Intertwined fractional quantum anomalous Hall states and charge density waves

Xue-Yang Song [1], Chao-Ming Jian [2], Liang Fu [1], and Cenke Xu [3]
[1] *Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2] *Department of Physics, Cornell University, Ithaca, New York 14853, USA*
[3] *Department of Physics, University of California, Santa Barbara, California 93106, USA*

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Motivated by the recent experimental breakthrough on the observation of the fractional quantum anomalous Hall (FQAH) effects in semiconductor and graphene moiré materials, we explore the rich physics associated with the coexistence of FQAH effect and the charge density wave (CDW) order that spontaneously breaks the translation symmetry. We refer to a state with both properties as "FQAH crystal." We show that the interplay between FQAH effect and CDW can lead to a rich phase diagram including multiple topological phases and topological quantum phase transitions at the same moiré filling. In particular, we demonstrate the possibility of direct quantum phase transitions from a FQAH crystal with Hall conductivity $\sigma_H = -2/3$ to a trivial CDW insulator with $\sigma_H = 0$ and, more interestingly, to a QAH crystal with $\sigma_H = -1$.

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I. INTRODUCTION

The recent advance in the fabrication and control of two-dimensional (2D) van der Waals heterostructures has enabled the development of moiré superlattices that feature tunable minibands. Topologically nontrivial minibands in moiré materials provide an ideal avenue to search for topological states of matter. As an example, evidence of fractional quantum Hall effect in fractionally filled moiré band was observed in twisted bilayer graphene under a magnetic field above $\sim$5 T or higher [1,2]. More recently, thermodynamic and transport measurements revealed the fractional quantum anomalous Hall (FQAH) effect at zero magnetic field in twisted TMD homobilayers [3–6] and rhombohedral pentalayer graphene/hBN superlattice [7].

The discovery of FQAH effect points towards a fertile ground for studying the strong interaction effect in topological moiré bands of 2D materials. Twisted TMD homobilayers feature spin-valley-locked moiré bands with opposite Chern numbers in the two valleys [8–10]. At finite carrier density, Coulomb interaction drives spontaneous valley polarization and FQAH effect is anticipated at fractional fillings of the valley-polarized Chern band [10–12]. Interaction induced FQAH states in Chern bands are also known as (zero-field) fractional Chern insulators in the literature [13–17]. The highly tunable nature of moiré systems, with abundant tuning knobs such as twist angle, displacement field, electrostatic doping, and gate screening, offers a large parameter space to explore FQAH states and proximate phases [18–30].

In this work, we explore the rich physics of a state with coexisting FQAH and CDW (a state that we refer to as FQAH crystal) and its proximate phases that occur at the same moiré band filling under zero magnetic field. Our study is motivated by the observed phase transition at hole filling $\nu = -2/3$ in twisted bilayer $MoTe_2$ under a displacement field from an FQAH state with quantized Hall conductance $\sigma_H = -2/3$ (in the unit of e^2/h) to an insulating state [5,6]. We consider a scenario in which the FQAH state spontaneously breaks the translational symmetry of the moiré lattice. We call this state a FQAH crystal, analogous to the notion of "Hall crystal" introduced in Ref. [31]. Our consideration of FQAH crystal is partly motivated by the recent numerical finding of a softened magnetoroton gap in $\sigma_H = -2/3$ FQAH states [20], which suggests incipient CDW order with tripled $\sqrt{3} \times \sqrt{3}$ moiré unit cells (as illustrated in Fig. 1) at experimentally relevant twist angles.

We show by field theory analysis that a variety of strongly correlated phases can be found in the vicinity of FQAH crystal. These include a trivial CDW insulator with $\sigma_H = 0$ and a $\sigma_H = -1$ QAH state with CDW order, which we call QAH crystal. While the QAH-crystal phase has been proposed in moiré systems under the name of topological charge density wave [26,32,33], its connection to FQAH physics [34–36] and phase transitions has received little attention before. All the phases considered in this work possess the same type of CDW order, but are distinguished by their different topological properties.

We further show that direct and (potentially) continuous phase transitions between these topologically distinct phases are theoretically allowed. Interestingly, these phase transitions can be described by (2 + 1)D quantum electrodynamics (QED) with a Chern-Simons (CS) term of the $U(1)$ gauge field coupled to either fermionic or bosonic charges. In our theory, the transition from the FQAH crystal with $\sigma_H = -2/3$ to the trivial CDW insulator is described by a fermionic QED with two flavors Dirac fermions at low energy coupled with a $U(1)$ gauge field with a CS term at level $1/2$. On the other hand, its transition to a QAH crystal with $\sigma_H = -1$ is described by either bosonic or fermionic QEDs. These two descriptions are *dual* to each other based on the boson-fermion duality web that was actively discussed in recent years [37–40].

We note that direct transitions between a standard FQAH state (without any spontaneous symmetry breaking) and exotic CDW* with extra topological order were studied in

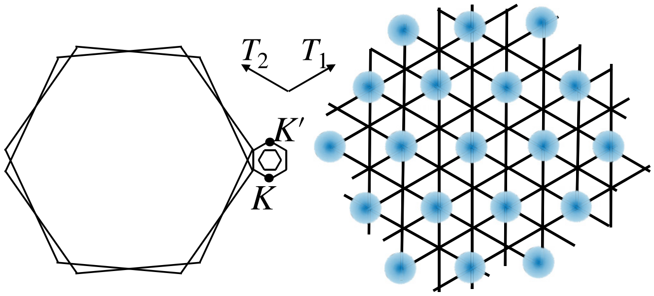

FIG. 1. (Left) Moiré Brillouin zone formed when twisting the original Brillouin zone (two large hexagons) of each TMD layer. When charge density wave forms that triples the unit cell, schematically shown on the right on the moiré superlattices, and the Brillouin zone is further folded to the center smallest hexagon on the left.

Ref. [28], whereas our work starts from a FQAH crystal (with $\sigma_H = -2/3$ and CDW order) and obtains different proximate phases. In particular, we highlight the possibility of a QAH crystal (with $\sigma_H = -1$ and CDW order) as a proximate phase and a direct phase transition between FQAH crystal and QAH crystal at $\nu = -2/3$.

II. PHASES AT $\nu = -2/3$

The most prominent state observed experimentally in the homobilayer TMD moiré system is the $\sigma_H = \pm 2/3$ FQAH state at hole filling $n_h = 2/3$, i.e., at hole density of 2/3 per moiré unit cell. Since this state is shown to be fully spin/valley polarized by magnetic circular dichroism measurements, in the following we consider a spinless electron system at charge density $\nu = -2/3$.

Throughout the discussion below we postulate the presence of a charge density wave (CDW) which triples the unit cell (examples shown in Fig. 1), such that the holes are at integer filling with respect to the enlarged moiré unit cell. We will show that under tripling of the unit cell, it is natural to construct phases with Hall conductivity $\sigma_H = -2/3, -1, 0$ in a unified formalism. Furthermore, there can be direct and (potentially) continuous quantum phase transitions between any of the two states mentioned above, though a direct transition between the $\sigma_H = -2/3$ state and the trivial insulator with $\sigma_H = 0$ requires certain discrete space-time symmetries.

For the purpose of constructing these phases and describing their properties, it is convenient to use the standard parton construction. One can formally write the hole operator as $c = \Phi f$, where the bosonic parton Φ carries the physical electric charge and the charge-neutral parton f is a fermion. The electric charge can actually be assigned arbitrarily between Φ and f, which should not change the final physics. The parton construction formally enlarges the Hilbert space of the holes, which can be remedied by coupling Φ and f both to an internal dynamical $U(1)$ gauge field a, with charge ± 1, respectively. The dynamical $U(1)$ gauge field enforces a local constraint which equates the local density of f to that of Φ. The physical state of holes is obtained by enforcing the relation of hole density to that of the partons, i.e., $\nu_h = \nu_\Phi = \nu_f = 2/3$ with respect to the original moiré unit cell. Importantly, in the presence of a CDW order that triples the unit cell, both the holes and partons are at integer fillings with respect to the enlarged unit cell.

A. Phases tuned by fermionic parton f

In the following we will construct a series of states by making Φ a bosonic fractional quantum Hall state with Hall conductivity $-1/2$. Each state can also be equivalently constructed employing the composite fermion picture through vortex attachment. As is well known in the context of Landau level systems, composite fermions experience a modified residual magnetic field and prominent fractional quantum Hall states are formed at integer filling of composite fermion Landau levels [41,42]. As we show later, in (moiré) lattice systems, the mean-field state of composite fermions allows much richer possibilities, leading to a series of new states [43,44].

$\sigma_H = -2/3$ state. The $\sigma_H = -2/3$ state at filling $\nu = -2/3$ is the most prominent state observed experimentally in the homobilayer TMD moiré system. This state can be constructed naturally using the parton formalism, where Φ and f each forms its own "mean-field state." With the assumption of the existence of a background charge density wave that triples the unit cell, the original Chern band in the moiré Brillouin zone (BZ) would split into three bands in the folded moiré BZ and the fermionic parton f would fill two out of the three bands due to its 2/3 filling. One natural way to construct the 2/3 state is for the bosonic parton Φ to form a $\nu = -1/2$ Laughlin state and at the mean-field level the fermionic parton f fills two low energy bands in the folded BZ with Chern numbers $+1, +1$. As we mentioned previously, the $\sigma_H = -2/3$ state constructed here has FQAH effect as well as spontaneous translation symmetry breaking, which we refer to as FQAH crystal. Later, we will demonstrate with a composite fermion construction that the existence of the $\sigma_H = -2/3$ state itself does not have to break the translation symmetry. However, all of the nearby states within our formalism must break the translation symmetry. This observation motivates us to focus on the scenario where the translation symmetry is already broken in the $\sigma_H = -2/3$ state.

Here we would like to demonstrate that the parton construction given above is a natural state for holes at filling 2/3 of the moiré unit cell. We note that the flux ϕ_Φ per moiré unit cell felt by the parton Φ is not necessarily equal to the physical flux seen by holes ϕ_h, due to the internal gauge field a coupled to both Φ and f. In the continuum, the total fluxes seen by the hole and the partons should in general obey the relation $\phi_h = \phi_\Phi + \phi_f$. In twisted semiconductor bilayers [8,9,45] and other continuum systems [46] where holes fill a valley polarized Chern band, despite being at zero magnetic field, the holes experience an effective flux $\phi_h = -1$ per moiré unit cell produced by the periodic skyrmion spin (or layer pseudospin) texture in real space [47] and we could further set $\phi_\Phi = -4/3$ to allow Φ to form a Laughlin $\nu = -1/2$ state. This leaves $\phi_f = 1/3$ and $\nu_f = 2/3$. The fermionic partons hence are naturally allowed to fill two Landau levels, equivalent to filling two bands with Chern number $+1$. In fact, with the CDW order that triples the unit cell that we postulate, the f fill $\tilde{\phi}_f = 1$ with a filling $\tilde{\nu}_f = 2$ per enlarged unit cell; hence f could naturally form an insulator with total Chern number $C = 2$.

In terms of the Chern-Simons theory, this state corresponds to the following Lagrangian:

$$\mathcal{L} = -\frac{2}{4\pi} b \wedge db + \frac{1}{2\pi} b \wedge da - \frac{1}{2\pi} a \wedge d(a_1 + a_2) + \sum_{i=1,2} \frac{1}{4\pi} a_i \wedge da_i. \tag{1}$$

The gauge field b is the "dual" of the current of the bosonic parton Φ; a_1 and a_2 are the dual of the fermionic parton f that fills the two Chern bands with Chern number $(+1, +1)$; a is the gauge field that couples to both Φ and f. The CS Lagrangian Eq. (1) can also be written in a more compact form using the K matrix [48]:

$$\mathcal{L} = \frac{1}{4\pi} K_{2/3,IJ} a^I \wedge da^J, \tag{2}$$

where

$$K_{2/3} = \begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \tag{3}$$

and $a^I = (b, a_1, a_2, a)$. We will hereafter abbreviate $a \wedge db$ as adb without loss of clarity. The topological ground state degeneracy is given by the determinant of $K_{2/3}$, which in this case is 3. To derive the Hall conductivity of this state, we need to couple a^I to the external electromagnetic field A, through a "charge vector" [48]. In the current construction, the charge vector is $v = (1, 0, 0, 0)$, meaning that only the bosonic parton Φ carries electric charge $+1$. By integrating out all the dynamical gauge field a^I, one can show that the total Hall conductivity of the state is $\sigma_H = -2/3$, i.e., $\sigma_H = vK^{-1}v^T = -2/3$.

An alternative picture is that the $2/3$ state can be viewed as holes at filling 1, forming a $\nu = -1$ integer quantum Hall state, together with electrons at filling $1/3$, forming a $\nu = 1/3$ Laughlin state. The K matrix for this construction is

$$K'_{2/3} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \tag{4}$$

with the charge vector $v = (1, -1)$. The first diagonal element of $K'_{2/3}$ describes the $\nu = -1$ quantum Hall state and the second diagonal element describes the $\nu = 1/3$ Laughlin state. The K matrix in Eq. (3) is related to the K' matrix in Eq. (4) (up to two extra fields that describe a trivial, neutral sector) by a similarity transformation in $SL(4, Z)$:

$$W^T K_{2/3} W = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$W = \begin{pmatrix} -1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 2 & 1 & -1 \end{pmatrix}. \tag{5}$$

The third picture of constructing the $2/3$ state is through the composite fermion (CF) and flux attachment. As we mentioned before, when the holes fill a valley polarized Chern band, the physics is topologically equivalent to an integer quantum Hall state where a hole "sees" a $\phi_h = -1$ magnetic flux quantum through each moiré unit cell. Then a composite fermion is constructed by binding the hole with two-flux quanta of a gauge field a, i.e., the composite fermions will "see" total gauge flux $\phi_{cf} = \phi_h + 2\rho_{cf}$. When the hole density is $2/3$ per moiré unit cell, the density of ϕ_{cf} would be $1/3$ flux quantum per moiré unit cell. Hence the composite fermions would naturally fill two Landau levels of ϕ_{cf} and form an integer quantum Hall state with composite fermion Hall conductivity $\sigma_{cf} = 2$. When inserting an extra flux density $\delta\phi_{cf}$ into the system, composite fermion density $\delta\rho_{cf} = \sigma_{cf}\delta\phi_{cf}$ will be accumulated and the total Hall conductivity is the ratio between the extra density of composite fermions and the density of extra magnetic flux: $\sigma_H = \delta\rho_{cf}/\delta\phi_h = \sigma_{cf}/(1 - 2\sigma_{cf}) = -2/3$. This composite fermion construction for the $2/3$ state does not break translation. The composite fermion picture for understanding the observed FQAH states in twisted bilayer $MoTe_2$ is strongly supported by recent numerical studies [23,24,27].

To formally implement this flux attachment [49], we introduce a noncompact gauge field b whose charge is the flux of a. We will demonstrate that a $U(1)_{-2}$ CS term for b attaches two units of fluxes of a to the composite fermion, which also carries charge under gauge field combination $a + A$. The Lagrangian of all the field mentioned above reads

$$\mathcal{L} = -\frac{2}{4\pi} bdb + \frac{1}{2\pi} bda + \mathcal{L}_{\mathrm{CF}}[\psi, a + A], \tag{6}$$

where the mutual CS term between b, a implies that the flux of a is charged under b and the last term of Eq. (6) is the CF Lagrangian capturing the physics that the CF (ψ) is coupled to $a + A$.

The equations of motion with respect to b and a lead to the following relations:

$$\frac{\delta\mathcal{L}}{\delta b_0} = 0 \ \rightarrow \ \frac{da}{2\pi} = \frac{2db}{2\pi},$$
$$\frac{\delta\mathcal{L}}{\delta a_0} = 0 \ \rightarrow \ \rho_{cf} = \frac{db}{2\pi}. \tag{7}$$

Combining the equations we obtain the relation $2\rho_{cf} = \frac{da}{2\pi}$, which corresponds to the picture of flux attachment: each CF is bound with two flux quanta of gauge field a.

When the CF fermion ψ fills Chern bands with total Chern number C_{cf} (a nonzero integer), we need to introduce $|C_{cf}|$ copies of gauge fields a_i, which are dual to the current of the CFs:

$$\mathcal{L}_{\mathrm{CF}}[\psi, a + A] = \mathrm{sgn}(C_{cf}) \sum_{i=1}^{C_{cf}} \frac{1}{4\pi} a_i da_i - \frac{1}{2\pi} a_i d(a + A), \tag{8}$$

where each self-CS term of a_i describes the CF filling a complete Landau level (or equivalently Chern band with Chern number 1). The flux current of a_i, which is the dual of the CF current, couples to $a + A$.

For $C_{cf} = 2$, after combining Eq. (6) and Eq. (8) we eventually arrive at *exactly the same* K matrix as that in Eq. (3), albeit the charge vector now is (0,1,1,0), which corresponds to shifting the electric charge from the bosonic parton Φ to

TABLE I. Summary of the three formalisms that describe the three states with $\sigma_H = -2/3, -1, 0$, respectively. When the composite fermions (CF) fill Chern bands with total Chern number C_{cf}, the physical Hall conductivity is $\sigma_H = \frac{C_{cf}}{1-2C_{cf}}$. The charge vector for the left K matrices are $(1, 0, 0, 0)^T$ for the first two rows and $(1, 0, 0)^T$ for the last one. The total composite fermion Chern number is given by the sum of diagonal elements (excluding the first and last rows) of the left K matrices.

Phase	CF	Parton f (Φ in Laughlin $-\frac{1}{2}$ state)	Electron+holes ($\nu = -1$ IQH)	K matrix
$\sigma_H = -\frac{2}{3}$	$C_{cf} = 2$	$C_f = 2$	Electron in $\frac{1}{3}$ Laughlin	$\begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \simeq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
$\sigma_H = 0$	$C_{cf} = 0$	$C_f = 0$	Electron in $\nu = 1$ IQH	$\begin{pmatrix} -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \simeq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
$\sigma_H = -1$	$C_{cf} = 1$	$C_f = 1$	Electron in trivial insulator	$\begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \simeq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

the fermionic parton f, and it still leads to $\sigma_H = -2/3$. The charge vector could be transformed to (1,0,0,0), by relabeling $a \to a - A$. Then the two formalisms based on parton and CF yield exactly identical K matrices and Hall conductivity.

$\sigma_H = 0$ and $\sigma_H = -1$ states. To construct a trivial insulator phase with $\sigma_H = 0$, we can still fix the bosonic parton at a $\nu = -1/2$ Laughlin state and let the fermionic parton f fill two bands with Chern numbers $+1, -1$, respectively. The K matrix of this state is similar to Eq. (3), with the diagonal component K_{33} changed to -1. This change will lead to the Hall conductivity $\sigma_H = 0$, without any topological degeneracy.

An integer QAH state with $\sigma_H = -1$ and coexisting CDW order (referred to as the QAH-crystal state) can be constructed by removing the row and column of the K matrix that involves the second band of the fermionic parton, meaning that the fermionic parton f now fills bands with total Chern numbers $+1$.

In the composite fermion picture, when the CF forms a $\nu = +1$ quantum Hall state, inserting a $+1$ ϕ_h-flux quantum is accompanied by -2 units of fluxes of a and accumulating -1 charge of CF since there is a total -1 unit of extra flux ϕ_{cf}, i.e., the CFs form a $\nu = -1$ state with respect to ϕ_h. This state eventually corresponds to the $\sigma_H = -1$ state. The K matrix can be similarly deduced and is equivalent to that deduced from the parton construction.

The trivial insulator with $\sigma_H = 0$ corresponds to the CF forming a trivial insulator, which leads to trivial electromagnetic response. The K matrices of the three states constructed in this subsection are summarized in Table I.

Translation breaking enforced by filling. Importantly, when the translation symmetry of the moiré superlattice is preserved, it is not possible (at least within the CF mean-field theory) to have a direct transition from a state that fills a CF band with Chern number $C_{cf} = 2$ and correspondingly Hall conductivity $\sigma_H = -2/3$ to states with $C_{cf} = 1, 0$ and $\sigma_H = -1, 0$. Since the effective field seen by the CFs is $\phi_{cf} = 1/3$ flux quanta through each moiré plaquette, the CFs obey the magnetic translation symmetry: $T_1 T_2 = T_1^{-1} T_2^{-1} e^{i2\pi/3}$ for the two elementary translations $T_{1,2}$ that enclose a moiré unit cell. The mean-field spectrum of the CFs will be threefold degenerate as the magnetic translation admits representations with minimal dimensions of 3 [50].

Therefore, the translation symmetry guarantees that the change of the Chern number ΔC_{cf} across a transition must have ΔC_{cf} equal to multiples of 3, as it is given by the number of gapless Dirac cones in the spectrum. Hence if the transitions $\sigma_H = -2/3 \to -1$ and $\sigma_H = -2/3 \to 0$ are described by changing the band Chern number of the composite fermions, they can only occur when the translation symmetry of the moiré lattice is broken, i.e., they can only occur when there is a background charge density wave order. In particular, the simplest scenario of CDW is to triple the unit cell, rendering the magnetic translation trivial and permitting a direct transition with $\Delta C_{cf} = -1, -2$, etc. In the next section, we shall demonstrate explicitly direct transition from $\sigma_H = -2/3$ FQAH crystal to either $\sigma_H = 0$ CDW or $\sigma_H = -1$ QAH crytsal.

B. Phases tuned by bosonic parton Φ

In all the three states constructed in the last subsection, the bosonic parton Φ always forms a $\nu = -1/2$ bosonic Laughlin state. Starting with the FQAH-crystal state with $\sigma_H = -2/3$, two more states can be constructed by changing the physics of the bosonic parton Φ. These states/phases are summarized in the global phase diagram Fig. 2.

One such state is an insulator without any Hall conductivity, but it has a neutral topological order and neutral chiral edge states, leading to quantized thermal Hall effect. This quantum thermal Hall insulator can be obtained from the FQAH crystal, by driving the bosonic parton Φ into a trivial insulator. When Φ is in a trivial insulator, there is no nontrivial response to the external electromagnetic field as Φ is the parton that carries the electric charge. However, this state must still have a nontrivial topological order, as the K matrix of this state corresponds to Eq. (3) after removing the components that involve gauge field b. The determinant of the remaining 3×3 K matrix is 2 and it is equivalent to a simple semion topological order. The semion topological order can also be revealed by integrating out a_1 and a_2, which yields a level-2

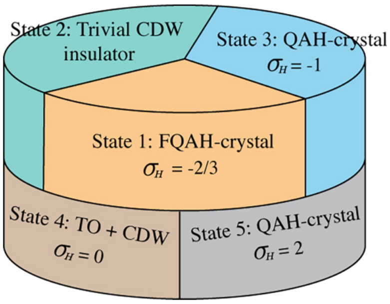

FIG. 2. Schematic global phase diagram in terms of the parton construction, tuned by both physics of bosonic parton Φ (vertical direction) and fermionic parton f (horizontal directions) starting from the FQAH crystal. We discuss interesting critical theories among the phases shown in Sec. III.

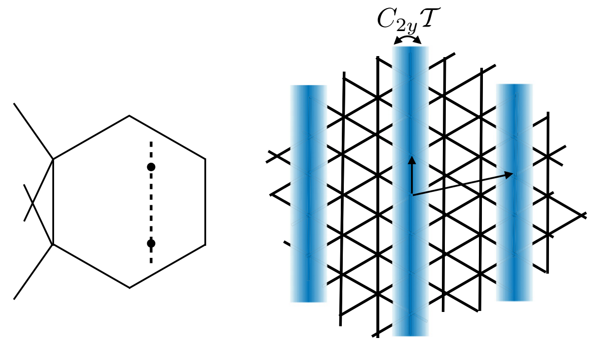

FIG. 3. (Left) Zoomed in moiré Brillouin zone with the symmetry $C_{2y}\mathcal{T}$ shown that protects degeneracy of two generic Dirac cones (black dots) aligned along the vertical axis. The arrows indicate the action of C_{2y}, which flips the horizontal axis. When combined with time reversal, $C_{2y}\mathcal{T}$ flips the vertical axis of the Brillouin zone and hence protects the degeneracy of the Dirac cone as plotted. When charge density wave forms a stripe pattern schematically shown on the right, the symmetry is preserved and a direct transition from $\sigma_H = -2/3 \to 0$ could be realized.

CS term for the gauge field a. This semion topological order with zero Hall conductivity is one of the states discussed in Ref. [28].

The other state is a QAH-crystal state with Hall conductivity $\sigma_H = +2$. This state can be constructed from FQAH crystal by driving the bosonic parton Φ into a "superfluid" state. In the condensate of Φ, the hole operator c is identified with f and, since f fills two bands with Chern number $+1$, this leads to a QAH-crystal state with Hall conductivity $\sigma_H = 2$. A QAH state without topological order is possible as we assumed a background CDW that triples the unit cell.

III. QUANTUM PHASE TRANSITIONS

So far, we have constructed five different states centered around the "2/3" state, in a phase diagram tuned by mean-field physics of the bosonic parton Φ and the fermionic parton f. These states can also be equally well constructed through other formalisms, including the composite fermions and flux attachment. In this section, we discuss the quantum phase transitions between these states. We use the parton formalism [e.g., Eq. (3)] in the following, while the composite fermion formalism with a different charge vector could yield identical results. Here we stress that the FQAH state we start with is in fact a FQAH crystal, while the starting point of Ref. [28] was a FQAH state without spontaneous translation symmetry breaking. Hence different proximate phases and phase transitions are obtained in these two papers. For example, in our current case the most natural $\sigma_H = 0$ insulator state next to the $\sigma_H = -2/3$ state is a trivial insulator with CDW, while in Ref. [28] the $\sigma_H = 0$ state has a topological order that would lead to a nontrivial thermal Hall signal.

A. $\sigma_H = -2/3 \to 0$ transition

To drive a transition between the FQAH-crystal state with $\sigma_H = -2/3$ and a trivial CDW insulator with $\sigma_H = 0$, we can keep the bosonic parton Φ in the $\nu = -1/2$ state unchanged and only change the total Chern number of the fermionic parton bands from $C = 2$ to $C = 0$, which can be realized by changing one of the occupied bands from $C = +1$ to $C = -1$. If there is a direct transition between these two states, it must involve two Dirac fermions at low energy. The complete critical theory reads

$$\mathcal{L}_{1-2} = \sum_{i=1,2} \bar{\psi}_i \gamma \cdot (\mathrm{i}\partial - a)\psi_i + m\bar{\psi}\psi - \frac{1}{2\pi} ad(a_1 - b) + \frac{1}{4\pi} a_1 da_1 - \frac{2}{4\pi} bdb - \frac{1}{2\pi} Adb, \tag{9}$$

where a_1, b are the dual fields associated with the filled $C = 1$ band of f [hence the $U(1)_1$ for a_1] and the boson Φ currents, respectively. Two Dirac fermions must exist at low energy at the transition for one f band to change from $C = 1$ ($m > 0$) to $C = -1$ ($m < 0$).

The theory Eq. (9) can be simplified at the cost of losing the proper quantization of the CS terms. Integrating out a_1 generates $-1/(4\pi)ada$ and integrating out b generates $1/(8\pi)(A - a)d(A - a)$. The simplified theory then reads

$$\mathcal{L}_{2;-\frac{1}{2}} = \sum_{i=1,2} \bar{\psi}_i \gamma \cdot (\mathrm{i}\partial - a)\psi_i + m\bar{\psi}\psi - \frac{1}{8\pi} ada - \frac{1}{4\pi} adA + \frac{1}{8\pi} AdA.$$

We use the notation $\mathcal{L}_{N_f;k}$ to label the QED Lagrangian with N_f flavors of Dirac fermions and a Chern-Simons term at level k.

The degeneracy of the two Dirac fermions can be guaranteed by extra discrete space-time symmetries. In the absence of displacement field, the entire homobilayer twisted TMD moiré system has a C_{2y} symmetry and a twofold rotation along the vertical axis in Fig. 1, as well as a time-reversal symmetry $\mathcal{T}$. Both symmetries exchange the two valleys. Hence each valley of the system holds a composite symmetry of $C_{2y}\mathcal{T}$. This composite symmetry sends $(k_x, k_y) \to (k_x, -k_y)$. The degeneracy of the two Dirac fermions that is needed for a direct "$\sigma_H = -2/3 \to \sigma_H = 0$" transition in our setup depends on the type of CDW order. For example, if the CDW is a stripe order along the y direction with modulation along the x direction as shown in Fig. 3, the two Dirac points could still be located at $(k_x, -k_y)$ and $(k_x, -k_y)$ points of the BZ and their degeneracy is still protected by the $C_{2y}\mathcal{T}$ symmetry. In contrast, if there is a $\sqrt{3}\times\sqrt{3}$ CDW order with C_3 symmetry shown in Fig. 1, there is no scenario where $C_3, C_{2y}\mathcal{T}$ together protect two and only two degenerate Dirac cones. Here we note that an out-of-plane displacement field in principle breaks the C_{2y} symmetry as it exchanges the two layers; hence under a displacement field the transition may split into two.

Although the microscopic symmetry C_{2y} of the system may be broken by a displacement field, extra effective symmetries may still exist in the physics of the moiré minibands. For example, there is an extra discrete symmetry of the Hamiltonian that describes one valley of the system [8], which is a composite of $R_x : y \to -y$, and a "time reversal" that acts on this one-valley Hamiltonian. This symmetry still exists with the presence of the displacement field. If this symmetry is a good approximate symmetry of the moiré miniband, it can also protect the degeneracy of two Dirac points and a direct transition of changing Chern number by 2, as was observed in model studies in Refs. [8,20].

An alternative description for the same transition from the $\sigma_H = -2/3$ state to the $\sigma_H = 0$ state can be constructed using the "electron-hole picture": the $\sigma_H = -2/3$ state can be viewed as a composition of holes at the $\nu = -1$ IQH state and electrons in the Laughlin $\nu = 1/3$ state. To drive a transition to the $\sigma_H = 0$ state we need a transition of the electrons from the $\nu = 1/3$ state to the $\nu = 1$ IQH state. The critical theory reads

$$\mathcal{L}_{1-2;eh} = \sum_{i=1,2}\bar{\psi}_i\gamma\cdot(\mathrm{i}\partial - a)\psi_i + m\bar{\psi}\psi + \frac{1}{2\pi}ad\tilde{b} + \frac{2}{4\pi}\tilde{b}d\tilde{b} + \frac{1}{2\pi}Ad\tilde{b} + \frac{1}{4\pi}AdA. \tag{10}$$

The last term $+\frac{1}{4\pi}AdA$ accounts for the hole $\nu = -1$ state that stays unchanged throughout the transition. The rest of the Lagrangian describes the transition of the electrons from the $\nu = 1/3$ state to a $\nu = 1$ IQH state. Note now $d\tilde{b}$ is dual to the bosonic parton of *electrons* (rather than *holes*), which carry charge -1; hence the coupling $+\frac{1}{2\pi}Ad\tilde{b}$ has an opposite sign from the coupling between db and A in Eq. (9). When the two Dirac cones are gapped out by a mass, integrating out the ψ's gives

$$\mathcal{L}_m = \frac{-\mathrm{sgn}(m)}{4\pi}ada + \frac{1}{2\pi}ad\tilde{b} + \frac{2}{4\pi}\tilde{b}d\tilde{b} + \frac{1}{2\pi}Ad\tilde{b} + \frac{1}{4\pi}AdA. \tag{11}$$

It is straightforward to verify that, for $m > 0$ ($m < 0$), the theory describes states with Hall responses $\sigma_H = -2/3$ ($\sigma_H = 0$), respectively. Starting with Eq. (10), the theory simplifies again to $\mathcal{L}_{2;-\frac{1}{2}}$ after integrating out $\tilde{b}$.

B. $\sigma_H = -2/3 \to -1$ transition

From the parton picture, this transition involves changing a fermion f's state from integer quantum Hall state with $\nu = +1$ to $\nu = 0$. This transition can be described by QED with one Dirac fermion in the infrared; the critical theory reads

$$\mathcal{L}_{1-3} = \bar{\psi}\gamma\cdot(\mathrm{i}\partial - a)\psi + m\bar{\psi}\psi - \frac{1}{8\pi}ada - \frac{1}{2\pi}ad(a_1 - b) + \frac{1}{4\pi}a_1da_1 - \frac{2}{4\pi}bdb - \frac{1}{2\pi}Adb. \tag{12}$$

We have added $-\frac{1}{8\pi}ada$ to properly regularize a single Dirac cone, which arises from another massive Dirac fermion which must exist in the same band as ψ. Here, the single Dirac fermion is written in the convention that, by changing the sign of m, ψ would generate a level $\pm 1/2$ CS term for a. Integrating out b, a_1, we obtain a simplified theory (again at the cost of not properly quantizing the CS term)

$$\mathcal{L}_{1;-1} = \bar{\psi}\gamma\cdot(\mathrm{i}\partial - a)\psi + m\bar{\psi}\psi - \frac{1}{4\pi}ada + \frac{1}{8\pi}AdA - \frac{1}{4\pi}adA. \tag{13}$$

Similarly, when the Dirac cone is gapped out by a *positive* mass term $m\bar{\psi}\psi$, integrating out the ψ's gives

$$\mathcal{L}_{m>0} = -\frac{3}{8\pi}ada + \frac{1}{8\pi}AdA - \frac{1}{4\pi}adA, \tag{14}$$

which generates Hall conductivity $\sigma_H = -2/3$. While $m < 0$ one has

$$\mathcal{L}_{m<0} = -\frac{1}{8\pi}ada + \frac{1}{8\pi}AdA - \frac{1}{4\pi}adA \tag{15}$$

and integrating out a leaves $\frac{1}{4\pi}AdA$, describing a state with $\sigma_H = -1$.

From standard boson-fermion duality [37,38], the critical theory $\mathcal{L}_{1;-1}$ is dual to

$$\mathcal{L}_{1;-1} \leftrightarrow |(\partial - \mathrm{i}\beta)\phi|^2 + \frac{1}{4\pi}\beta d\beta - \frac{1}{2\pi}ad\beta - \frac{1}{8\pi}ada + \frac{1}{8\pi}AdA - \frac{1}{4\pi}adA.$$

Here β is another gauge field that couples to the dual bosonic field ϕ. One could verify that the massive and condensed phase of ϕ corresponds to $\mathcal{L}_{m>(<)0}$, respectively, yielding $\sigma_H = -2/3, -1$. One can also directly perform the duality transformation from Eq. (12).

Integrating a in the dual bosonic theory leaves

$$\mathcal{L}_{1;-1} \leftrightarrow |(\partial - \mathrm{i}\beta)\phi|^2 + \frac{3}{4\pi}\beta d\beta + \frac{1}{2\pi}\beta dA + \frac{1}{4\pi}AdA. \tag{16}$$

This Chern-Simons-matter theory with ϕ coupled to a $U(1)$ gauge field with a CS term with level 3 is the standard theory that describes a transition between a trivial insulator and a fractional quantum Hall state with threefold topological degeneracy [51]. Combined with the last term $\frac{1}{4\pi}AdA$, which corresponds to an extra $\nu = -1$ IQH layer, the theory describes a transition between states with $\sigma_H = -2/3$ and $\sigma_H = -1$. This FQAH-crystal to QAH-crystal transition also admits another description in terms of bosonic partons, which is a modified version of the FQAH to QAH+CDW transition discussed in Ref. [28] driven by the condensation of three "vortex" fields coupled to a $U(1)_3$ Chern-Simons term. It is worth noting that this modified vortex condensation theory takes the same form as Eq. (16) [52].

C. $\sigma_H = -2/3 \to +2$ transition

Another potentially direct transition is between states 1 and 5, i.e., a transition from a FQAH-crystal state with $\sigma_H = -2/3$ to a QAH-crystal state with $\sigma_H = +2$. In the parton construction, this requires changing the state of Φ from a $\nu = -1/2$ Laughlin state to a "superfluid" state. This transition of Φ was discussed in Refs. [49,53] and it is described by a QED with two flavors of Dirac fermions and a CS term at level -1. In our notation, the critical theory of Φ is described by a Lagrangian $\mathcal{L}_{2;-1}$ and the Dirac fermions are charges of the gauge field b, i.e., the dual of the current of Φ. To describe the transition between states 1 and 5, we need to couple b to several other gauge fields a, a_i as in Eq. (1). To wit, the critical theory reads

$$\mathcal{L} = \sum_{i=1,2} \bar{\chi}_i \gamma \cdot (\mathrm{i}\partial - b)\chi_i + m\bar{\chi}\chi - \frac{1}{4\pi} bdb - \frac{1}{2\pi} Adb + \frac{1}{2\pi} b \wedge da - \frac{1}{2\pi} a \wedge d(a_1 + a_2) + \sum_{i=1,2} \frac{1}{4\pi} a_i \wedge da_i,$$

where the first line is the critical theory of Laughlin to superfluid transition for Φ and the second line describes the fermion partons and their coupling to Φ.

After integrating out the a and a_i, we arrive at the critical theory between states 1 and 5:

$$\mathcal{L}_{1-5} = \tilde{\mathcal{L}}_{2;-1/2} = \sum_{i=1,2} \bar{\chi}_i \gamma \cdot (\mathrm{i}\partial - b)\chi_i + m\bar{\chi}\chi - \frac{1}{8\pi} bdb - \frac{1}{2\pi} Adb. \tag{17}$$

We note that here the Dirac fermion χ_i is different from the fermions in the previous sections, as it is charged under b [hence the critical theory $\tilde{\mathcal{L}}_{2,-1/2}$ is different from $\mathcal{L}_{2;-1/2}$ in Eq. (10) with different charge assignment]. The degeneracy of two Dirac cones is again protected by $C_{2y}\mathcal{T}$ in a stripe order.

Integrating out the fermion χ_i, we obtain the following action:

$$\mathcal{L}_m = -\frac{\mathrm{sgn}(m)}{4\pi} bdb - \frac{1}{8\pi} bdb - \frac{1}{2\pi} Adb. \tag{18}$$

It is straightforward to verify that, for $m > 0$ ($m < 0$), the final Hall conductivity is $\sigma_H = -2/3$ ($\sigma_H = +2$).

D. Transitions involving quantum thermal Hall insulator

Reference [28] proposed a proximate insulating phase of the $\sigma_H = -2/3$ FQAH state to be one with vanishing Hall response and a neutral topological order (TO) described by the $U(1)_2$ CS terms. In the current framework, this state could be obtained by putting the bosonic parton Φ in a Mott insulator state. Formally, this amounts to eliminating the dual b of the boson current, i.e., setting $b = 0$, in the construction of the 2/3 state in Eq. (1). Physically, it means that the bosonic sectors are trivially gapped in the low energy. Integrating out a then sets $a_1 = -a_2$ and one gets a $U(1)_2$ CS coupling of an internal gauge field, describing the neutral TO, signified by a quantized thermal Hall response nevertheless.

The transition from the quantum thermal Hall insulator to the $\sigma_H = -2/3$ FQAH crystal or $\sigma_H = +2$ QAH crystal can hence be obtained by tuning the bosonic parton Φ out of the Mott phase to a Laughlin $-1/2$ state (for state 1) or a superfluid (for state 5), respectively.

The "Laughlin $-1/2$ to Mott" transition of the bosonic parton is realized by condensing the "vortices" of the bosonic partons in the Laughlin $-1/2$ state [28]. The critical theory describing the transition from state 1 to state 4 hence reads

$$\mathcal{L}_{1-4} = |(\partial - \mathrm{i}b)\Phi_v|^2 - \frac{2}{4\pi} bdb - \frac{1}{2\pi} Adb + \frac{1}{2\pi} ad(b - a_1 - a_2) + \sum_{i=1,2} \frac{1}{4\pi} a_i da_i. \tag{19}$$

The first term describes the condensation of the vortices Φ_v, which is indicated by its coupling to b, whose flux equals the boson density. As Φ_v condenses, the field b acquires a gap and could be ignored. Hence one arrives at the quantum thermal Hall insulator with $\sigma_H = 0$. An insulator phase of Φ_v just leaves the Lagrangian for the 2/3 state Eq. (1). A similar transition was discussed in Ref. [28], albeit with three vortex fields enforced by fractional filling 2/3.

The transition from quantum thermal Hall insulator to QAH crystal with $\sigma_H = +2$ is given by the standard boson Mott-superfluid transition,

$$\mathcal{L}_{4-5} = |(\partial + \mathrm{i}a - \mathrm{i}A)\Phi|^2 - \frac{1}{2\pi} ad(a_1 + a_2) + \sum_{i=1,2} \frac{1}{4\pi} a_i da_i. \tag{20}$$

When Φ is gapped, the remaining last two terms describe the neutral topological order. The condensation of Φ sets $a = A$ and the CS terms of a_i's then give a Hall conductivity $\sigma_H = 2$.

As for possible tuning parameters that could induce the transitions of Φ among Mott, superfluid, and Laughlin phases, we highlight bandwidth, controllable by twist angle, pressure, displacement field, etc., and interaction, controllable by gate screening. A less flat band may favor Φ to condense as to minimize kinetic energy, while a longer range interaction will favor Mott insulators over Laughlin states, which are exact ground states for a Hamiltonian with contact interactions [54]. Also, the state with $\sigma_H = -2/3$ can be tuned to the states with either $\sigma_H = -1$ or 0 by the displacement field [3,55].

IV. SUMMARY

In this work we discussed the phase diagram centered around a FQAH-crystal state with $\sigma_H = -2/3$ state at filling $-2/3$, motivated by recent experiments. Various phases and phase transitions can be obtained by tuning the physics of the bosonic and fermionic partons, including a direct transition between the $\sigma_H = -2/3$ state and a trivial insulating state with $\sigma_H = 0$ observed in recent experiments. Interestingly, we also find a direct transition between the $\sigma_H = -2/3$ FQAH state and a $\sigma_H = -1$ QAH state.

Our formalism and conclusions can easily be generalized to other FQAH states. For example, if the bosonic parton Φ forms a $\nu = -1/p$ Laughlin state (with even integer p) and the fermionic parton (or composite fermion) f fills Chern bands with total Chern number C_{cf}, we would end up with an FQAH state with Hall conductivity

$$\sigma_H = \frac{C_{cf}}{1 - pC_{cf}}. \tag{21}$$

For example, when $p = -2, C_{cf} = 1$, i.e., the composite fermions fill a Chern band with $C_{cf} = 1$, one constructs a Laughlin state with $\sigma_H = 1/3$. When the CFs go through a transition from $C_{cf} = 1 \rightarrow 0$, the electronic state transitions from $\sigma_H = 1/3 \rightarrow 0$. The critical theory is similar to that of $\sigma_H = -2/3 \rightarrow -1$ with the Lagrangian $\mathcal{L}_{1;-1}$ in Eq. (13), with an only difference of an integer quantum Hall layer described by $-1/(4\pi)AdA$. The critical theory for the transition hence reads

$$\mathcal{L}_{1/3\rightarrow 0} = \mathcal{L}_{1;-1} - \frac{1}{4\pi}AdA. \tag{22}$$

More states and phase transitions can be constructed by changing the "mean-field states" of Φ and f. This general construction will be useful to understand the growing number of FQAH states [7] observed in this rapidly developing field.

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[1] Y. Xie, A. T. Pierce, J. M. Park, D. E. Parker, E. Khalaf, P. Ledwith, Y. Cao, S. H. Lee, S. Chen, P. R. Forrester *et al.*, Nature (London) **600**, 439 (2021).

[2] E. M. Spanton, A. A. Zibrov, H. Zhou, T. Taniguchi, K. Watanabe, M. P. Zaletel, and A. F. Young, Science **360**, 62 (2018).

[3] J. Cai, E. Anderson, C. Wang, X. Zhang, X. Liu, W. Holtzmann, Y. Zhang, F. Fan, T. Taniguchi, K. Watanabe, Y. Ran, T. Cao, L. Fu, D. Xiao, W. Yao, and X. Xu, Nature (London) **622**, 63 (2023).

[4] Y. Zeng, Z. Xia, K. Kang, J. Zhu, P. Knüppel, C. Vaswani, K. Watanabe, T. Taniguchi, K. F. Mak, and J. Shan, Nature (London) **622**, 69 (2023).

[5] H. Park, J. Cai, E. Anderson, Y. Zhang, J. Zhu, X. Liu, C. Wang, W. Holtzmann, C. Hu, Z. Liu, T. Taniguchi, K. Watanabe, J.-h. Chu, T. Cao, L. Fu, W. Yao, C.-Z. Chang, D. Cobden, D. Xiao, and X. Xu, Nature (London) **622**, 74 (2023).

[6] F. Xu, Z. Sun, T. Jia, C. Liu, C. Xu, C. Li, Y. Gu, K. Watanabe, T. Taniguchi, B. Tong *et al.*, Phys. Rev. X **13**, 031037 (2023).

[7] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, Nature **626**, 759 (2024).

[8] F. Wu, T. Lovorn, E. Tutuc, I. Martin, and A. H. MacDonald, Phys. Rev. Lett. **122**, 086402 (2019).

[9] H. Yu, M. Chen, and W. Yao, Natl. Sci. Rev. **7**, 12 (2020).

[10] T. Devakul, V. Crépel, Y. Zhang, and L. Fu, Nat. Commun. **12**, 6730 (2021).

[11] H. Li, U. Kumar, K. Sun, and S.-Z. Lin, Phys. Rev. Res. **3**, L032070 (2021).

[12] V. Crépel and L. Fu, Phys. Rev. B **107**, L201109 (2023).

[13] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. **106**, 236803 (2011).

[14] E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011).

[15] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).

[16] D. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat. Commun. **2**, 389 (2011).

[17] N. Regnault and B. A. Bernevig, Phys. Rev. X **1**, 021014 (2011).

[18] N. Morales-Durán, J. Wang, G. R. Schleder, M. Angeli, Z. Zhu, E. Kaxiras, C. Repellin, and J. Cano, Phys. Rev. Res. **5**, L032022 (2023).

[19] C. Wang, X.-W. Zhang, X. Liu, Y. He, X. Xu, Y. Ran, T. Cao, and D. Xiao, Phys. Rev. Lett. **132**, 036501 (2024).

[20] A. P. Reddy, F. Alsallom, Y. Zhang, T. Devakul, and L. Fu, Phys. Rev. B **108**, 085117 (2023).

[21] W.-X. Qiu, B. Li, X.-J. Luo, and F. Wu, Phys. Rev. X **13**, 041026 (2023).

[22] V. Crépel, N. Regnault, and R. Queiroz, arXiv:2305.10477.

[23] H. Goldman, A. P. Reddy, N. Paul, and L. Fu, Phys. Rev. Lett. **131**, 136501 (2023).

[24] J. Dong, J. Wang, P. J. Ledwith, A. Vishwanath, and D. E. Parker, Phys. Rev. Lett. **131**, 136502 (2023).

[25] X. Liu, C. Wang, X.-W. Zhang, T. Cao, and D. Xiao, arXiv:2308.07488.

[26] C. Xu, J. Li, Y. Xu, Z. Bi, and Y. Zhang, arXiv:2308.09697.

[27] A. P. Reddy and L. Fu, Phys. Rev. B **108**, 245159 (2023).

[28] X.-Y. Song, Y.-H. Zhang, and T. Senthil, arXiv:2308.10903.

[29] J. Yu, J. Herzog-Arbeitman, M. Wang, O. Vafek, B. A. Bernevig, and N. Regnault, Phys. Rev. B **109**, 045147 (2024).

[30] A. Abouelkomsan, A. P. Reddy, L. Fu, and E. J. Bergholtz, arXiv:2309.16548.

[31] Z. Tešanović, F. Axel, and B. I. Halperin, Phys. Rev. B **39**, 8525 (1989).

[32] H. Polshyn, Y. Zhang, M. A. Kumar, T. Soejima, P. Ledwith, K. Watanabe, T. Taniguchi, A. Vishwanath, M. P. Zaletel, and A. F. Young, Nat. Phys. **18**, 42 (2022).

[33] H. Pan, M. Xie, F. Wu, and S. Das Sarma, Phys. Rev. Lett. **129**, 056804 (2022).

[34] A. Kol and N. Read, Phys. Rev. B **48**, 8890 (1993).

[35] S. Kourtis and M. Daghofer, Phys. Rev. Lett. **113**, 216404 (2014).

[36] J. Dong, P. J. Ledwith, E. Khalaf, J. Y. Lee, and A. Vishwanath, Phys. Rev. Res. **5**, 023166 (2023).

[37] W. Chen, M. P. A. Fisher, and Y.-S. Wu, Phys. Rev. B **48**, 13749 (1993).
[38] N. Seiberg, T. Senthil, C. Wang, and E. Witten, Ann. Phys. (NY) **374**, 395 (2016).
[39] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, Phys. Rev. X **7**, 031051 (2017).
[40] X.-Y. Song and Y.-H. Zhang, SciPost Phys. **15**, 215 (2023).
[41] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[42] Y. Zhang and J. Shi, Phys. Rev. Lett. **113**, 016801 (2014).
[43] J. Y. Lee, C. Wang, M. P. Zaletel, A. Vishwanath, and Y.-C. He, Phys. Rev. X **8**, 031015 (2018).
[44] X.-Y. Song, H. Goldman, and L. Fu, Phys. Rev. B **108**, 205123 (2023).
[45] N. Morales-Durán, N. Wei, and A. H. MacDonald, arXiv:2308.03143.
[46] N. Paul, Y. Zhang, and L. Fu, Sci. Adv. **9**, eabn1401 (2023).
[47] We are not discussing a tight-binding model on the moiré scale; hence the magnetic flux cannot be removed through large gauge transformation.
[48] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[49] M. Barkeshli and J. McGreevy, Phys. Rev. B **86**, 075136 (2012).
[50] A. A. Burkov and L. Balents, Phys. Rev. B **72**, 134502 (2005).
[51] X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. **70**, 1501 (1993).
[52] The multiple vortex fields are required as the $2/3$ state there in Ref. [28] preserved translation; hence vortex degeneracy is enforced by fractional filling. When CDW orders are assumed here and integer filling is reached for an enlarged unit cell, the transition is modified to a single vortex ϕ condensation transition discussed here.
[53] M. Barkeshli and J. McGreevy, Phys. Rev. B **89**, 235116 (2014).
[54] S. A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).
[55] X. Xu (private communication).