Fractal subsystem symmetries, 't Hooft anomalies, and UV/IR mixing

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In this work, we study unconventional anisotropic topologically ordered phases in 3*d* that manifest type-II fractonic physics along submanifolds. While they behave as usual topological order along a preferred spatial direction, their physics along perpendicular planes is dictated by the presence of fractal subsystem symmetries, completely restricting the mobility of anyonic excitations and their bound states. We consider an explicit lattice model realization of such phases and proceed to study their properties under periodic boundary conditions and, later, in the presence of boundaries. We find that for specific lattice sizes, the system possesses line and fractal membrane symmetries that are mutually anomalous, resulting in a nontrivially gapped ground state space. This amounts to the spontaneous breaking of the fractal symmetries, implying a subextensive ground state degeneracy. For the remaining system sizes the fractal symmetries are explicitly broken by the periodic boundary conditions, which is intrinsically related to the uniqueness of the ground state. Despite that, the system is still topologically ordered since locally created quasiparticles have nontrivial mutual statistics and, in the presence of boundaries, it still presents anomalous edge modes. The intricate symmetry interplay dictated by the lattice size is a wild manifestation of ultraviolet/infrared (UV/IR) mixing.

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I. INTRODUCTION

It is quite remarkable that certain elementary systems, containing simple degrees of freedom that interact only locally, can give rise to exotic forms of matter [1]. Topologically ordered systems are among the most prominent examples, with emergent quasiparticle excitations carrying anyonic statistics. Fractonic topological ordered systems are even more exotic [2–15], with excitations that are intrinsically devoid of mobility due to the existence of generalized forms of symmetries known as subsystem symmetries [16–19].

Subsystem symmetries are generated by conserved charges along rigid spatial subdimensional manifolds, which tend to be extremely sensible to the underlying lattice geometry. Compliance with such conservation laws imposes severe restrictions on the mobility of the excitations. This contrasts with usual topological order, which is in general characterized by topologically deformable symmetry generators, known as higher-form symmetries, and poses no constrains on excitation's mobility [20–24].

Abelian topological order and fractonic phases may be rephrased in terms of spontaneous breaking of, respectively, finite higher-form and subsystem symmetries [25–27]. This follows from the definition of topological order in terms of local indistinguishability of ground states [28,29]. Additionally, any two ground states are connected by extended symmetry operators, which precisely fits the notion of spontaneous symmetry breaking in the context of a generalized symmetry [30]. Generalized symmetries appear to distinguish themselves from usual symmetries, as they can be present in low-energy states even when the Hamiltonian is not symmetric [30–32]. Furthermore, although in this work we only discuss systems containing Abelian anyons—usually associated to higher-form and subsystem symmetries—it is worth mentioning that non-Abelian anyons can also be casted in the language of generalized noninvertible symmetries [33–37].

An alternative way of understanding the nontriviality of the ground state space is through the nontrivial braiding statistics among excitations, which signals the presence of 't Hooft anomalies in Abelian topological order. The matching of anomalies from the ultraviolet (UV) to the infrared (IR) imply that the ground state must be nontrivially gapped [38].

It is common for d-dimensional systems invariant under subsystem symmetries, with support in dimensions smaller than d, to have a subextensive number of conserved charges. Typical examples are type-I fracton systems that, when defined on a 3d $L \times L \times L$ system, possess charge conservation laws in $O(L)$ individual planes [14,15]. This leads to a macroscopically large amount of symmetry generators, which in turn implies an enormous degeneracy of states. In particular, the ground state degeneracy (IR feature) is sensitive to the number of sites of the system (UV feature)—a phenomenon called UV/IR mixing.

The emergence of UV/IR mixing from simple bosonic local theories has challenged our understanding of effective theories and renormalization group. This follows from the fact that the IR physics depends sensitively on the UV details of the theory, and has been a major point of investigation [12,39–42]. In the context of generalized symmetries, several exactly solvable models have shed light on the origin of such phenomenon. In the context of gapped

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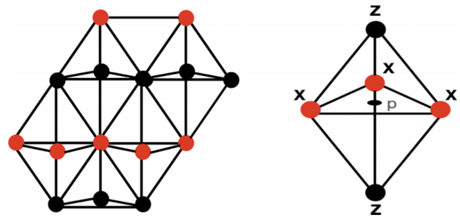

FIG. 1. Three-dimensional hexagonal close-packed lattice and a TD operator.

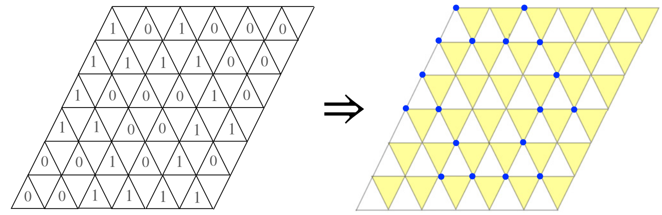

FIG. 2. Membrane operator constructed from a constraint.

dipole moments (and higher-multipole momenta) conserving gauge theories, emergent higher-form symmetries obey twisted boundary conditions in order for holonomy operators to close onto themselves [43–46]. For subsystem symmetries, which are closely related with systems studied in the work, the UV/IR mixing emerges from the fact that the system possesses an increasing number of symmetries as the system size grows [3,16,47].

A more dramatic manifestation of UV/IR mixing shows up in the case of subsystem symmetries where the charges are supported in submanifolds of fractal dimensions [39,48–54]. These symmetries are intimately related to type-II fracton physics [4], where no excitations are allowed to move. In this work we study fractal symmetries with support on Sierpinski triangles, which have Hausdorff dimension $\log(3)/\log(2)$. As a consequence, the low-energy properties are extremely sensitive to the lattice details and typically there is no uniform dependence of the ground state degeneracy with the lattice size as in the case of type-I fractons. Additionally, we find that the fractal symmetries are always broken. Whether it is spontaneously or explicitly broken depends on the lattice linear size L.

II. MODEL

The phases we are interested in are captured by the low-energy states of the fractal model introduced in [49] in the context of topological quantum glassiness. The degrees of freedom of this model correspond to qubits located at the sites of a hexagonal close-packed lattice. The lattice is defined as the interpolating stack (along the $\hat{z}$ direction) of $d = 2$ triangular lattice planes that are dislocated in relation to each other by $a_0(\hat{x}/2 + \sqrt{3}\hat{y}/3 + \hat{z}/2)$, where a_0 is the lattice spacing. We now focus on the lattice composed of centers p of triangles, which can be decomposed into two sublattices $\Lambda = \Lambda_1 \oplus \Lambda_2$, as shown in Fig. 1. The Hamiltonian is

$$H = -J \sum_{a=1,2} \sum_{p\in\Lambda_a} \mathcal{O}_p, \qquad J > 0, \tag{1}$$

where the triangular dipyramidal (TD) operators $\mathcal{O}_p$ are given in terms of Pauli matrices X and Z according to

$$\mathcal{O}_p \equiv Z_{p+\frac{1}{2}\hat{z}} X_{p+\frac{1}{2}\hat{x}-\frac{1}{2\sqrt{3}}\hat{y}} X_{p-\frac{1}{2}\hat{x}-\frac{1}{2\sqrt{3}}\hat{y}} X_{p+\frac{1}{\sqrt{3}}\hat{y}} Z_{p-\frac{1}{2}\hat{z}}, \tag{2}$$

where p is the center of the TD operator, as in Fig. 1.

A key property of the model (1) is that it is written in terms of commuting projectors, i.e., $[\mathcal{O}_p, \mathcal{O}_{p'}] = 0$ for any pair of operators, even when they share sites. Indeed, two neighboring TD operators can share one or two sites, and it is simple to see that in both cases they commute. Therefore, the energy spectrum is gapped and the energy levels are given by the sum of the eigenvalues of $\mathcal{O}_p$. As $\mathcal{O}_p^2 = \mathbb{1}$, the eigenvalues of $\mathcal{O}_p$ are ± 1 and ground states obey $\mathcal{O}_p\,|GS\rangle = +1\,|GS\rangle$ for all p.

III. GROUND STATE DEGENERACY

Each lattice site hosts a two-dimensional Hilbert space, so that the specification of a state in a lattice with N sites requires 2^N labels. As the total number of TD operators is equal to the number of lattice sites when the system is defined with periodic boundary conditions, it seems that the eigenvalues of the operators $\mathcal{O}_p$ are able to provide exactly the 2^N labels for the states. However, not all TD operators are independent. There are certain constraints that reduce the number of available labels and consequently increase the degeneracy. Let N_c be the total number of constraints in the system. Then, the total number of available labels is 2^{N-N_c}, so that the ground state degeneracy is $GSD = 2^N/2^{N-N_c} = 2^{N_c}$.

We can write the constraints as

$$\prod_{p\in\Lambda} \mathcal{O}_p^{t_p} = \mathbb{1}, \tag{3}$$

involving a set of weights t_p mod 2. The dimension of the set $\{t_p\}$ gives the number of independent constraints N_c of the model.

Let us consider a periodic lattice with size $L \times L \times L_z$. For linear sizes $L = 2^n - 2^m$, with integers $n \geqslant 2$ and $0 \leqslant m < n$, the number of constraints is $2^n - 2^{m+1}$ for each one of the sublattices [55]. Furthermore, the constraints produce fractal Sierpinski structures in the xy planes. An example for $L = 2^3 - 2^1 = 6$ is shown in the left of Fig. 2.

Lattice sizes of the form $L = 2^n - 2^m$ are not the only ones that possess nontrivial ground state degeneracy, as we can construct larger systems simply by taking multiple copies of the size L, namely, we can consider systems with sizes of the form $L' = k(2^n - 2^m) \neq 2^{n'} - 2^{m'}$. In this case, the ground state degeneracy is that of the building-block copy $L = 2^n - 2^m$, since it provides nontrivial solutions for t's that are compatible with the periodic boundary conditions. As an example, take the size $L = 9$. It is not of the form $2^n - 2^m$, but it can be expressed as $L = 3(2^2 - 2^0)$. The ground state degeneracy is the same as that one of the case $L = 2^2 - 2^0 = 3$.

It may happen that there is more than one way to express certain size in terms of copies, i.e., $L = k(2^n - 2^m) = k'(2^{n'} - 2^{m'})$. In this case, we shall look for the building-block copy that provides the maximum number of nontrivial solutions for t's, namely, which number among $2^n - 2^{m+1}$ and $2^{n'} - 2^{m'+1}$ is greater. This occurs for the smallest value of

k [55]. Taking into account that there are two sublattices, we obtain the total number of constraints $N_c = 2(2^n - 2^{m+1})$. With this in mind, we can express in a unified way the ground state degeneracy associated with sizes

$$L = k(2^n - 2^m), \qquad 0 \leqslant m < n-1, \quad \text{and} \quad k \geqslant 1 \tag{4}$$

that is given by

$$GSD = 2^{(2^{n+1} - 2^{m+2})}, \tag{5}$$

with n and m associated with the copy with the lowest value of k. Of course, whenever a size can be expressed with $k = 1$, then it dictates the degeneracy. An example is $L = 18$, which cannot be expressed with $k = 1$. There are three ways to obtain $L = 18$, namely $3(2^3 - 2^1)$, $6(2^2 - 1)$, and $9(2^2 - 2)$. According to the above discussion, the ground state degeneracy is given by (5) with $n = 3$ and $m = 1$, which is constituted of three copies ($k = 3$) of the size $L = 6$.

The GSD in Eq. (5) does not depend on L_z due to the simple structure of the TD operators along the z direction. Since it has two Z Pauli matrices, one on top and one at the bottom, all the constraints involve a product of TD operators along the whole z direction. This is analogous to usual topological constrains, as in the toric code [56]. Despite of this, the third dimension is crucial to ensure topological ordering.

For system sizes $L \times L \times L_z$ in which L cannot be expressed in the form (4), the ground state is unique since there are no nontrivial solutions for t's. We shall discuss these cases later. The intricate dependence of the ground state degeneracy on the size of the system is a severe manifestation of the UV/IR mixing. To stress this, consider $m = 0$ and a very large value of n. In this case, the system with the size $L = 2^n - 1$ has a huge degeneracy $GSD = 2^{(2^{n+1} - 2^2)}$, whereas the system with size just one unit larger $L = 2^n$ has a unique ground state.

For general $L_x \neq L_y$ it is not clear the ground state degeneracy. However, for certain classes of system sizes, for example $L_x = qL_y$, for an integer q and $L_x = 2^n - 2^m$, we can show that ground state degeneracy is $2^{2(L_y - 2^m)}$.

IV. SYMMETRY OPERATORS AND MIXED 'T HOOFT ANOMALIES

A nontrivial ground state degeneracy is a reflection of a nontrivial algebra among certain symmetry operators, which indicates a nontrivial mutual statistics among anyons. For sizes of the form (4), there are two types of subsystem symmetry operators: fractal membrane operators disposed in the xy plane and Wilson line operators extended along the z direction.

Membrane operators are intimately related to the constraint structure (3) in the xy plane (see relation (10) of the Supplemental Material [55]). Consider $\prod_z \prod_{j \in \mathcal{M}_I^a(z)} \mathcal{O}_j = \mathbb{1}$, where j runs over all the sites belonging to $\mathcal{M}_I^a$ of the sublattice $a = 1, 2$ in the xy plane, with $I = 1, \ldots, 2^n - 2^{m+1}$ specifying the constraint. A subset of this product, ranging from $z = z_1 + a_0/2$ to $z = z_2 - a_0/2$, results in two membranes,

$$\prod_{z=z_1}^{z_2} \prod_{j \in \mathcal{M}_I^a(z)} \mathcal{O}_j = M_I^b(z_1) M_I^b(z_2), \qquad b \neq a, \tag{6}$$

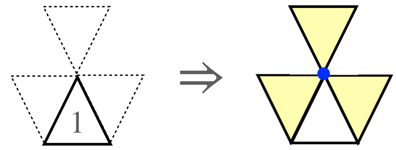

FIG. 3. Association of an operator $\mathcal{O}$ partaking the constraint in the xy plane in a sublattice with a Z operator (blue dot) in the other sublattice (yellow).

where, M_I^a is defined as

$$M_I^a \equiv \prod_{j \in \mathcal{M}_I^a} Z_j. \tag{7}$$

Each of these closed membrane operators individually commute with the Hamiltonian, since they contain either zero or two Z operators acting on each TD operator (see Fig. 2). Notably, while two membranes are obtained as a product of $\mathcal{O}_j$, an individual M_I^a is a nontrivial symmetry [57]. As the constraints exhibit fractal Sierpinski structures in the xy plane, the membrane operators resulting from this association will also enjoy such fractal patterns. It's worth mentioning that the membranes defined in Eq. (6) are associated with the opposite sublattice of the constraint, as illustrated in Fig. 2 by the blue dots acting on the yellow sublattice.

Consistent with Eq. (7), the membranes can be obtained by associating to each TD operator partaking the constraint and belonging to one of the sublattices a Z operator acting in a site of the other sublattice, according to the map in Fig. 3. This is reminiscent of the duality between dynamics and interactions [48].

There is a one-to-one correspondence among the constraints, associated with nontrivial solutions for $\{t_p\}$, and the closed membrane operators. Accordingly, there is a total of $N_c = 2(2^n - 2^{m+1})$ linearly independent membrane operators. Although for every z coordinate we can define a membrane operator as in Eq. (7), they are not regarded as independent. This is so because any two membranes $M_I^a(z_1)$ and $M_I^a(z_2)$, belonging to the same sublattice and with the same I, can be connected to each other through a product of TD operators. Thus, unless there is a defect, $\mathcal{O}_p |\psi\rangle = -|\psi\rangle$, between the planes at z_1 and z_2, the eigenvalues of $M_I^a(z_1)$ and $M_I^a(z_2)$ are constrained to be the same. Also, the membrane operators can be topologically deformed in the z direction through local products of TD operators.

Wilson lines along the z direction are constructed as $W_i^a = \prod_{j \in \mathcal{L}_i^a} X_j$, where $\mathcal{L}_i^a$ stands for a line along the z axis crossing xy planes at the site $i = (x, y)$, belonging to the sublattice a. If the line $\mathcal{L}_i^a$ is closed, the corresponding operator commutes with the Hamiltonian and then it is a symmetry operator.

Wilson line operators are rigid and, in principle, for each sublattice, there are L^2 line operators, one for each $i = (x, y)$ in the xy plane. However, not all of them are independent, as products of three lines can also be reduced to products of TD operators, which act as the identity on the ground state. The number of independent Wilson lines is $2^n - 2^{m+1}$ for each sublattice and that they are in one-to-one correspondence with the membrane operators [55]. Accordingly, we label the Wilson lines with the same type of index I as the membrane operators,

$$W_I^a = \prod_{j\in\mathcal{L}_I^a} X_j, \quad (8)$$

where $\mathcal{L}_I^a$, with $I = 1, \ldots, 2^n - 2^{m+1}$, corresponds to each one of the independent Wilson lines. We can pair line and membrane operators such that a particular W_I^a commutes with all other membrane operators except the one labeled by the same numbers M_I^a,

$$M_I^a W_J^b = (-1)^{\delta_{IJ}\delta_{ab}} W_J^b M_I^a, \quad (9)$$

with $a, b = 1, 2$ and $I, J = 1, \ldots, 2^n - 2^{m+1}$ [55]. This algebra leads to the ground state degeneracy $GSD = (2^{2^n - 2^{m+1}})^2$, reproducing (5). Such a nontrivial algebra among pairs of membrane and Wilson operators can be rephrased in terms of a mixed 't Hooft anomaly between the subsystem symmetries, preventing the ground state manifold to be trivially gapped.

We recall that a mixed 't Hooft anomaly is an obstruction to simultaneously gauging the corresponding symmetries. This can be understood in an intuitive way. Consider a single pair of symmetry operators satisfying (9) with $MW = -WM$, as well as states $|\psi'\rangle$ and $|\psi''\rangle$ constructed as

$$|\psi'\rangle = MW\,|\psi\rangle \quad \text{and} \quad |\psi''\rangle = WM\,|\psi\rangle\,. \quad (10)$$

The commutation relation between M and W implies $|\psi''\rangle = -|\psi'\rangle$. As the two states differ by a phase, they belong to the same ray and are identified actually as the same state. Now, if we try to gauge both symmetries, which means that the physical states must be invariant under the action of M and W, we get a contradiction $|\psi\rangle = -|\psi\rangle$. In other words, there are no physical states in the gauged theory and the partition function vanishes.

V. EXCITATIONS AND MUTUAL STATISTICS

Excitations correspond to any state with $\mathcal{O}_p\,|\psi\rangle = -|\psi\rangle$, for some TD operators. Such configurations are either created in pairs at the endpoints of open Wilson lines or in groups of three at the corners of open fractal membranes. The rigidity of string and membrane operators is reflected in the allowed dynamics for the quasiparticles excitations. While open lines W_I^a are associated with the free transport of excitations along the z direction, open M_I^a cannot be used to move particles in the xy plane. This is because M_I^a maps a single particle state into a two-particle state, a process that has a high energy cost $\Delta E \sim J$. Open membranes M_I^a can be used, though, to map three-particle states into three-particle states with no energetic cost, through the growth of the underlying fractal membrane $\mathcal{M}_I^a$. However, such a process passes through highly energetic intermediate states, making the corresponding tunneling probability very small. This implies that all excitations are immobile along the xy plane in finite times, which is a manifestation of type-II fracton physics [4].

We can define the notion of mutual statistics among a single excitation and a three-particle composite excitation [58,59]. Let us denote such a state as $|1, 3\rangle$, corresponding to the configuration in 1 of Fig. 4. First, we consider the transport of the single excitation along the z direction through the application of a Wilson line operator without intercepting the triangular region defined by the three-particle excitation. Then, we consider the application of an open membrane operator to put apart the three excitations, enlarging the region defined by them. Carrying the reverse transport of the single excitation along the z direction, now the line operator intersects the membrane and produces a minus sign. The final step is to apply the membrane operator again to return the three excitations to their initial position. This sequence of operations is depicted in Fig. 4. In algebraic terms, it reads

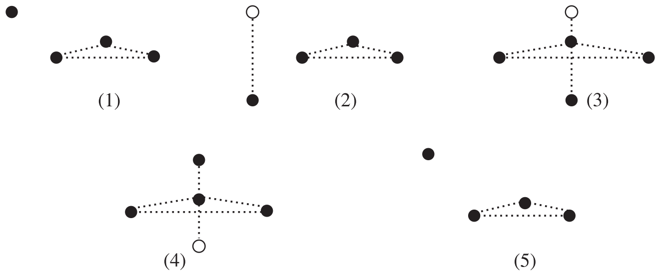

FIG. 4. Sequence of steps leading to the notion of mutual statistics.

$$M_I^{a\dagger} W_I^{a\dagger} M_I^a W_I^a\,|1,3\rangle = -\,|1,3\rangle\,, \quad (11)$$

where we have used the algebra (9). The minus sign on the right hand side implies nontrivial mutual anyonic statistics among the involved excitations, which is a signature of long-range entanglement of topological ordered phases.

VI. SPONTANEOUS BREAKING OF SUBSYSTEM SYMMETRIES

It is enlightening to view fractal topological order from the perspective of spontaneous breaking of subsystems symmetries. Topological order is characterized by the indistinguishability of the ground states, which means that for any local operator Φ, it follows that

$$\langle GS, a|\,\Phi\,|GS, b\rangle = C\delta_{ab}, \quad (12)$$

where C is a constant independent of the particular ground state $|GS, a\rangle$. Distinct ground states are connected by extended symmetry operators. This is precisely the notion of spontaneous symmetry breaking, but for a generalized symmetry, where extended symmetry operators act nontrivially on the ground states, taking from one to another.

In order to ensure the property (12), we need two generalized symmetries mixed by a 't Hooft anomaly. Let us consider a specific pair of line W and membrane M operators, satisfying $[H, W] = [H, M] = 0$ and $[M, W] \neq 0$, with $W^2 = M^2 = \mathbb{1}$. We choose the ground states $|GS, a\rangle$, $a = 1, 2$, as simultaneous eigenstates of H and W,

$$H\,|GS, a\rangle = E_0\,|GS, a\rangle \quad \text{and} \quad W\,|GS, a\rangle = \lambda_a\,|GS, a\rangle\,, \quad (13)$$

where $\lambda_a = \pm 1$ and $\lambda_2 = -\lambda_1$. Then, $|GS, 1\rangle$ and $|GS, 2\rangle$ are connected by the membrane operator, $|GS, 2\rangle = M\,|GS, 1\rangle$. Now let us see how the two symmetries W and M lead to (12). We start by computing $\langle GS, 2|\,\Phi\,|GS, 2\rangle = \langle GS, 1|\,M\Phi M\,|GS, 1\rangle$. As the membrane operator M is mobile

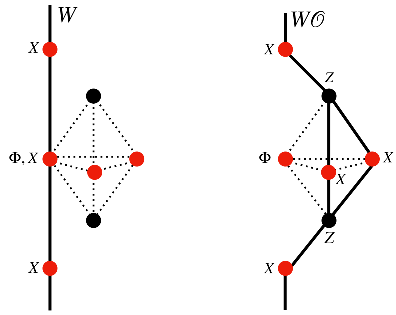

FIG. 5. Local splitability of the Wilson line.

along the z direction, we can move it in order to avoid the position of the local operator, which means that M commutes with Φ. This implies that $\langle GS, 2|\, \Phi\, |GS, 2\rangle = \langle GS, 1|\, \Phi\, |GS, 1\rangle$.

It remains to show that $\langle GS, 2|\, \Phi\, |GS, 1\rangle = 0$. We start with the left hand side of this equation and use (13) to write it as $\langle GS, 2|\, \Phi\, |GS, 1\rangle = \lambda_1^{-1}\lambda_2^{-1}\, \langle GS, 2|\, W\, \Phi\, W\, |GS, 1\rangle$. We cannot use the same argument as before to justify the commutation of W and Φ in the case where they intercept because W is rigid. However, such a line operator enjoys a slightly different property, which we refer to as local splitability. This is a kind of nontopological deformation in the sense that it does not preserve the form of a line. Nervertheless, this can be used to avoid the point of support of the local operator. Supposing that Φ is located at a site j, we can use a TD operator $\mathcal{O}$ containing an X_j to define the new object $\tilde{W} \equiv \mathcal{O}W = W\mathcal{O}$, which avoid the site j at the price of splitting the line around it, as shown in Fig. 5. With this,

$$\begin{aligned}\langle GS, 2|\, \Phi\, |GS, 1\rangle &= \lambda_1^{-1}\lambda_2^{-1}\, \langle GS, 2|\, W\, \Phi\, W\, |GS, 1\rangle \\ &= -\, \langle GS, 2|\, \mathcal{O}\, W\, \Phi\, W\, \mathcal{O}\, |GS, 1\rangle \\ &= -\, \langle GS, 2|\, \Phi\, |GS, 1\rangle\,, \end{aligned} \tag{14}$$

where we have used that $\mathcal{O}\, |GR, a\rangle = |GR, a\rangle$. Therefore, we obtain $\langle GS, 2|\, \Phi\, |GS, 1\rangle = 0$. In conclusion, the two subsystem symmetries possessing a mixed 't Hooft anomaly ensure the condition of indistinguishability of the ground states required for topological order, which in turn is equivalent to the spontaneous breaking of the subsystem symmetries.

VII. EXPLICIT BREAKING OF FRACTAL SUBSYSTEM SYMMETRIES

For sizes L as in Eq. (4) with $n = m + 1$ or sizes that cannot be expressed in the form of Eq. (4), the ground state is unique since there are no nontrivial solutions for t's. Accordingly, there are no membrane operators that commute with the Hamiltonian, since they do not close. In other words, the fractal subsystem symmetries are explicitly broken. Membrane operators that do not commute with a single TD operator can be used to create local excitations. This implies that there is only a single global anyonic superselection sector, as this type of membrane can be used to create and destroy single particles. The corresponding phases, however, are also topologically ordered as the notion of mutual statistics in Eq. (11) is still present.

While spontaneous symmetry breaking of discrete subsystem symmetry produces topological ordered phases with nontrivial ground state degeneracy, it is not a necessary condition (instead, it is a sufficient condition) as long as we define topological order in terms of long-range entanglement [1]. This is suitable for phases with a unique ground state, since the indistinguishability condition (12) is trivial in such cases. On the other hand, the mutual statistics in (11) implies long-range entanglement and, consequently, topological order.

Every Wilson line can be expressed in terms of a product of TD operators for these system sizes. Consider a membrane that creates a local excitation at the site i, M_i^a. According to the correspondence shown in Fig. 2, we can construct from such a membrane an operator involving the product of $\mathcal{O}$'s, whose result is an isolated X at the site i and a set of Z's in neighboring planes above and below X. The product of this structure along the z direction annihilates the Z's so that we end up with a Wilson line, $W_i^a = \prod_{m\in z}\prod_{p\in\mathcal{M}_i^a}\mathcal{O}_p^m$ [55]. Acting on the ground state all these operators become the identity. These are trivial symmetry operators in the sense that they involve only operators that are present in the Hamiltonian.

If we cut open the system along the xy plane at fixed coordinates $z = z_1$ and $z = z_2$, the TD operators at the cut planes will split in two and become boundary operators, $\mathcal{O} \to \mathcal{B}^{z_1}\,\mathcal{B}^{z_2}$. In this case, Wilson lines need to be attached to boundary operators,

$$W_i^1 = \underbrace{\left(\prod_{m\in z\neq z_0}\ \prod_{p\in\mathcal{M}_i^1}\mathcal{O}_p^m\right)}_{\text{bulk}}\underbrace{\left(\prod_{p\in\mathcal{M}_i^1} Z_p^{L-\frac{a_0}{2}}\,(XXX)_p^L\right)}_{\text{boundary at } z_1}\underbrace{\left(\prod_{p\in\mathcal{M}_i^1}(XXX)_p^{z_0}\,Z_p^{z_0-\frac{a_0}{2}}\right)}_{\text{boundary at } z_2} \tag{15}$$

and

$$W_i^2 = \underbrace{\left(\prod_{m\in z-a_0/2}\ \prod_{p\in\mathcal{M}_i^2}\mathcal{O}_p^m\right)}_{\text{bulk}}\underbrace{\left(\prod_{p\in\mathcal{M}_i^2} Z_p^{L}\right)}_{\text{boundary at } z_1}\underbrace{\left(\prod_{p\in\mathcal{M}_i^2} Z_p^{z_0}\right)}_{\text{boundary at } z_2}, \tag{16}$$

for the lines in the two sublattices. These symmetry operators are no longer constrained to act as the identity on the ground state because the presence of the boundary operators.

This leads to the existence of protected edge modes as long as the extended symmetries W are preserved [60]. While the Wilson lines (15) and (16) involving both bulk and boundary operators do commute among themselves, there is a nontrivial algebra among boundary operators. In other words, while the symmetries are realized exactly in the whole system (bulk+boundary), they are anomalous when considering only the boundary. Consider, for example, the boundary symmetry operators at $z = z_0$, $\mathcal{B}_i^1 \equiv \prod_{p\in\mathcal{M}_i^1} Z_p^{z_0-\frac{a}{2}}(XXX)_p^{z_0}$, and $\mathcal{B}_i^2 \equiv Z_i^{z_0}$. They satisfy $\mathcal{B}_i^1\,\mathcal{B}_j^2 = (-1)^{\delta_{ij}}\mathcal{B}_j^2\,\mathcal{B}_i^1$, which leads to a nontrivial degeneracy of the boundary modes. Through the application of TD operators, these boundary operators can be stretched into the bulk, so that the nontrivial algebra flows to the bulk (anomaly inflow).

VIII. CONCLUSIONS

We have reported on a model that presents exotic topological order due to an intricate interplay between fractal subsystem symmetries and the lattice size. The fractal subsystem symmetries are quite sensitive to the lattice size and exist only for linear sizes of the form $L = k(2^n - 2^m)$, with integers $k \geqslant 1$, $n > 2$, and $m \geqslant 0$, satisfying $n > m + 1$. The fractal symmetries possess mixed 't Hooft anomalies with the line subsystem symmetries, which lead to a nontrivial ground state degeneracy. This amounts to the spontaneous breaking of the fractal symmetry. For the remaining sizes, the fractal symmetry is explicitly broken and there are no fractal membrane operators that commute with the Hamiltonian. Consequently, there are no mixed 't Hooft anomalies and the ground state is unique. Despite the unique global anyonic superselection sector, such cases are topologically ordered, since the ground state is still long-range entangled.

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