Frustrated magnets in the limit of infinite dimensions: Dynamics and disorder-free glass transition

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(Received 16 November 2023; accepted 27 March 2024; published 18 April 2024)

We study the statistical mechanics and the equilibrium dynamics of a system of classical Heisenberg spins with frustrated interactions on a d-dimensional simple hypercubic lattice, in the limit of infinite dimensionality $d \to \infty$. In the analysis we consider a class of models in which the matrix of exchange constants is a linear combination of powers of the adjacency matrix. This choice leads to a special property: the Fourier transform of the exchange coupling $J(\mathbf{k})$ presents a $(d-1)$-dimensional surface of degenerate maxima in momentum space. Using the cavity method, we find that the statistical mechanics of the system presents for $d \to \infty$ a paramagnetic solution which remains locally stable at all temperatures down to $T = 0$. To investigate whether the system undergoes a glass transition we study its dynamical properties assuming a purely dissipative Langevin equation, and mapping the system to an effective single-spin problem subject to a colored Gaussian noise. The conditions under which a glass transition occurs are discussed including the possibility of a local anisotropy and a simple type of anisotropic exchange. The general results are applied explicitly to a simple model, equivalent to the isotropic Heisenberg antiferromagnet on the d-dimensional face-centered-cubic lattice with first- and second-nearest-neighbor interactions tuned to the point $J_1 = 2J_2$. In this model, we find a dynamical glass transition at a temperature T_g separating a high-temperature liquid phase and a low-temperature vitrified phase. At the dynamical transition, the Edwards-Anderson order parameter presents a jump demonstrating a first-order phase transition.

DOI: 10.1103/PhysRevB.109.144414

I. INTRODUCTION

Many important phenomena in magnetism are controlled by frustration. Striking examples are spin systems on triangular, kagome, pyrochlore, face-centered-cubic (fcc), and other geometrically frustrated lattices, which present an extremely rich and complex phenomenology [1]. In many of these systems, frustration forces every plaquette to present multiple degenerate minima. In the lattice, when plaquettes share corners or edges, this leads often to striking collective behaviors.

At the same time, frustration is a key ingredient in systems displaying rugged energy landscapes, slow relaxation, and glassy dynamics. Prototype examples are spin glasses [2,3], controlled by a random distribution of exchange couplings. However, frustration has a central role also in one of the theories for the vitrification of liquids [4,5], in Coulomb-frustrated models [4–14], and in systems which present random patterns of stripes [15], relevant, for example, in Coulomb-frustrated charge separation [6], ferromagnetic thin films [16], and possibly even biology [17]. More precisely, we are talking about models in which frustration arises from the competition between short- and long-range interactions (the latter can be Coulomb, dipole-dipole, or can have another nature). Here in the paper we will use the term "Coulomb frustrated" to refer to such models, just for brevity. These models are closely connected to the phenomenon of avoided criticality [5]. In Coulomb-frustrated theories, complementary theoretical analyses [4–14] and numerical simulations [11] have indicated of a glassy behavior even in absence of a quenched disorder. Similar behavior was shown to happen in magnetic systems with dipole-dipole interactions [16] and in Brazovskii-type models with a line of minima of exchange energy [15,18].

A natural question is therefore as follows: Under which conditions is frustration alone sufficient to generate a glassy state? This question has been a subject of investigations for decades. Many theoretical models have been analyzed, for example, fully frustrated spin systems [19–22], kinetically constrained models [23], and models of Josephson-junction arrays [24–26]. In Ref. [27], a spin model with three-spin interactions has been shown to undergo a glass transition even in presence of a purely ferromagnetic coupling. In this case, frustration is not present at the level of ground state, but a "dynamical" frustration present in finite-temperature configurations was shown to be sufficient to produce a glass phase.

In the context of geometrically frustrated magnets, the possibility of a glass transition in absence of disorder has attracted interest as an explanation to the behavior experimentally oberved in some kagome and pyrochlore materials [28–31]. We note, for example, that a glassy "spin jam" state has been predicted in a Heisenberg model on a lattice consisting of a triangular network of bipyramids, a structure realized in $SrCr_{9p}Ga_{12-9p}O_{19}$ [32]. A key element of the theory of Ref. [32] is the prediction that, in this structure, a quantum order-by-disorder effect can generate nonperiodic metastable

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states separated by large barriers, and that the number of metastable states grows exponentially with the perimeter, rather than exponentially with the area.

Recently, systems which combine a geometrically frustrated lattice with a long-range interaction have attracted attention. In Ref. [33], it has been shown by numerical methods that an electron system with Coulomb repulsion on the triangular lattice undergoes an "electron glass" transition at quarter filling. The mechanism proposed for the emergence of glassiness is that the long-range force lifts the degeneracy between the many classical ground states on the triangular lattice, introducing large energy barriers and a rugged landscape. The disorder-free glass transition detected theoretically has been proposed as an explanation for the slow dynamics observed in an organic conductor [33].

Another theory analyzed recently which combines similarly a frustrated geometry and a long-range coupling is the Ising model with dipolar interaction on the kagome net. For this model, Ref. [34] found numerical evidence of a glass transition and a slow relaxation consistent with the Vogel-Fulcher law. The model has been analyzed in Ref. [35] by an analytical method, based on the Bethe-Peierls approximation. The results confirmed the existence of a glass transition, although the second-order transition found in Ref. [35] indicates a divergence of the relaxation time different from the Vogel-Fulcher law, and similar to that of disordered spin glasses.

On the experimental side, two recent works [36,37] indicated that a disorder-free glass transition can occur even in an elemental solid: the rare-earth neodymium (Nd). In particular, Refs. [36,37] reported evidence of aging, ultraslow dynamics, and complex amorphous structures in nanoscale magnetization patterns imaged by spin-polarized scanning tunneling microscopy on the surface of Nd. It was shown that the randomness of the observed patterns becomes enhanced, and not reduced, when the defect concentration is lowered. This points to a theoretical explanation in terms of a "self-induced spin glass" ("self-induced" means an assumption that frustrations alone are sufficient for the spin-glass behavior in deterministic systems) [15,16]. This scenario has been corroborated by the observation of slow relaxation in spin dynamics simulations, performed using exchange constants calculated from first principles.

The results found recently motivate us to investigate an exactly solvable spin model exhibiting a disorder-free glass transition. In particular, we revisit a model studied by Lopatin and Ioffe [38], which has as its key ingredient a frustrated antiferromagnetic interaction on a large-dimensional hypercubic lattice. In the work of Ref. [38], the model was introduced as a lattice theory for the glass transition of supercooled liquids, and presented as fundamental degrees of freedom a set of Ising-type binary variables representing particles ($\rho_i = 1$) and holes ($\rho_i = 0$). The interaction was constructed assuming a matrix of exchange couplings of the form $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$ with $f(x) = -ux^2$ and $\hat{t}$ the adjacency matrix (a matrix t_{ij} such that $t_{ij} = 1$ if i and j are nearest neighbors and $t_{ij} = 0$ otherwise). Reference [38] solved the problem for an arbitrary chemical potential using a replica method. Despite the simplicity of the model, it was shown that the system undergoes a dynamical and a static glass transition.

In this work, we study a similar model in the case in which the degrees of freedom are classical spin vectors $\mathbf{S}_i$. The continuous nature of the degrees of freedom allows us to study the possibility of a vitrification transition from a dynamical analysis, based on a continuous Langevin equation. In the main part of the work, we derive a solution of the model, exact in the limit $d \to \infty$, including the possibility of anisotropy. Eventually, we apply the results to a special case: an isotropic model with $f(x) = J(x^2 - 1)$, $J < 0$. After separation of the hypercubic lattice into sublattices, this model is equivalent to two decoupled Heisenberg models on the fcc structure. In particular, the model has two nonzero couplings: a nearest-neighbor interaction of strength $J_1 = J/d$ and a second-nearest-neighbor interaction of magnitude $J_2 = J/(2d)$. For this model we identify a dynamical first-order glass transition, signaled by a jump of the Edwards-Anderson order parameter. The transition occurs even in absence of quenched disorder: the glass phase is self-induced.

The class of interactions $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$ has a special property, which has not been emphasized in Ref. [38]: for this special form of interaction, the Fourier transform $J(\mathbf{k})$ of the exchange coupling develops a $(d-1)$-dimensional surface of degenerate maxima in momentum space [39,40].

Similar degenerate surfaces appear in the Brazovskii model and in Coulomb-frustrated theories [4–16]. In Coulomb-frustrated models, the emergence of a glass phase has been investigated intensively using broken replica symmetry, with the specific tools borrowed from the theory of conventional, that is, disordered, spin glasses [41]. In particular, replica analyses based on the self-consistent screening approximation (SCSA) [8] and on local mean-field approximations [10,14] predicted a glassy behavior of the random first-order type. The presence of a glassy dynamical arrest has been corroborated by numerical simulations, and by an analytical study based on a mode-coupling theory and a dynamic SCSA [11,12]. In dipole-frustrated [16] and for Brazovskii-type models [14,15] analog glass transitions have been predicted within analyses by the replica method.

In Coulomb-frustrated models the surface of maxima occurs usually in the region of small momenta, at characteristic wavelengths much larger than the atomic spacing. In addition, the interactions have an infinite range. These factors introduce differences with respect to the model studied here, where the interactions are short ranged and the degenerate surface lies in a region of large wave vectors, comparable with the size of the Brillouin zone. Despite these differences, there remain similarities. This suggests that the exactly solvable model which we study exemplifies some of the features of the "stripe-glass" behavior, at a dynamical mean-field level.

The methodology used in our work allows to study vitrification in a physically transparent approach, based on explicit consideration of spin dynamics at large timescale, in analogy with the pioneering work by Edwards and Anderson on spin glasses [2] and to the mode-coupling approach by Götze in the theory of structural glasses [42].

The range of applications of the model is even broader. The theory analyzed here allows to investigate, in infinite dimensions, the model of a frustrated magnet with a degenerate surface of helical states. Spin systems with degenerate manifolds of spiral ground states space have attracted an extensive interest and have been predicted to host spiral spin-liquid phases (see Refs. [39,40,43–45] for theoretical analyses and for experimental evidences of spiral spin-liquid states in cubic spinels).

The paper is organized as follows. In Sec. II, we introduce the model analyzed in this work. In Sec. III, we derive an exact solution of the equilibrium statistical mechanics of the system in the limit $d \to \infty$. The solution shows that renormalizations stabilize the paramagnetic solution at all temperatures down to $T = 0$, a situation analog to the "avoided criticality" [7]. In Sec. IV, we study the dynamics of the system, assuming a purely dissipative Langevin equation. In the large-d limit, we show that the problem maps to an effective single-site Langevin equation with a colored noise, self-consistently determined by a set of consistency relations. The conditions under which the dynamics develops a glass transition are discussed in Sec. V. The general results are eventually applied in the case of an isotropic Heisenberg model, and for a simple interaction $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$ with $f(x) = J(x^2 - 1)$. The results for this model are presented in Sec. VI. We conclude the paper with a brief summary in Sec. VII.

A. Notations

Throughout the paper we will use the following notations. The symbol $\int_{\mathbf{S}_i}$ stands for an integral over all values of the spin $\mathbf{S}_i$ (in spherical coordinates $\int_{\mathbf{S}_i} = \int_0^{\pi} \sin\theta_i d\theta_i \int_0^{2\pi} d\varphi_i$). $\delta_S(\mathbf{S} - \mathbf{S}')$ is a delta function in spin space, defined according to the invariant measure [in spherical coordinates $\delta_S(\mathbf{S} - \mathbf{S}') = \delta(\theta - \theta')\delta(\varphi - \varphi')/\sin\theta$]. 3×3 matrices carrying spin indices such as $f^{\alpha\beta}(x)$ or $\chi^{\alpha\beta}$ are written in matrix notation with a symbol $\underline{\underline{f}}(x)$, $\underline{\underline{\chi}}$, etc. Matrices carrying lattice indices, such as t_{ij} and $J_{ij}^{\alpha\beta}$, are denoted as $\hat{t}$, $\hat{J}$. Matrix multiplications and inverses are defined assuming contraction of all internal spin and lattice indices ($[\hat{A}\hat{B}]_{ij}^{\alpha\beta} = \sum_k A_{ik}^{\alpha\gamma} B_{kj}^{\gamma\beta}$, $[\hat{A}^{-1}\hat{A}]_{ij}^{\alpha\beta} = \sum_k A^{-1\,\alpha\gamma}_{ik} A_{kj}^{\gamma\beta} = [\hat{1}]_{ij}^{\alpha\beta} = \delta^{\alpha\beta}\delta_{ij}$ for matrices with both spin and lattice indices; $[AB]_{ij} = \sum_k A_{ik}B_{kj}$ for matrices in site space only). We assume everywhere summation over repeated spin (greek) indices: $A^{\alpha\gamma}B^{\gamma\beta} = \sum_\gamma A^{\alpha\gamma}B^{\gamma\beta}$.

II. MODEL

Although we eventually apply the results explicitly to a particular isotropic model, in the main part of the work we keep the discussion more general, and analyze spin systems subject to a (possibly anisotropic) exchange interaction $J_{ij}^{\alpha\beta}$ and an onsite anisotropy $V(\mathbf{S}_i)$. We thus assume a Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}^{\alpha\beta} S_i^{\alpha} S_j^{\beta} + \sum_i V(\mathbf{S}_i). \tag{1}$$

The degrees of freedom $\mathbf{S}_i$ are classical spins: three-dimensional vectors with Cartesian coordinates S_i^{α}, $\alpha = x, y, z$, and with the constraint of unit length $\mathbf{S}_i^2 = S_i^{\alpha} S_i^{\alpha} = 1$. The labels i, j run over the sites of a hypercubic lattice in d dimensions. In the analysis we take eventually the limit of large dimensionality $d \to \infty$. The $d \to \infty$ approximation [46] has demonstrated its power for the consideration of lattice fermionic problems where it is the base of dynamical mean-field theory (DMFT) (for reviews see Refs. [47–49]).

The anisotropy energy $V(\mathbf{S}_i)$ in the model may be arbitrary. We make the only assumption that it is even [$V(\mathbf{S}_i) = V(-\mathbf{S}_i)$] in such way that H is invariant under time reversal, that is, spin reversal. For the exchange interaction we consider instead a coupling of a special form: we assume that $J_{ij}^{\alpha\beta} = [f^{\alpha\beta}(\hat{t}/\sqrt{2d})]_{ij}$, with $f^{\alpha\beta}(x)$ a 3×3 matrix of functions. This definition is a generalization of the interaction $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$, which was introduced and analyzed in the case of Ising variables by Lopatin and Ioffe [38]. In the definition, $\hat{t}$ is the adjacency matrix and powers of $\hat{t}$ are intended in the sense of matrix multiplication. For example, the function $f^{\alpha\beta}(x) = J_0^{\alpha\beta} + J_1^{\alpha\beta}x + J_2^{\alpha\beta}x^2 + J_4^{\alpha\beta}x^4$ corresponds to the coupling

$$J_{ij}^{\alpha\beta} = J_0^{\alpha\beta}\delta_{ij} + \frac{J_1^{\alpha\beta}}{\sqrt{2d}} t_{ij} + \frac{J_2^{\alpha\beta}}{2d}\sum_k t_{ik}t_{kj} + \frac{J_4^{\alpha\beta}}{4d^2}\sum_{k,l,m} t_{ik}t_{kl}t_{lm}t_{mj}, \tag{2}$$

an interaction which is nonzero between sites ij having a Manhattan distance $\ell_{ij} = \sum_{a=1}^{d} |i_a - j_a|$ at most equal to 4. In general, the exchange coupling is constructed from linear combinations of the matrix elements $[\hat{t}^n]_{ij} = \sum_{k_1,\dots,k_{n-1}} t_{ik_1}t_{k_1k_2}\dots t_{k_{n-2}k_{n-1}}t_{k_{n-1}j}$. These matrix elements are equal to the number of different paths which start at i, end at j, and are composed by a sequence of n steps along the bonds of the hypercubic lattice. If $f^{\alpha\beta}(x)$ is a polynomial in x the corresponding interaction is short ranged and the degree of the polynomial is equal to the range (measured according to the Manhattan distance).

For convenience, we assume that the onsite exchange vanishes: $J_{ii}^{\alpha\beta} = 0$. This is convenient for the subsequent mean-field calculations and does not imply a loss of generality because any local interaction can be absorbed in a redefinition of $V(\mathbf{S}_i)$. Imposing the condition $J_{ii}^{\alpha\beta} = 0$ requires to tune the constant term $J_0^{\alpha\beta}$ as a function of $J_2^{\alpha\beta}$, $J_4^{\alpha\beta}$, ... in such way as to cancel the contributions of the onsite matrix elements $[\hat{t}^{2n}/(2d)^n]_{ii}$ [50].

When the model is analyzed in an arbitrary dimension d and eventually in the limit $d \to \infty$, it is essential that the coupling constants are scaled in such way to ensure a limiting large-d model which is well defined and nontrivial. By taking $J_{ij}^{\alpha\beta} = [f^{\alpha\beta}(\hat{t}/\sqrt{2d})]_{ij}$ we have assumed, following Ref. [38], that the relevant scaling consists in assigning a factor of order $O(d^{-1/2})$ to every power of the adjacency matrix. The analyses in the next sections show indeed that the limit $d \to \infty$ at fixed $f^{\alpha\beta}(x)$ is well defined.

The consistency of the limit, however, does not occur for arbitrary $f^{\alpha\beta}(x)$ but requires that the function $f^{\alpha\beta}(x)$ satisfies certain conditions, which qualitatively correspond to the presence of a strong frustration. In fact the scaling $\approx d^{-1/2}$ is analog to the scaling needed in mean-field models of spin glasses [3] and in other strongly frustrated infinite-dimensional models [22,38]. If $f^{\alpha\beta}(x)$ described a nonfrustrated interaction, instead, a different scaling would be required. For example, in the case of a nearest-neighbor interaction, the exchange coupling would have to be scaled as $1/d$, and not as $d^{-1/2}$, to ensure a finite $d \to \infty$ limit [22,51].

Before discussing the conditions which $f^{\alpha\beta}(x)$ must satisfy in order to generate a model consistent with the $d^{-1/2}$ scaling, it is useful to describe some essential properties of the exchange coupling $J_{ij}^{\alpha\beta}$ which follow from the construction $J_{ij}^{\alpha\beta} = [f^{\alpha\beta}(\hat{t}/\sqrt{2d})]_{ij}$. As shown by a direct calculation, the Fourier transform of $J_{ij}^{\alpha\beta}$ is

$$J^{\alpha\beta}(\mathbf{k}) = \sum_i e^{-i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} J_{ij}^{\alpha\beta} = f^{\alpha\beta}(\varepsilon_{\mathbf{k}}), \tag{3}$$

with $\varepsilon_{\mathbf{k}} = \sqrt{2/d}\sum_{a=1}^{d}\cos(k_a)$. Thus, $J^{\alpha\beta}(\mathbf{k})$ depends on the wave vector $\mathbf{k}$ only through $\varepsilon_{\mathbf{k}}$. This implies that, for many choices of the function $f^{\alpha\beta}(x)$ the interaction can develop degenerate surfaces of maxima in momentum space. More precisely, consider the case in which the exchange is isotropic, $f^{\alpha\beta}(x) = \delta^{\alpha\beta} f(x)$. In dimension d, the variable $\varepsilon_{\mathbf{k}}$ is confined to the interval $-\sqrt{2d} \leqslant \varepsilon_{\mathbf{k}} \leqslant \sqrt{2d}$. As a result, two situations can occur, depending on the form of the function $f(x)$. If $f(x)$ reaches its maximum at one of the boundaries $\pm\sqrt{2d}$, the maxima of $J(\mathbf{k})$ occur at isolated points in momentum space [the center of the Brillouin zone $\Gamma = (k_1, \ldots, k_d) = (0, \ldots, 0)$, corresponding to $\varepsilon_\Gamma = \sqrt{2d}$, or the zone corner $R = (k_1, \ldots, k_d) = (\pi, \ldots, \pi)$ with $\varepsilon_R = -\sqrt{2d}$]. However, if $f(x)$, as a function of x, has maxima within the interval $-\sqrt{2d} < x < \sqrt{2d}$ then $J(\mathbf{k})$ reaches its maximum value on a surface in momentum space [the surface defined by the relation $\varepsilon_{\mathbf{k}} = \varepsilon_{\max}$, with $\varepsilon_{\max}$ the point, or points, at which f reaches its maximum; this defines a $(d-1)$-dimensional manifold in reciprocal space]. In the anisotropic case, the same analysis remains valid since the Fourier transform has constant values on surfaces in momentum space. For example, the maximum eigenvalue of $J^{\alpha\beta}(\mathbf{k})$ occurs at a $(d-1)$ manifold in $\mathbf{k}$ space, unless it occurs at the Γ or R points.

We now return to the discussion of the conditions under which the model admits a well-defined limit for $d \to \infty$. For large dimensionality the interval $[-\sqrt{2d}, \sqrt{2d}]$ on which ε is defined becomes unbounded and $\varepsilon_{\mathbf{k}}$ can assume arbitrary real values. If the eigenvalues of $f^{\alpha\beta}(x)$ have no upper bound on the real axis $-\infty < x < \infty$, the Fourier transform $J^{\alpha\beta}(\mathbf{k})$ can be made arbitrarily large by choosing $\mathbf{k}$ near the zone center Γ or the zone corner R. This then implies a ferromagnetic or antiferromagnetic instability at a critical temperature which diverges for $d \to \infty$ (see Sec. II A). The large-d limit, in this case, is inconsistent with the assumed $d^{-1/2}$ scaling. If instead, the eigenvalues of $f^{\alpha\beta}(x)$ are bounded from above, the maximum eigenvalue of $J(\mathbf{k})$ occurs on a $(d-1)$-dimensional surface in momentum space and remains finite when $d \to \infty$. This is the case which will be considered throughout the rest of the paper, and which leads at large d to a well-defined limit. Throughout all the rest of this work, we assume therefore that the boundedness condition is satisfied: the eigenvalues of $f^{\alpha\beta}(x)$ are assumed to be bounded from above on the real axis.

The boundedness condition on f has a simple interpretation in terms of the exchange couplings in real space. Suppose that the interaction has a range R, and, thus, that $f^{\alpha\beta}(\varepsilon) = \sum_{m=0}^{R} J_m^{\alpha\beta}\varepsilon^m$ is a polynomial of degree R. If the maximum degree R is odd, it is clear that the eigenvalues cannot be bounded: depending on the signs of $J_R^{\alpha\beta}$, the system presents ferromagnetic or antiferromagnetic instabilities. Thus, R must be even. In addition, since the interaction is required to have an upper bound, the interaction of maximum range $J_R^{\alpha\beta}$ must be negative (antiferromagnetic) [52]. To interpret this condition it is useful to note that the hypercubic lattice in d dimensions is bipartite and can be partitioned into two sublattices: the sublattice A composed of the sites such that the sum $\sum_{a=1}^{d} x_a$ of the coordinates $(x_1, \ldots, x_d)$ is an even integer and the complementary sublattice B where the sum $\sum_{a=1}^{d} x_a$ is an odd integer (see Fig. 1). The sublattices A and B have the structure of a generalized fcc lattice in d dimensions [53]. After this partitioning, it is clear that interactions with even range couple the spins within the same sublattice (A-A or B-B), while interactions with an odd range induce a coupling between different sublattices (A-B and B-A).

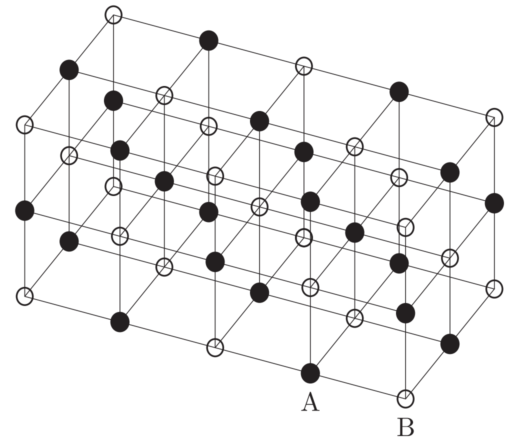


FIG. 1. Partitioning of the cubic lattice into two interpenetrating fcc sublattices (A and B) for $d = 3$. The sublattices A and B can be identified as the sets of lattice points $(x_1, \ldots, x_d)$ for which $\sum_{a=1}^{d} x_a$ is, respectively, even and odd. The separation of even and odd sites can be applied in arbitrary dimension d to separate the hypercubic lattice into fcc sublattices [53].

Thus, the conditions for a well-defined $d \to \infty$ limit are that the coupling of longest range is antiferromagnetic and takes place within the fcc sublattices, and not across the two different sublattices. This shows that the well-known frustration of the fcc structure [21,39,40,53] plays a role in the model studied here. We note, however, that in the family of interactions which we study the frustration is not purely geometrical because all models which we analyze have nonzero couplings beyond the nearest neighbors. For example, in Sec. VI we study the isotropic model with $V(\mathbf{S}) = 0$, $f^{\alpha\beta}(x) = J\delta^{\alpha\beta}(x^2 - 1)$, $J < 0$. In this model, since f is even in x, the interactions take place entirely within the fcc sublattices, and the intersublattice coupling vanishes. Thus, the theory describes effectively to two independent copies of a model on the fcc lattice. The model, however, has two nonzero couplings on the fcc lattice: a nearest-neighbor (NN) coupling of magnitude J/d and a second-NN interaction of magnitude $J/(2d)$. For this reason, the resulting problem differs significantly from the NN fcc model studied in Ref. [21]. The model with interaction $f(x) = J(x^2 - 1)$, $J <$

0, rather, provides a direct analog in infinite dimensions of the three-dimensional Heisenberg fcc antiferromagnet with first- and second-nearest-neighbor interactions tuned to $J_1 = 2J_2$. This model has been analyzed in Refs. [39,40] as a candidate for hosting spiral-spin-liquid behavior [43–45].

In general, all models analyzed here have the property of displaying surfaces of degeneracy in the Fourier transform of $J(\mathbf{k})$. Thus, the models which we discuss can be viewed as a large-dimensional mean-field theories of spiral-spin-liquid models. We note, however, that the special form $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$ of the coupling which we analyze is not the most general which is known to produce degenerate surfaces in momentum space. Models with an interaction constructed via powers of the adjacency matrix have been analyzed in the context of spiral spin liquids [44] but different theories, in which $\hat{J}$ is not reducible to a function of $\hat{t}$ have been studied [39,43].

A. Analysis within the Weiss mean-field approximation

Although the exact large-d results presented in the next sections lead to results which differ qualitatively from those of Weiss mean-field theory (MFT), a first understanding of the model, and in particular of the role of frustration for $d \to \infty$, can be obtained by analysis at the level of the MFT approximation. To illustrate this with an example, we consider first the case of the isotropic model $[V(\mathbf{S}) = 0]$ with $f^{\alpha\beta}(x) = \delta^{\alpha\beta} f(x)$, $f(x) = J(x^2 - 1)$ (this model will be addressed in more detail in Sec. VI using the complete large-d results). As mentioned above, the interaction $f(x) = J(x^2 - 1)$ leads to two decoupled Heisenberg models on two independent fcc sublattices. A given site $i = (x_1, x_2, \ldots, x_d)$ is coupled with an exchange coupling of strength $J_1 = J/d$ with the $2d(d-1)$ sites $j = (x_1 \pm 1, x_2 \pm 1, x_3, \ldots, x_d)$, $j = (x_1 \pm 1, x_2, x_3 \pm 1, \ldots, x_d)$, $j = (x_1, x_2 \pm 1, x_3 \pm 1, x_4, \ldots, x_d)$, which are NN on the fcc lattice, and with an exchange of magnitude $J/2d = J_1/2$ to the $2d$ sites $j = (x_1 \pm 2, x_2, \ldots, x_d)$, $(x_1, x_2 \pm 2, x_3, \ldots)$, ..., which belong to the second-NN shell on the fcc structure. The -1 in the definition $f(x) = J(x^2 - 1)$ is needed to ensure that $J_{ii} = 0$: it is required to cancel the onsite matrix element $[\hat{t}^2/(2d)]_{ii} = 1$. Within MFT, the susceptibility in the paramagnetic phase is given by

$$\chi^{-1\alpha\beta}(\mathbf{k}) = \chi_0^{-1\alpha\beta} - J^{\alpha\beta}(\mathbf{k}), \tag{4}$$

where $\chi_0^{\alpha\beta} = \delta^{\alpha\beta}/(3k_B T)$ is the susceptibility of a single spin (in the isotropic case). The MFT predicts as a result a second-order transition at a critical temperature $T_{\text{MFT}} = J_{\max}/(3k_B)$ where $J_{\max}$ is the maximum value of $J(\mathbf{k})$. By Eq. (3), $J_{\max}$ simply equals to the maximum $f_{\max}$ of the function $f(x)$. If the coupling is ferromagnetic ($J > 0$), $f(\varepsilon)$ is not bounded from above. In this case the interaction is not frustrated and the critical temperature T_{MFT} diverges for $d \to \infty$. If instead $J < 0$, $J_{\max} = f_{\max} = |J|$ and the Weiss transition temperature $T_{\text{MFT}} = |J|/(3k_B)$ remains finite.

The fact that T_{MFT} remains of $O(1)$ when $d \to \infty$ signals physically that the interaction is very strongly frustrated. In fact, the interaction in the model has a magnitude of order J/d but has a coordination of $z \approx d^2$ for d large. If the interactions were not strongly frustrated, we would expect a critical temperature which diverges for $d \to \infty$ as $z|J_2|d^{-1} \approx d^2 \times d^{-1} \propto d$. The fact that this divergence does not occur signifies that the local fields $\sum_j J_{ij}\mathbf{S}_j$ which act on a site i do not scale linearly with the dimension d but only as their square root $\sqrt{d}$. This behavior is analog to that of fully connected spin glasses [3] and of other mean-field frustrated models [22,38]. (See Ref. [20] for an analysis of the large-d scaling of the local fields in the fully frustrated hypercubic Ising model.)

The same considerations apply for any $f(\varepsilon)$ and in the anisotropic case. If the eigenvalues of $f^{\alpha\beta}(\varepsilon)$ are not bounded from above, the large-d limit is unstable and the MFT transition temperature diverges. If instead the eigenvalues of $f^{\alpha\beta}(\varepsilon)$ are bounded from above, the interaction is strongly frustrated the finite T_{MFT} indicates a stable large-d limit. The finiteness of T_{MFT} is again a sign of a very strong frustration. In fact, the expansion of $J_{ij}^{\alpha\beta} = [f^{\alpha\beta}(\hat{t}/\sqrt{2d})]_{ij}$ generates a coupling $J_{ij}^{\alpha\beta}$ of order $d^{-\ell_{ij}/2}$ where ℓ_{ij} is the Manhattan distance between i and j. The suppression factor $d^{-\ell_{ij}/2}$ does not compensate the growth of the coordination numbers $z_\ell \propto d^\ell$. Thus, in absence of frustration, we would expect a divergence $T_{\text{MFT}} \approx z_R|J_R|d^{-R/2} \propto d^R \times d^{-R/2} \approx d^{R/2}$, where R is the maximum range of the interaction.

As a remark, we note that the MFT analysis above relied, although in an indirect way, on the scaling factors $d^{-1/2}$ assigned to each power of the adjacency matrix. The role of the scaling factors has been to ensure that the condition $J_{ii}^{\alpha\beta} = 0$ can be imposed while keeping $f^{\alpha\beta}(x)$ fixed and finite as $d \to \infty$ [50].

B. Ground states in the isotropic case

In the isotropic case and when the function $f(x)$ is bounded from above, the interaction $J_{ij} = [f(\hat{t}/\sqrt{2d})]_{ij}$ admits always helical ground states of the form $\mathbf{S}_i = \mathbf{A}\cos(\mathbf{k}\cdot\mathbf{x}_i) + \mathbf{B}\sin(\mathbf{k}\cdot\mathbf{x}_i)$, with $\mathbf{A}^2 = \mathbf{B}^2 = 1$, $(\mathbf{A}\cdot\mathbf{B}) = 0$ (these are exact ground states and can be derived using the Luttinger-Tisza method) [7,39,40,44]. The modulation vector $\mathbf{k}$ is any wave vector belonging to the surface $\varepsilon_{\mathbf{k}} = \varepsilon_{\max}$, with $\varepsilon_{\max}$ a point at which $f(\varepsilon)$ is maximal. The exact ground-state energy per particle is therefore $E/N = -f(\varepsilon_{\max})/2 = -f_{\max}/2$ in isotropic models.

For some interactions, the single helices are not the only type of ground states. For example, if $f(x) = f(-x)$, so that the system breaks into two decoupled fcc sublattices, it is possible to construct ground states by taking independent helices, with arbitrary wave vectors $\mathbf{k}_A$ and $\mathbf{k}_B$ and arbitrary magnetization vectors $\mathbf{A}_A$, $\mathbf{B}_A$, $\mathbf{A}_B$, $\mathbf{B}_B$ on the two sublattices.

III. STATIC PROPERTIES

As in other mean-field frustrated models [3,38], the Weiss mean-field approximation is not exact for $d \to \infty$. The exact large-d solution is, rather, similar to the dynamical mean-field theory of correlated electron systems [51], and has to include local self-energy effects. In this section we derive an exact $d \to \infty$ solution of the statistical mechanical properties of the system, assuming a homogeneous state with unbroken translational and spin-inversion symmetries. In contrast with the behavior predicted by the MFT approximation, the exact large-d solution shows that the symmetric state remains locally stable at all temperatures down to $T = 0$.

To derive the large-d solution, we use an approach based on the cavity method (in the context of DMFT for fermions, see Ref. [51]). In particular we use as a starting point a methodology analog to the cavity approach applied in Ref. [3] to the Sherrington-Kirkpatrick (SK) spin-glass model. A key idea of this approach consists in analyzing the joint probability $P(\mathbf{S}_i, \mathbf{b}_i)$ of the spin $\mathbf{S}_i$ and the internal field $b_i^\alpha = \sum_j J_{ij}^{\alpha\beta} S_j^\beta$ acting at the same site i. In equilibrium at temperature T [54]

$$P(\mathbf{S}_i, \mathbf{b}_i) = Z^{-1} \int_{\bar{\mathbf{S}}_{i_1}} \cdots \int_{\bar{\mathbf{S}}_{i_N}} [\delta_S(\mathbf{S}_i - \bar{\mathbf{S}}_i)\delta(\mathbf{b}_i - \bar{\mathbf{b}}_i) \times \mathrm{e}^{-\beta H(\bar{\mathbf{S}}_{i_1},\ldots,\bar{\mathbf{S}}_{i_N})}], \tag{5}$$

where Z is the partition function, $\beta = 1/(k_B T)$, $\bar{b}_i^\alpha = \sum_j J_{ij}^{\alpha\beta} \bar{S}_j{}^\beta$, and the integral is over all possible configurations of the N spins. $P(\mathbf{S}_i, \mathbf{b}_i)$ can be interpreted, simply, as the probability that, extracting a random configuration at temperature T, the spin and the field at site i have given values $\mathbf{S}_i$ and $\mathbf{b}_i$.

The reason why P provides a particularly convenient framework for the analysis is that it describes efficiently a property of the large-d limit: for $d \to \infty$ the field $\mathbf{b}_i$ has strong correlations with the spin $\mathbf{S}_i$ at the same site, but weaker correlations with the spins $\mathbf{S}_j$ at the other sites $j \neq i$. Considering the joint distribution P allows to isolate the strong correlation of $\mathbf{b}_i$ with $\mathbf{S}_i$, by factorizing

$$P(\mathbf{S}_i, \mathbf{b}_i) = Z_i' Z^{-1} \mathrm{e}^{\beta(\mathbf{b}_i \cdot \mathbf{S}_i) - \beta V(\mathbf{S}_i)} p'(\mathbf{b}_i). \tag{6}$$

Here Z_i' and $p'(\mathbf{b}_i)$ are, respectively, the partition function and the distribution of the field $\mathbf{b}_i$ in a cavity system in which the site i is removed [an $(N-1)$-body system containing all spins apart from i]. Explicitly,

$$Z_i' = \int_{\mathbf{S}_{i_2}} \cdots \int_{\mathbf{S}_{i_N}} \mathrm{e}^{-\beta H_i'(\mathbf{S}_{i_1},\ldots,\mathbf{S}_{i_N})} \tag{7}$$

and

$$p'(\mathbf{b}_i) = Z_i'^{-1} \int_{\mathbf{S}_{i_2}} \cdots \int_{\mathbf{S}_{i_N}} \left[\delta\left(b_i^\alpha - \textstyle\sum_j J_{ij}^{\alpha\beta} S_j^\beta\right) \times \mathrm{e}^{-\beta H_i'(\mathbf{S}_{i_1},\ldots,\mathbf{S}_{i_N})}\right], \tag{8}$$

where

$$H_i' = -\frac{1}{2} \sum_{j\neq i, k\neq i} J_{jk}^{\alpha\beta} S_j^\alpha S_k^\beta + \sum_{j \neq i} V(\mathbf{S}_j) \tag{9}$$

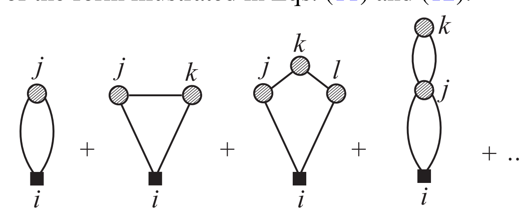
is the Hamiltonian of the cavity system. The cavity distribution $p'(\mathbf{b}_i)$ describes the probability to find a given value of $b_i^\alpha = \sum_j J_{ij}^{\alpha\beta} S_j^\beta$ when extracting a random configuration $\mathbf{S}_{i_2}$,..., $\mathbf{S}_{i_N}$ in the cavity system, in which the effects of the interactions with $\mathbf{S}_i$ are absent and, thus, is by construction uncorrelated to $\mathbf{S}_i$.

Thanks to the fact that the correlation between $\mathbf{b}_i$ and $\mathbf{S}_i$ is subtracted in $p'(\mathbf{b}_i)$, the cavity distribution has simple statistical properties in large dimensions. In the limit $d \to \infty$, as shown below, $p'(\mathbf{b}_i)$ is simply a Gaussian:

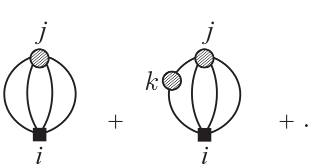
$$p'(\mathbf{b}_i) = \frac{\exp\left[-L^{-1\alpha\beta} b_i^\alpha b_i^\beta / 2\right]}{\sqrt{(2\pi)^3 \det \underline{\underline{L}}}}. \tag{10}$$

Here $\underline{\underline{L}}$ is a 3×3 matrix defining the covariance of the cavity distribution and needs to be determined as a function of the temperature T. Since we assumed a symmetric solution, with unbroken translation and spin-inversion symmetries, the Gaussian distribution is centered and $\underline{\underline{L}}$ does not depend on the lattice site i.

The result that the cavity distribution $p'(\mathbf{b}_i)$ is Gaussian can be derived using a perturbative analysis which is analog, from the diagrammatic point of view, to DMFT [51]. Let us discuss, in brief, the derivation. The ℓth cumulant $C_\ell'^{\alpha_1\ldots\alpha_\ell}$ of $p'(\mathbf{b}_i)$ is related to the connected moments of the cavity spin distribution via $C_\ell'^{\alpha_1\ldots\alpha_\ell} = \sum_{j_1,j_2,\ldots,j_\ell} J_{ij_1}^{\alpha_1\beta_1} \ldots J_{ij_\ell}^{\alpha_\ell\beta_\ell} \langle\langle S_{j_2}^{\beta_2} \ldots S_{j_\ell}^{\beta_\ell} \rangle\rangle'^{(i)}$. Here $\langle\langle \ldots \rangle\rangle'^{(i)}$ are connected averages computed in the cavity system [the symbol $\langle\langle \ldots \rangle\rangle$ is used to denote connected averages, and the sign $'(i)$ reminds that the averaging is computed in a system with one cavity at site i]. The C_ℓ', in turn, can be computed by a perturbative expansion in powers of $J_{ij}^{\alpha\beta}$ [a high-temperature expansion with fixed $\beta V(\mathbf{S})$]. The corresponding diagrams are linked-cluster graphs (see, for example, Refs. [55,56] for discussions of the linked-cluster expansion in spin systems). As a result, the diagrams contributing, for example, to the the cumulants C_2' and C_4' have a structure of the form illustrated in Eqs. (11) and (12):

$C_2'^{\alpha\beta} =$ 

$$= \sum_{j,k} J_{ij}^{\alpha\gamma} J_{ik}^{\delta\beta} \langle\langle S_j^\gamma S_k^\delta \rangle\rangle'^{(i)}$$
$$= \sum_{j\neq i} J_{ij}^{\alpha\gamma} \rho_2^{\gamma\delta} J_{ji}^{\delta\beta} + \sum_{j,k\ \neq i} \beta J_{ij}^{\alpha\gamma} \rho_2^{\gamma\mu} J_{jk}^{\mu\nu} \rho_2^{\nu\delta} J_{ki}^{\delta\beta}$$
$$+ \beta^2 \sum_{j,k,l\ \neq i} J_{ij}^{\alpha\gamma} \rho_2^{\gamma\mu} J_{jk}^{\mu\nu} \rho_2^{\nu\rho} J_{kl}^{\rho\sigma} \rho_2^{\sigma\delta} J_{li}^{\delta\beta}$$
$$+ \frac{\beta^2}{2} \sum_{j,k\ \neq i} J_{ij}^{\alpha\gamma} J_{ij}^{\beta\delta} \rho_4^{\gamma\delta\mu\nu} J_{jk}^{\mu\rho} \rho_2^{\rho\sigma} J_{kj}^{\sigma\nu} + \ldots, \tag{11}$$

$C_4'^{\alpha\beta\gamma\delta} =$ 

$$= \sum_{j,k,l,m\ \neq i} J_{ij}^{\alpha\mu} J_{ik}^{\beta\nu} J_{il}^{\gamma\rho} J_{im}^{\delta\sigma} \langle\langle S_j^\mu S_k^\nu S_l^\rho S_m^\sigma \rangle\rangle'^{(i)}$$
$$= \sum_{j\neq i} J_{ij}^{\alpha\mu} J_{ij}^{\beta\nu} J_{ij}^{\gamma\rho} J_{ij}^{\delta\sigma} \rho_4^{\mu\nu\rho\sigma}$$
$$+ 4\beta \sum_{j,k\ \neq i} J_{ik}^{\alpha\lambda} \rho_2^{\lambda\omega} J_{kj}^{\omega\mu} J_{ij}^{\gamma\rho} J_{ij}^{\delta\sigma} \rho_4^{\mu\nu\rho\sigma} + \ldots \tag{12}$$

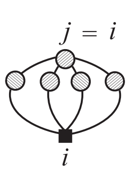
Here $\rho_2^{\alpha\beta} = \langle\langle S_i^\alpha S_i^\beta \rangle\rangle^{(0)}$ and $\rho_4^{\alpha\beta\gamma\delta} = \langle\langle S_i^\alpha S_i^\beta S_i^\gamma S_i^\delta \rangle\rangle^{(0)}$ are the second and fourth cumulants of the unperturbed spin distribution. These are equal to the cumulants of the single-site distribution $\rho(\mathbf{S}) = \mathrm{e}^{-\beta V(\mathbf{S})} / \int_{\bar{\mathbf{S}}} \mathrm{e}^{-\beta V(\bar{\mathbf{S}})}$: $\rho_2^{\alpha\beta} = \int_{\mathbf{S}} \rho(\mathbf{S}) S^\alpha S^\beta$, $\rho_4^{\alpha\beta\gamma\delta} = \int_{\mathbf{S}} \rho(\mathbf{S}) S^\alpha S^\beta S^\gamma S^\delta - \rho_2^{\alpha\beta}\rho_2^{\gamma\delta} - \rho_2^{\alpha\gamma}\rho_2^{\beta\delta} - \rho_2^{\alpha\delta}\rho_2^{\beta\gamma}$, and are represented by vertices with 2 and 4 incoming lines in the graphs. At higher orders the expansion involves vertices of arbitrary order (vertices with 6, 8, ... incoming lines), representing the mth-order cumulants of the noninteracting distribution $\rho_m^{\alpha_1 \ldots \alpha_m} = \partial^m [\ln \int_{\mathbf{S}} \rho(\mathbf{S}) \mathrm{e}^{\mathbf{x}\cdot\mathbf{S}}]/(\partial x^{\alpha_1} \ldots \partial x^{\alpha_m})|_{\mathbf{x}=0}$. The lines in the graphs, connecting different vertices, represent the interactions $J_{jk}^{\alpha\beta}$, $J_{ij}^{\alpha\beta}$.

In general, the expansion of any C'_ℓ with ℓ arbitrary can be represented in terms of graphs with the same structure of Eqs. (11) and (12). The graphs contributing to C'_ℓ have ℓ lines attached to the "origin" i, representing the ℓ factors $J_{ij}^{\alpha\beta}$ in $C_\ell'^{\alpha_1 \ldots \alpha_n} = \sum_{j_1,\ldots,j_\ell \neq i} J_{ij_1}^{\alpha_1\beta_1} \ldots J_{ij_\ell}^{\alpha_\ell\beta_\ell} \langle\langle S_{j_1}^{\beta_1} \ldots S_{j_\ell}^{\beta_\ell} \rangle\rangle'^{(i)}$, and any number of "internal" vertices and lines describing the perturbative expansion of the cavity correlations. Importantly, since C'_ℓ are moments of the cavity system, the summations over the lattice sites j, k, l range over all lattice points *except* the site i. The rules to associate a diagram with a perturbative term require to multiply a given term by a numerical prefactor counting the multiplicity of the diagram. The numerical prefactors, however, are irrelevant to the arguments below.

From the point of view of the large-d behavior, the graphs are completely analog to the diagrams of the cavity method in DMFT [51]. The perturbative expansion in $J_{ij}^{\alpha\beta}$, in particular, is identical diagrammatically to the expansion around the atomic limit (the exchange coupling $J_{ij}^{\alpha\beta}$ plays the role of the electron hopping amplitude t_{ij}).

Applying well-known arguments of power counting in $1/d$, it can be shown that C'_2 is of order $O(1)$ for $d \to \infty$, whereas all higher-order cumulants C'_ℓ with $\ell \geqslant 4$ vanish in the large-d limit [51]. In brief, for the interaction considered here, this can be shown using that the couplings $J_{ij}^{\alpha\beta}$ are of order $O(d^{-\ell_{ij}/2})$ where ℓ_{ij} is the Manhattan distance between i and j. As a consequence of this scaling, the diagrams contributing to C'_2 can be shown to be finite: the summations over internal coordinates compensate the suppression $O(d^{-\ell_{ij}/2})$ coming from the scaling of the interaction. The graphs for C'_4 or higher cumulants C'_ℓ, $\ell \geqslant 4$, instead, are suppressed in the large-d limit because they involve more than two lines connected to the site i and thus, more powers of $d^{-\ell_{ij}/2}$.

A crucial subtlety is that the vanishing of the higher cumulants C'_ℓ $\ell \geqslant 4$ relies essentially on the cavity construction. In fact, a diagram such as



would give a finite, $O(1)$, contribution if the summation over j was allowed to run over all the N sites of the system, including the site i. As is well known by standard power-counting arguments, the $O(1)$ contribution of the graph would arise entirely from the term $j = i$, in which the internal vertex j is "collapsed" to the origin [51], whereas the terms $j \neq i$ give contributions which vanish at $d \to \infty$. The cavity construction forces all internal summations to range over sites $j \neq i$ and thus suppresses the $O(1)$ contribution. As a result C'_4 picks up only contributions which vanish for d large. The same is valid for arbitrary diagrams and for any $\ell \geqslant 4$.

Since all cumulants C'_ℓ with ℓ odd vanish by spin-inversion symmetry, and all cumulants with $\ell \geqslant 4$ are negligible for $d \to \infty$, $p'(\mathbf{b}_i)$ is a centered Gaussian distribution, with $L^{\alpha\beta} = C'_\ell$.

Having derived Eq. (10), we can use Eq. (8) to reconstruct the distribution

$$P(\mathbf{S}_i, \mathbf{b}_i) = \frac{\exp\left[\beta(\mathbf{b}_i \cdot \mathbf{S}_i) - \beta V(\mathbf{S}_i) - \frac{1}{2} L^{-1\alpha\beta} b_i^\alpha b_i^\beta\right]}{Z_1 \sqrt{(2\pi)^3 \det \underline{\underline{L}}}}, \quad (13)$$

which characterizes the statistical mechanics of the full system (without cavities). The normalization in Eq. (13) is

$$Z_1 = Z/Z'_i = \int_{\mathbf{S}} \mathrm{e}^{-\beta V(\mathbf{S}) + \frac{1}{2}\beta^2 L^{\alpha\beta} S^\alpha S^\beta}, \quad (14)$$

and is fixed by the condition $1 = \int_{\mathbf{S}} \int \mathrm{d}^3\mathbf{b}\, P(\mathbf{S}, \mathbf{b})$.

The distribution (13) has a form analog to the single-site distribution $P(S_i, b_i)$ of the SK model in the paramagnetic phase and for zero external fields. In addition to the local potential $V(\mathbf{S}_i)$, there are two factors, both of order 1 for large d: the fluctuations of the cavity and the term $\exp[\beta(\mathbf{b}_i \cdot \mathbf{S}_i)]$, which encodes the correlations between $\mathbf{S}_i$ and $\mathbf{b}_i$. However in the SK model the variance of the cavity distribution is temperature independent [equal to $J^2(1-q) = J^2$ [3] because the overlap q vanishes in the paramagnetic state], whereas here $L^{\alpha\beta}$ has a nontrivial temperature dependence, which has to be analyzed explicitly.

In principle, $L^{\alpha\beta}$ could be computed directly from the second moment $C_2'^{\alpha\beta} = L^{\alpha\beta}$. This, however, requires to compute correlation functions in the cavity system. In the following, we use a different approach, based on self-consistency conditions analog to the self-consistency of effective single-site problems in DMFT and other field-theoretical approaches [14,38,51,57].

In particular, we fix $L^{\alpha\beta}$ by matching two alternative expressions for the site-diagonal elements $c^{\alpha\beta} = C_{ii}^{\alpha\beta}$, $a^{\alpha\beta} = A_{ii}^{\alpha\beta}$, $\lambda^{\alpha\beta} = \Lambda_{ii}^{\alpha\beta}$ of the full two-point correlation functions $C_{ij}^{\alpha\beta} = \langle S_i^\alpha S_j^\beta \rangle$, $A_{ij}^{\alpha\beta} = \langle S_i^\alpha b_j^\beta \rangle$, $\Lambda_{ij}^{\alpha\beta} = \langle b_i^\alpha b_j^\beta \rangle$ (calculated in the complete system, without cavities).

For coincident sites $i = j$ these correlations can be calculated directly from the single-site distribution (13) as

$$\begin{aligned} C_{ii}^{\alpha\beta} &= c^{\alpha\beta} = \int_{\mathbf{S}} \int \mathrm{d}^3\mathbf{b}\, S^\alpha S^\beta\, P(\mathbf{S}, \mathbf{b}), \\ A_{ii}^{\alpha\beta} &= a^{\alpha\beta} = \int_{\mathbf{S}} \int \mathrm{d}^3\mathbf{b}\, S^\alpha b^\beta\, P(\mathbf{S}, \mathbf{b}), \quad (15) \\ \Lambda_{ii}^{\alpha\beta} &= \lambda^{\alpha\beta} = \int_{\mathbf{S}} \int \mathrm{d}^3\mathbf{b}\, b^\alpha b^\beta\, P(\mathbf{S}, \mathbf{b}). \end{aligned}$$

Due to the special form of Eq. (13), they satisfy the "equations of motion"

$$\lambda^{\alpha\beta} = L^{\alpha\beta} + \beta^2 L^{\alpha\gamma} c^{\gamma\delta} L^{\delta\beta}, \quad a^{\alpha\beta} - \beta c^{\alpha\gamma} L^{\gamma\beta} = 0. \tag{16}$$

Equations (16) reflect the fact that for fixed $\mathbf{S}_i$, the distribution of the local field $P(\mathbf{S}_i, \mathbf{b}_i)$ is a Gaussian with covariance $L^{\alpha\beta}$ and mean $\langle b_i^\alpha \rangle|_{\mathbf{S}_i \text{fixed}} = \beta L^{\alpha\beta} S_i^\beta$. After setting $b_i^\alpha = \beta L^{\alpha\beta} S_i^\beta + \zeta_i^\alpha$, ζ_i^α is a centered Gaussian uncorrelated from S_i^α so that $\lambda^{\alpha\beta} = \langle b_i^\alpha b_i^\beta \rangle = \beta^2 L^{\alpha\gamma} L^{\beta\delta} \langle S_i^\gamma S_i^\delta \rangle + \langle \zeta_i^\alpha \zeta_i^\beta \rangle = \beta^2 L^{\alpha\gamma} c^{\gamma\delta} L^{\delta\beta} + L^{\alpha\beta}$, $a^{\alpha\beta} = \beta L^{\beta\gamma} \langle S_i^\alpha S_i^\gamma \rangle + \langle S_i^\alpha \zeta_i^\beta \rangle = \beta c^{\alpha\gamma} L^{\gamma\beta}$. The first of Eqs. (16), in particular, has a simple interpretation. The covariance of the local field $\mathbf{b}_i$ receives two contributions: a part $L^{\alpha\beta}$, which originates from ζ_i^α and which describes the fluctuations of the cavity, and a part due to the bias $\beta L^{\alpha\gamma} S_i^\gamma$, which is related to the Onsager reaction field, and which fluctuates when averaged over random orientations of $\mathbf{S}_i$.

The self-consistency conditions are expressed by requiring that the fluctuations computed from Eqs. (15) match with a separate calculation of the second-order correlation functions. To obtain an independent set of expressions for $C_{ij}^{\alpha\beta}, A_{ij}^{\alpha\beta}, \Lambda_{ij}^{\alpha\beta}$ we use that at noncoincident sites $i \neq j$, these correlations can be calculated via a generalized cavity method, with two cavities [3]: one at i and one at j. As for the single-cavity method, the calculation of the correlations can be approached by analyzing the probability $P(\mathbf{S}_i, \mathbf{b}_i; \mathbf{S}_j, \mathbf{b}_j)$ to find simultaneously, given values of the variables $\mathbf{S}_i, \mathbf{b}_i, \mathbf{S}_j, \mathbf{b}_j$ at the two sites i and j. Factorizing the Gibbs weights, this probability can be written as

$$\begin{aligned} P(\mathbf{S}_i, \mathbf{b}_i; \mathbf{S}_j, \mathbf{b}_j) = Z''_{ij} Z^{-1} \exp\big\{\beta\big[(\mathbf{b}_i \cdot \mathbf{S}_i) + (\mathbf{b}_j \cdot \mathbf{S}_j) \\ - J_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta - V(\mathbf{S}_i) - V(\mathbf{S}_j)\big]\big\} \\ \times p''_{ij}\big(b_i^\alpha - J_{ij}^{\alpha\beta} S_j^\beta, b_j^\alpha - J_{ji}^{\alpha\beta} S_i^\beta\big). \end{aligned} \tag{17}$$

The terms in the first two lines describe the Gibbs-Boltzmann weight associated with the energy $V(\mathbf{S}_i) + V(\mathbf{S}_j) - (\mathbf{b}_i \cdot \mathbf{S}_i) - (\mathbf{b}_j \cdot \mathbf{S}_j) + J_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta$ (the last term cancels the double counting of the direct i-j interaction). $p''_{ij}(\mathbf{X}_1, \mathbf{X}_2)$ is the distribution of the fields $X_1^\alpha = b_i^\alpha - J_{ij}^{\alpha\beta} S_j^\beta = \sum_{k \neq i,j} J_{ik}^{\alpha\beta} S_k^\beta$ and $X_2^\alpha = b_j^\alpha - J_{ji}^{\alpha\beta} S_i^\beta = \sum_{k \neq i,j} J_{jk}^{\alpha\beta} S_k^\beta$ calculated in the two-cavity system (with the sites i and j removed). The normalization factor Z''_{ij} is the partition function of the two-cavity system.

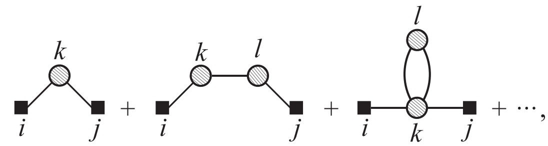
As in the case of the single-cavity distribution $p'(\mathbf{b})$, we can analyze $p''_{ij}(\mathbf{X}_1, \mathbf{X}_2)$ by studying perturbatively its cumulants

$$\begin{aligned} C''^{\,\alpha_1 \ldots \alpha_\ell \beta_1 \beta_m}_{\ell,m} &= \big\langle\big\langle X_1^{\alpha_1} \ldots X_1^{\alpha_\ell} X_2^{\beta_1} \ldots X_2^{\beta_m} \big\rangle\big\rangle''^{(ij)} \\ &= \sum_{k_1,\ldots,k_\ell,l_1,\ldots,l_m \neq i,j} J_{ik_1}^{\alpha_1\gamma_1} \ldots J_{ik_\ell}^{\alpha_\ell\gamma_\ell} J_{jl_1}^{\beta_1\delta_1} \ldots J_{jl_m}^{\beta_m\delta_m} \\ &\quad \times \big\langle\big\langle S_{k_1}^{\gamma_1} \ldots S_{k_\ell}^{\gamma_\ell} S_{l_1}^{\delta_1} S_{l_m}^{\delta_m} \big\rangle\big\rangle''^{(ij)}. \end{aligned} \tag{18}$$

Here $\langle\langle \ldots \rangle\rangle''^{(ij)}$ are connected averages in the system with two cavities at i and j.

In the $d \to \infty$ limit we find that the cumulants $C''^{\,\alpha_1\ldots\alpha_\ell}_{\ell,0}$ and $C''^{\,\alpha_1\ldots\alpha_\ell}_{0,\ell}$, which involve only one of the two sites are equal, up to negligible corrections, to the cumulants $C'^{\,\alpha_1\ldots\alpha_\ell}_{\ell}$ of the system with a single cavity. This is due to the fact that the presence of a second cavity at a point j which is *away* from the origin i has only a small effect on the diagrams for the cumulants of X_1 (similarly the cavity at i has small effects on the cumulants of X_2).

Turning to the cross correlations between X_1 and X_2, an analysis of the mixed cumulants shows that the diagrams for $C''^{\,\alpha\beta}_{1,1}$,

$C''^{\,\alpha\beta}_{1,1} =$ 

are of order $d^{-\ell_{ij}/2}$ while the graphs for cumulants of higher order vanish in a faster way for large d.

We thus find that, at leading order for d large, $p''(\mathbf{X}_1, \mathbf{X}_2)$ is Gaussian and has the form

$$\begin{aligned} p''_{ij}(\mathbf{X}_1, \mathbf{X}_2) \simeq [(2\pi)^3 \det \underline{\underline{L}}]^{-1} \exp\big\{-\big[\big(L^{-1\alpha\beta} X_1^\alpha X_1^\beta \\ + L^{-1\alpha\beta} X_2^\alpha X_2^\beta\big)/2 + M_{ij}^{\alpha\beta} X_1^\alpha X_2^\beta\big]\big\}. \end{aligned} \tag{19}$$

The off-diagonal term $M_{ij}^{\alpha\beta}$ is of order $d^{-\ell_{ij}/2}$ ($M_{ij}^{\alpha\beta} \simeq -L^{-1\alpha\gamma} C''^{\,\gamma\delta}_{1,1} L^{-1\delta\beta}$).

The correlations $C_{ij}^{\alpha\beta}$, $A_{ij}^{\alpha\beta}$, and $\Lambda_{ij}^{\alpha\beta}$ at leading order can be calculated by plugging Eq. (19) into (17), by substituting $Z''_i/Z \simeq 1/Z_1^2$, and by expanding to first order in $J_{ij}^{\alpha\beta}$ and $M_{ij}^{\alpha\beta}$ [which are both $O(d^{-\ell_{ij}/2})$]. After an explicit calculation we find the relations

$$\begin{aligned} C_{ij}^{\alpha\beta} &= \beta c^{\alpha\gamma} J_{ij}^{\gamma\delta} c^{\delta\beta} - a^{\alpha\gamma} M_{ij}^{\gamma\delta} a^{\beta\delta}, \\ A_{ij}^{\alpha\beta} &= c^{\alpha\gamma} J_{ij}^{\gamma\beta} + \beta c^{\alpha\gamma} J_{ij}^{\gamma\delta} a^{\delta\beta} - a^{\alpha\gamma} M_{ij}^{\gamma\delta} \lambda^{\delta\beta}, \\ \Lambda_{ij}^{\alpha\beta} &= J_{ij}^{\alpha\gamma} a^{\gamma\beta} + a^{\gamma\alpha} J_{ij}^{\gamma\beta} + \beta a^{\gamma\alpha} J_{ij}^{\gamma\delta} a^{\delta\beta} - \lambda^{\alpha\gamma} M_{ij}^{\gamma\delta} \lambda^{\delta\beta}, \end{aligned} \tag{20}$$

which are exact at leading order for $d \to \infty$.

Using Eqs. (16) and (20), we can eliminate $M_{ij}^{\alpha\beta}$ and obtain the following relations, valid for arbitrary i and j (coincident or noncoincident):

$$\begin{aligned} C_{ij}^{\alpha\beta} &= -(k_B T)^2 \sigma^{\alpha\beta} \delta_{ij} - k_B T A_{ij}^{\alpha\gamma} \sigma^{\gamma\beta}, \\ \Lambda_{ij}^{\alpha\beta} &= -k_B T J_{ij}^{\alpha\beta} - \beta \sigma^{-1\alpha\gamma} A_{ij}^{\gamma\beta}. \end{aligned} \tag{21}$$

Here $\sigma^{\alpha\beta} = \lambda^{-1\alpha\beta} - L^{-1\alpha\beta}$.

Equations (21) now do not make any reference to the cavity system; they express relations between the "true" correlation functions C, A, and Λ. The calculation can be completed using that the correlations have to satisfy by definition $A_{ij}^{\alpha\beta} = \sum_k C_{ik}^{\alpha\gamma} J_{kj}^{\gamma\beta}$, $\Lambda_{ij}^{\alpha\beta} = \sum_k J_{ik}^{\alpha\gamma} A_{kj}^{\gamma\beta}$.

As a result we obtain

$$\begin{aligned} \hat{C} &= -(\beta^2 \hat{\sigma}^{-1} + \beta \hat{J})^{-1}, \quad \hat{A} = \hat{C}\hat{J}, \\ \hat{\Lambda} &= \hat{J}\hat{C}\hat{J} = -k_B T \hat{J} + (\hat{\sigma} + \beta \hat{J}^{-1})^{-1}. \end{aligned} \tag{22}$$

Here the matrix $\hat{\sigma}$ is site diagonal, and is equal to $\sigma_{ij}^{\alpha\beta} = \delta_{ij}\sigma^{\alpha\beta}$. It plays the role of the local self-energy of DMFT [51].

The self-consistency conditions can finally be imposed by requiring that the onsite elements $C_{ii}^{\alpha\beta}$, $A_{ii}^{\alpha\beta}$, and $\Lambda_{ii}^{\alpha\beta}$ computed from Eqs. (22) coincide with the correlations $c^{\alpha\beta}$, $a^{\alpha\beta}$, $\lambda^{\alpha\beta}$ of the single-site problem, computed from Eqs. (15) and the distribution (13). Using Eqs. (16) it can be checked that the three matching conditions $C_{ii}^{\alpha\beta} = c^{\alpha\beta}$, $A_{ii}^{\alpha\beta} = a^{\alpha\beta}$, and $\Lambda_{ii}^{\alpha\beta} = \lambda^{\alpha\beta}$ are equivalent, so $L^{\alpha\beta}$ can be determined by imposing any one of them.

This completes the solution of the problem for $d \to \infty$, at least in the case of a symmetric solution, with unbroken symmetries. Since the two-site correlation $C_{ij}^{\alpha\beta}$ is connected to the static susceptibility $\chi_{ij}^{\alpha\beta}$ by the thermodynamic relation $\chi_{ij}^{\alpha\beta} = \beta C_{ij}^{\alpha\beta}$, the first of Eqs. (22) can be written as $\hat{\chi}^{-1} = -\beta\hat{\sigma}^{-1} - \hat{J}$, and, after a Fourier transform

$$\chi^{-1\alpha\beta}(\mathbf{k}) = -\beta\sigma^{-1\alpha\beta} - J^{\alpha\beta}(\mathbf{k}). \tag{23}$$

The result is common in systems with large coordination numbers: the susceptibility has the same structure as the Weiss mean-field susceptibility $\chi^{-1\alpha\beta} = \chi_0^{-1\alpha\beta} - J^{\alpha\beta}(\mathbf{k})$ [Eq. (4)], but $\chi_0^{-1\alpha\beta}$ is now replaced by a renormalized term $-\beta\sigma^{-1\alpha\beta}$, local in real space, and determined by self-consistency conditions. The term $-\beta\sigma^{-1\alpha\beta}$ plays the role of the "locator matrix" [38], here in the special case of a state with zero magnetization.

A. Internal energy and free energy

The effective single-site problem and the self-consistency equations give access to the thermodynamic properties of the system. In fact, since the energy can be recast as $H = \sum_i V(\mathbf{S}_i) + \frac{1}{2}\sum_i(\mathbf{b}_i \cdot \mathbf{S}_i)$, the thermodynamic internal energy per site can be computed as

$$\frac{E(\beta)}{N} = \langle V(\mathbf{S})\rangle_1 - \frac{1}{2}a^{\alpha\alpha}. \tag{24}$$

Here $\langle V(\mathbf{S})\rangle_1 = Z_1^{-1}\int_{\mathbf{S}} V(\mathbf{S})\exp[-\beta V(\mathbf{S}) + \beta^2 L^{\alpha\beta}S^\alpha S^\beta/2]$ is the average anisotropy energy computed with the distribution (13) and $a^{\alpha\alpha}$ is the trace of $a^{\alpha\beta} = \langle S^\alpha b^\beta\rangle$. Integrating over temperatures and using the self-consistency conditions we find that the free energy is given by an expression with a standard "tr ln" form [38]

$$\begin{aligned}\frac{F(\beta)}{N} &= -TS_\infty + k_\mathrm{B}T\int_0^\beta \mathrm{d}\beta'\, E(\beta')\\ &= -k_\mathrm{B}T\,\ln Z_1(\beta, L^{\alpha\beta}) - \frac{k_\mathrm{B}T}{2}\mathrm{tr}\,\ln[-\beta^2(\underline{\underline{\sigma}}^{-1} + \underline{\underline{L}})]\\ &\quad + \frac{k_\mathrm{B}T}{2}\mathrm{tr}\,\ln[-(\beta^2\hat{\sigma}^{-1} + \beta\hat{J})].\end{aligned} \tag{25}$$

Here $Z_1(\beta, L^{\alpha\beta}) = \int_{\mathbf{S}}\exp[-\beta V(\mathbf{S}) + \beta^2 L^{\alpha\beta}S^\alpha S^\beta/2]$ is the single-site partition function [Eq. (14)] and the integration constant S_∞ is equal to the entropy per site in the infinite-temperature limit. In the classical model analyzed here S_∞ is an arbitrary constant with no physical meaning (only entropy differences are meaningful). In the second line of Eq. (25), we have chosen $S_\infty = k_\mathrm{B}\ln(4\pi)$.

We also note that the self-consistency equations can be obtained from a variational principle on F. If we regard the free energy as a function $F(\beta, L^{\alpha\beta}, \sigma^{\alpha\beta})$ of three independent variables β, $L^{\alpha\beta}$, and $\sigma^{\alpha\beta}$, defined by Eq. (25), then the self-consistency conditions are equivalent to the requirements that F is stationary with respect to variations of $L^{\alpha\beta}$ and $\sigma^{\alpha\beta}$. In fact, the relations $\partial F/\partial L^{\alpha\beta}|_{\beta,\sigma} = 0$ and $\partial F/\partial\sigma^{\alpha\beta}|_{\beta,L} = 0$ give $c^{\alpha\beta} = C_{ii}^{\alpha\beta}$ and

$$\beta^2 c^{\alpha\beta} = -(\underline{\underline{\sigma}}^{-1} + \underline{\underline{L}})^{-1\alpha\beta}, \tag{26}$$

a relation which, using Eqs. (16), can be shown to be equivalent to the condition $\sigma^{\alpha\beta} = \lambda^{-1\alpha\beta} - L^{-1\alpha\beta}$.

B. Local stability of the symmetric solution

In the derivation, the system has been assumed to present unbroken spin-inversion and translation symmetries. In other words, the analysis described a system which, starting from a high-temperature paramagnetic phase, is continuously cooled down. This symmetric solution can be consistent only if the susceptibility matrix $\chi_{ij}^{\alpha\beta}$ is positive definite, in such way that the system is locally stable against a spontaneous modulation. In this section, we show that this local stability condition is satisfied at any temperature, down to $T = 0$: the disordered phase is never destabilized, at arbitrarily low T.

To analyze stability, consider first the isotropic case. In this case, $c^{\alpha\beta} = \delta^{\alpha\beta}/3$, $L^{\alpha\beta} = L\delta^{\alpha\beta}$, $\sigma^{\alpha\beta} = \sigma\delta^{\alpha\beta}$, $\lambda^{\alpha\beta} = \lambda\delta^{\alpha\beta}$, and several of the expressions simplify. In particular, expressing $C_{ii}^{\alpha\beta}$ as a Fourier integral and using the infinite-dimensional density of states [38,51]

$$\nu(\varepsilon) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\mathrm{d}k_1\ldots\int_{-\pi}^{\pi}\mathrm{d}k_d\,\delta(\varepsilon - \varepsilon_\mathbf{k})\underset{d\to\infty}{\to}\frac{\mathrm{e}^{-\varepsilon^2/2}}{\sqrt{2\pi}}, \tag{27}$$

the self-consistency condition $c^{\alpha\beta} = C_{ii}^{\alpha\beta}$ reduces to the scalar equation

$$\frac{\beta}{3} = -\int_{-\infty}^{\infty}\mathrm{d}\varepsilon\,\frac{\nu(\varepsilon)}{\beta/\sigma + f(\varepsilon)} = I(\beta/\sigma). \tag{28}$$

As discussed in Sec. II, $f(\varepsilon)$ is assumed to be bounded from above, with a maximum value $\max_\varepsilon[f(\varepsilon)] = f_\mathrm{max}$. The integral $I(\beta/\sigma)$ therefore is well defined for $\beta/\sigma < -f_\mathrm{max}$. Since $I(\beta/\sigma)$ tends to 0 for $\beta/\sigma \to -\infty$ and to $+\infty$ for β/σ to $-f_\mathrm{max}$, there is always a value of β/σ in the interval $-\infty < \beta/\sigma < -f_\mathrm{max}$ which solves Eq. (28), at any temperature T. Thus, the self-consistency relations admit a solution at any T. In addition, the solution corresponds always to a locally stable state because the susceptibility $\chi^{-1}(\mathbf{k}) = -\beta/\sigma - J(\mathbf{k}) = -\beta/\sigma - f(\varepsilon_\mathbf{k}) \geqslant f_\mathrm{max} - f(\varepsilon_\mathbf{k})$ is automatically positive for all $\mathbf{k}$.

We see therefore that the second-order transition predicted by the Weiss theory (Sec. II A) disappears in the exact large-d solution. The reason why the instability is suppressed at arbitrarily low T can be traced to the fact that $J(\mathbf{k}) = f(\varepsilon_\mathbf{k})$ has its maximum value f_max on an entire surface in momentum space, and not on isolated $\mathbf{k}$ points. Due to this geometrical property, there are a large number of modes near the degenerate surface $\varepsilon_\mathbf{k} = \varepsilon_\mathrm{max}$ which become simultaneously soft as β/σ approaches $-f_\mathrm{max}$. The fluctuation of these modes makes the integral $I(\beta/\sigma)$ diverge for $\beta/\sigma \to -f_\mathrm{max}$, ensuring that Eq. (28) has always a solution.

The mechanism at play is closely analog to that occurring in the Brazovskii model, and in other models with Coulomb- or dipole-frustrated interactions [7,9,14–16], where the modes near a degenerate surface remove a second-order instability.

Before continuing the discussion, let us analyze the properties of the solution in more detail. The condition $J_{ii} = 0$ implies that $\int_{-\infty}^{\infty} d\varepsilon\, \nu(\varepsilon) f(\varepsilon) = J_{ii} = 0$. This, in particular, shows that $f_{\max}$ is always positive and $\beta/\sigma < -f_{\max}$ is always negative. Using $\int_{-\infty}^{\infty} d\varepsilon\, \nu(\varepsilon) = 1$ and analyzing Eq. (28) it can then be shown that $\beta/\sigma < \min(\{-f_{\max}, -3/\beta\})$. At high temperatures $\beta/\sigma \approx -3/\beta + O(\beta^{-3})$. The effective single-atom susceptibility $\tilde{\chi} = -\sigma/\beta$ is approximately equal to its Weiss mean-field value $\chi_0 = \beta/3$. At low temperatures, however, $\tilde{\chi}$ becomes much smaller than χ_0 and eventually saturates to a finite value $\tilde{\chi} = -\sigma/\beta \approx 1/f_{\max}$ when $T \to 0$. The temperature dependence of $\tilde{\chi}$ is such that $\tilde{\chi}^{-1}$ is always larger than $J(\mathbf{k})$ at all wavelengths, so that the system is locally stable at all T.

In the anisotropic case, the analysis is more complex. However, we expect that from the point of view of the local stability of the disordered solution, the result is the same. When all eigenvalues of $f^{\alpha\beta}(x)$ are bounded from above, we expect that the paramagnetic state remains locally stable at all T.

This local stability analysis rules out that the system may order at low temperatures via a continuous transition. It is not excluded, however, that the model may undergo a first-order transition into an ordered state. In this case, the symmetric solution which we presented corresponds to the one of lowest free energy only for temperatures higher than the temperature $T_{\text{f.o.}}$ of the transition, while for $T < T_{\text{f.o.}}$ it has to be interpreted as a metastable supercooled phase ($T_{\text{f.o.}}$ denotes the temperature of the first-order transition).

A study of possible first-order transitions requires an analysis of the broken-symmetry solutions of the large-d model, which is beyond the scope of this work [58]. We give, however, some remarks on this question. In the context of continuous field theories with degenerate surfaces of soft modes, the Brazovskii model with scalar degrees of freedom has been predicted to present a fluctuation-induced first-order transition to a modulated phase [5,18]. Coulomb-frustrated models with vector degrees of freedom and $O(n)$ symmetry have been argued, instead, to behave differently. In particular, the analyses in Refs. [5,7] indicate that for $n \geqslant 3$ Coulomb-frustrated models present no ordered phases when there is an exact degeneracy of low-lying modes at a momentum-space surface (although a lifting of the degeneracy due to lattice effects beyond the continuum limit induce a finite-temperature transition).

In the models studied here, the mechanism by which the symmetric state is stabilized at arbitrarily low T is analog to that occurring in Coulomb-frustrated models since it originates from the anomalous density of states near a surface in momentum space. It should be noted, however, that the lattice spin systems which we study differ from continuous field theories because of a significantly different shape of the degenerate surface. In Coulomb-frustrated models, degenerate surfaces of soft modes arise due to competing interactions on different length scales. In the limit of small frustration, the surface of soft modes occurs in a region of wavelengths much larger than the lattice spacing. In the lattice systems of interest in this work, instead, frustration occurs at the microscopic scale and the surface of soft modes occurs at large wave vectors, comparable to the size of the Brillouin zone. The degenerate surface of soft modes, as a result, has a highly nonspherical shape, determined by the condition $\varepsilon_{\mathbf{k}} = \varepsilon_{\max}$.

Due to the nonspherical shape, it is possible that a mechanism of *order by disorder* [43] selects at low temperatures a modulated phase. Order by disorder has indeed been found in the $J_1 = 2J_2$ fcc antiferromagnet in three dimensions: in this model Ref. [39] predicted the entropic selection at low temperatures of an ordered helical state, with a modulation vector oriented along a high-symmetry direction [39]. It is likely therefore that, also in the large-d models studied here, an ordered state is selected entropically as the most stable low-temperature phase.

Since, however, we showed that the disordered phase is locally stable in the $d \to \infty$ model, we can consistently study its properties at all T (in the region $T < T_{\text{f.o.}}$ the analysis then describes a metastable, supercooled phase).

In the following sections we will focus on the disordered solution and, using a dynamical analysis, we will explore the possibility that, at a certain temperature, it may undergo a dynamical glass transition, and a consequent breaking of ergodicity. If the glass transition competes with a first-order transition and $T_{\text{f.o.}} > T_{\text{g}}$ we assume that the system can be supercooled down to T_{g}.

In other words, the disordered solution represents the state of the system in a spin-liquid phase [43], and by the dynamical analysis in the next sections we study whether this liquid state can freeze into a glass. A first-order transition into a modulated phase is instead analog to a crystallization. The assumption which we make is that the crystallization, if present, can be avoided by supercooling.

C. Distribution of the internal field and low-temperature limit of the internal energy in isotropic models

Before analyzing dynamical properties, we discuss some additional properties of the static solution focusing on the isotropic case. In isotropic models, the cavity distribution $p'(\mathbf{b})$ is, at any temperature, a rotationally invariant Gaussian of width L. The variance of the cavity field distribution L is related to the self-energy σ by the relation $L = -3/\beta^2 - 1/\sigma$ [which follows from Eqs. (16) using that in the rotationally invariant case $c^{\alpha\beta} = \delta^{\alpha\beta}/3$ at all T]. At high temperatures, L is controlled by the leading orders of perturbation theory and is approximately $L \approx \mathcal{J}_2^2/3$, with $\mathcal{J}_2^2 = \int_{-\infty}^{\infty} d\varepsilon\, \nu(\varepsilon) f^2(\varepsilon) = \sum_k (J_{ik})^2$. When the temperature is lowered, L decreases. Eventually in the limit $T \to 0$, L vanishes as $L \approx -1/\sigma \approx k_{\text{B}} T f_{\max}$.

Consider now the complete distribution of the field $p(\mathbf{b}) = \int_{\mathbf{S}} P(\mathbf{S}, \mathbf{b})$ [3]. At high temperatures, the reaction field is weak and the distribution $p(\mathbf{b})$ is approximately equal to the cavity distribution $p'(\mathbf{b})$. In particular, the variance λ of $p(\mathbf{b})$ is approximately the same as the variance L of $p'(\mathbf{b})$.

When the temperature is lowered, the correlation between $\mathbf{b}_i$ and $\mathbf{S}_i$ becomes more important and the distribution $p(\mathbf{b}_i)$

becomes significantly different from $p'(\mathbf{b}_i)$. Eventually in the limit of low temperatures, the distribution of the field $\mathbf{b}_i$ is completely dragged by the spin $\mathbf{S}_i$. In fact, using $L \approx k_B T f_{\max}$ we find

$$P(\mathbf{S}, \mathbf{b}) \approx \frac{\exp\{\beta[(\mathbf{b}\cdot\mathbf{S}) - b^2/(2f_{\max}) - f_{\max}/2]\}}{4\pi\sqrt{(2\pi)^3 \det \underline{\underline{L}}}} \underset{\beta\to\infty}{\to} \frac{1}{4\pi}\delta(\mathbf{b} - f_{\max}\mathbf{S}). \tag{29}$$

In all relevant configurations, $\mathbf{b}_i$ is equal to $f_{\max}\mathbf{S}_i$ up to a small fluctuation (with root mean square $\sqrt{L} \approx \sqrt{k_B T f_{\max}}$). The probability $p(\mathbf{b}) = \int_{\mathbf{S}} P(\mathbf{S}, \mathbf{b})$ is then concentrated within a narrow spherical surface of radius $f_{\max}$. The variance of the distribution $\lambda = 1/\sigma + \beta^2/(3\sigma^2)$ remains finite and approaches $\lambda \approx f_{\max}^2/3$ for $T \to 0$.

Equation (29) implies, in addition, that the internal energy per site for $T \to 0$ tends to $E_0 = -Na^{\alpha\alpha}/2 = -N f_{\max}/2$ in isotropic models. E_0 is exactly equal to the energy of the spiral ground states, which is the exact ground-state energy of the system. This suggests that the configurations contributing to the $T \to 0$ limit of the symmetric state are built predominantly from a superposition of Fourier modes with $\mathbf{k}$ lying near the degenerate surface. However, we expect that the correlations in this disordered phase remain short ranged since the correlation function $C_{ij}^{\alpha\beta}$ evolves smoothly when the system is cooled down starting from high temperatures.

We note as a remark that E_0 describes the $T \to 0$ limit of the *equilibrium* internal energy. This energy may be unreachable if the system remains trapped in a glassy state at a vitrification temperature T_g. The state in which the system freezes at T_g may have an energy larger than E_0 when it is eventually cooled down to $T \to 0$. In this case, the relevant thermodynamic properties should be analyzed taking into account the trapping into a metastable state (see Ref. [38] for a replica theory of the thermodynamics within a glass phase).

IV. DYNAMICAL PROPERTIES

The analysis in Sec. III described the properties of the disordered phase from the point of view of equilibrium statistical mechanics. In this section, we extend the analysis and study the equilibrium dynamics of the system. The results of this dynamical analysis will be used in Sec. V to analyze the conditions under which the disordered phase can become nonergodic, and develop a glassy behavior.

To study the system from a dynamical point of view, we assume a purely dissipative Langevin equation

$$\dot{\mathbf{S}}_i = -\mathbf{S}_i \times [\mathbf{S}_i \times (\mathbf{N}_i + \boldsymbol{\nu}_i)] = \mathbf{N}_i + \boldsymbol{\nu}_i - \mathbf{S}_i[\mathbf{S}_i \cdot (\mathbf{N}_i + \boldsymbol{\nu}_i)]. \tag{30}$$

Here $\mathbf{N}_i$ is the vector

$$\mathbf{N}_i = -\frac{\partial H}{\partial \mathbf{S}_i} = \mathbf{b}_i + \mathbf{F}_i, \tag{31}$$

$b_i^\alpha = \sum_j J_{ij}^{\alpha\beta} S_j^\beta$ is the instantaneous internal field, and $F^\alpha(\mathbf{S}_i) = -\partial V(\mathbf{S}_i)/\partial S_i^\alpha$ is a contribution from the onsite anisotropy. $\boldsymbol{\nu}_i(t)$ is a random torque, considered as a white Gaussian noise with zero mean and

$$\langle \nu_i^\alpha(t)\nu_j^\beta(t')\rangle = 2k_B T \delta^{\alpha\beta}\delta_{ij}\delta(t-t'). \tag{32}$$

For simplicity we have adopted a rescaled set of units, absorbing the gyromagnetic ratio and the Gilbert damping constant in a redefinition of the timescale.

This purely dissipative dynamics is a particular case, in the limit of strong damping, of the stochastic Landau-Lifshitz-Gilbert (LLG) equation [59,60]. We restrict the analysis to the overdamped case to simplify the presentation, but the derivations could be adapted to include a precession term. We expect that the emergence of vitrification depends on the structure of the energy landscape, and not on the specific nature of the dynamical equations. Thus, it seems natural to assume that a more detailed, precessional dynamics, would give the same conditions for the emergence of glassiness.

In the analysis, we focus on equilibrium dynamics: we consider correlation functions averaged over the Gaussian distribution of the noises $\boldsymbol{\nu}_i$ *and* over the initial conditions $\mathbf{S}_{i_1 0},\ldots,\mathbf{S}_{i_N 0}$ at time $t = 0$, assigning to the initial conditions a weight given by the equilibrium Gibbs distribution $Z^{-1}\exp[-\beta H(\mathbf{S}_{i_1 0},\ldots,\mathbf{S}_{i_N 0})]$.

As in the case of the static properties, the $d \to \infty$ limit allows to reduce the dynamics of the N-body problem to an effective single-site problem and a set of self-consistency equations. We present detailed derivations of the dynamical properties in Secs. IV A–IV C, and in Appendix B, but for compactness we summarize the main results throughout this and the next page.

In the limit $d \to \infty$ we find that the dynamics of a single spin can be replaced by a non-Markovian Langevin equation

$$\dot{\mathbf{S}}_i = -\mathbf{S}_i \times \{\mathbf{S}_i \times [\mathbf{b}_i(t) + \mathbf{F}_i + \boldsymbol{\nu}_i(t)]\}, \tag{33}$$

subject to a time-dependent field $\mathbf{b}_i(t)$ given by

$$b_i(t) = \zeta_i^\alpha(t) + \beta l^{\alpha\beta}(t) S_{i0}^\beta + \int_0^t dt'\, K^{\alpha\beta}(t-t') S_i^\beta(t'). \tag{34}$$

In Eq. (34), $\boldsymbol{\zeta}_i(t)$ is a colored Gaussian noise with zero mean and correlation $\langle \zeta_i^\alpha(t)\zeta_i^\beta(t')\rangle = l^{\alpha\beta}(t-t')$. The kernel $K^{\alpha\beta}(t-t')$ is related to the spectrum of the noise by the fluctuation-dissipation relation

$$K^{\alpha\beta}(t-t') = -\beta\Theta(t-t')\frac{d}{dt} l^{\alpha\beta}(t-t'). \tag{35}$$

Finally, the term $\beta l^{\alpha\beta}(t) S_{i0}^\beta$ in Eq. (34) is controlled by the same function $l^{\alpha\beta}(t)$ which defines the correlation of $\zeta_i^\alpha(t)$, and depends on $\mathbf{S}_{i0}$, which is the initial condition of the spin $\mathbf{S}_i$ at time $t = 0$.

Equations (33) and (34) define a single-site problem which encodes all time-dependent correlation functions of the spin $\mathbf{S}_i$ and the field $\mathbf{b}_i$ at a single site i. In other words, instead, of solving the N-body dynamics, the local correlations $\langle S_i^{\alpha_1}(t_1)\ldots S_i^{\alpha_n}(t_n) b_i^{\beta_1}(t_1')\ldots b_i^{\beta_\ell}(t_\ell')\rangle$ can be calculated equivalently by solving the single-site dynamics, and by averaging the solutions of the one-site Langevin equations over the realizations of $\boldsymbol{\nu}_i(t)$, $\boldsymbol{\zeta}_i(t)$, and over the initial conditions of $\mathbf{S}_{i0}$.

In this procedure, the initial conditions $\mathbf{S}_{i0}$ must be averaged with the probability distribution

$$\begin{aligned} P_{1\text{eq}}(\mathbf{S}_{i0}) &= Z^{-1}\int_{\mathbf{S}_{i_2}}\cdots\int_{\mathbf{S}_{i_N}} \mathrm{e}^{-\beta H(\mathbf{S}_{i0},\mathbf{S}_{i_2},\dots,\mathbf{S}_{i_N})} \\ &= \int \mathrm{d}^3\mathbf{b}\, P(\mathbf{S}_{i0},\mathbf{b}) \\ &= Z_1^{-1}\mathrm{e}^{-\beta V(\mathbf{S}_{i0})+\frac{1}{2}\beta^2 L^{\alpha\beta}S_{i0}^{\alpha}S_{i0}^{\beta}} \end{aligned} \tag{36}$$

which is the Gibbs probability of the spin $\mathbf{S}_{i0}$.

The non-Markovian single-site problem is entirely specified by the correlation function $l^{\alpha\beta}(t-t')$. At equal times $t=t'$, $l^{\alpha\beta}(t=t')=l^{\alpha\beta}(0)$ can be shown to be equal to the matrix $L^{\alpha\beta}$ found in the static analysis of Sec. III. For arbitrary times, $l^{\alpha\beta}(t-t')$ can be fixed by a set of self-consistency conditions satisfied by the correlations $C_{ij}^{\alpha\beta}(t-t')=\langle S_i^{\alpha}(t)S_j^{\beta}(t')\rangle$, $A_{ij}^{\alpha\beta}(t-t')=\langle S_i^{\alpha}(t)b_j^{\beta}(t')\rangle$, $\Lambda_{ij}^{\alpha\beta}(t-t')=\langle b_i^{\alpha}(t)b_j^{\beta}(t')\rangle$ and their site-diagonal elements $c^{\alpha\beta}(t-t')=C_{ii}^{\alpha\beta}(t-t')$, $a^{\alpha\beta}(t-t')=A_{ii}^{\alpha\beta}(t-t')$, $\lambda^{\alpha\beta}(t-t')=\Lambda_{ii}^{\alpha\beta}(t-t')$.

In particular, we find that the self-consistency relations can be reduced to a compact form when expressed in terms of the time derivatives $\dot{C}_{+ij}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}C_{ij}^{\alpha\beta}(t-t')/\mathrm{d}t$, $\dot{A}_{+ij}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}A_{ij}^{\alpha\beta}(t-t')/\mathrm{d}t$, $\dot{\Lambda}_{+ij}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}\Lambda_{ij}^{\alpha\beta}(t-t')/\mathrm{d}t$, restricted to the retarded region $t>t'$. $\dot{C}_+$ is simply related to the response function $G_{ij}^{\alpha\beta}(t-t')=\langle\delta S_i^{\alpha}(t)/\delta\nu_j^{\beta}(t')\rangle$ by the fluctuation-dissipation theorem (FDT) $G_{ij}^{\alpha\beta}(t-t')=-\beta\dot{C}_{+ij}^{\alpha\beta}(t-t')$ (see Appendix A).

In the limit $d\to\infty$, we find that $\dot{C}_+$, $\dot{A}_+$, $\dot{\Lambda}_+$ satisfy the relations

$$\begin{aligned} &\dot{C}_{+ij}^{\alpha\beta}(t-t')-\delta_{ij}\dot{c}_{+}^{\alpha\beta}(t-t') \\ &\quad=\beta\int_{t'}^{t}\mathrm{d}t''\int_{t'}^{t''}\mathrm{d}t'''\big[\dot{C}_{+ij}^{\alpha\gamma}(t-t'') \\ &\qquad\times K^{\gamma\delta}(t''-t''')\dot{c}_{+}^{\delta\beta}(t'''-t')\big] \\ &\qquad-\beta\int_{t'}^{t}\mathrm{d}t''\,\dot{A}_{+ij}^{\alpha\gamma}(t-t'')\dot{c}_{+}^{\gamma\beta}(t''-t'), \end{aligned} \tag{37}$$

$$\begin{aligned} \dot{A}_{+ij}^{\alpha\beta}(t-t') &= \dot{c}_{+}^{\alpha\gamma}(t-t')J_{ij}^{\gamma\beta} \\ &\quad-\beta\int_{t'}^{t}\mathrm{d}t''\dot{c}_{+}^{\alpha\gamma}(t-t'')\bigg[\dot{\Lambda}_{+ij}^{\gamma\beta}(t''-t') \\ &\quad-\int_{t'}^{t''}\mathrm{d}t'''\,K^{\gamma\delta}(t''-t''')\dot{A}_{+ij}^{\delta\beta}(t'''-t')\bigg], \end{aligned} \tag{38}$$

or, in frequency space,

$$\begin{aligned} \dot{C}_{+}^{\alpha\beta}(\mathbf{k},\omega)-\dot{c}_{+}^{\alpha\beta}(\omega) &= \beta\dot{C}_{+}^{\alpha\gamma}(\mathbf{k},\omega)K^{\gamma\delta}(\omega)\dot{c}_{+}^{\delta\beta}(\omega) \\ &\quad-\beta\dot{A}_{+}^{\alpha\gamma}(\mathbf{k},\omega)\dot{c}_{+}^{\gamma\beta}(\omega), \\ \dot{A}_{+}^{\alpha\beta}(\mathbf{k},\omega) &= \dot{c}_{+}^{\alpha\gamma}(\omega)J^{\gamma\beta}(\mathbf{k})-\beta\dot{c}_{+}^{\alpha\gamma}(\omega)\dot{\Lambda}_{+}^{\gamma\beta}(\mathbf{k},\omega) \\ &\quad+\beta\dot{c}_{+}^{\alpha\gamma}(\omega)K^{\gamma\delta}(\omega)\dot{A}_{+}^{\delta\beta}(\mathbf{k},\omega). \end{aligned} \tag{39}$$

Here $\dot{c}_{+}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}c^{\alpha\beta}(t-t')/\mathrm{d}t$ is the time derivative of the single-site correlation $\langle S^{\alpha}(t)S^{\beta}(t')\rangle$, related to the single-site response function $g^{\alpha\beta}(t-t')=\langle\delta S_i^{\alpha}(t)/\delta\nu_i^{\beta}(t')\rangle$ by the FDT $g^{\alpha\beta}(t-t')=-\beta\dot{c}_{+}^{\alpha\beta}(t-t')$. $\dot{c}_{+}^{\alpha\beta}(\omega)=\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{i\omega t}\dot{c}_{+}^{\alpha\beta}(t-t')$ is the Fourier transform of $\dot{c}_+$.

Together with the relations $\dot{A}_{+}^{\alpha\beta}(\mathbf{k},\omega)=\dot{C}_{+}^{\alpha\gamma}(\mathbf{k},\omega)J^{\gamma\beta}(\mathbf{k})$, $\dot{\Lambda}_{+}^{\alpha\beta}(\omega,\mathbf{k})=J^{\alpha\gamma}(\mathbf{k},\omega)\dot{A}_{+}^{\gamma\beta}(\mathbf{k},\omega)$, Eqs. (39) fix the correlations as functionals of K and $\dot{c}_+$.

The solutions are

$$\begin{aligned} \dot{C}_{+}^{\alpha\beta}(\mathbf{k},\omega) &= [(\underline{\underline{\dot{c}_{+}}}(\omega))^{-1}-\beta\underline{\underline{K}}(\omega)+\beta\underline{\underline{J}}(\mathbf{k})]^{-1\alpha\beta}, \\ \dot{A}_{+}^{\alpha\beta}(\mathbf{k},\omega) &= \dot{C}_{+}^{\alpha\gamma}(\mathbf{k},\omega)J^{\gamma\beta}(\mathbf{k}), \\ \dot{\Lambda}_{+}^{\alpha\beta}(\mathbf{k},\omega) &= J^{\alpha\gamma}(\mathbf{k})\dot{C}_{+}^{\gamma\delta}(\mathbf{k},\omega)J^{\delta\beta}(\mathbf{k}). \end{aligned} \tag{40}$$

This result has a DMFT-like form, as expected due to the limit of large dimensionality: the self-energy depends on the frequency ω but not on the momentum $\mathbf{k}$ [51].

The self-consistency conditions can be imposed by requiring that the site-diagonal correlations $\dot{C}_{+ii}^{\alpha\beta}(t-t')$, $\dot{A}_{+ii}^{\alpha\beta}(t-t')$, $\dot{\Lambda}_{+ii}^{\alpha\beta}(t-t')$ computed by Fourier transformation from Eqs. (40) coincide with the corresponding quantities $\dot{c}_{+}^{\alpha\beta}(t-t')$, $\dot{a}_{+}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}a^{\alpha\beta}(t-t')/\mathrm{d}t=\Theta(t-t')\mathrm{d}\langle S_i^{\alpha}(t)b_i^{\beta}(t')\rangle/\mathrm{d}t$, $\dot{\lambda}_{+}^{\alpha\beta}(t-t')=\Theta(t-t')\mathrm{d}\lambda^{\alpha\beta}(t-t')/\mathrm{d}t=\Theta(t-t')\mathrm{d}\langle b_i^{\alpha}(t)b_i^{\beta}(t')\rangle/\mathrm{d}t$ computed from the single-site Langevin equation.

These conditions can be expressed in frequency space as

$$\begin{aligned} \underline{\underline{\dot{c}_{+}}}(\omega) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}^d k}{(2\pi)^d}[(\underline{\underline{\dot{c}_{+}}}(\omega))^{-1}-\beta\underline{\underline{K}}(\omega)+\beta\underline{\underline{J}}(\mathbf{k})]^{-1}, \\ \underline{\underline{\dot{a}_{+}}}(\omega) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}^d k}{(2\pi)^d}\{[(\underline{\underline{\dot{c}_{+}}}(\omega))^{-1}-\beta\underline{\underline{K}}(\omega)+\beta\underline{\underline{J}}(\mathbf{k})]^{-1}\underline{\underline{J}}(\mathbf{k})\}, \\ \underline{\underline{\dot{\lambda}_{+}}}(\omega) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}^d k}{(2\pi)^d}\{\underline{\underline{J}}(\mathbf{k})[(\underline{\underline{\dot{c}_{+}}}(\omega))^{-1}-\beta\underline{\underline{K}}(\omega) \\ &\qquad+\beta\underline{\underline{J}}(\mathbf{k})]^{-1}\underline{\underline{J}}(\mathbf{k})\}. \end{aligned} \tag{41}$$

The three matching conditions for $\dot{c}_+$, $\dot{a}_+$, $\dot{\lambda}_+$ are equivalent to each other because Eqs. (37), (38), and thus Eqs. (41), are consistent by construction with the relations

$$\begin{aligned} \dot{a}_{+}^{\alpha\beta}(\omega) &= \dot{c}_{+}^{\alpha\gamma}(\omega)K^{\gamma\beta}(\omega), \\ \dot{\lambda}_{+}^{\alpha\beta}(\omega) &= -\frac{1}{\beta}K^{\alpha\beta}(\omega)+K^{\alpha\gamma}(\omega)\dot{c}_{+}^{\gamma\delta}(\omega)K^{\delta\beta}(\omega), \end{aligned} \tag{42}$$

which are automatically satisfied by the effective single-site Langevin equation (see Sec. IV B). Thus, we can choose equivalently to impose any one of the three relations (41) to fix the self-consistency.

As a remark, note that the self-consistency equations fix the time derivatives and not directly the correlation functions. However, since the equal-time correlations must be equal to the static correlations, computed in Sec. III, we can deduce any time-dependent average by integrating over time (using the static averages of Sec. III as a boundary condition at $t=t'$).

A. Derivation: Effective single-site problem

To derive the dynamical results we used, as in Sec. III, a combination of perturbation theory and the cavity method [3,51]. The starting point is a methodology analog to the cavity approach used in Ref. [3] for the dynamics of the SK model. A fundamental idea of this cavity approach is that a single spin $\mathbf{S}_i(t) = \mathbf{S}_{i_1}(t)$ has only a weak effect on the trajectories of the other $N-1$ spins $\mathbf{S}_{i_2}(t)$,..., $\mathbf{S}_{i_N}(t)$. For each set of initial conditions $\mathbf{S}_{i0}$, $\mathbf{S}_{i_20}$,..., $\mathbf{S}_{i_N0}$ and for each realization of the noise $\boldsymbol{\nu}_i(t)$, $\boldsymbol{\nu}_{i_2}(t)$,..., $\boldsymbol{\nu}_{i_N}(t)$, this allows to expand the trajectories of the $N-1$ spins i_2, ..., i_N [3]:

$$S_j^\alpha(t) = S_j^{\alpha\prime(i)}(t) + \sum_k \int_0^t \mathrm{d}t'\, G_{jk}^{(1)\alpha,\mu\prime(i)}(t,t') J_{ki}^{\mu\beta} S_i^\beta(t') + \frac{1}{2}\sum_{k,l}\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, G_{j,kl}^{(2)\ \alpha,\mu\nu\prime(i)}(t;t',t'') \times J_{ki}^{\mu\beta} J_{li}^{\nu\gamma} S_i^\beta(t') S_i^\nu(t'') + \cdots, \tag{43}$$

and the magnetic field $b_i^\alpha = \sum_j J_{ij}^{\alpha\beta} S_j^\beta(t)$ acting on site $\mathbf{S}_i$,

$$b_i^\alpha(t) = \eta_i^\alpha(t) + \int_0^t \mathrm{d}t'\, K^{(1)\alpha,\beta}(t,t') S_i^\beta(t') + \frac{1}{2}\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t''\, K^{(2)\alpha,\beta\gamma}(t;t',t'') S_i^\beta(t') S_i^\gamma(t'') + \cdots \tag{44}$$

as power series in the interaction between the spin $\mathbf{S}_i$ and the cavity.

The zero-order term in Eq. (43), denoted as $S_j^{\alpha\prime(i)}(t)$, is a solution of the Langevin equations of the cavity system [Eqs. (30) in absence of the site i, that is, $\dot{\mathbf{S}}_j^{\prime(i)} = -\mathbf{S}_j^{\prime(i)} \times (\mathbf{S}_j^{\prime(i)} \times (\mathbf{N}_j^{\prime(i)} + \boldsymbol{\nu}_j))$ with $\mathbf{N}_j^{\prime(i)} = \mathbf{F}(\mathbf{S}_j^{\prime(i)}) + \mathbf{b}_j^{\prime(i)}$ and $b_j^{\alpha\prime(i)} = \sum_{k\neq i} J_{jk}^{\alpha\beta} S_k^{\beta\prime(i)}$]. $G^{(n)}$ are the nth-order response functions

$$G_{j,k_1,\dots,k_n}^{(n)\alpha,\mu_1\mu_n\prime(i)}(t;t_1,\dots,t_n) = \frac{\delta S_j^{\alpha\prime(i)}(t)}{\delta \nu_{k_1}^{\mu_1}(t_1)\dots\delta \nu_{k_n}^{\mu_n}(t_n)}, \tag{45}$$

calculated along the trajectory $S_j^{\prime\alpha}(t)$ ($G^{(n)}$ are trajectory dependent).

In Eq. (44), $\eta_i^\alpha(t) = \sum_j J_{ij}^{\alpha\beta} S_j^{\beta\prime(i)}(t)$ is the magnetic field at zero order. The kernels $K^{(n)}(t)$ describe the nth-order corrections to the instantaneous field due to the interactions of the cavity with $\mathbf{S}_i$, and are related to the response functions via

$$K^{(n)\alpha,\mu_1\dots\mu_n}(t;t_1,\dots,t_n) = \sum_{j,k_1,\dots,k_n} J_{ij}^{\alpha\beta} \times G_{j,k_1\dots k_n}^{(n)\beta,\nu_1\dots\nu_n\prime(i)}(t;t_1,\dots,t_n) J_{k_1 i}^{\nu_1\mu_1}\dots J_{k_n i}^{\nu_n\mu_n}. \tag{46}$$

By construction $S_j^{\alpha\prime(i)}(t)$, $G^{(n)}$, $\boldsymbol{\eta}_i(t)$, and $K^{(n)}$ do not depend explicitly on $\mathbf{S}_{i0}$ and $\boldsymbol{\nu}_i$, but only on the noises $\boldsymbol{\nu}_{i_2}(t)$,..., $\boldsymbol{\nu}_{i_N}(t)$ and the initial conditions $\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}$ of the cavity. For varying initial conditions and realizations of the noises, they become random variables.

In order to derive Eq. (34), it is necessary to show that the complex expression (44) for the instantaneous field simplifies for $d\to\infty$. In particular we need to derive four properties: (1) The nonlinear response terms have a negligible effect at large d, so that the series in Eq. (44) can be truncated keeping only the term $\eta_i^\alpha(t)$ and the linear response part $\int_0^t \mathrm{d}t' K^{(1)\alpha\beta}(t,t') S_i^\beta(t')$. (2) The linear response kernel $K^{(1)}(t,t')$ is effectively a deterministic quantity, with a negligible variance. For any trajectory at temperature T, $K^{(1)\alpha\beta}(t,t')$ is then equal to its thermal average $K^{\alpha\beta}(t-t') = \langle K^{(1)\alpha\beta}(t,t')\rangle$, up to negligible fluctuations. (3) The random field $\eta_i(t)$ has for $d\to\infty$ a Gaussian distribution, with mean $\beta l^{\alpha\beta}(t) S_{i0}^\beta$ and correlation $l^{\alpha\beta}(t-t')$. (4) The kernel $K^{\alpha\beta}(t-t')$ is related to $l^{\alpha\beta}(t-t')$ by the FDT.

These results are similar to those valid in the SK model [3], and can be derived by studying in a perturbative expansion the cumulants of $\boldsymbol{\eta}_i(t)$, and the $K^{(n)}$. Let us consider first the cumulants of $\boldsymbol{\eta}_i(t)$. In equilibrium, the initial conditions $\mathbf{S}_{i_20}$,..., $\mathbf{S}_{i_N0}$ are drawn with a distribution which, for each $\mathbf{S}_{i0}$, is given by

$$p(\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}|\mathbf{S}_{i0}) = \frac{Z^{-1}}{P_{1\mathrm{eq}}(\mathbf{S}_{i0})} \mathrm{e}^{-\beta H(\mathbf{S}_{i0},\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0})}. \tag{47}$$

Thus, we can study the fluctuations of $\boldsymbol{\eta}$ by analyzing the cumulants

$$C_{\ell|\mathbf{S}_{i0}}^{\prime\alpha_1,\dots,\alpha_n} = \langle\langle \eta_i^{\alpha_1}(t_1)\dots\eta_i^{\alpha_\ell}(t_\ell)\rangle\rangle_{\boldsymbol{\nu}_2,\dots,\boldsymbol{\nu}_N;\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}|\mathbf{S}_{i0}}. \tag{48}$$

Here $\langle\langle\dots\rangle\rangle_{\boldsymbol{\nu}_2,\dots,\boldsymbol{\nu}_N;\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}|\mathbf{S}_{i0}}$ are connected averages over $\boldsymbol{\nu}_2$,..., $\boldsymbol{\nu}_N$, and the initial conditions $\mathbf{S}_{i_20}$,..., $\mathbf{S}_{i_N0}$, weighted with the distribution (47).

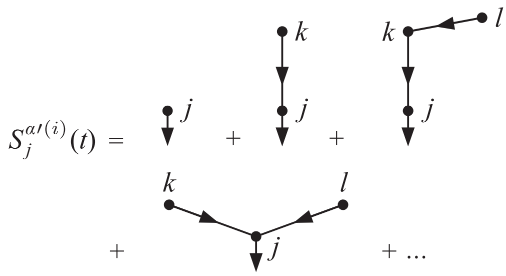
For large d, $p(\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}|\mathbf{S}_{i0})$ is close to the cavity distribution $Z_i^{\prime -1}\exp[-\beta H_i'(\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0})]$ [the difference between $p(\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}|\mathbf{S}_{i0})$ and the cavity distribution comes from the interaction $-\sum_j J_{ij}^{\alpha\beta} S_{i0}^\alpha S_{j0}^\beta$ which is small for large d].

Thus, we can at first ignore the corrections due to $\mathbf{S}_{i0}$ and calculate the cumulants

$$C_\ell^{\prime\alpha_1,\dots,\alpha_n} = \langle\langle \eta_i^{\alpha_1}(t_1)\dots\eta_i^{\alpha_\ell}(t_\ell)\rangle\rangle'^{(i)}_{\boldsymbol{\nu}_2,\dots,\boldsymbol{\nu}_N;\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}} = \sum_{j_1,\dots,j_\ell} J_{ij_1}^{\alpha_1\beta_1}\dots J_{ij_\ell}^{\alpha_\ell\beta_\ell} \langle\langle S_{j_1}^{\beta_1\prime(i)}(t_1) \times\dots\times S_{j_\ell}^{\beta_\ell\prime(i)}(t_\ell)\rangle\rangle'^{(i)}_{\boldsymbol{\nu}_2,\dots,\boldsymbol{\nu}_N;\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0}}, \tag{49}$$

assuming that the averages over initial conditions are weighted simply with the Gibbs distribution $Z_i^{\prime -1}\exp[-\beta H_i'(\mathbf{S}_{i_20},\dots,\mathbf{S}_{i_N0})]$ of the cavity. After this simplification, the cumulants in Eq. (49) reduce to averages calculated in the equilibrium dynamics of the cavity system.

The C_ℓ' can be studied perturbatively by expanding at the same time the solutions of the cavity Langevin equations and the distribution of initial conditions $Z_i^{\prime -1}\exp(-\beta H_i')$ as a series in the interactions $J_{jk}^{\alpha\beta}$ acting within the cavity system. To this end, we note that the expansion of the solutions $\mathbf{S}'^{(i)}(t)$, for any fixed realization of the initial conditions $\mathbf{S}_{i_20}$,..., $\mathbf{S}_{i_N0}$ and of the noises $\boldsymbol{\nu}_{i_2}(t)$,..., $\boldsymbol{\nu}_{i_N}(t)$, can be represented in terms of tree diagrams of the form



$$= S_j^{\alpha\,(0)}(t) + \sum_k \int_0^t \mathrm{d}t'\; \mathcal{G}_j^{(1)\,\alpha\beta}(t,t') J_{jk}^{\beta\gamma} S_k^{\gamma\,(0)}(t')$$
$$+ \sum_{k,l} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\; \mathcal{G}_j^{(1)\,\alpha\beta}(t,t') J_{jk}^{\beta\gamma} \times \mathcal{G}_k^{(1)\,\gamma\delta}(t',t'') J_{kl}^{\delta\mu} S_l^{\mu\,(0)}(t'')$$
$$+ \sum_{k,l} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\; \mathcal{G}_j^{(2)\,\alpha\beta\gamma}(t;t',t'') \times J_{jk}^{\beta\delta} S_k^{(0)\,\delta}(t') J_{jl}^{\gamma\mu} S_l^{(0)\,\mu}(t'') + \dots \qquad (50)$$

The zero-order term in Eq. (50) is simply the solution $S_j^{\alpha(0)}(t)$ of the noninteracting Langevin equation $\dot{\mathbf{S}}_j = -\mathbf{S}_j \times \{\mathbf{S}_j \times [\mathbf{F}_j + \boldsymbol{\nu}_j(t)]\}$, in absence of exchange interactions. The diagrammatic corrections can be visualized as a stream of processes acting one after the other in time. The first correction, represented in the second graph of (50), describes the first-order perturbation of the trajectory $\mathbf{S}_j$ due to the field of the other spins $\mathbf{S}_k$, and involves the linear response function $\mathcal{G}_j^{(1)\alpha\beta}(t,t') = \delta S_j^{(0)\alpha}(t)/\delta\nu_j^{\beta}(t')$ of the spin $\mathbf{S}_j$, computed in the noninteracting problem. The third graph describes two perturbative processes acting in chain: the field due to $\mathbf{S}_l^{(0)}(t'')$ polarizes the spin $\mathbf{S}_k(t')$ at a later time t' and the correction induced on $\mathbf{S}_k(t')$ in turn polarizes $\mathbf{S}_j(t)$ at t. The last graph in Eq. (50) describes the second-order nonlinear response of $\mathbf{S}_j$ to the field induced on it by $\mathbf{S}_k^{(0)}$ and $\mathbf{S}_l^{(0)}$.

In general, terms of higher order involve an arbitrary number of interaction lines and nonlinear response functions of arbitrary order. In the diagrammatic representation, we have used the following graphical conventions. The solution $\mathbf{S}_j^{(0)}(t)$ of the noninteracting Langevin equations of site j is represented by a dot with one outgoing line, located at the lattice site j. The noninteracting response functions $\mathcal{G}_j^{(n)\alpha,\beta_1\dots\beta_n}(t;t_1,\dots,t_n)$ of $\mathbf{S}_j$ are represented by dots with n incoming lines and one outgoing line, located at j. Finally, the interactions $J_{jk}^{\alpha\beta}$ are represented by lines connecting different sites.

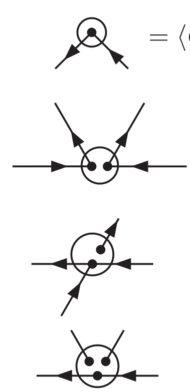
Every interaction $J_{jk}^{\alpha\beta}$ describes the effect of $\mathbf{S}_k$ as a source of perturbation to the trajectory $\mathbf{S}_j$ or vice versa. This implies that all lines in the graph connect one outgoing leg in a vertex to one ingoing leg of a different vertex. As a result, the arrows distinguishing ingoing and outgoing directions can be drawn directly on the midpoints of the interaction lines, as in the example (50).

Every term represented in a graph has to be summed over all internal sites, and integrated over all times. Since all response functions $\mathcal{G}^{(n)}$ are causal, the outward line at each vertex has a time t which is larger than the times $t_1,\dots, t_n$ of the incoming lines. The orientation of the arrows thus describes the direction of growing times.

The expansion (50) can be used to describe the perturbative solution of $\mathbf{S}'_j(t)$ for any given noise realization and for any fixed initial condition. To determine cumulants of $\boldsymbol{\eta}_i(t)$ the solution must be averaged over the $\boldsymbol{\nu}_j(t)$ and the $\mathbf{S}_{j0}$. In order to calculate the cumulant $\langle\langle \eta_i^{\alpha_1}(t_1)\dots\eta_i^{\alpha_\ell}(t_\ell)\rangle\rangle$ perturbatively we need to draw ℓ tree diagrams of the type (50), representing the solutions $S_{j_1}^{\prime\beta_1(i)}(t_1)$,..., $S_{j_n}^{\prime\beta_\ell(i)}(t_\ell)$, add ℓ lines representing the factors $J_{ij_1}^{\alpha_1\beta_1}$,...,$J_{ij_\ell}^{\alpha_n\beta_\ell}$ in Eq. (49), and average the resulting product of graphs.

The perturbative terms needed in this expansion involve averages of products of $\mathbf{S}_j^{(0)}(t)$, of noninteracting response functions $\mathcal{G}^{(n)}$, and of initial conditions $\mathbf{S}_{j0}$, calculated in the nonperturbed problem (with $J_{jk}^{\alpha\beta} = 0$). As in the linked-cluster expansion, the perturbative terms are conveniently handled by separating these averages, which we denote as $\langle \mathbf{S}_{j_1}^{(0)}\dots\mathbf{S}_{j_n}^{(0)}\dots\mathcal{G}_{k_1}^{(n_1)}\dots\mathcal{G}_{k_m}^{(n_m)}\dots\mathbf{S}_{l_1 0}\dots\mathbf{S}_{l_p 0}\rangle^{(0)}$, as sums of connected averages (cumulants).

This leads to an expansion analog to the LCE in which the vertices are connected averages of the type $\langle\langle \mathbf{S}_j^{(0)}\dots\mathbf{S}_j^{(0)}\dots\mathcal{G}_j^{(n_1)}\dots\mathcal{G}_j^{(n_m)}\dots\mathbf{S}_{j0}\dots\mathbf{S}_{j0}\rangle\rangle^{(0)}$, computed at zero order (with $J_{ij}^{\alpha\beta} = 0$). These vertices are fully local because at zero order all spins are noninteracting, and all connected averages which are not site diagonal vanish. In the following the vertices will be represented graphically in the form



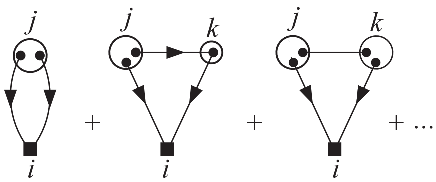
$$= \langle G^{(1)\alpha,\beta}(t,t')\rangle^{(0)}$$
$$= \langle\langle \mathcal{G}^{(1)\alpha,\beta}(t_1,t_1)\, \mathcal{G}^{(1)\gamma,\delta}(t_3,t_4)\rangle\rangle^{(0)}$$
$$= \langle\langle \mathcal{G}^{(2)\alpha,\beta\gamma}(t_1;t_2,t_3)\; S^{\delta(0)}(t_4)\rangle\rangle^{(0)}$$
$$= \langle\langle \mathcal{G}^{(1)\alpha,\beta}(t,t')\, S_0^{\gamma} S_0^{\delta}\rangle\rangle^{(0)}\,. \qquad (51)$$

Here the dots with incoming and outgoing arrows have the same meaning as before: they represent the solutions of the noninteracting Langevin equations and the nth-order noninteracting response functions $\mathcal{G}_j^{(n)}$. Dots connected to lines without arrows, instead, are introduced to represent factors of S_{j0}^{α}, where $\mathbf{S}_{j0}$ is the initial condition of $\mathbf{S}_j$ at time $t = 0$. The circle stands for a connected average of all quantities inside it, over $\boldsymbol{\nu}_j(t)$, and $\mathbf{S}_{j0}$, calculated using the zero-order Langevin dynamics and the zero-order ensemble $P(\mathbf{S}_{j0}) = \rho(\mathbf{S}_{j0}) = \exp[-\beta V(\mathbf{S}_{j0})]/\int_{\mathbf{S}}\exp[-\beta V(\mathbf{S})]$. Since the system is homogeneous, the value of the vertices does not depend on the site j. In addition, since $P(\mathbf{S}_{j0}) = \rho(\mathbf{S}_{j0})$ is the equilibrium probability for the noninteracting Langevin equation, the vertices are time-translation invariant.

The perturbative expansion involves in general vertices of the type (51), with arbitrarily many "dots" representing the connected average of arbitrary products of $S^{(0)}(t)$, $\mathcal{G}^{(n)}$, and $\mathbf{S}_{i0}$.

Using this graphical representation, the first few terms of the cumulant $C_2^{\prime\alpha\beta}(t-t') = \langle\langle \eta_i^\alpha(t)\eta_i^\beta(t')\rangle\rangle'^{(i)}_{\boldsymbol{\nu}_{i_2},\ldots,\boldsymbol{\nu}_{i_N};\mathbf{S}_{i_2 0},\ldots,\mathbf{S}_{i_N 0}}$ can be represented as

$$C_2^{\prime\alpha\beta}(t-t') =$$



$$
\begin{aligned}
&= \sum_j J_{ij}^{\alpha\gamma} J_{ij}^{\beta\delta} \langle S_j^{\beta\,(0)}(t) S_k^{\gamma\,(0)}(t')\rangle^{(0)} \\
&+ \sum_{j,k} J_{ij}^{\alpha\gamma} J_{jk}^{\delta\mu} J_{ik}^{\beta\nu} \int_0^{t'} \mathrm{d}t'' \Big[\langle S_j^{(0)\,\gamma}(t) S_j^{(0)\,\delta}(t'')\rangle^{(0)} \\
&\quad \times \langle \mathcal{G}_k^{(1)\,\mu\nu}(t',t'')\rangle^{(0)} \Big] \\
&+ \sum_{j,k} \beta J_{ij}^{\alpha\gamma} J_{jk}^{\delta\mu} J_{ik}^{\beta\nu} \langle S_j^{(0)\,\gamma}(t) S_{j\,0}^{\delta}\rangle^{(0)} \\
&\quad \times \langle S_k^{(0)\,\gamma}(t) S_{k\,0}^{\delta}\rangle^{(0)} + \ldots
\end{aligned} \tag{52}
$$

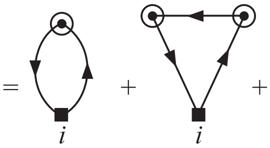
Here the two "external" lines connected to the origin i correspond to the factors $J_{ij}^{\alpha\beta}$, $J_{ik}^{\alpha\beta}$ in the definition of the local field $\eta_i^\alpha(t) = \sum_j J_{ij}^{\alpha\beta} S_j^{\beta\prime(i)}(t)$. The inner part of the graphs describes the average of the product of $S_j^{\beta\prime(i)}(t)$ and $S_k^{\beta\prime(i)}(t')$. The third graph contains a term coming from the first-order expansion of the distribution $Z_i'^{-1}\mathrm{e}^{-\beta H_i'(\mathbf{S}_{i_2 0},\ldots,\mathbf{S}_{i_N 0})}$ of the initial conditions. The contribution of the corresponding interaction is represented by a "static line" (illustrated in the diagram as a line without an arrow).

At higher order the diagrams involve arbitrary numbers of lines with arrows (representing the perturbative solution of the Langevin equations) and without arrows (representing the expansion of the Gibbs distribution of the cavity). The expansion of cumulants C_ℓ' with arbitrary ℓ is given by diagrams with the same structure, but with ℓ lines connected to the origin i.

We can now discuss the scaling of diagrams in the large-d limit. Although the vertices of the dynamic problem are more complex than those of the static calculation, the power counting for $d\to\infty$ is the same. As a result, it turns out that the only finite cumulant for $d\to\infty$ is $C_2^{\prime\alpha\beta}(t-t')$ because it is represented by graphs in which two lines are attached to i. All higher cumulants of $\boldsymbol{\eta}(t)$, instead, are represented by graphs in which more than two lines are attached to the origin i. Since by the cavity construction the internal sites are summed over all lattice sites except i, this implies, as in Sec. III, that all cumulants beyond the second are suppressed for $d\to\infty$. This shows that $\boldsymbol{\eta}_i(t)$ has a Gaussian distribution in large dimension. We identify the cumulant $C_2^{\prime\alpha\beta}(t-t')$ with the correlation function $l^{\alpha\beta}(t-t')$ describing the fluctuation of the noise in the effective single-site dynamics.

The same power-counting analysis which has been used for correlations of $\boldsymbol{\eta}$ can be applied to any cumulant involving both $\boldsymbol{\eta}$ and the kernels $K^{(n)}$. The result is the same: the only terms of order 1 are those for which the corresponding perturbative graphs have two lines, and no more, connected to i. Using this, we see that the average of the linear kernel $K^{(1)\alpha\beta}(t,t')$ is of order 1 because it has an expansion given by diagrams

$$K^{\alpha\beta}(t-t') =$$



$$
\begin{aligned}
&= \sum_j J_{ij}^{\alpha\gamma} J_{ij}^{\beta\delta} \langle \mathcal{G}^{(1)\,\gamma\delta}(t,t')\rangle^{(0)} \\
&+ \sum_{j,k} \int_{t'}^{t} \mathrm{d}t'' \,\Big(J_{ij}^{\alpha\gamma} J_{jk}^{\delta\mu} J_{ki}^{\nu\beta} \\
&\quad \times \langle \mathcal{G}_j^{(1)\,\gamma\delta}(t,t'')\rangle^{(0)} \langle \mathcal{G}_k^{(1)\,\mu\nu}(t,t'')\rangle^{(0)} \Big) + \ldots
\end{aligned} \tag{53}
$$

All higher response functions ($K^{(n)}$, $n \geqslant 2$) are, instead, negligible for $d\to\infty$ because their averages and cumulants are represented by graphs in which more than two lines are attached to i. In particular, the variance of $K^{(1)}$ is given by a graph with four lines connected to the origin i, and is negligible. Thus, $K^{(1)}$ is deterministic: it has negligible fluctuations in the limit $d\to\infty$.

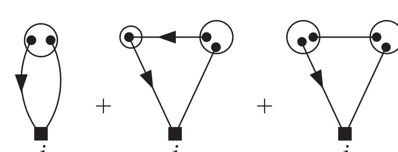
In addition, we note that since the fluctuation $l^{\alpha\beta}(t-t') = C_2^{\prime\alpha\beta}(t-t')$ and the kernel $K^{\alpha\beta}(t-t')$ are related to the correlation and the linear response functions of the cavity system in equilibrium. Thus, it can be shown that they are related by the FDT [Eq. (35)] (see Appendix A).

To complete the calculation we have to discuss one last point: the fact that the correct averages over initial conditions should be computed with the weight (47) and not with the cavity distribution $Z_i'^{-1}\exp(-\beta H_i')$. The difference between the two distributions leads to negligible corrections to all cumulants, apart from one: the mean $\langle\boldsymbol{\eta}(t)\rangle_{\nu_2,\ldots,\nu_N,\mathbf{S}_{i_2},\ldots,\mathbf{S}_{i_N}|\mathbf{S}_{i0}}$. In fact, the average of $\boldsymbol{\eta}(t)$ computed with the cavity distribution is zero by inversion symmetry. The mean $\langle\boldsymbol{\eta}(t)\rangle_{\nu_2,\ldots,\nu_N,\mathbf{S}_{i_2},\ldots,\mathbf{S}_{i_N}|\mathbf{S}_{i0}}$ at fixed $\mathbf{S}_{i0}$, instead, receives a $O(1)$ contribution. For $d\to\infty$ this contribution can be calculated expanding the distribution (47) to first order in the coupling $\beta\sum_j J_{ij}^{\alpha\beta} S_{i0}^\alpha S_{j0}^\beta$. As a result we find

$$
\begin{aligned}
&\big\langle \eta_i^\alpha(t)\big\rangle_{\boldsymbol{\nu}_{i_2},O\boldsymbol{\nu}_{i_N},\mathbf{S}_{i_2 0},\ldots,\mathbf{S}_{i_N 0}|\mathbf{S}_{i0}} \\
&\quad \underset{d\to\infty}{\to} \sum_{j,k} \beta J_{ij}^{\alpha\gamma} J_{ik}^{\beta\delta} \big\langle\!\big\langle S_j^{\gamma\prime(i)}(t) S_{k0}^{\delta}\big\rangle\!\big\rangle'^{(i)}_{\boldsymbol{\nu}_{i_2}\ldots\,\boldsymbol{\nu}_{i_N},\mathbf{S}_{i_2 0}\ldots\mathbf{S}_{i_N 0}} \\
&\quad = \beta l^{\alpha\beta}(t) S_{i0}^\beta.
\end{aligned} \tag{54}
$$

Diagrammatically this term is represented by

$$\langle \eta_i(t)\rangle =$$



(55),

which, again, are graphs in which only two lines are connected to the origin. One of the two lines is now a static line, representing the term $\beta J_{ij}^{\alpha\beta} S_{i0}^\alpha S_{j0}^\beta$ in the distribution (47).

In all other terms, the difference between the distribution (47) and the cavity distribution is negligible. In particular, the variance $l^{\alpha\beta}(t-t')$ and the kernel $K^{\alpha\beta}(t-t')$ calculated above remain exact for $d\to\infty$. This can be seen systematically expanding

$$\langle\langle\eta^{\alpha_1}\ldots\eta^{\alpha_n}K^{(n_1)}\ldots K^{(n_m)}\rangle\rangle_{\boldsymbol{\nu}_{i_2},\ldots,\boldsymbol{\nu}_{i_N},\mathbf{S}_{i_2 0},\ldots,\mathbf{S}_{i_N 0}|\mathbf{S}_{i0}} = \sum_{m=0}^{\infty}\frac{\beta^m}{m!}\sum_{j_1,\ldots,j_m}J_{ij_1}^{\beta_1\gamma_1}\ldots J_{ij_m}^{\beta_m\gamma_m}S_{i0}^{\beta_1}\ldots S_{i0}^{\beta_m}\langle\langle S_{j_1 0}^{\gamma_1}\ldots S_{j_m 0}^{\gamma_m} \times\eta^{\alpha_1}\ldots\eta^{\alpha_n}K^{(n_1)}\ldots K^{(n_m)}\rangle\rangle'^{(i)}_{\boldsymbol{\nu}_{i_2}\ldots\boldsymbol{\nu}_{i_N},\mathbf{S}_{i_2 0},\mathbf{S}_{i_N 0}} \quad (56)$$

as a series of connected averages with insertions of $\mathbf{S}_{j0}$ calculated in the cavity ensemble. Every additional interaction with $\mathbf{S}_{i0}$ carries an additional external static line attached to i, and suppresses the graph.

This concludes the derivation. Introducing $\zeta_i^\alpha(t)=\eta_i^\alpha(t)-\beta l^{\alpha\beta}(t)$, and using Eq. (44) we find that the instantaneous magnetic field is for large d: $b_i^\alpha(t)=\zeta_i^\alpha(t)+\beta l^{\alpha\beta}(t)S_{i0}^\beta+\int_0^t \mathrm{d}t\,K^{\alpha\beta}(t-t')S_i^\beta(t')$, with $\boldsymbol{\zeta}$ a zero-mean Gaussian noise having $\langle\zeta_i^\alpha(t)\zeta_i^\beta(t')\rangle=l^{\alpha\beta}(t-t')$.

B. General properties of the single-site dynamical problem

The non-Markovian Langevin equations (33) and (34) derive from the equilibrium dynamics of the system and thus satisfy time-translation invariance (TTI) and the fluctuation-dissipation theorem. As a result, the single-site response function $g^{\alpha\beta}(t-t')=\langle\delta S_i^\alpha(t)/\delta\nu_i^\beta(t')\rangle$ is related to the single-site correlation $c^{\alpha\beta}(t-t')=\langle S_i^\alpha(t)S_i^\beta(t')\rangle$ via the FDT $g^{\alpha\beta}(t-t')=-\beta\Theta(t-t')\mathrm{d}c^{\alpha\beta}(t-t')/\mathrm{d}t=\dot{c}_+^{\alpha\beta}(t-t')$. The TTI is not immediately manifest in the single-site equations because the term $\beta l^{\alpha\beta}(t)S_{i0}^\beta$ and the integral $\int_0^t \mathrm{d}t' K^{\alpha\beta}(t-t')S_i^\beta(t')$ seem to depend explicitly on the choice of the origin of time $t_0=0$. However, the two noninvariant terms [the truncation of the integral at the lower limit and the field $\beta l^{\alpha\beta}(t)S_{i0}^\beta$] compensate each other, leading to results which do not depend on the origin of time. This can be verified, for example, by mapping the Langevin dynamics to an equivalent Markovian problem, in which the colored noise is mimicked by a bath of harmonic oscillators [61,62]. Integrating out the bath leads to an equation of motion in which $\mathbf{b}_i(t)$ has the form (34).

In addition to TTI and the FDT, another essential property of the single-site Langevin problem is that it is consistent with the static results of Sec. III. All equal-time correlations of $\mathbf{S}_i(t)$ and $\mathbf{b}_i(t)$ coincide with the static correlations described by the distribution (13), when the matrix $L^{\alpha\beta}$ is identified with the instantaneous correlation $l^{\alpha\beta}(0)$. This can be seen conveniently by studying the correlations at time $t=0$, where the field (34) reduces to $b_i^\alpha(0)=\zeta_i^\alpha(0)+\beta l^{\alpha\beta}(0)S_{i0}^\beta$. $\zeta_i^\alpha(0)$ is a Gaussian field with $\langle\zeta_i^\alpha(0)\zeta_i^\beta(0)\rangle=l^{\alpha\beta}(0)$. Identifying $l^{\alpha\beta}(0)=L^{\alpha\beta}$ and using that $\mathbf{S}_{i0}$ is weighted by the distribution (36), it is simple to verify that the joint distribution of $\mathbf{S}_{i0}$ and $\mathbf{b}_i(0)$ is exactly the static distribution $P(\mathbf{S}_i,\mathbf{b}_i)$ in Eq. (13). TTI implies that $P(\mathbf{S}_i(t),\mathbf{b}_i(t))$ remains the same at all later times.

We finally discuss two equations of motion, which are useful in the subsequent derivations. Using the property of Gaussian averages [63] $\langle S_i^\alpha(t)\zeta_i^\beta(t')\rangle=\int_0^\infty \mathrm{d}t'' l^{\beta\gamma}(t'-t'')\langle\delta S_i^\alpha(t)/\delta\zeta_i^\gamma(t'')\rangle$ and the fluctuation-dissipation relations $g^{\alpha\beta}(t-t')=-\beta\dot{c}_+^{\alpha\beta}(t-t')$, $K^{\alpha\beta}(t-t')=-\beta\Theta(t-t')\mathrm{d}l^{\alpha\beta}(t-t')/\mathrm{d}t$ we find, for $t\geqslant t'$,

$$a^{\alpha\beta}(t-t')=\langle S_i^\alpha(t)b_i^\beta(t')\rangle=\beta c^{\alpha\gamma}(t-t')l^{\gamma\beta}(0)+\int_{t'}^t \mathrm{d}t'' g^{\alpha\gamma}(t-t'')l^{\gamma\beta}(t''-t'), \quad (57)$$

$$\begin{aligned}\lambda^{\alpha\beta}(t-t')&=\langle b_i^\alpha(t)b_i^\beta(t')\rangle\\&=l^{\alpha\beta}(t-t')+\beta^2 l^{\alpha\gamma}(t-t')c^{\gamma\delta}(0)l^{\delta\beta}(0)\\&\quad+\beta\int_{t'}^t \mathrm{d}t''\,K^{\alpha\gamma}(t-t'')c^{\gamma\delta}(t''-t')l^{\delta\beta}(0)\\&\quad+\int_{t'}^t \mathrm{d}t''\int_{t'}^{t''}\mathrm{d}t'''\,[K^{\alpha\gamma}(t-t'')\\&\quad\times g^{\gamma\delta}(t''-t''')l^{\delta\beta}(t'''-t')].\end{aligned} \quad (58)$$

The $\lambda^{\alpha\beta}(t-t')$ correlation in the region $t<t'$ follows from Eq. (58) by symmetry.

For equal times, these relations reduce to the static equations (16). Taking a time derivative and using the fluctuation-dissipation relations we find after a Fourier transform Eqs. (42). We note also that Eqs. (57) and (58) are fully consistent with the time-translation invariance of the colored single-site dynamics.

C. Self-consistency equations

To conclude, we discuss the derivation of Eqs. (37) and (38), which are needed to fix the self-consistency of the single-site problem. When $i=j$, Eqs. (37) and (38) can be shown to be equivalent to Eqs. (42), and, thus, are satisfied automatically.

For noncoincident sites $i\neq j$, the correlations can be studied by a two-cavity method, with two cavities at i and j. In this framework, the correlations are analyzed by studying an effective two-body Langevin equation, in which the effects of the remaining $N-2$ spins are described via a fluctuating bath.

At leading order for $d\to\infty$, the effective two-spin problem is equivalent to two independent copies of the single-site Langevin equations [Eqs. (33) and (34)]. Nontrivial correlations between the two sites arise from the leading corrections, which are of order $O(d^{-\ell_{ij}/2})$, and which couple the motion of the two spins. These corrections are due to (1) the direct contribution of $\mathbf{S}_i$ to the field $\mathbf{b}_j$ (and, vice versa of $\mathbf{S}_j$ to $\mathbf{b}_i$) and (2) indirect contributions, mediated by the bath.

A diagrammatic analysis analog to that discussed in Sec. IV A shows that the two-site equations can be written in the form

$$\begin{aligned}\dot{\mathbf{S}}_i&=-\mathbf{S}_i\times\{\mathbf{S}_i\times[\mathbf{F}_i+\mathbf{b}_i(t)+\boldsymbol{\nu}_i(t)]\},\\\dot{\mathbf{S}}_j&=-\mathbf{S}_j\times\{\mathbf{S}_j\times[\mathbf{F}_j+\mathbf{b}_j(t)+\boldsymbol{\nu}_j(t)]\},\end{aligned} \quad (59)$$

where the components of the fields $b_i^\alpha(t)$ and $b_j^\alpha(t)$ are

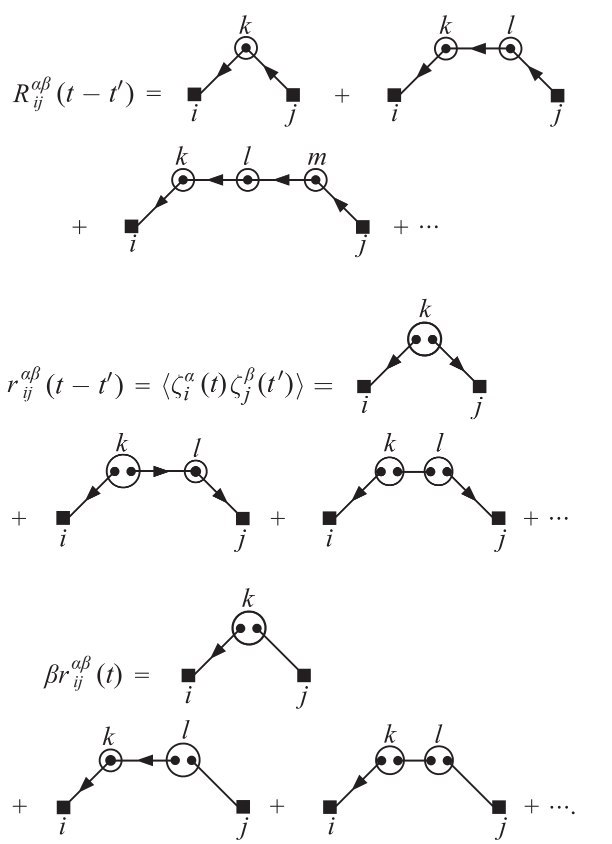
$$b_i^\alpha(t)=\zeta_i^\alpha(t)+\beta l^{\alpha\beta}(t)S_{i0}^\beta+\int_0^t \mathrm{d}t'\,K^{\alpha\beta}(t-t')S_i^\beta(t') + J_{ij}^{\alpha\beta}S_j^\beta(t)+\beta r_{ij}^{\alpha\beta}(t)S_{j0}^\beta+\int_0^t \mathrm{d}t'\,R_{ij}^{\alpha\beta}(t-t')S_j^\beta(t'),$$

$$b_j^\alpha(t) = \zeta_j^\alpha(t) + \beta l^{\alpha\beta}(t) S_{j0}^\beta + \int_0^t \mathrm{d}t'\, K^{\alpha\beta}(t-t') S_j^\beta(t') + J_{ji}^{\alpha\beta} S_i^\beta(t) + \beta r_{ji}^{\alpha\beta}(t) S_{i0}^\beta + \int_0^t \mathrm{d}t'\, R_{ji}^{\alpha\beta}(t-t') S_i^\beta(t'). \tag{60}$$

Here, $\boldsymbol{\zeta}_i(t)$ and $\boldsymbol{\zeta}_j(t')$ are Gaussian noises with zero mean. The site-diagonal correlations $\langle \zeta_i^\alpha(t)\zeta_i^\beta(t')\rangle = \langle \zeta_j^\alpha(t)\zeta_j^\beta(t')\rangle$ are of order 1 and are equal, up to negligible corrections to the correlation $l^{\alpha\beta}(t-t')$ defining the single-site Langevin equation. In addition, the random fields have a correlation $\langle \zeta_i^\alpha(t)\zeta_j^\beta(t')\rangle = r_{ij}^{\alpha\beta}(t-t')$.

The kernels, similarly, have site-diagonal parts $K^{\alpha\beta}(t-t')$ which are equal for d large to the kernels $K^{\alpha\beta}(t-t')$ entering the single-site equation (34). In addition, there are off-diagonal response terms $R_{ij}^{\alpha\beta}(t-t')$, $R_{ji}^{\alpha\beta}(t-t')$.

Diagrammatically, the off-diagonal corrections encoded in Eq. (60) correspond to graphs of the form



The diagrams imply that correlation and response functions r_{ij}, R_{ij} are of order $d^{-\ell_{ij}/2}$, with ℓ_{ij} the Manhattan distance between i and j. At this order, no other diagram contributes, and thus Eqs. (60) are exact.

The intersite terms are related via $r_{ij}^{\alpha\beta}(t-t') = \sum_{k,l\neq i,j} J_{ik}^{\alpha\gamma} J_{jl}^{\beta\delta} \langle\langle S_k^{\gamma\prime\prime(ij)}(t) S_l^{\delta\prime\prime(ij)}(t')\rangle\rangle^{\prime\prime(ij)}$, $R_{ij}^{\alpha\beta}(t-t') = \sum_{k,l\neq i,j} J_{ik}^{\alpha\gamma} J_{jl}^{\beta\delta} \langle\langle G^{(1)\gamma,\delta\prime\prime(ij)}(t,t')_{kl}\rangle\rangle^{\prime\prime(ij)}$, $R_{ji}^{\beta\alpha}(t-t') = \sum_{k,l\neq i,j} J_{ik}^{\alpha\gamma} J_{jl}^{\beta\delta} \langle\langle G_{lk}^{(1)\delta,\gamma\prime\prime(ij)}(t,t')\rangle\rangle^{\prime\prime(ij)}$ to the equilibrium correlations $\langle\langle S_k^{\gamma\prime\prime(ij)}(t) S_l^{\delta\prime\prime(ij)}(t')\rangle\rangle^{\prime\prime(ij)}$ and to the response functions $\langle\langle G^{(1)\gamma,\delta\prime\prime(ij)}(t,t')_{kl}\rangle\rangle^{\prime\prime(ij)}$ of the two-cavity system. Thus, they satisfy the fluctuation-dissipation relations

$$R_{ij}^{\alpha\beta}(t-t') = -\beta\Theta(t-t')\frac{\mathrm{d}}{\mathrm{d}t} r_{ij}^{\alpha\beta}(t-t'),$$
$$R_{ji}^{\beta\alpha}(t-t') = -\beta\Theta(t-t')\frac{\mathrm{d}}{\mathrm{d}t} r_{ji}^{\beta\alpha}(t-t'). \tag{61}$$

The two-site equations (59) and (60) allow to calculate the correlations $C_{ij}^{\alpha\beta}(t-t')$, $A_{ij}^{\alpha\beta}(t-t')$, $\Lambda_{ij}^{\alpha\beta}(t-t')$. The explicit calculations are presented in Appendix B, and use that for d large, the equations for the two spins can be solved at first order in r_{ij}, R_{ij}, and J_{ij}. As a result we find

$$\dot{C}_{+ij}^{\alpha\beta}(\omega) = -\beta \dot{c}_+^{\alpha\gamma}(\omega)\big(J_{ij}^{\gamma\delta} + R_{ij}^{\gamma\delta}(\omega)\big)\dot{c}_+^{\delta\beta}(\omega), \tag{62}$$

$$\dot{A}_{+ij}^{\alpha\beta}(\omega) = \dot{c}_+^{\alpha\gamma}(\omega)\big(J_{ij}^{\gamma\beta} + R_{ij}^{\gamma\beta}(\omega)\big) + \dot{C}_{+ij}^{\alpha\gamma}(\omega)K^{\gamma\beta}(\omega),$$
$$\dot{\Lambda}_{+ij}^{\alpha\beta}(\omega) = -\frac{1}{\beta}R_{ij}^{\alpha\beta}(\omega) + K^{\alpha\gamma}(\omega)\dot{C}_{+ij}^{\gamma\delta}(\omega)K^{\delta\beta}(\omega) + \big(J_{ij}^{\alpha\gamma} + R_{ij}^{\alpha\gamma}(\omega)\big)\dot{c}_+^{\gamma\delta}(\omega)K^{\delta\beta}(\omega) + K^{\alpha\gamma}(\omega)\dot{c}_+^{\gamma\delta}(\omega)\big(J_{ij}^{\delta\beta} + R_{ij}^{\delta\beta}(\omega)\big). \tag{63}$$

Equation (62) has a very simple interpretation: the response function at noncoincident sites $i \neq j$ receives two contributions of the same order of magnitude ($\approx d^{-\ell_{ij}/2}$): one from the direct interaction $J_{ij}^{\alpha\beta}$ and one mediated by the cavity.

Combining Eqs. (62) and (63), and using Eqs. (42) we find relations which do not depend on R_{ij}, and which are valid both for $i = j$ and for $i \neq j$:

$$\dot{C}_{+ij}^{\alpha\beta}(\omega) - \delta_{ij}\dot{c}_+^{\alpha\beta}(\omega) = \beta\dot{C}_{+ij}^{\alpha\gamma}(\omega)K^{\gamma\delta}(\omega)\dot{c}_+^{\delta\beta}(\omega) - \beta\dot{A}_{+ij}^{\alpha\gamma}(\omega)\dot{c}_+^{\gamma\beta}(\omega),$$
$$\dot{A}_{+ij}^{\alpha\beta}(\omega) = \dot{c}_+^{\alpha\gamma}J_{ij}^{\gamma\beta} - \beta\dot{c}_+^{\alpha\gamma}(\omega)\dot{\Lambda}_{+ij}^{\gamma\beta}(\omega) + \beta\dot{c}_+^{\alpha\gamma}(\omega)K^{\gamma\delta}(\omega)\dot{A}_{+ij}^{\delta\beta}(\omega). \tag{64}$$

These are equivalent to Eqs. (39), and in real time, to Eqs. (37) and (38).

V. GLASS TRANSITION

The dynamical analysis allows to distinguish under which conditions the paramagnetic phase has a liquid-like or a glasslike behavior. The liquid and the glass phases can be discerned by the large-time behavior of the time-dependent correlations: $\lim_{|t-t'|\to\infty} l^{\alpha\beta}(t-t') = L_2^{\alpha\beta}$, $\lim_{|t-t'|\to\infty} c^{\alpha\beta}(t-t') = c_2^{\alpha\beta}$, $\lim_{|t-t'|\to\infty} C_{ij}^{\alpha\beta}(t-t') = C_{2ij}^{\alpha\beta}$, $\lim_{|t-t'|\to\infty} A_{ij}^{\alpha\beta}(t-t') = A_{2ij}^{\alpha\beta}$… . In the ergodic phase all correlations decay to zero at large times. In the glass phase, instead, L_2, c_2, λ_2, C_{2ij} are different from zero. In particular, $c_2^{\alpha\beta}$, the large-time limit of the spin correlation, can be identified with the Edwards-Anderson order parameter: $c_2^{\alpha\beta} = q_{\mathrm{EA}}^{\alpha\beta}$.

To locate the occurrence of a glass transition, it is not necessary to solve the dynamics in detail. In analogy with the theory of supercooled liquids in $d \to \infty$ [62], the order parameters L_2, c_2, a_2, λ_2 are fixed by closed equations which make reference only to the static averages and to the long-time limit of the correlations, and not to their transient behavior.

To derive these equations, we need to discuss the $|t-t'| \to \infty$ limit of both the single-site problem and the self-consistency relations. We begin by discussing the single-site problem. When ergodicity is broken and $L_2^{\alpha\beta} \neq 0$, the field $\mathbf{b}_i(t)$ develops a static component, which remains constant for an infinitely long time. The single-site colored Langevin equation in presence of this quenched component can be analyzed by methods analog to the approach used in Ref. [62] for the vitrification of a supercooled liquid in infinite dimensions. In presence of a quenched component, $\boldsymbol{\zeta}_i(t)$ can be separated as the sum $\boldsymbol{\zeta}_i(t) = \boldsymbol{\zeta}_{1i}(t) + \boldsymbol{\zeta}_{2i}$ of two independent parts $\boldsymbol{\zeta}_{1i}$ and $\boldsymbol{\zeta}_{2i}$, both having Gaussian distributions with zero mean. $\boldsymbol{\zeta}_{1i}(t)$ is a dynamic noise, with a correlation $\langle \zeta_{1i}^\alpha(t)\zeta_{1i}^\beta(t')\rangle = l^{\alpha\beta}(t-t') - L_2^{\alpha\beta} = l_1^{\alpha\beta}(t-t')$ which decreases to zero at large times. $\boldsymbol{\zeta}_{2i}$ is instead a static noise, with $\langle \zeta_{2i}^\alpha \zeta_{2i}^\beta\rangle = L_2^{\alpha\beta}$ describing the time-persistent part of the random field $\boldsymbol{\zeta}_i$. The dynamic and static parts of the noise are mutually uncorrelated: $\langle \zeta_{1i}^\alpha(t)\zeta_{2i}^\beta\rangle = 0$.

The field $\mathbf{b}_i(t)$ which enters in the single-site Langevin equations breaks similarly into static and dynamic parts:

$$b_i^\alpha(t) = b_{2i}^\alpha + \zeta_{1i}^\alpha(t) + \beta l_1^{\alpha\beta}(t) S_{i0}^\beta + \int_0^t \mathrm{d}t'\, K_1^{\alpha\beta}(t-t') S_i^\beta(t'), \tag{65}$$

with $b_{2i}^\alpha = \zeta_{2i}^\alpha + \beta L_2^{\alpha\beta} S_{i0}^\beta$ and $K_1^{\alpha\beta}(t-t') = -\beta\Theta(t-t')\mathrm{d}l_1^{\alpha\beta}(t-t')/\mathrm{d}t = K^{\alpha\beta}(t-t')$ (the constant $L_2^{\alpha\beta}$ does not contribute to the time derivative defining the memory kernel $K^{\alpha\beta}$).

We can assume that the dynamical noises $\boldsymbol{\nu}_i$ and $\boldsymbol{\zeta}_1$ drive the system into an equilibrium state dependent on the static field mean field $\mathbf{b}_{2i}$. In other words, we assume that after running the Langevin dynamics from the initial time $t = 0$ to a large time, the system relaxes to a Gibbs probability, subject to the static field $\mathbf{b}_{2i}$.

Analyzing the colored Langevin equations with the methodology of Refs. [61,62], we find that the asymptotic large-time distribution is

$$P_1(\mathbf{S}_i|\mathbf{b}_{2i}) = \frac{\mathrm{e}^{-\beta V(\mathbf{S}_i) + \beta(\mathbf{b}_{2i}\cdot\mathbf{S}_i) + \frac{1}{2}\beta^2 L_1^{\alpha\beta} S_i^\alpha S_i^\beta}}{\int_{\bar{\mathbf{S}}} \mathrm{e}^{-\beta V(\bar{\mathbf{S}}) + \beta(\mathbf{b}_{2i}\cdot\bar{\mathbf{S}}) + \frac{1}{2}\beta^2 L_1^{\gamma\delta}\bar{S}^\gamma \bar{S}^\delta}}, \tag{66}$$

with $L_1^{\alpha\beta} = l_1^{\alpha\beta}(0) = L^{\alpha\beta} - L_2^{\alpha\beta}$. This distribution can be interpreted as the probability of a single spin within an individual metastable state. The quenched component $\mathbf{b}_{2i}$ represents a mean field, present within the metastable state, and acting on the spin $\mathbf{S}_i$. When the Gibbs distribution breaks into a superposition of many nonergodic sectors, the field $\mathbf{b}_{2i}$ is itself a random variable. The probability to extract a value of $\mathbf{b}_{2i}$ can be calculated using the Gaussian distribution of $\boldsymbol{\zeta}_{2i}$ and the equilibrium distribution $P_{1\mathrm{eq}}(\mathbf{S}_{i0})$ of the initial conditions $\mathbf{S}_{i0}$. The result is

$$\begin{aligned} P_{\mathrm{slow}}(\mathbf{b}_{2i}) &= \int_{\mathbf{S}_0}\int \mathrm{d}^3\boldsymbol{\zeta}_2 \Bigg\{ \delta\big(b_{2i}^\alpha - \beta L_2^{\alpha\beta} S_0^\beta - \zeta_2^\alpha\big) \\ &\quad\times \frac{\exp\big(-L_2^{-1\alpha\beta}\zeta_2^\alpha\zeta_2^\beta/2\big)}{\sqrt{(2\pi)^3 \det \underline{\underline{L_2}}}} \frac{1}{Z_1} \exp\big[-\beta V(\mathbf{S}_0) \\ &\quad + \beta^2 L^{\alpha\beta} S_0^\alpha S_0^\beta/2\big]\Bigg\} \\ &= \frac{1}{Z_1\sqrt{(2\pi)^3 \det \underline{\underline{L_2}}}} \int_{\mathbf{S}_0} \exp\big[-\beta V(\mathbf{S}_0) \\ &\quad + \beta(\mathbf{b}_{2i}\cdot\mathbf{S}_0) - L_2^{-1\alpha\beta} b_{2i}^\alpha b_{2i}^\beta/2 + \beta^2 L_1^{\alpha\beta} S_0^\alpha S_0^\beta/2\big], \end{aligned} \tag{67}$$

and can be interpreted as the probability to find a given value of the mean field, selecting a metastable state at random in the Gibbs ensemble.

Since $P_{1\mathrm{eq}}(\mathbf{S}_i) = \int \mathrm{d}^3\mathbf{b}_{2i} P_{\mathrm{slow}}(\mathbf{b}_{2i}) P_1(\mathbf{S}_i|\mathbf{b}_{2i})$, the equal-time averages are consistent with the static analysis. The separation of the average into a quenched and a vibrational part, however, allows also to calculate the correlations at large time separations. In particular, the Edwards-Anderson order parameter can be calculated as

$$\begin{aligned} q_{\mathrm{EA}}^{\alpha\beta} &= \lim_{|t-t'|\to\infty} \big\langle S_i^\alpha(t) S_i^\beta(t')\big\rangle \\ &= \int \mathrm{d}^3\mathbf{b}_{2i}\, P_{\mathrm{slow}}(\mathbf{b}_{2i})\, \big\langle S_i^\alpha\big\rangle\big|_{\mathbf{b}_{2i}} \big\langle S_i^\beta\big\rangle\big|_{\mathbf{b}_{2i}} \\ &= \overline{\big\langle S_i^\alpha\big\rangle\big|_{\mathbf{b}_{2i}} \big\langle S_i^\beta\big\rangle\big|_{\mathbf{b}_{2i}}}. \end{aligned} \tag{68}$$

Here $\langle \bullet \rangle|_{\mathbf{b}_{2i}} = \int_{\mathbf{S}} \bullet\, P(\mathbf{S}|\mathbf{b}_{2i})$ is an average calculated in the ensemble (66), with $\mathbf{b}_{2i}$ fixed, and the overline symbol $\overline{\bullet} = \int \mathrm{d}^3\mathbf{b}_{2i} \bullet P_{\mathrm{slow}}(\mathbf{b}_{2i})$ is an average over $\mathbf{b}_{2i}$, weighted with the distribution $P_{\mathrm{slow}}(\mathbf{b}_{2i})$.

More generally, arbitrary averages $\langle \Phi_1(\mathbf{S}(t_1))\, \Phi_2(\mathbf{S}(t_2))\ldots\Phi_n(\mathbf{S}(t_n))\rangle$ in the limit in which all time differences are large can be calculated as

$$\overline{\langle \Phi_1(\mathbf{S}(t_1))\rangle|_{\mathbf{b}_{2i}} \ldots \langle \Phi_n(\mathbf{S}(t_n))\rangle|_{\mathbf{b}_{2i}}}. \tag{69}$$

Equation (69) assumes that within a single state the dynamics loses correlation at large time, so that for a fixed $\mathbf{b}_{2i}$, $\langle \Phi_1(\mathbf{S}(t_1))\ldots\Phi_n(\mathbf{S}(t_n))\rangle|_{\mathbf{b}_{2i}} \approx \langle \Phi_1(\mathbf{S}(t_1))\rangle|_{\mathbf{b}_{2i}} \times \cdots \times \langle \Phi_n(\mathbf{S}(t_n))\rangle|_{\mathbf{b}_{2i}}$ when the time separations are large. A frozen correlation, which persists for large times, emerges after averaging over the static field $\mathbf{b}_{2i}$.

Equation (68) provides an equation linking the Edwards-Anderson order parameter to $L^{\alpha\beta}$, $L_2^{\alpha\beta}$, and $L_1^{\alpha\beta}$. To fix the order parameters, we also need to discuss the large-time limit of the self-consistency equations. To this end, it is convenient to integrate Eqs. (37) over time. In the limit $t - t' \to +\infty$ we find

$$\begin{aligned} &\int_{t'}^{t} \mathrm{d}t''\, \dot{C}_{+ij}^{\alpha\beta}(t''-t') \\ &\quad= -C_{1ij}^{\alpha\beta} = -\delta_{ij} c_1^{\alpha\beta} - \beta\int_{t'}^{t}\mathrm{d}t_1\, \big\{\big[A_{ij}^{\alpha\gamma}(t-t_1) - A_{ij}^{\alpha\gamma}(0)\big] \\ &\qquad\times \dot{c}_+^{\gamma\beta}(t_1-t')\big\} + \beta\int_{t'}^{t}\mathrm{d}t_1\int_{t'}^{t_1}\mathrm{d}t_2 \big[C_{ij}^{\alpha\gamma}(t-t_1) \\ &\qquad - C_{ij}^{\alpha\gamma}(0)\big] K^{\gamma\delta}(t_1-t_2)\dot{c}_+^{\delta\beta}(t_2-t'), \end{aligned} \tag{70}$$

where $C_{1ij}^{\alpha\beta} = C_{ij}^{\alpha\beta}(0) - C_{2ij}^{\alpha\beta}$ and $c_1^{\alpha\beta} = c^{\alpha\beta}(0) - c_2^{\alpha\beta}$. Since $\dot{c}_+^{\alpha\beta}(t_1 - t)$ and $K^{\gamma\delta}(t_1 - t_2)$ are defined by time derivatives of correlations, they do not contain quenched parts. Thus, $\dot{c}_+^{\alpha\beta}(t_1 - t')$ and $\int_{t'}^{t_1} \mathrm{d}t_2 K^{\gamma\delta}(t_1 - t_2)\dot{c}_+^{\delta\beta}(t_2 - t')$ vanish for $|t_1 - t'| \to \infty$. As a result the integrals are dominated by the region in which t_1 is near t' and far from t. This allows to replace in the integrals $A_{ij}^{\alpha\gamma}(t-t_1) = \lim_{|t-t_1|\to\infty} A^{\alpha\gamma}(t-t_1) = A_{2ij}^{\alpha\beta}$, $C_{ij}^{\alpha\gamma}(t-t_1) = \lim_{|t-t_1|\to\infty} = C_{2ij}^{\alpha\gamma}$. Introducing $C_{1ij}^{\alpha\beta} = C_{ij}^{\alpha\beta}(0) - C_{2ij}^{\alpha\beta}$, $A_{1ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(0) - A_{2ij}^{\alpha\beta}$ and using the FDT we find that Eq. (70) behaves for $|t-t'|$ large as

$$\begin{aligned}\beta^2 C_{1ij}^{\alpha\gamma} &\int_{t'}^{t} \mathrm{d}t_2\, \dot{c}_+^{\delta\beta}(t_2-t')[l^{\gamma\delta}(t-t_2) - l^{\gamma\delta}(0)] \\ &= \beta^2 C_{1ij}^{\alpha\gamma} L_1^{\gamma\delta} c_1^{\delta\beta} = -C_{1ij}^{\alpha\beta} + \delta_{ij} c_1^{\alpha\beta} + \beta A_{1ij}^{\alpha\gamma} c_1^{\gamma\beta}. \qquad (71)\end{aligned}$$

In Eq. (71) we have used again that the integration is dominated by the region in which t_2 is far from t.

Analyzing in a similar way Eqs. (38), (57), and (58), and using that the equal-time correlations satisfy the static relations (16) we find

$$\begin{aligned} A_{1ij}^{\alpha\beta} + \beta^2 c_1^{\alpha\gamma} L_1^{\gamma\delta} A_1^{\delta\beta} &= c_1^{\alpha\gamma} J_{ij}^{\gamma\beta} + \beta c_1^{\alpha\gamma} \Lambda_{1ij}^{\gamma\beta}, \\ a_1^{\alpha\beta} &= \beta c_1^{\alpha\gamma} L_1^{\gamma\beta}, \qquad (72) \\ \lambda_1^{\alpha\beta} &= L_1^{\alpha\beta} + \beta^2 L_1^{\alpha\gamma} c_1^{\gamma\delta} L_1^{\delta\beta}, \end{aligned}$$

where $\Lambda_{1ij}^{\alpha\beta} = \Lambda_{ij}^{\alpha\beta}(0) - \Lambda_{2ij}^{\alpha\beta}$, $a_1^{\alpha\beta} = a^{\alpha\beta}(0) - a_2^{\alpha\beta}$, $\lambda_1^{\alpha\beta} = \lambda^{\alpha\beta}(0) - \lambda_2^{\alpha\beta}(t-t')$, $\Lambda_{2ij}^{\alpha\beta} = \lim_{|t-t'|\to\infty} \Lambda_{ij}^{\alpha\beta}(t-t')$, $a_2^{\alpha\beta} = \lim_{|t-t'|\to\infty}\langle S_i^\alpha(t) b_i^\beta(t')\rangle$, $\lambda_2^{\alpha\beta}(t-t') = \lim_{|t-t'|\to\infty}\langle b_i^\alpha(t) b_i^\beta(t')\rangle$. Introducing the "self-energy" $\sigma_{1ij}^{\alpha\beta} = \delta_{ij}(\lambda_1^{-1\alpha\beta} - L_1^{-1\alpha\beta})$ the solutions can be written as

$$\begin{aligned} \hat{C}_1 &= -(\beta^2 \hat{\sigma}_1^{-1} + \beta \hat{J})^{-1}, \\ \hat{A}_1 &= \hat{C}_1 \hat{J}, \qquad (73) \\ \hat{\Lambda}_1 &= \hat{J}\hat{C}_1\hat{J} = -k_B T \hat{J} + (\hat{\sigma}_1 + \beta \hat{J})^{-1}. \end{aligned}$$

These equations are identical in form to the relations (22). They show that the vibrational parts $\hat{C}_1$, $\hat{A}_1$, $\hat{\Lambda}_1$ satisfy the same relations of the full correlations $\hat{C}$, $\hat{A}$, $\hat{\Lambda}$, but with a different self-energy.

Combining Eqs. (68) and (73) we can write the self-consistency equations for the order parameters as

$$\begin{aligned} c_1^{\alpha\beta} &= c^{\alpha\beta} - q_{\mathrm{EA}}^{\alpha\beta} \\ &= -\int_{-\pi}^{\pi} \frac{\mathrm{d}^d k}{(2\pi)^d} \left[\left(\beta^2 \underline{\underline{\sigma_1}}^{-1} + \beta \underline{\underline{J}}(\mathbf{k})\right)^{-1} \right]^{\alpha\beta} \\ &= -\int_{-\infty}^{\infty} \mathrm{d}\varepsilon\, \nu(\varepsilon) \left[\left(\beta^2 \underline{\underline{\sigma_1}}^{-1} + \beta \underline{\underline{f}}(\varepsilon)\right)^{-1} \right]^{\alpha\beta}. \qquad (74) \end{aligned}$$

In the paramagnetic phase, the solution is trivial: $q_{\mathrm{EA}}^{\alpha\beta} = 0$, $c_1^{\alpha\beta} = c^{\alpha\beta}$, $\sigma_1^{\alpha\beta} = \sigma^{\alpha\beta}$, $\lambda_1^{\alpha\beta} = \lambda^{\alpha\beta}$, ... In the glass phase, a solution with $q_{\mathrm{EA}} \neq 0$ appears. In this case the static correlations $C_{ij}^{\alpha\beta}$ differ from $C_{1ij}^{\alpha\beta}$. The statistical mechanical fluctuations calculated in Sec. III do not have a completely vibrational origin but, rather, a part of the fluctuations becomes configurational.

VI. APPLICATION: ISOTROPIC HEISENBERG MODEL ON THE INFINITE-DIMENSIONAL fcc LATTICE

In this section we apply the results to a specific case: an isotropic antiferromagnet with $V(\mathbf{S}) = 0$, $f^{\alpha\beta}(x) = f(x)\delta^{\alpha\beta} = J(x^2-1)\delta^{\alpha\beta}$, and $J < 0$. As discussed in Sec. II, this interaction couples only the spins which reside in the same fcc sublattice (A or B) of the simple hypercubic crystal and, as a result, the model can be viewed as describing two independent copies of a spin system on the fcc lattice [53]. The interaction $f(x) = J(x^2-1)$, in particular, is equivalent to a model on the fcc lattice with two nonzero antiferromagnetic couplings: a nearest-neighbor interaction $J_1 = J/d$ and a second-nearest-neighbor interaction $J_2 = J_1/2 = J/(2d)$.

Since the maximum of $f(x)$ occurs at $x = 0$, the system admits a degenerate manifold of helical ground states, with modulation vectors belonging to the $(d-1)$-dimensional surface defined by the condition $\varepsilon_{\mathbf{k}} = 0$.

The static properties of the model, computed from the equilibrium Gibbs distribution as discussed in Sec. III, are shown in Figs. 2 and 3. In particular, Fig. 2 shows the temperature dependence of L and λ, calculated numerically using Eq. (28),

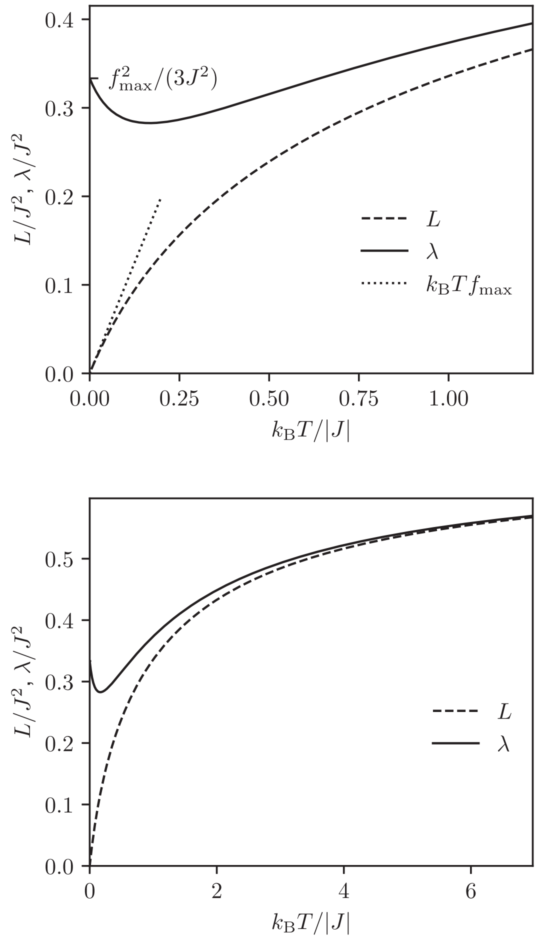


FIG. 2. Temperature dependence of the variance of the cavity distribution L (dashed line) and of the full equilibrium distribution λ (solid line), in the isotropic model with $V(\mathbf{S}) = 0$, $f^{\alpha\beta}(x) = J(x^2-1)\delta^{\alpha\beta}$. The top panel shows the low-temperature region, the bottom panel a wider range of temperatures. At low T, $L(T) \approx k_B T f_{\max} = k_B T |J|$. (The dotted line is a guide to the eye highlighting the asymptotic behavior at $T \to 0$.) The variance λ, instead, approaches a finite limit $f_{\max}^2/(3J^2)$. The temperature dependence $\lambda(T)$ is nonmonotonic: $\lambda(T)$ has a minimum at $T \simeq 0.167|J|$. At higher temperatures the curves $L(T)$ and $\lambda(T)$ approach each other, and eventually saturate to $L(T) \simeq \lambda(T) \simeq 2J^2/3$ in the limit $T \to \infty$.

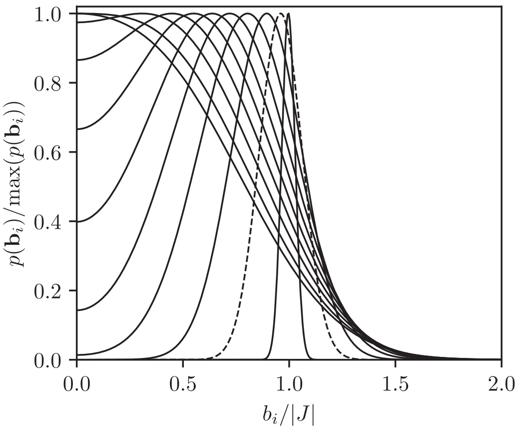


FIG. 3. Equilibrium distribution of the field $\mathbf{b}_i$ for 10 different temperatures, equally spaced between $10^{-3}|J|/k_B$ and $0.25|J|/k_B$ (solid lines). The 10 values of the temperature are, with three-digit precision, $k_BT/|J| = 0.001, 0.029, 0.056, 0.084, 0.112, 0.139, 0.167, 0.195, 0.222, 0.25$. The dashed line shows the equilibrium distribution of the field $\mathbf{b}_i$ at the glass temperature $T_g \simeq 0.0103|J|/k_B$. For the higher temperatures in the plot, the distribution differs from a Gaussian, but has a bell-like shape with a maximum at $b_i = 0$. At lower temperatures, the maximum shifts away from $b_i = 0$. For $T \to 0$, the distribution becomes concentrated on a narrow shell near the spherical surface $|\mathbf{b}_i| = |J| = f_{\max}$.

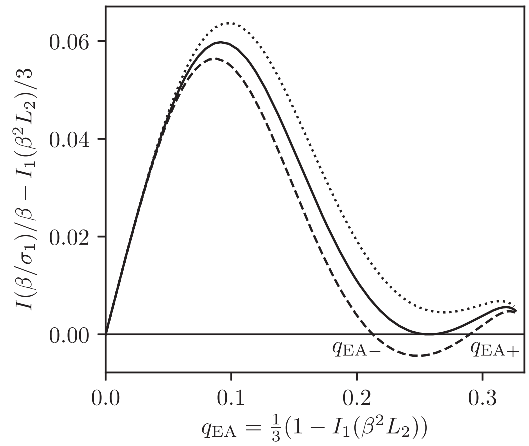


FIG. 4. Numerical analysis of Eqs. (75). The curves show the function $I(\beta/\sigma_1)/\beta - I_1(\beta^2L_2)/3$ calculated substituting $\beta/\sigma_1 = -1/(\beta c_1) - \beta L_1 = -3/[\beta I_1(\beta^2 L_2)] + \beta L_2 - \beta L$. The x axis is parametrized by the order parameter $q_{EA}(\beta^2L_2) = [1 - I_1(\beta^2L_2)]/3$. The dotted, solid, and dashed curves represent the curves calculated at temperatures respectively equal to $T = 0.012|J|/k_B$, $0.0103|J|/k_B$, and $0.009|J|/k_B$. The solutions of the self-consistency problem are defined by the points at which the curves cross 0. For high temperature the curve intersects 0 only at the trivial point $q_{EA} = 0$ (dotted line). A nontrivial solution appears at $T_g \simeq 0.0103|J|/k_B$. Below T_g, the equations have two solutions $q_{EA\pm}$. The physical solution is q_{EA+}, the one with the largest order parameter.

and the relations $L = -3/\beta^2 - 1/\sigma$, $\lambda = 1/\sigma + \beta^2/(3\sigma^2)$. In the limit of small temperatures, the variance of the cavity distribution $L = \langle \mathbf{b}_i^2\rangle'^{(i)}/3$ decreases with T and vanishes for $T \to 0$ as $L(T) \approx k_BTf_{\max} = k_BT|J|$. The full variance of the equilibrium distribution $\lambda = \langle \mathbf{b}_i^2\rangle/3$, instead, tends to $f_{\max}^2/3 = |J|^2/3$ for $T \to 0$. In the opposite limit of high temperatures, $L \approx \lambda \approx \mathcal{J}_2^2/3 = \sum_j J_{ij}^2/3 = 2|J|^2/3$.

The numerical solution shows a further interesting feature: the dependence of λ on temperature is nonmonotonic. λ reaches a minimum at a temperature $T \simeq 0.167|J|$ and, for lower temperatures, starts to *grow* when T is decreased.

The behavior $d\lambda/dT < 0$ occurs in a region of temperatures for which the equilibrium distribution of the field $p(\mathbf{b}_i) = \int_{\mathbf{S}} P(\mathbf{S}_i, \mathbf{b}_i) = \sinh(\beta|\mathbf{b}_i|)\exp[-(\beta^2L + L^{-1}\mathbf{b}_i^2)/2]/[(2\pi L)^{3/2}\beta|\mathbf{b}_i|]$ is significantly different from a Gaussian function. The shape of the distribution $p(\mathbf{b}_i)$ is shown in Fig. 3 for various temperatures.

To investigate the occurrence of a glass transition in the model, we study the self-consistent equations for the order parameter [Eqs. (68) and (74)], which in the isotropic case reduce to

$$\begin{aligned} c_1 &= \frac{1}{3} - q_{EA} = \frac{1}{\beta} I(\beta/\sigma_1), \\ \beta^2 c_1 &= -1/\left(\sigma_1^{-1} + L_1\right), \\ q_{EA} &= \frac{1}{3}\,\overline{\langle S^\alpha\rangle\langle S^\alpha\rangle} = \frac{1}{3}\overline{(\mathcal{L}(\beta|\mathbf{b}_2|))^2} \\ &= \frac{1}{3}[1 - I_1(\beta^2 L_2)], \\ L &= L_1 + L_2. \end{aligned} \tag{75}$$

In these relations $I(x) = -\int_{-\infty}^{\infty} d\varepsilon\, \nu(\varepsilon)/[x + f(\varepsilon)]$ is the integral introduced in Eq. (28), $\mathcal{L}(x) = 1/\tanh(x) - 1/x$ is the Langevin function, and

$$I_1(x) = \int_{-\infty}^{\infty} dy\, \frac{y[1 - (\mathcal{L}(y))^2]}{(2\pi x^3)^{1/2}} \exp\left(-\frac{(y - x)^2}{2x}\right). \tag{76}$$

The results of a numerical study of these equations are presented in Fig. 4. At high temperatures, the only solution is the trivial one. However, at a temperature $T_g \simeq 0.0103|J|/k_B$ a nontrivial solution appears. We interpret T_g as the temperature of a dynamical glass transition, at which the system vitrifies. It is interesting to note that T_g is an order of magnitude smaller than the temperature scales which characterize the behavior of the static solution. For example, T_g is much smaller than the temperature $T \simeq 0.167|J|/k_B$ at which the derivative $d\lambda/dT$ changes sign.

At T_g, the Edwards-Anderson order parameter q_{EA} jumps discontinuously from 0 to $q_{EA}(T_g) \simeq 0.2575$. This value corresponds to an average local magnetization at T_g equal to $|\mathbf{m}| = [\overline{\langle S^\alpha\rangle\langle S^\alpha\rangle}]^{1/2} = \sqrt{3q_{EA}(T_g)} \simeq 0.879$. We interpret this as the average magnitude of the local moments in an amorphous state in which the system vitrifies at T_g.

Below T_g Eqs. (75) present two solutions $q_{EA\pm}(T)$, which bifurcate from the transition point. The branch with smaller overlap, which we denote as $q_{EA-}(T)$, is an unphysical solution because it leads to an order parameter which decreases when T is lowered. The temperature dependence of the solution $q_{EA}(T) = q_{EA+}(T)$, with larger overlap, is illustrated in Fig. 5. Starting from the transition point, $q_{EA}(T_g)$ grows at small T and eventually $3q_{EA+}$ reaches 1 for $T \to 0$.

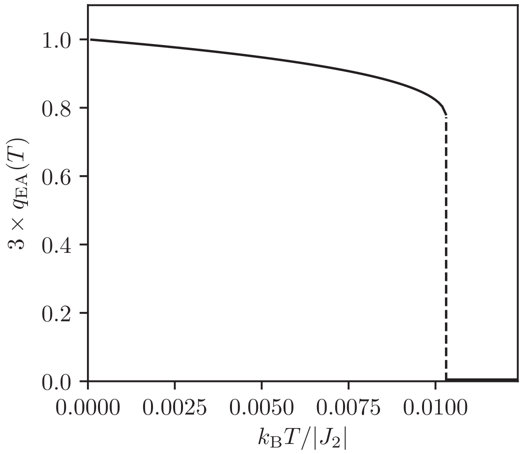


FIG. 5. Equilibrium average of the Edwards-Anderson order parameter $q_{\mathrm{EA}}(T)$ obtained from a numerical solution of Eqs. (75). The order parameter represented in the figure describes the large-time limit $3q_{\mathrm{EA}}(T) = \lim_{|t-t'|\to\infty}\langle S^\alpha(t)S^\alpha(t')\rangle$, with an averaging $\langle\dots\rangle$ weighted with the equilibrium Gibbs distribution at temperature T. This equilibrium q_{EA} may differ in general from the average order parameter which the system presents after a cooling from the paramagnetic phase $T > T_{\mathrm{g}}$ because the spins remain trapped in a single metastable state at T_{g} and fall out of equilibrium at lower temperatures.

The transition is of a dynamical first-order type, similar to the transition which occurs in supercooled liquids in $d \to \infty$ [62] and which was predicted in stripe systems [8–14]. At T_{g}, the equilibrium statistical properties remain smooth, without any signature of a phase transition, but the system undergoes a dynamical arrest. This contrasts, for example, with the second-order transitions found in random spin glasses [3].

Below T_{g} the system remains trapped into a single metastable state and falls out of equilibrium. After cooling from the high-temperature liquid regime $T > T_{\mathrm{g}}$ to the glassy region $T < T_{\mathrm{g}}$, we can expect that the physical thermal properties are not controlled by the equilibrium distribution at T, but by the distribution of metastable states at the temperature at T_{g}, at which the system fell out of equilibrium for the first time. The replica theory is useful to discuss this regime [38].

The analogy with the theory of stripe glasses and with the Ising model analyzed in Ref. [38] suggests that the system may present an ideal glass transition at a temperature $T_{\mathrm{s}} < T_{\mathrm{g}}$, signaled by the vanishing of the configurational entropy. We leave open the question whether this transition is present in the Heisenberg model studied here.

In conclusion, we note that the model analyzed in this section is a direct extension in infinite dimensions of the fcc J_1-J_2 Heisenberg antiferromagnet with $J_1 = 2J_2$. In three dimensions this antiferromagnetic fcc model has been studied by Balla *et al.* [39,40]. The analysis of Ref. [39] showed that the three-dimensional (3D) J_1-J_2 fcc model ultimately orders in the limit $T \to 0$, due to an order-by-disorder mechanism. Our analysis in infinite dimensions predicts that the disordered phase is locally stable, so that ordering must occur via a first-order transition. The dynamics within the disordered solution, in addition, undergoes a dynamical glass transition at a finite T.

VII. CONCLUSIONS

In this paper, we have analyzed a class of infinite-dimensional frustrated spin models, characterized by the property that their interaction $J(\mathbf{k})$ presents degenerate surfaces in momentum space. Using the cavity method, we derived in the limit of infinite dimensionality $d \to \infty$ the exact solution of the statistical mechanical properties within the disordered, paramagnetic phase. By a study of the equilibrium dynamics of the system, we derived a set of consistency equations for the glass order parameters. The theory was applied explicitly to the case of an isotropic model, equivalent after bipartition of the hypercubic lattice to a Heisenberg model on the fcc lattice. For this model we identified the temperature of dynamical vitrification. The transition is of a dynamical first-order type and is similar to the vitrification predicted in the theory of stripe glassiness. Although the exact results were obtained in the limit of infinite dimensions, the power demonstrated by DMFT methods in describing correlated electron systems [47–49] indicates the relevance of the results also for the realistic case of a system in three dimensions.

In comparison to the previous works hypothesizing self-induced spin-glass states in deterministic frustrated spin models [15,16] our approach seems to have two advantages. First, it has an explicit small parameter, albeit formal, and second it does not use the replica trick, whose applicability for deterministic systems is not obvious. Instead, we study the large-time behavior of correlation functions, in the spirit of the original approach by Edwards and Anderson in the theory of spin glasses, and of the Götze mode-coupling theory in the context of structural glasses. Importantly, both approaches lead to the same conclusion that frustration only can be sufficient for vitrification. This qualitatively supports the interpretation of the experimental data of Refs. [36,37] as an observation of a self-induced spin-glass state in elemental neodymium at low temperatures. Another interesting technical question concerning the relation between dynamical and replica approaches will be considered elsewhere.

ACKNOWLEDGMENTS

This work was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program, Grant Agreement No. 854843-FASTCORR and by the Dutch Research Council (NWO) via the Spinoza Prize. We are thankful to A. Khajetoorians, L. Niggli, and T. Westerhout for stimulating discussions.

APPENDIX A: FOKKER-PLANCK EQUATION AND FLUCTUATION-DISSIPATION THEOREM

The Fokker-Planck (FP) equation associated with the Brownian motion of magnetic moments has been discussed in several works (see, for example, Refs. [59,64–66]).

Here we give a brief derivation of the FP equation, and discuss the FDT and the Hilbert space representation of the dynamics. To derive the equations, we parametrize the spins by two coordinates (θ_i^1, θ_i^2). A natural choice are the spherical coordinates $\theta_i^1 = \theta_i$, $\theta_i^2 = \varphi_i$, $\mathbf{S}(\theta^1, \theta^2) = (\sin\theta^1\cos\theta^2, \sin\theta^1\sin\theta^2, \cos\theta^1)$, but an arbitrary parametrization can be used. In the following we denote as θ_i^μ, $\mu = 1, 2$, the coordinates. To avoid confusion, letters from the beginning of the greek alphabet α, β, γ, δ are used to denote the Cartesian components of the spins, as in the main text; letters from the final part of the alphabet μ, ν, ρ, ... are used, instead, to denote the two-dimensional parametrization θ^μ.

In terms of the θ_i^μ, the Langevin equation can be written as

$$\dot{\theta}_i^\mu = g^{\mu\nu}(\boldsymbol{\theta}_i)[\mathbf{u}_\nu(\boldsymbol{\theta}_i)\cdot\mathbf{E}_i], \tag{A1}$$

where $\mathbf{E}_i = -\mathbf{S}_i \times [\mathbf{S}_i \times (\mathbf{N}_i + \boldsymbol{\nu}_i)]$ is the right-hand side of the equation of motion (30), $\mathbf{u}_\mu(\boldsymbol{\theta}) = \partial\mathbf{S}/\partial\theta^\mu$ are the tangent vectors, and $g^{\mu\nu}(\boldsymbol{\theta})$ is the inverse of the metric tensor $g_{\mu\nu}(\boldsymbol{\theta}) = [\mathbf{u}_\mu(\boldsymbol{\theta})\cdot\mathbf{u}_\nu(\boldsymbol{\theta})]$.

Using the theory of Langevin processes on manifolds [63] and introducing the reparametrization-invariant distribution

$$\rho(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N) = \frac{\left\langle\prod_{i=1}^N \delta[\boldsymbol{\theta}_i - \boldsymbol{\theta}_i(t)]\right\rangle_{\boldsymbol{\nu}}}{\prod_{j=1}^N\sqrt{g(\boldsymbol{\theta}_j)}} \tag{A2}$$

we find after some computations

$$\frac{\partial\rho}{\partial t} + \sum_{i=1}^N \frac{1}{\sqrt{g(\boldsymbol{\theta}_i)}}\frac{\partial}{\partial\theta_i^\mu}\left\{\sqrt{g(\boldsymbol{\theta}_i)}g^{\mu\nu}(\boldsymbol{\theta}_i) \times\left[[\mathbf{u}_\nu(\boldsymbol{\theta}_i)\cdot\mathbf{N}_i]\rho - k_\mathrm{B}T\frac{\partial\rho}{\partial\theta_i^\nu}\right]\right\} = 0. \tag{A3}$$

In Eq. (A2) the factor $1/\prod_{j=1}^N\sqrt{g(\boldsymbol{\theta}_j)}$, $g(\boldsymbol{\theta}_i) = \det g_{\mu\nu}(\boldsymbol{\theta}_i)$, is introduced in order to make the probability invariant. With this normalization ρ is the invariant probability distribution. It gives the probability to find the system in an infinitesimal element $\mathrm{d}\Omega$ of the configuration space: $\mathrm{d}P = \rho(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)\mathrm{d}\Omega = \rho(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)\prod_i\sqrt{g(\boldsymbol{\theta}_i)}\mathrm{d}^2\theta_i$.

In spherical coordinates $g(\boldsymbol{\theta}_i) = \sin^2\theta_i$ and the infinitesimal element is the usual $\mathrm{d}\Omega = \prod_{i=1}^N \sin\theta_i\mathrm{d}\theta_i\mathrm{d}\varphi_i$. Equation (A3) becomes in these coordinates

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \sum_{i=1}^N\Bigg\{&\frac{1}{\sin\theta_i}\frac{\partial}{\partial\theta_i}\Bigg[\sin\theta_i\Bigg(\big(N_i^x\cos\theta_i\cos\varphi_i\\ &+ N_i^y\cos\theta_i\sin\varphi_i - N_i^z\sin\theta_i\big)\rho - k_\mathrm{B}T\frac{\partial\rho}{\partial\theta_i}\Bigg)\Bigg]\\ &+\frac{1}{\sin^2\theta_i}\frac{\partial}{\partial\varphi_i}\Bigg[\big(-N_i^x\sin\varphi_i + N_i^y\cos\varphi_i\big)\sin\theta_i\rho\\ &- k_\mathrm{B}T\frac{\partial\rho}{\partial\varphi_i}\Bigg]\Bigg\} = 0.\end{aligned} \tag{A4}$$

As a remark, we note that the methodology which leads to Eq. (A3) assumes the Stratonovich stochastic calculus. Thus, in Eq. (30) we implicitly assume a Langevin equation (30) defined according to the Stratonovich prescription (see Ref. [66] for different prescriptions).

The time evolution can be written more compactly introducing the angular momentum operators [65], which in a general frame of coordinates can be written as

$$\begin{aligned}\mathscr{L}_i^\alpha\bullet &= -i\epsilon^{\alpha\beta\gamma}g^{\mu\nu}(\boldsymbol{\theta}_i)\big[S_i^\beta u_\mu^\gamma(\boldsymbol{\theta}_i)\big]\frac{\partial}{\partial\theta_i^\nu}\bullet\\ &= -\frac{i\epsilon^{\alpha\beta\gamma}}{\sqrt{g(\boldsymbol{\theta}_i)}}\frac{\partial}{\partial\theta_i^\nu}\sqrt{g(\boldsymbol{\theta}_i)}g^{\mu\nu}(\boldsymbol{\theta}_i)S_i^\beta u_\mu^\gamma(\boldsymbol{\theta}_i)\bullet.\end{aligned} \tag{A5}$$

These are generators of spin rotations and satisfy the commutation relations $[\mathscr{L}_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma}S_i^\gamma$, $[\mathscr{L}_i^\alpha, \mathscr{L}_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma}\mathscr{L}_i^\gamma$. Note that $[S_i^\alpha, S_j^\beta] = 0$ because the spins here are simply classical variables.

In terms of the $\mathscr{L}_i^\alpha$, the FP equation (A3) can be recast as

$$\frac{\partial\rho}{\partial t} + \mathscr{H}\rho = 0, \tag{A6}$$

with

$$\mathscr{H} = \sum_{i=1}^N \mathscr{L}_i^\alpha\big(i\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma + k_\mathrm{B}T\mathscr{L}_i^\alpha\big) \tag{A7}$$

and the components of the angular momentum operators read as usual

$$\begin{aligned}\mathscr{L}_i^x &= i\sin\varphi_i\frac{\partial}{\partial\theta_i} + \frac{i\cos\varphi_i}{\tan\theta_i}\frac{\partial}{\partial\varphi_i},\\ \mathscr{L}_i^y &= -i\cos\varphi_i\frac{\partial}{\partial\theta_i} + \frac{i\sin\varphi_i}{\tan\theta_i}\frac{\partial}{\partial\varphi_i},\\ \mathscr{L}_i^z &= -i\frac{\partial}{\partial\varphi_i}.\end{aligned} \tag{A8}$$

Using that $\mathscr{L}_i^\alpha H = i\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma$, it is simple to show that the operator $i\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma + k_\mathrm{B}T\mathscr{L}_i^\alpha$ annihilates the Gibbs distribution $\rho_\mathrm{G} = Z^{-1}\mathrm{e}^{-\beta H}$. Thus, ρ_G is stationary in time: $\partial\rho_\mathrm{G}/\partial t = -\mathscr{H}\rho_\mathrm{G} = 0$.

Time-dependent correlation functions can be studied by setting up a Hilbert-space representation [63,67]. Here, in order to maintain a notation which is manifestly reparametrization invariant, we find it convenient to define the scalar product of two functions $|\Phi_1\rangle = \Phi_1(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)$, $|\Phi_2\rangle = \Phi_2(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)$ as $\langle\Phi_1|\Phi_2\rangle = \int\prod_j \mathrm{d}^2\theta_j\sqrt{g(\boldsymbol{\theta}_j)}\Phi_1^*(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)\Phi_2(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)$, including in the definition of the product the measure $\sqrt{g}$. [In spherical coordinates $\sqrt{g(\theta,\varphi)} = \sin\theta$ and the integrals are the usual $\int \mathrm{d}^2\theta_j\sqrt{g(\boldsymbol{\theta}_j)} = \int_0^\pi\mathrm{d}\theta_j\int_0^{2\pi}\mathrm{d}\varphi_j\sin\theta_j$.]

If the distribution of the initial conditions at time $t = 0$ is $|\rho\rangle$, the average of a product of spin variables at later times can be represented as

$$\begin{aligned}\big\langle S_{i_1}^{\alpha_1}(t_1)\dots S_{i_l}^{\alpha_l}(t_l)\big\rangle &= \langle 1|S_{i_1}^{\alpha_1}\mathrm{e}^{-\mathscr{H}(t_1-t_2)}\\ &\quad\times\dots\times\mathrm{e}^{-\mathscr{H}(t_{l-1}-t_l)}S_{i_l}^{\alpha_l}\mathrm{e}^{-\mathscr{H}t_l}|\rho\rangle,\end{aligned} \tag{A9}$$

where it is assumed that the times are arranged in the order as $t_1 > t_2 > \dots > t_l > 0$. $|1\rangle$ is simply a function equal to 1, so that the scalar product $\langle 1|\Phi\rangle$ is the integral of $\Phi(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)$ over the configuration space, computed with the invariant measure $\prod_j \mathrm{d}^2\theta_j\sqrt{g(\boldsymbol{\theta}_j)}$. In particular, the condition that $|\rho\rangle$ is normalized is expressed by $\langle 1|\rho\rangle = 1$.

It can be verified that $\langle 1|\mathscr{L}_i^\alpha = 0$. In fact, for any function $|\Phi\rangle$, Eq. (A5) implies

$$\begin{aligned}\langle 1|\mathscr{L}_i^\alpha|\Phi\rangle = -i\epsilon^{\alpha\beta\gamma}\int\prod_{j=1}^N \mathrm{d}^2\theta_j\frac{\partial}{\partial\theta_i^\nu}\big[\sqrt{g}(\boldsymbol{\theta}_i)\\ \times g^{\mu\nu}(\boldsymbol{\theta}_i)S_i^\beta u_\mu^\gamma\Phi(\boldsymbol{\theta}_1,\dots,\boldsymbol{\theta}_N)\big] = 0,\end{aligned} \tag{A10}$$

which vanishes because it is a total divergence, and the configuration space has no boundary. The relation $\langle 1|\mathscr{L}_i^\alpha$ implies that $\langle 1|\mathscr{H}=0$, that is, $\langle 1|$ is a left eigenvector with zero eigenvalue. This implies in particular that $\langle 1|\mathrm{e}^{-\mathscr{H}t}=\langle 1|$, which is the conservation of probability: $\langle 1|\rho(t)\rangle=\langle 1|\mathrm{e}^{-\mathscr{H}t}|\rho\rangle=\langle 1|\rho\rangle=1$.

Similarly, the Gibbs distribution $|\rho_{\mathrm{G}}\rangle$ is a right eigenvector with zero eigenvalue: $\mathscr{H}|\rho_{\mathrm{G}}\rangle=0$. In addition for all i, $(i\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma+k_{\mathrm{B}}T\mathscr{L}_i^\alpha)|\rho_{\mathrm{G}}\rangle=0$. These relations allow to write the correlations in the form [63]

$$\left\langle S_{i_1}^{\alpha_1}(t_1)\dots S_{i_l}^{\alpha_l}(t_l)\right\rangle=\langle 1|T\left\{\mathscr{S}_{i_1}^{\alpha_1}(t_1)\dots\mathscr{S}_{i_l}^{\alpha_l}(t_l)\right\}|\rho_{\mathrm{G}}\rangle,\quad\text{(A11)}$$

where $\mathscr{S}_i^\alpha(t)=\mathrm{e}^{\mathscr{H}t}S_i^\alpha\mathrm{e}^{-\mathscr{H}t}$ and T is the time-ordering operator.

In this representation, the FDT is encoded by the fact that the time derivative of the correlation function, in the region $t>t'$, is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S_i^\alpha(t)S_j^\beta(t')\right\rangle\\&=\langle 1|\left[\mathscr{H},\mathscr{S}_i^\alpha(t)\right]\mathscr{S}_j^\beta(t')|\rho_{\mathrm{G}}\rangle\\&=-\langle 1|\mathscr{S}_i^\alpha(t-t')\left[\mathscr{H},S_j^\beta\right]|\rho_{\mathrm{G}}\rangle\\&=ik_{\mathrm{B}}T\epsilon^{\beta\gamma\delta}\langle 1|\mathscr{S}_i^\alpha(t-t')\mathscr{L}_j^\gamma(0)\mathscr{S}_j^\delta(0)|\rho_{\mathrm{G}}\rangle,\end{aligned}\quad\text{(A12)}$$

and is proportional to the Kubo formula for the linear response function $G_{ij}^{\alpha\beta}=\langle\delta S_i^\alpha(t)/\delta\nu_j^\beta(t')\rangle$:

$$\begin{aligned}&G_{ij}^{\alpha\beta}(t-t')\\&=-i\Theta(t-t')\epsilon^{\beta\gamma\delta}\langle 1|\mathscr{S}_i^\alpha(t-t')\mathscr{L}_j^\gamma(0)\mathscr{S}_j^\delta(0)|\rho_{\mathrm{G}}\rangle\\&=-\beta\Theta(t-t')\frac{\mathrm{d}}{\mathrm{d}t}\left\langle S_i^\alpha(t)S_j^\beta(t')\right\rangle.\end{aligned}\quad\text{(A13)}$$

Equation (A13) follows from the general formulas of linear response theory. It can be derived by using that in presence of a small external field $\delta\nu_j^\alpha(t)$ the FP evolution equations become $\partial\rho/\partial t+\mathscr{H}\rho+\mathscr{H}_1(t)\rho=0$ with $\mathscr{H}_1(t)=i\epsilon^{\alpha\beta\gamma}\sum_{i=1}^N\mathscr{L}_i^\alpha S_i^\beta\delta\nu_i^\gamma(t)$. The perturbation $\mathscr{H}_1(t)$ describes the effect on the motion of the field $\delta\nu_i^\alpha(t)$ which modifies $\mathbf{N}_i\to\mathbf{N}_i+\boldsymbol{\delta\nu}_i$.

In analogy with other purely dissipative equations [63,67], it is also useful to discuss the FP equation by performing the transformation $\rho\to\tilde{\rho}$, $\rho=\mathrm{e}^{-\beta H/2}\tilde{\rho}$. The transformed operators are

$$\begin{aligned}\tilde{\mathscr{L}}_i^\alpha&=\mathrm{e}^{\beta H/2}\mathscr{L}_i^\alpha\mathrm{e}^{-\beta H/2}=\mathscr{L}_i^\alpha-\frac{i}{2}\beta\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma,\\\tilde{S}_i^\alpha&=\mathrm{e}^{\beta H/2}\mathscr{L}_i^\alpha\mathrm{e}^{-\beta H/2}=S_i^\alpha,\\\tilde{\mathscr{H}}&=\mathrm{e}^{\beta H/2}\mathscr{H}\mathrm{e}^{-\beta H/2}=k_{\mathrm{B}}T\sum_{i=1}^N\tilde{\mathscr{L}}_i^\alpha\tilde{\mathscr{L}}_i^{\alpha+}.\end{aligned}\quad\text{(A14)}$$

Here $\tilde{\mathscr{L}}_i^{\alpha+}=\mathscr{L}_i^\alpha+i\beta\epsilon^{\alpha\beta\gamma}S_i^\beta N_i^\gamma/2$ is the adjoint of $\tilde{\mathscr{L}}_i^\alpha$ ($\mathscr{L}_i^\alpha$ and S_i^α are self-adjoint). In this representation $\tilde{\mathscr{H}}$ is Hermitian, and has the same left and right eigenvectors. By contrast, the evolution operator $\mathscr{H}$ before the transformation is not Hermitian [63]. Equation (A14) also shows that $\tilde{\mathscr{H}}$ is positive semidefinite: all eigenvalues of $\tilde{\mathscr{H}}$ are $\geqslant 0$. The Gibbs distribution $|\tilde{\rho}_{\mathrm{G}}\rangle=\mathrm{e}^{\beta H/2}|\rho_{\mathrm{G}}\rangle$ is a ground state and satisfies $\tilde{\mathscr{L}}_i^{\alpha+}|\tilde{\rho}_{\mathrm{G}}\rangle=0$.

All results presented in this Appendix, clearly, remain valid when considering the equilibrium dynamics of a cavity system.

APPENDIX B: DYNAMICAL TWO-POINT CORRELATIONS

The derivation in Sec. IV C implies that any dynamical correlation of $\mathbf{S}_i$, $\mathbf{b}_i$, $\mathbf{S}_j$, $\mathbf{b}_j$ can be calculated replacing the definitions $b_i^\alpha(t)=\sum_k J_{ik}^{\alpha\beta}S_k^\beta(t)$, $b_j^\alpha(t)=\sum_k J_{ik}^{\alpha\beta}S_k^\beta(t)$, with Eqs. (60). In more detail, we can calculate the second-order correlations as

$$\begin{aligned}C_{ij}^{\alpha\beta}(t-t')&=\left\langle S_i^\alpha(t)S_i^\beta(t')\right\rangle_{\boldsymbol{\nu}_i,\boldsymbol{\nu}_j,\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j,\mathbf{S}_{i0},\mathbf{S}_{j0}},\\A_{ij}^{\alpha\beta}(t-t')&=\left\langle S_i^\alpha(t)b_i^\beta(t')\right\rangle_{\boldsymbol{\nu}_i,\boldsymbol{\nu}_j,\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j,\mathbf{S}_{i0},\mathbf{S}_{j0}},\\\Lambda_{ij}^{\alpha\beta}(t-t')&=\left\langle b_i^\alpha(t)b_i^\beta(t')\right\rangle_{\boldsymbol{\nu}_i,\boldsymbol{\nu}_j,\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j,\mathbf{S}_{i0},\mathbf{S}_{j0}},\end{aligned}\quad\text{(B1)}$$

where $\mathbf{S}_i(t)$, $\mathbf{S}_j(t)$, $\mathbf{b}_i(t)$, $\mathbf{b}_j(t)$ are solutions of the two-site Langevin equations $\dot{\mathbf{S}}_i=-\mathbf{S}_i\times\{\mathbf{S}_i\times[\mathbf{F}(\mathbf{S}_i)+\mathbf{b}_i+\boldsymbol{\nu}_i]\}$, $\dot{\mathbf{S}}_j=-\mathbf{S}_j\times\{\mathbf{S}_j\times[\mathbf{F}(\mathbf{S}_j)+\mathbf{b}_j+\boldsymbol{\nu}_j]\}$, with the fields $\mathbf{b}_i$ and $\mathbf{b}_j$ in Eqs. (60). The correlations are determined by averaging the solutions over the noises $\boldsymbol{\nu}_i$, $\boldsymbol{\nu}_j$, $\boldsymbol{\zeta}_i$, $\boldsymbol{\zeta}_j$, and over the initial conditions $\mathbf{S}_{i0}$, $\mathbf{S}_{j0}$ of the two spins.

Let us first analyze the correlation $C_{ij}^{\alpha\beta}(t-t')$. Since r_{ij}, R_{ij}, and J_{ij} are small ($\approx d^{-\ell_{ij}/2}$), the spins are only weakly coupled. For any fixed realization of the noises and the initial conditions the trajectories can be expanded to first order as

$$\begin{aligned}S_i^\alpha(t)&=\tilde{S}_i^\alpha(t)+\int_0^t\mathrm{d}t''\,\tilde{G}_i^{\alpha,\gamma}(t,t'')\Delta b_i^\gamma(t''),\\S_j^\alpha(t)&=\tilde{S}_j^\alpha(t)+\int_0^t\mathrm{d}t''\,\tilde{G}_j^{\alpha,\gamma}(t,t'')\Delta b_j^\gamma(t''),\\\Delta b_i^\gamma(t'')&=J_{ij}^{\gamma\delta}\tilde{S}_j^\delta(t'')+\beta r_{ij}^{\gamma\delta}(t'')S_{j0}^\delta\\&\quad+\int_0^{t''}\mathrm{d}t'''\,R_{ij}^{\gamma\delta}(t''-t''')\tilde{S}_j^\delta(t'''),\\\Delta b_j^\gamma(t'')&=J_{ji}^{\gamma\delta}\tilde{S}_i^\delta(t'')+\beta r_{ji}^{\gamma\delta}(t'')S_{i0}^\delta\\&\quad+\int_0^{t''}\mathrm{d}t'''\,R_{ji}^{\gamma\delta}(t''-t''')\tilde{S}_i^\delta(t''').\end{aligned}\quad\text{(B2)}$$

Here $\tilde{\mathbf{S}}_i(t)$, $\tilde{\mathbf{S}}_j(t)$ are solutions of the single-site Langevin equations

$$\begin{aligned}\dot{\tilde{\mathbf{S}}}_i&=-\tilde{\mathbf{S}}_i\times\{\tilde{\mathbf{S}}_i\times[\mathbf{F}(\tilde{\mathbf{S}}_i)+\tilde{\mathbf{b}}_i(t)+\boldsymbol{\nu}_i]\},\\\dot{\tilde{\mathbf{S}}}_j&=-\tilde{\mathbf{S}}_j\times\{\tilde{\mathbf{S}}_j\times[\mathbf{F}(\tilde{\mathbf{S}}_j)+\tilde{\mathbf{b}}_j(t)+\boldsymbol{\nu}_j]\},\\\tilde{b}_i^\alpha(t)&=\boldsymbol{\zeta}_i(t)+\beta l^{\alpha\beta}S_{i0}^\beta+\int_0^{t''}K^{\alpha\gamma}(t-t'')\tilde{S}_i^\beta(t''),\\\tilde{b}_j^\alpha(t)&=\boldsymbol{\zeta}_j(t)+\beta l^{\alpha\beta}S_{j0}^\beta+\int_0^{t''}K^{\alpha\gamma}(t-t'')\tilde{S}_j^\beta(t''),\end{aligned}\quad\text{(B3)}$$

and $\tilde{G}_i^{\alpha\beta}(t,t')$, $\tilde{G}_i^{\alpha\beta}(t,t')$ are the corresponding response functions.

To find $C_{ij}^{\alpha\beta}(t-t')$ we need to discuss the averaging over initial conditions and over the noise. Since we are studying equilibrium correlations, the initial conditions must be weighted with the joint probability $P_2(\mathbf{S}_{i0},\mathbf{S}_{j0})$ to extract given values $\mathbf{S}_{i0}$ and $\mathbf{S}_{j0}$ of two spins in the Gibbs distribution. This probability can be calculated as

$$P_2(\mathbf{S}_{i0},\mathbf{S}_{j0}) = \int \mathrm{d}^3\bar{\mathbf{b}}_i \int \mathrm{d}^3\bar{\mathbf{b}}_j P(\mathbf{S}_{i0},\bar{\mathbf{b}}_i;\mathbf{S}_{j0},\bar{\mathbf{b}}_j), \tag{B4}$$

where $P(\mathbf{S}_i,\mathbf{b}_i;\mathbf{S}_j,\mathbf{b}_j)$ is the distribution found in the static computation [Eqs. (17) and (19)]. At leading order for $d\to\infty$, it is sufficient to expand the distribution (19) at first order in $J_{ij}^{\alpha\beta}$ and $M_{ij}^{\alpha\beta}$. After integration over the magnetic fields we find

$$P_2(\mathbf{S}_{i0},\mathbf{S}_{j0}) = P_1(\mathbf{S}_{i0})P_{1\mathrm{eq}}(\mathbf{S}_{j0})\big(1+\beta J_{ij}^{\alpha\beta}S_{i0}^{\alpha}S_{j0}^{\beta} - \beta^2 L^{\alpha\gamma}M_{ij}^{\gamma\delta}L^{\delta\beta}S_{i0}^{\alpha}S_{j0}^{\beta}\big). \tag{B5}$$

The static equation (19), on the other hand, implies that the fields $X_1^\alpha=\sum_{k\neq i,j}J_{ik}^{\alpha\beta}S_k^\beta$, $X_2^\alpha=\sum_{k\neq i,j}J_{jk}^{\alpha\beta}S_k^\beta$ have a correlation $\langle\langle X_1^\alpha X_2^\alpha\rangle\rangle''^{(ij)}\approx -L^{\alpha\gamma}M_{ij}^{\gamma\delta}L^{\delta\beta}$ when they are averaged with the statistical distribution of the cavity system. X_1^α and X_2^α play the same role of ζ_i^α and ζ_j^α, the Gaussian noises which enter the dynamical equations (60). We can thus identify $\langle\langle X_1^\alpha X_2^\alpha\rangle\rangle''^{(ij)}$ with the equal-time element $r_{ij}^{\alpha\beta}(0)$ of the correlation $r_{ij}^{\alpha\beta}(t-t')$. As a result, we can write the equilibrium distribution of the initial conditions as $P_2(\mathbf{S}_{i0},\mathbf{S}_{j0})=P_{1\mathrm{eq}}(\mathbf{S}_{i0})P_{1\mathrm{eq}}(\mathbf{S}_{j0})[1+\beta J_{ij}^{\alpha\beta}S_{i0}^\alpha S_{j0}^\beta+\beta^2 r_{ij}^{\alpha\beta}(0)S_{i0}^\alpha S_{j0}^\beta]$, where $P_{1\mathrm{eq}}(\mathbf{S}_{i0})$ is the single-site probability [Eq. (36)].

Consider now the distribution of the noise. We can formally write the Gaussian probability $P_{2\mathrm{noise}}[\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j]$ of a given realization $\boldsymbol{\zeta}_i(t)$, $\boldsymbol{\zeta}_j(t)$ in the form

$$P_{2\mathrm{noise}}[\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j] = \mathcal{N}_2 \exp\left[-\frac{1}{2}\int_0^\infty\int_0^\infty \mathrm{d}t\,\mathrm{d}t' \sum\nolimits_{a=i,j}\sum\nolimits_{b=i,j} l_{ab}^{-1\alpha\beta}(t-t')\zeta_a^\alpha(t)\zeta_b^\beta(t')\right], \tag{B6}$$

where $l_{ab}^{-1\alpha\beta}(t-t')$ is the inverse of the matrix

$$l_{ab}^{\alpha\beta}(t-t') = \begin{Vmatrix} l_{ii}^{\alpha\beta}(t-t') & l_{ij}^{\alpha\beta}(t-t') \\ l_{ji}^{\alpha\beta}(t-t') & l_{jj}^{\alpha\beta}(t-t')\end{Vmatrix} = \begin{Vmatrix} l^{\alpha\beta}(t-t') & r_{ij}^{\alpha\beta}(t-t') \\ r_{ji}^{\alpha\beta}(t-t') & l^{\alpha\beta}(t-t')\end{Vmatrix}, \tag{B7}$$

and $\mathcal{N}_2$ is a normalization constant. The inverse is intended in the sense of kernels: $\sum_c\int_0^\infty \mathrm{d}t''\, l_{ac}^{-1\alpha\gamma}(t-t'')l_{cb}^{\gamma\beta}(t''-t') = \delta^{\alpha\beta}\delta_{ab}\delta(t-t')$.

Expanding to first order in r_{ij} we find

$$P_{2\mathrm{noise}}[\boldsymbol{\zeta}_i,\boldsymbol{\zeta}_j] \simeq P_{1\mathrm{noise}}[\boldsymbol{\zeta}_1]P_{1\mathrm{noise}}[\boldsymbol{\zeta}_1]\left[1+\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}t''\int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\big(l^{-1\alpha\gamma}(t-t_1)r_{ij}^{\gamma\delta}(t_1-t_2)l^{-1\delta\beta}(t_2-t')\zeta_i^\alpha(t)\zeta_j^\beta(t')\big)\right], \tag{B8}$$

where

$$P_{1\mathrm{noise}}[\boldsymbol{\zeta}] = \mathcal{N}_1 \exp\left[-\frac{1}{2}\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}t' l^{-1\alpha\beta}(t-t')\zeta^\alpha(t)\zeta^\beta(t')\right] \tag{B9}$$

is the noise distribution of the single-site problem.

We can now calculate the correlation. We get three terms at leading order. The first, which we denote as $C_{1ij}^{\alpha\beta}(t-t')$, comes from the correction of the trajectories [Eq. (B2)] calculated in the unperturbed ensemble. Using that $\langle G_i^{(n)\alpha\beta}(t,t')\rangle = \langle G^{(n)\alpha\beta}(t,t')\rangle = g^{\alpha\beta}(t-t') = -\beta\dot{c}_+^{\alpha\beta}(t-t')$ is the single-site response function we get

$$C_{1ij}^{\alpha\beta}(t-t') = \int_0^t \mathrm{d}t''\, g^{\alpha\gamma}(t-t'')\big\langle \Delta b_i^\gamma(t'')S_j^\beta(t')\big\rangle + \int_0^{t'} \mathrm{d}t''\, g^{\beta\gamma}(t'-t'')\big\langle \Delta b_j^\gamma(t'')S_i^\alpha(t)\big\rangle. \tag{B10}$$

A second term, $C_2^{\alpha\beta}(t-t')$, comes from the distribution of initial conditions:

$$C_2^{\alpha\beta}(t-t') = \beta\big[J_{ij}^{\gamma\delta}+\beta r_{ij}^{\gamma\delta}(0)\big]\big\langle S_i^\alpha(t)S_{i0}^\gamma\big\rangle\big\langle S_j^\beta(t')S_{j0}^\delta\big\rangle = \beta\big[J_{ij}^{\gamma\delta}+\beta r_{ij}^{\gamma\delta}(0)\big]c^{\alpha\gamma}(t)c^{\beta\delta}(t'). \tag{B11}$$

Finally, we have a third term arising from the correlation of the noise:

$$C_{3ij}^{\alpha\beta}(t-t') = \int_0^\infty \mathrm{d}t_1\int_0^\infty \mathrm{d}t_2\int_0^\infty \mathrm{d}t_3\int_0^\infty \mathrm{d}t_4 l^{-1\gamma\mu}(t_1-t_3)r_{ij}^{\mu\nu}(t_3-t_4)l^{-1\nu\delta}(t_4-t_2)\big\langle S_i^\alpha(t)\zeta_i^\gamma(t_1)\big\rangle\big\langle S_j^\beta(t')\zeta_j^\delta(t_2)\big\rangle, \tag{B12}$$

which, using the properties of Gaussian averages [63], can be rewritten as

$$C_{3ij}^{\alpha\beta}(t-t') = \int_0^\infty \mathrm{d}t''\int_0^\infty \mathrm{d}t'''\,\big[g^{\alpha\gamma}(t-t'')g^{\beta\delta}(t'-t''')r_{ij}^{\gamma\delta}(t''-t''')\big]. \tag{B13}$$

Collecting all terms $C_{ij}^{\alpha\beta} = C_{1ij}^{\alpha\beta} + C_{2ij}^{\alpha\beta} + C_{3ij}^{\alpha\beta}$, and using the FDT, we find and expression manifestly consistent with TTI:

$$C_{ij}^{\alpha\beta}(t-t') = \Theta(t-t')\Big\{\big[J_{ij}^{\gamma\delta} + \beta r_{ij}^{\gamma\delta}(0)\big]\Big[\beta c^{\beta\delta}(0)c^{\alpha\gamma}(t-t') + \int_{t'}^{t} dt''\, g^{\alpha\gamma}(t-t'')c^{\beta\delta}(t'-t'')\Big] + \int_{t'}^{t} dt'' \int_{t'}^{t''} dt'''\, g^{\alpha\gamma}(t-t'')r_{ij}^{\gamma\delta}(t''-t''')g^{\delta\beta}(t'''-t')\Big\} + \Theta(t'-t)\Big\{[J_{ij}^{\gamma\delta} + \beta r_{ij}^{\gamma\delta}(0)]\Big[\beta c^{\alpha\gamma}(0)c^{\beta\delta}(t'-t) + \int_{t}^{t'} dt''\, c^{\alpha\gamma}(t-t'')g^{\beta\delta}(t'-t'')\Big] + \int_{t}^{t'} dt''' \int_{t}^{t'''} dt'' g^{\gamma\alpha}(t''-t)r_{ij}^{\gamma\delta}(t''-t''')g^{\beta\delta}(t'-t''')\Big\}. \tag{B14}$$

Finally, differentiating over time we find

$$\dot{C}_{+ij}^{\alpha\beta}(t-t') = -\beta\Theta(t-t')\Big\{\int_{t'}^{t} dt''\, \dot{c}_{+}^{\alpha\gamma}(t-t'')\Big[J_{ij}^{\gamma\delta}\dot{c}_{+}^{\delta\beta}(t''-t') + \int_{t'}^{t''} dt'''\, \big(\dot{c}_{+}^{\alpha\gamma}(t-t'')R_{ij}^{\gamma\delta}(t''-t''')\dot{c}_{+}^{\delta\beta}(t'''-t')\big)\Big]\Big\}, \tag{B15}$$

an expression which in frequency space becomes Eq. (62) in the main text.

Note in passing that for equal times $t = t'$, Eq. (B14) reduces to $C_{ij}^{\alpha\beta} = \beta c^{\alpha\gamma}[J_{ij}^{\gamma\delta} + \beta r_{ij}^{\gamma\delta}(0)]c^{\delta\beta}$, which is equivalent to the first of the static equations (20) after the identification $r_{ij}^{\alpha\beta}(0) = -L^{\alpha\gamma}M_{ij}^{\gamma\delta}L^{\delta\beta}$.

We now consider $A_{ij}^{\alpha\beta}(t-t')$ and $\Lambda_{ij}^{\alpha\beta}(t-t')$. It is not necessary to go through a detailed calculation, as we did for $C_{ij}^{\alpha\beta}(t-t')$, because equations of motion analog to Eqs. (42) relate A and Λ to C.

Writing compactly the fields as

$$b_a^{\alpha}(t) = \zeta_a^{\alpha}(t) + \sum_{b=i,j}\Big[\beta l_{ab}^{\alpha\gamma}(t)S_{b0}^{\gamma} + J_{ab}^{\alpha\beta}S_b^{\beta}(t) + \int_0^t dt''\, K_{ab}^{\alpha\gamma}(t-t')S_b^{\gamma}(t')\Big], \tag{B16}$$

and introducing

$$K_{ab}^{\alpha\beta}(t-t') = -\beta\Theta(t-t')\frac{d}{dt}l_{ab}^{\alpha\beta}(t-t') = \begin{Vmatrix} K^{\alpha\beta}(t-t') & R_{ij}^{\alpha\beta}(t-t') \\ R_{ji}^{\alpha\beta}(t-t') & K^{\alpha\beta}(t-t') \end{Vmatrix} \tag{B17}$$

and

$$\dot{C}_{+ab}^{\alpha\beta}(t-t') = \begin{Vmatrix} \dot{C}_{+ii}^{\alpha\beta}(t-t') & \dot{C}_{+ij}^{\alpha\beta}(t-t') \\ \dot{C}_{+ji}^{\alpha\beta}(t-t') & \dot{C}_{+jj}^{\alpha\beta}(t-t') \end{Vmatrix} = \begin{Vmatrix} \dot{c}_{+}^{\alpha\beta}(t-t') & \dot{C}_{+ij}^{\alpha\beta}(t-t') \\ \dot{C}_{+ji}^{\alpha\beta}(t-t') & \dot{c}_{+}^{\alpha\beta}(t-t') \end{Vmatrix} \tag{B18}$$

we find, using the properties of Gaussian averages and the FDT,

$$\dot{A}_{+ab}^{\alpha\beta}(t-t') = \sum_{c=i,j}\dot{C}_{+ac}^{\alpha\gamma}(t-t')J_{cb}^{\gamma\beta} + \sum_{c=i,j}\int_{t'}^{t} dt''\, \dot{C}_{+ac}^{\alpha\gamma}(t-t'')K_{cb}^{\gamma\beta}(t''-t'), \tag{B19}$$

$$\dot{\Lambda}_{+ab}^{\alpha\beta}(t-t') = -\frac{1}{\beta}K_{ab}^{\alpha\beta}(t-t') + \sum_{c,d=i,j}\Big\{\int_{t'}^{t} dt'' \int_{t'}^{t''} dt'''\big[K_{ac}^{\alpha\gamma}(t-t'')\dot{C}_{+cd}^{\gamma\delta}(t''-t''')K_{db}^{\delta\beta}(t'''-t')\big] + J_{ac}^{\alpha\gamma}\dot{C}_{+cd}^{\gamma\delta}(t-t')J_{db}^{\delta\beta} + J_{ac}^{\alpha\gamma}\int_{t'}^{t} dt''\dot{C}_{+cd}^{\gamma\delta}(t-t'')K_{db}^{\delta\beta}(t''-t') + J_{db}^{\delta\beta}\int_{t'}^{t} dt''K_{ac}^{\alpha\gamma}(t-t'')\dot{C}_{+cd}^{\gamma\delta}(t''-t')\Big\}. \tag{B20}$$

$\dot{A}_{+ab}^{\alpha\beta}(t-t')$ and $\dot{\Lambda}_{+ab}^{\alpha\beta}(t-t')$ are given by convolutions of retarded response functions, acting in sequence. All time integrals can be extended to $-\infty$ to $+\infty$ because the $\Theta(t-t')$ factors in the definition of $\dot{C}_{+}$ and K automatically restrict the integration to the causal region.

This allows to write Eqs. (B19) and (B20) in frequency space as

$$\dot{A}_{+ab}^{\alpha\beta}(\omega) = \sum_{c=i,j}\dot{C}_{+ac}^{\alpha\gamma}(\omega)\big[J_{cb}^{\gamma\beta} + K_{cb}^{\gamma\beta}(\omega)\big], \quad \dot{\Lambda}_{+ab}^{\alpha\beta}(\omega) = -\frac{1}{\beta}K_{ab}^{\alpha\beta}(\omega) + \sum_{c,d=i,j}\big[\big[J_{ac}^{\alpha\gamma} + K_{ac}^{\alpha\gamma}(\omega)\big]\dot{C}_{+cd}^{\gamma\delta}(\omega)\big[J_{db}^{\delta\beta} + K_{db}^{\delta\beta}(\omega)\big]\big]. \tag{B21}$$

We now take the (ij) matrix element of the equations and keep only terms of linear order in J_{ij}, r_{ij}, R_{ij}. Using that $J_{ii} = J_{jj} = 0$ we find

$$\dot{A}_{+ij}^{\alpha\beta}(\omega) = \dot{c}_{+}^{\alpha\gamma}(\omega)\big[J_{ij}^{\gamma\beta} + R_{ij}^{\gamma\beta}(\omega)\big] + \dot{C}_{+ij}^{\alpha\gamma}(\omega)K^{\gamma\beta}(\omega), \tag{B22}$$

$$\dot{\Lambda}_{+ij}^{\alpha\beta}(\omega) = -\frac{1}{\beta}R_{ij}^{\alpha\beta}(\omega) + K^{\alpha\gamma}(\omega)\dot{C}_{+ij}^{\gamma\delta}(\omega)K^{\delta\beta}(\omega) + \big[J_{ij}^{\alpha\gamma} + R_{ij}^{\alpha\gamma}(\omega)\big]\dot{c}_{+}^{\gamma\delta}(\omega)K^{\delta\beta}(\omega) + K^{\alpha\gamma}(\omega)\dot{c}_{+}^{\gamma\delta}(\omega)\big[J_{ij}^{\delta\beta} + R_{ij}^{\delta\beta}(\omega)\big], \tag{B23}$$

which are Eqs. (63) in the main text.

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