon_{n'}\frac{\epsilon_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}, \quad \phi_n = \frac{\nu}{2}\int d\epsilon_{n'}\frac{\Delta_{n'}}{\left(\Delta^2_{n'}+\epsilon^2_{n'}\right)^{1/2}}. \tag{B33}$$

Comparing Eqs. (B32) and (B33), we see that the relations (B28) are satisfied. These relations are spin Ward identities for a superconductor with frequency-dependent gap function.

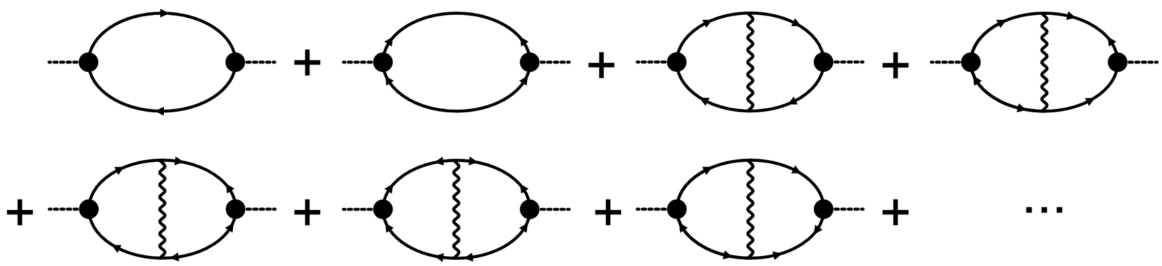

FIG. 9. Ladder series contributing to the spin and charge correlators to first order in the $\mathbf{q}=0$, $i\Omega_m \to 0$ limit in terms of Gorkov's normal and anomalous functions. For the spin correlator the side vertex is σ^z while for the charge correlator is σ^0. This causes terms containing anomalous propagators at only one side vertex to differ in sign between the spin and charge series since $[\hat{F}, \hat{\sigma}_0]=0$ while $\{\hat{F}, \hat{\sigma}^z\}$.

We next use these Ward identities to prove that $\Pi^{\mathrm{sp},ii}(q=0,\ \Omega_m \to 0)$ vanishes, as is required by global spin conservation. To first order in the interaction $V^{l=0}_{n-n'}$, the ladder diagrams for $\Pi^{\mathrm{sp},ii}(q,\Omega_m)$ in a superconductor are shown in Fig. 9. The full spin-polarization bubble, expressed in terms of renormalized vertices, is shown in Fig. 10. In analytical form,

$$\Pi^{\mathrm{sp},ii}(q=0,\Omega_m \to 0)=\frac{\nu}{\pi}\int d\epsilon_n \int_{-\Lambda}^{\Lambda} d\xi_k \frac{\xi_k^2 - Z_n^2\big[(\epsilon_n^2-\Delta_n^2)(1+d\Sigma_n/d\epsilon_n)+2\epsilon_n\Delta_n d\phi_n/d\epsilon_n\big]}{\big[\xi_k^2+Z_n^2(\Delta_{n'}^2+\epsilon_{n'}^2)\big]^2}. \tag{B34}$$

Integrating over ξ_k in the same was as we did in the normal state and then taking the limit $\Lambda \to \infty$, we obtain

$$\Pi^{\mathrm{sp},ii}(q=0,\Omega_m \to 0)=\nu\left[2-\int d\epsilon_n \frac{\Delta_n^2(1+d\Sigma_n/d\epsilon_n)-\Delta_n\epsilon_n d\phi_n/d\epsilon_n}{Z_n(\Delta_n^2+\epsilon_n^2)^{3/2}}\right]. \tag{B35}$$

Expressing $d\phi_n/d\epsilon_n$ via Eq. (B29), we rewrite Eq. (B35) as

$$\Pi^{\mathrm{sp},ii}(q=0,\Omega_m \to 0)=\nu\left[2-\int d\epsilon_n \frac{\Delta_n\big(\Delta_n-\epsilon_n\frac{d\Delta_n}{d\epsilon_n}\big)}{\big(\Delta_n^2+\epsilon_n^2\big)^{3/2}}\right]. \tag{B36}$$

Using Eq. (B31) we reexpress the integrand in the right-hand of Eq. (B36) as the full derivative:

$$\Pi^{\mathrm{sp},ii}(q=0,\Omega_m \to 0)=2\nu\left[1-\int_0^{\infty} d\epsilon_n \frac{d}{d\epsilon_n}\frac{\epsilon_n}{\big(\Delta_n^2+\epsilon_n^2\big)^{1/2}}\right]. \tag{B37}$$

We see that $\Pi^{\mathrm{sp},ii}(q=0,\ \Omega_m \to 0)$ indeed vanishes.

c. Ward identity for $\mathbf{\Gamma}^{ch}_{n+m,n}$

We now repeat the calculations of the previous section in the charge channel, for the vertex $\hat{\Gamma}^{\mathrm{ch}}_{n+m\alpha,n\beta}=\delta_{\alpha\beta}\hat{\Gamma}_{n+m,n}$, $\hat{\Gamma}_{n+m,n}=\hat{\tau}_3(\Gamma^{(0)}_{n+m,n}\hat{\tau}_0+\Gamma^{(1)}_{n+m,n}\hat{\tau}_1)$. The Bethe-Salpeter equation for $\hat{\Gamma}_{n+m,n}$ is

$$\hat{\Gamma}_{n+m,n}=1+\nu T\sum_{n'} V^{l=0}_{n-n'}\int d\xi_k \hat{\mathcal{G}}_{n'+m}(\xi_k)\hat{\tau}_3\hat{\Gamma}_{n'+m,n'}\hat{\mathcal{G}}_{n'}(\xi_k)\hat{\tau}_3. \tag{B38}$$

Again, working in the small-Ω_m limit we can express this in components, as schematically depicted in Fig. 8,

$$\Gamma^{(0)}_{n+m,n}=1+\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(0)}_{n'+m,n'}\frac{\xi_{\mathbf{k}}^2-\tilde{\Sigma}_n^2-\phi_n^2}{\big(\tilde{\Sigma}_n^2+\phi_n^2+\xi_{\mathbf{k}}^2\big)^2}V^{l=0}_{n-n'}+i\Omega_m\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(1)}_{n'+m,n'}\frac{\phi_n\big[1+\frac{d\Sigma_n}{d\epsilon_n}\big]-\tilde{\Sigma}_n\frac{d\phi_n}{d\epsilon_n}}{\big(\tilde{\Sigma}_n^2+\phi_n^2+\xi_{\mathbf{k}}^2\big)^2}V^{l=0}_{n-n'},$$

$$\Gamma^{(1)}_{n+m,n}=\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(1)}_{n'+m,n'}\frac{1}{\tilde{\Sigma}_n^2+\phi_n^2+\xi_{\mathbf{k}}^2}V^{l=0}_{n-n'}-i\Omega_m\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda} d\xi_k \Gamma^{(0)}_{n'+m,n'}\frac{\phi_n\big[1+\frac{d\Sigma_n}{d\epsilon_n}\big]-\tilde{\Sigma}_n\frac{d\phi_n}{d\epsilon_n}}{\big(\tilde{\Sigma}_n^2+\phi_n^2+\xi_{\mathbf{k}}^2\big)^2}V^{l=0}_{n-n'}. \tag{B39}$$

Integrating over $\xi_{\mathbf{k}}$, we find

$$\Gamma^{(0)}_{n+m,n}=1+i\Omega_m\frac{\nu}{4}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{\Delta_{n'}\Big[1+\frac{d\Sigma_{n'}}{d\epsilon_{n'}}\Big]-\epsilon_{n'}\frac{d\phi_{n'}}{d\epsilon_{n'}}}{Z_n^2\big(\epsilon_{n'}^2+\Delta_{n'}^2\big)^{3/2}}V^{l=0}_{n-n'},\ \Gamma^{(1)}_{n+m,n}=\frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{1}{Z_{n'}\sqrt{\epsilon_{n'}^2+\Delta_{n'}^2}}V^{l=0}_{n-n'}$$

$$-i\Omega_m\frac{\nu}{4}\int d\epsilon_{n'}\Gamma^{(0)}_{n'+m,n'}\frac{\Delta_{n'}\Big[1+\frac{d\Sigma_{n'}}{d\epsilon_{n'}}\Big]-\epsilon_{n'}\frac{d\phi_{n'}}{d\epsilon_{n'}}}{Z_n^2\big(\epsilon_{n'}^2+\Delta_{n'}^2\big)^{3/2}}V^{l=0}_{n-n'}. \tag{B40}$$

Note that there is no term with $\Gamma^{(0)}$ in the right-hand side of the first equation. We again use Eq. (27) to rewrite

$$\Delta_{n'}\left[1+\frac{d\Sigma_{n'}}{d\epsilon_{n'}}\right]-\epsilon_{n'}\frac{d\phi_{n'}}{d\epsilon_{n'}}=Z_n\left(\Delta_n-\frac{d\Delta_n}{d\epsilon_n}\right). \tag{B41}$$

Substituting into Eq. (B40), we obtain

$$\Gamma^{(0)}_{n+m,n}=1+i\Omega_m\frac{\nu}{4}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{\Delta_{n'}-\frac{d\Delta_{n'}}{d\epsilon_{n'}}}{Z_{n'}\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'},$$
$$\Gamma^{(1)}_{n+m,n}=\frac{\nu}{2}\int d\epsilon_{n'}\Gamma^{(1)}_{n'+m,n'}\frac{1}{Z_{n'}\sqrt{\epsilon^2_{n'}+\Delta^2_{n'}}}V^{l=0}_{n-n'}-i\Omega_m\frac{\nu}{4}\int d\epsilon_{n'}\Gamma^{(0)}_{n'+m,n'}\frac{\Delta_{n'}-\frac{d\Delta_{n'}}{d\epsilon_{n'}}}{Z_{n'}\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'}. \tag{B42}$$

One can straightforwardly verify that at $\Omega_m\to 0$, the solution of these equations is

$$\lim_{\Omega_m\to 0}\Gamma^{(0)}_{n+m,n}=1+\frac{\nu}{2}\int d\epsilon_{n'}\Delta_{n'}\frac{\Delta_{n'}-\frac{d\Delta_{n'}}{d\epsilon_{n'}}}{\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}V^{l=0}_{n-n'}=1+\frac{d\Sigma_n}{d\epsilon_n},\quad \lim_{\Omega_m\to 0}\Gamma^{(1)}_{n+m,n}=\frac{\nu}{i\Omega_m}\int d\epsilon_{n'}\frac{\Delta_{n'}}{\sqrt{\epsilon^2_{n'}+\Delta^2_{n'}}}V^{l=0}_{n-n'}=\frac{2\phi_n}{i\Omega_m}. \tag{B43}$$

Note that for this solution the last term for Γ^1 can be dropped in Eq. (B43).

We can now verify that $\Pi^{ch}(\mathbf{q}=0,i\Omega_m\to 0)$ vanishes as required by global charge conservation. The ladder diagrams contributing to the bubble are still given by Figs. 9 and 10, but the side vertices are now spin δ functions. Analytically, the full $\Pi^{ch}(\mathbf{q}=0,i\Omega_m\to 0)$ is expressed as

$$\Pi^{ch}(\mathbf{q}=0,i\Omega_m\to 0)=\frac{\nu}{2\pi}\int d\epsilon_{n'}\int_{-\Lambda}^{\Lambda}d\xi_k 2\left[\left(1+\frac{d\Sigma_{n'}}{d\epsilon_{n'}}\right)\frac{\left(\xi_k^2-\tilde{\Sigma}^2_{n'}-\phi^2_{n'}\right)}{\left(\tilde{\Sigma}^2_{n'}+\phi^2_{n'}+\xi^2_{\mathbf{k}}\right)^2}+\frac{2\phi_{n'}}{i\Omega_m}\frac{i\Omega_m}{2}\frac{Z_{n'}\left(\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}\right)}{\left(\tilde{\Sigma}^2_{n'}+\phi^2_{n'}+\xi^2_{\mathbf{k}}\right)^2}\right]. \tag{B44}$$

Substituting $\Gamma^{(0)}_{n+m,n}$ and $\Gamma^{(1)}_{n+m,n}$ at $\Omega_m\to 0$ from Eq. (B43) and doing the integrals as in the normal state, we find

$$\Pi^{ch}(\mathbf{q}=0,i\Omega_m\to 0)=\nu\left(-2+\int d\epsilon_{n'}\Delta_{n'}\frac{\Delta_{n'}-\epsilon_{n'}\frac{d\Delta_{n'}}{d\epsilon_{n'}}}{\left(\epsilon^2_{n'}+\Delta^2_{n'}\right)^{3/2}}\right)=0. \tag{B45}$$

In the last line we used that the integrand is a total derivative. We thus verify that the particle density is conserved if the relation between the vertex function and the self-energy is given by Eq. (B43).

With some extra effort, one can extend the analysis to finite Ω_m and show that the solution of Eq. (B42) is

$$\hat{\Gamma}_{n+m,n}=\hat{\tau}_0+i\frac{\hat{\tau}_3\hat{\Sigma}_{n+m}\hat{\tau}_3-\hat{\Sigma}_n}{\Omega_m} \tag{B46}$$

or in components

$$\Gamma^{(0)}_{n+m,n}=1+\frac{\Sigma_{n+m}-\Sigma_n}{\Omega_m},\quad \Gamma^{(1)}_{n+m,n}=-i\frac{\phi_{n+m}+\phi_n}{\Omega_m}. \tag{B47}$$

This result has been obtained by Nambu [50] by different means. As we said, at $\Omega_m\to 0$ these relations reduce to

$$\Gamma^{(0)}_{n+m,n}\xrightarrow{\Omega_m\to 0}1+\frac{d\Sigma_n}{d\epsilon_n},\qquad \Gamma^{(1)}_{n+m,n}\xrightarrow{\Omega_m\to 0}-i\frac{2\phi_n}{\Omega_m}. \tag{B48}$$

Comparing Ward identities for spin and charge vertices, Eqs. (B28) and (B48), we see that the ones for $\Gamma^{(0)}$ are identical, while the ones for $\Gamma^{(1)}$ are different. In particular, for the spin vertex $\Gamma^{(1)}$ vanishes for frequency-independent gap, while for the charge vertex it remains finite and moreover is singular at $\Omega_m\to 0$. As we said, the origin for the difference is in the fact that charge fluctuations couple linearly to massless phase fluctuations and spin fluctuations do not couple to phase fluctuations.

d. Ward identity for momentum conservation

For completeness, we also consider the Ward identity associated with translational invariance, i.e., with conservation of

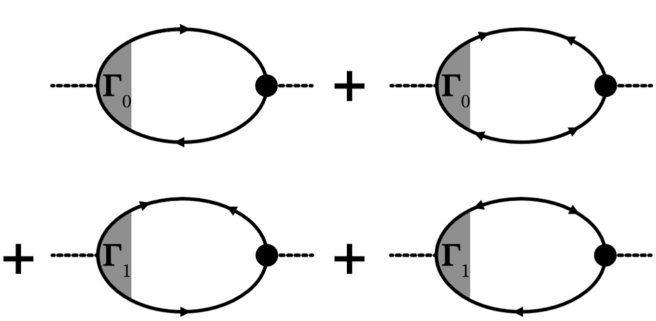

FIG. 10. Bubble diagram for the renormalized spin or charge correlator at $\mathbf{q}=0$. The normal Γ^0 and anomalous Γ^1 vertices are solutions of the Bethe-Salpeter equations (B24) and (B38) for spin and charge, respectively. The diagrammatic formulation of the vertex renormalization is shown in Fig. 8.

the total momentum. Let us consider a model with action

$$S = \sum_{n,\mathbf{p}} \bar{\Psi}(\mathbf{p},\tau)[-\partial_\tau + \hat{H}(\mathbf{p})]\Psi(\mathbf{p},\tau) + \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \int d\tau\, d\tau' V(t-t',\mathbf{q})\bar{\Psi}\Big(\mathbf{k}+\frac{\mathbf{q}}{2},\tau\Big)\hat{\tau}_3 \times \Psi\Big(\mathbf{k}-\frac{\mathbf{q}}{2},\tau\Big)\bar{\Psi}\Big(\mathbf{k}'-\frac{\mathbf{q}}{2},\tau'\Big)\hat{\tau}_3\Psi\Big(\mathbf{k}+\frac{\mathbf{q}}{2},\tau'\Big), \tag{B49}$$

where Ψ are Nambu spinors and we allow for the possibility of a time-dependent interaction due to exchange of bosons. The local symmetry transformation associated with translational invariance and appearing in Noether's theorem is $\Psi(\mathbf{p},\tau) = e^{i\boldsymbol{\alpha}(\tau)\cdot\mathbf{p}}\Psi'(\mathbf{p},\tau)$ for the Nambu spinors. Under such a change of variables the action changes as

$$S[\bar{\psi},\psi] = S[\bar{\psi}',\psi'] + \delta S[\bar{\psi}',\psi] + O(\alpha^2), \tag{B50}$$

where

$$\delta S = \int d\tau\, \boldsymbol{\alpha}(\tau)\cdot\partial_t \sum_{\mathbf{p}} \mathbf{p}\bar{\psi}(\mathbf{p},\tau)\psi(\mathbf{p},\tau) - i\frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} d\tau\, d\tau' \mathbf{q}\cdot[\boldsymbol{\alpha}(\tau)-\boldsymbol{\alpha}(\tau')]V(t-t',\mathbf{q})\bar{\psi} \times \Big(\mathbf{k}+\frac{\mathbf{q}}{2},\tau\Big)\hat{\tau}_3\psi\Big(\mathbf{k}-\frac{\mathbf{q}}{2},\tau\Big)\bar{\psi}\Big(\mathbf{k}'-\frac{\mathbf{q}}{2},\tau'\Big) \times \hat{\tau}_3\psi\Big(\mathbf{k}+\frac{\mathbf{q}}{2},\tau'\Big). \tag{B51}$$

This defines, through Noether's theorem, the total momentum of the system $\langle \mathbf{P}(\tau)\rangle$ via

$$\delta S \equiv i\int d\tau\, \boldsymbol{\alpha}(\boldsymbol{\tau})\cdot\partial_t\mathbf{P}(\tau). \tag{B52}$$

Note that *only for an instantaneous interaction is the total fermionic momentum separately conserved*. This simply reflects the fact that the bosons mediating the interaction may carry momentum too.

In the usual fashion [54] one may obtain a Ward identity by considering such a symmetry transformation within the functional integral. Specifically consider the following expectation value, where we perform a change of coordinates in the functional integral:

$$\begin{aligned}\langle\Psi(\mathbf{k},\tau)\bar{\Psi}(\mathbf{k}',\tau)\rangle &= \frac{1}{Z}\oint \mathcal{D}[\bar{\Psi},\Psi]\Psi(\mathbf{k},\tau)\bar{\Psi}(\mathbf{k}',\tau)e^{-S[\bar{\Psi},\Psi]}\\ &= \frac{1}{Z}\oint \mathcal{D}[\bar{\Psi},\Psi][1+i\boldsymbol{\alpha}(\tau)\cdot\mathbf{k}]\\ &\quad\times \Psi'(\mathbf{k},\tau)\bar{\Psi}(\mathbf{k}',\tau)[1-i\boldsymbol{\alpha}(t')\cdot\mathbf{k}']\\ &\quad\times e^{-S[\bar{\Psi}',\Psi']}(1-\delta S[\bar{\Psi}',\Psi']) + O(\alpha^2).\end{aligned} \tag{B53}$$

Using the fact that the measure is invariant under the change of variables $\Psi \to \Psi'$ we then obtain

$$\begin{aligned}&[i\boldsymbol{\alpha}(\tau)\cdot\mathbf{k} - i\boldsymbol{\alpha}(t')\cdot\mathbf{k}']\langle\Psi(\mathbf{k},\tau)\bar{\Psi}(\mathbf{k}',\tau)\rangle + O(\alpha^2)\\ &\quad= \langle\Psi(\mathbf{k},\tau)\bar{\Psi}(\mathbf{k}',\tau)\delta S[\bar{\Psi},\Psi]\rangle.\end{aligned} \tag{B54}$$

The expectation value on the left-hand side is simply the Green's function, while the right-hand side is related to the vertex function via the usual rule, expressed here in terms of Matsubara frequencies,

$$\Big\langle \Psi_n(\mathbf{k})\bar{\Psi}_{n'}(\mathbf{k}')\sum_{n'',\mathbf{p}}\bar{\Psi}_{n''}(\mathbf{p})\hat{\gamma}_{n''}(\mathbf{p})\Psi_{n''}(\mathbf{p})\Big\rangle \equiv \hat{\mathcal{G}}_n(\mathbf{k})\hat{\Gamma}_{n,n'}(\mathbf{k})\hat{\mathcal{G}}_{n'}(\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}'), \tag{B55}$$

where $\hat{\Gamma}$ is the fully renormalized vertex corresponding to $\hat{\gamma}$. Using these expectation values and writing the correlators in terms of Matsubara frequencies we then can rewrite Eq. (B54) as

$$\begin{aligned}&T\sum_n[i\boldsymbol{\alpha}_m\cdot\mathbf{k}(-\hat{\mathcal{G}}_{n-m}(\mathbf{k})\beta\delta_{n-n'-m}\delta(\mathbf{k}-\mathbf{k}'))]\\ &\quad- T\sum_n[i\boldsymbol{\alpha}_m\cdot\mathbf{k}'(-\hat{\mathcal{G}}_n(\mathbf{k}')\beta\delta_{n-n'-m}\delta(\mathbf{k}-\mathbf{k}'))]\\ &\quad= i\sum_m(i\Omega_m)\boldsymbol{\alpha}_m\hat{\mathcal{G}}_n(\mathbf{k})\hat{\Gamma}^{\text{mom}}_{n,n-m}(\mathbf{k})\hat{\mathcal{G}}_{n-m}(\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{B56}$$

We now use the fact that α_m is arbitrary and additionally act on both sides with the inverse of the Green's functions to arrive at the Ward identity for the momentum vertex

$$i\Omega_m\hat{\Gamma}^{\text{mom}}_{n+m,n}(\mathbf{k}) = \mathbf{k}\big(\hat{\mathcal{G}}^{-1}_{n+m} - \hat{\mathcal{G}}^{-1}_n\big) \tag{B57}$$

or

$$\hat{\Gamma}^{\text{mom}}_{n+m,n}(\mathbf{k}) = \mathbf{k}\left(1 + i\frac{\hat{\Sigma}^{-1}_{n+m} - \hat{\Sigma}^{-1}_n}{\Omega_m}\right). \tag{B58}$$

This relation holds for both Galilean-invariant and non-Galilean-invariant systems. However, in general this does not determine the form of the current vertex. Only *in a Galilean-invariant system* is the current given by

$$\mathbf{J}(\tau) = \frac{e}{m}\mathbf{P}(\tau) = \sum_{\mathbf{p}} e\frac{\mathbf{p}}{m}\bar{\Psi}(\mathbf{p},\tau)\Psi(\mathbf{p},\tau) \tag{B59}$$

and thus the renormalized current vertex is determined directly from Eq. (B58).

APPENDIX C: ξ-INTEGRATED GREEN'S FUNCTION WITH SUPERCURRENT

For a translationally invariant state, we can express the inverse Green's function to second order in $\mathbf{Q}$ as

$$\hat{\mathcal{G}}^{-1}_k = i\tilde{\Sigma}_n - \frac{\mathbf{k}}{m_1}\cdot\mathbf{Q} - \left(\xi_{\mathbf{k}} + \frac{Q^2}{2m_2} + \chi_k\right)\hat{\tau}_3 - \phi_k\hat{\tau}_1. \tag{C1}$$

The ξ-integrated Green's function is then, according to Eq. (9),

$$\hat{g}_n(\mathbf{k}_F) = \frac{1}{\pi}\int d\xi \frac{[\tilde{\Sigma}_n(\mathbf{k}_F) + i\mathbf{v}_F\cdot\mathbf{Q}]\hat{\tau}_3 - i\big[\xi + \frac{Q^2}{2m_2} + \chi_n(\mathbf{k}_F)\big] + \phi_n(\mathbf{k}_F)\hat{\tau}_2}{[\tilde{\Sigma}_n(\mathbf{k}_F) + i\mathbf{v}_F\cdot\mathbf{Q}]^2 + \big[\xi + \frac{Q^2}{2m_2} + \chi_n(\mathbf{k}_F)\big]^2 + |\phi_n(\mathbf{k}_F)|^2}. \tag{C2}$$

As long as the quasiclassical approximation holds, we can shift ξ to eliminate $(Q^2/2m_2) + \chi$ leaving simply

$$\hat{g}_n(\mathbf{k}_F) = \frac{[\tilde{\Sigma}_n(\mathbf{k}_F) + i\mathbf{v}_F\cdot\mathbf{Q}]\hat{\tau}_3 + \phi_n(\mathbf{k}_F)\hat{\tau}_2}{\sqrt{[\tilde{\Sigma}_n(\mathbf{k}_F) + i\mathbf{v}_F\cdot\mathbf{Q}]^2 + |\phi_n(\mathbf{k}_F)|^2}}. \tag{C3}$$

APPENDIX D: QUASICLASSICAL FREE-ENERGY FUNCTIONAL FOR ELIASHBERG THEORY

Starting with the inverse Green's function

$$\hat{\mathcal{G}}_k^{-1} = i\tilde{\Sigma}_n - \frac{\mathbf{k}}{m_1}\cdot\mathbf{Q} - \left(\xi_{\mathbf{k}} + \frac{Q^2}{2m_2} + \chi_k\right)\hat{\tau}_3 - \phi_k\hat{\tau}_1 \tag{D1}$$

we can evaluate the quasiclassical free energy as a sum of a kinetic term

$$F_{\text{kin}} = -T\ \ln\big[-\det\big(-\beta\hat{\mathcal{G}}_k^{-1}\big)\big] \tag{D2}$$

and a potential term

$$F_{\text{pot}} = -\frac{1}{2}T^2\sum_{k,k'}V_{k-k'}\text{tr}[\hat{\tau}_3\hat{\mathcal{G}}_k\hat{\tau}_3\hat{\mathcal{G}}_{k'}]. \tag{D3}$$

Note the presence of an additional minus sign inside the logarithm of the kinetic term, coming from the Nambu spinor measure

$$\begin{aligned} d\bar{\psi}_{k\uparrow}d\psi_{k\uparrow}d\bar{\psi}_{-k\downarrow}d\psi_{-k\downarrow} &= -d\bar{\psi}_{k\uparrow}d\psi_{k\uparrow}d\psi_{-k\downarrow}d\bar{\psi}_{-k\downarrow} \\ &= d\bar{\Psi}_k d\Psi_k. \end{aligned} \tag{D4}$$

1. Kinetic term

For the kinetic term we can start by writing

$$F_{\text{kin}} = -T\sum_k \ln\big(-\beta^2\det\hat{\mathcal{G}}_k^{-1}\big). \tag{D5}$$

The determinant is

$$\begin{aligned} D_k &\equiv -\det\hat{\mathcal{G}}_k^{-1} \\ &= \left(\tilde{\Sigma}_n + i\frac{\mathbf{k}}{m_1}\cdot\mathbf{Q}\right)^2 + \left(\xi_{\mathbf{k}} + \frac{Q^2}{2m_2} + \chi_k\right)^2 + \phi_k^2. \end{aligned} \tag{D6}$$

To regulate the sum, we will first integrate over momentum within finite limits and then take the limits to infinity at the end. Defining $S_n(\mathbf{k}_F)^2 = (\tilde{\Sigma}_n + i\frac{\mathbf{k}}{m_1}\cdot\mathbf{Q})^2 + \phi_k^2$,

$$\begin{aligned} F_{\text{kin}} =& -\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\int_{-\Lambda}^{\Lambda}d\xi\frac{S_n(\mathbf{k})^2 + \left(\xi_{\mathbf{k}} + \frac{Q^2}{2m_2} + \chi_k\right)^2}{T^2} \\ \equiv& -\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}I_n(\mathbf{k}_F). \end{aligned} \tag{D7}$$

Let us define dimensionless variables

$$r \equiv \frac{\chi + \frac{Q^2}{2m_1}}{\Lambda},\ \ s \equiv \frac{S}{\Lambda},\ \ z \equiv \frac{\xi}{\Lambda}, \tag{D8}$$

which lets us write

$$\begin{aligned} I_n(\mathbf{k}_F) &= \Lambda\int_{-1}^{1}dz\ \ln\left[\frac{\Lambda^2}{T^2}[(z+r)^2 + s^2]\right] \\ &= 4\Lambda\ \ln\frac{\Lambda}{T} + \Lambda\int_{-1+r}^{1+r}dz\ \ln(z^2+s^2). \end{aligned} \tag{D9}$$

We can expand Taylor in the limits of the integral

$$\int_{-1+r}^{1+r}dx\, f(x) = \int_{-1}^{1}dx\, f(x) + r[f(1) - f(-1)] + O(r^2) \tag{D10}$$

and we find

$$I_n(\mathbf{k}_F) = 4\Lambda\ \ln\frac{\Lambda}{T} + \Lambda\int_{-1}^{1}dz\ \ln(z^2+s^2) + O(r^2). \tag{D11}$$

We can thus safely neglect χ and $Q^2/(2m_2)$ since they are, by assumption, much smaller than Λ. We evaluate the remaining integral using integration by parts:

$$\begin{aligned} \frac{I_n(\mathbf{k}_F)}{\Lambda} &= 4\ \ln\frac{\Lambda}{T} + +\cancel{z\ln(z^2+s^2)|_{-1}^{1}} - \int_{-1}^{1}dz\frac{2z^2}{z^2+s^2} \\ &= 4\ \ln\frac{\Lambda}{T} + -2\int_{-1}^{1}dz\frac{z^2+s^2-s^2}{z^2+s^2} \\ &= 4\ \ln\frac{\Lambda}{T} + -4 + 2\int_{-1/s}^{1/s}dy\frac{1}{1+y^2} \\ &= 4\ \ln\frac{\Lambda}{eT} + 2s\tan^{-1}y\big|_{-1/s}^{1/s} = 4\ \ln\frac{\Lambda}{eT} + 2\pi s + O(s^2). \end{aligned} \tag{D12}$$

In the limit of $\Lambda\to\infty$, the integral $I_n(\mathbf{k}_F)$ consists of an (infinite) constant term, which is irrelevant for the response of the system plus a a term of order Λ^0:

$$\lim_{\Lambda\to\infty}I_n(\mathbf{k}_F) = C_\Lambda + 2\pi S_n(\mathbf{k}_F). \tag{D13}$$

We thus arrive at the expression for the kinetic part of the quasiclassical free energy

$$\begin{aligned} F_{\text{kin}} =& -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}} \\ &\times\sqrt{[\tilde{\Sigma}_n(\mathbf{k}_F) + i\mathbf{v}_F\cdot\mathbf{Q}]^2 + \phi_n(\mathbf{k}_F)^2}. \end{aligned} \tag{D14}$$

2. Potential term

The potential term is straightforwardly simplified using the definition of the ξ-integrated Green's function (9):

$$\begin{aligned} F_{\text{pot}} = & -\frac{1}{2}T^2\sum_{n,n'}\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_{n-n'}(|\mathbf{k}_F-\mathbf{k}'_F|)\\ & \times \nu^2\int d\xi\int d\xi'\text{tr}[\hat{\tau}_3\hat{\mathcal{G}}_n(\xi,\mathbf{k}_F)\hat{\tau}_3\hat{\mathcal{G}}_{n'}(\xi',\mathbf{k}'_F)]\\ = & \,\nu^2\pi^2T^2\sum_{n,n'}\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_{n-n'}(|\mathbf{k}_F-\mathbf{k}'_F|)\\ & \times[g_n(\mathbf{k}_F)g_{n'}(\mathbf{k}'_F)+f_n(\mathbf{k}_F)f_{n'}(\mathbf{k}'_F)]. \end{aligned} \tag{D15}$$

3. Total quasiclassical expression

In combining the two terms we note that we can rewrite the kinetic part

$$\begin{aligned} F_{\text{kin}} = & -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\frac{\Upsilon_n(\mathbf{k}_F)^2+\phi_n(\mathbf{k}_F)^2}{S_n(\mathbf{k}_F)}\\ = & -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\frac{\varpi_n(\mathbf{k}_F)\Upsilon_n(\mathbf{k}_F)+\Delta_n(\mathbf{k}_F)\phi_n(\mathbf{k}_F)}{S_n(\mathbf{k}_F)}\\ = & -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}[\Upsilon_n(\mathbf{k}_F)g_n(\mathbf{k}_F)+\phi_n(\mathbf{k}_F)f_n(\mathbf{k}_F)]\\ = & -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\varpi_n(\mathbf{k}_F)g_n(\mathbf{k}_F)\\ & -2\nu^2\pi^2T^2\sum_{n,n'}\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_{n-n'}(|\mathbf{k}_F-\mathbf{k}'_F|)\\ & \times[g_n(\mathbf{k}_F)g_{n'}(\mathbf{k}'_F)+f_n(\mathbf{k}_F)f_{n'}(\mathbf{k}'_F)], \end{aligned} \tag{D16}$$

where in the last equality we used the gap equation. We see that the second term is just $-2F_{\text{pot}}$ and thus we have

$$\begin{aligned} F = & -2\pi\nu T\sum_n\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\varpi_n(\mathbf{k}_F)g_n(\mathbf{k}_F)\\ & -\nu^2\pi^2T^2\sum_{n,n'}\oint_{\text{FS}}\frac{d\mathbf{k}_F}{S_{d-1}}\oint_{\text{FS}}\frac{d\mathbf{k}'_F}{S_{d-1}}V_{n-n'}(|\mathbf{k}_F-\mathbf{k}'_F|)\\ & \times[g_n(\mathbf{k}_F)g_{n'}(\mathbf{k}'_F)+f_n(\mathbf{k}_F)f_{n'}(\mathbf{k}'_F)]. \end{aligned} \tag{D17}$$

APPENDIX E: EVALUATION OF DYNAMIC COEFFICIENT

The integral of ξ may be performed immediately in Eq. (79). By rescaling the integration variable, we may then express the frequency integral as

$$\kappa = 2\nu\lim_{\Lambda\to\infty}\int\frac{dz}{2\pi}\frac{2}{1+z^2Z^2(\Lambda z)}. \tag{E1}$$

In general $Z(\epsilon)$ has the following properties:

(i) Z is an even function of frequency;
(ii) at large frequencies Z goes to 1, i.e., $\exists 0<\Omega_{\text{FL}}\ll\Lambda$ s.t. $\forall\,|\epsilon|>\Omega_{\text{FL}}, Z_\epsilon-1\leqslant(\Omega_{\text{FL}}/\Lambda)$.

With this in mind, we split the integration into a low-energy and a high-energy part:

$$\begin{aligned} \kappa & \approx 2\nu\int\frac{dz}{2\pi}\frac{2}{1+z^2Z^2(\Lambda z)}\\ & = 4\nu\left(\int_0^{\Omega_{\text{FL}}/\Lambda}+\int_{\Omega_{\text{FL}}/\Lambda}^{\infty}\right)\frac{dz}{2\pi}\frac{2}{1+z^2Z^2(\Lambda z)}. \end{aligned} \tag{E2}$$

The first term can then be bounded by

$$\int_0^{\Omega_{\text{FL}}/\Lambda}dz\frac{1}{1+z^2Z^2(\Lambda z)}\leqslant\frac{\Omega_{\text{FL}}}{\Lambda}, \tag{E3}$$

while for the second term

$$\begin{aligned} & \int_{\Omega_{\text{FL}}/\Lambda}^{\infty}dz\frac{1}{1+z^2Z^2(\Lambda z)}\\ & = \int_{\Omega_{\text{FL}}/\Lambda}^{\infty}dz\frac{1}{1+z^2}\frac{1}{1+\frac{z^2}{1+z^2}[Z^2(\Lambda z)-1]}\\ & \approx \int_{\Omega_{\text{FL}}/\Lambda}^{\infty}dz\frac{1}{1+z^2}\left(1-\frac{z^2}{1+z^2}[Z^2(\Lambda z)-1]+\cdots\right)\\ & \approx \int_0^{\infty}dz\frac{1}{1+z^2}+O\left(\frac{\Omega_{\text{FL}}}{\Lambda}\right). \end{aligned} \tag{E4}$$

We thus arrive at

$$\kappa = 2\frac{\nu}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+z^2} = 2\nu, \tag{E5}$$

where all other terms vanish in the $\Lambda\to\infty$ limit.

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