Hidden quasilocal charges and Gibbs ensemble in a Lindblad system

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(Received 11 May 2023; revised 24 January 2024; accepted 30 January 2024; published 23 February 2024)

We consider spin-1/2 chains with external driving that breaks the continuous symmetries of the Hamiltonian. We introduce a family of models described by the Lindblad equation with local jump operators. The models have hidden strong symmetries in the form of quasilocal charges, leading to multiple nonequilibrium steady states. We compute them exactly in the form of matrix-product operators and argue that they are the analogues of quantum many-body scars in the Lindbladian setting. We observe that the dynamics leads to the emergence of a Gibbs ensemble constructed from the hidden charges.

DOI: 10.1103/PhysRevB.109.054311

I. INTRODUCTION

If a small physical system makes contact with a much larger system (the bath), which is itself in thermal equilibrium, then the interaction with the bath will typically induce thermalization of the small system: in the long-time limit, all details of its initial state will be washed away and its emerging steady state will be determined by the thermodynamical state functions of the bath [1]. This is a general phenomenon in both the classical and in the quantum world, and it is essential for the formulation of statistical physics and thermodynamics.

A similar phenomenon also happens in situations with external driving [2,3]. Typically, there is a unique steady state whose properties depend only on the parameters of the driving, and all properties of the initial states are eventually lost during time evolution. Quantum many-body systems with driving (or simply in contact with their environment) can often be described by the Lindblad equation [4], and generic Lindblad systems have a unique nonequilibrium steady state (NESS) [5].

Models with a nonunique NESS are exceptional: they conserve additional information about the initial state [6]. They are analogous to isolated systems with ergodicity breaking, which have been well studied in the last two decades. Today, various mechanisms leading to ergodicity breaking are known [7–9] and all of them are associated with exotic symmetries of the system.

We focus on the following question: What are possible ways to have multiple NESS in a many-body Lindblad system? Similar to ergodicity breaking, nonuniqueness of the NESS is associated with the presence of extra conservation laws. In Lindblad systems, conserved quantities can be constructed if the model has so-called strong symmetries [10–13].

In this work, we uncover a different mechanism leading to unexpected degenerate NESS in a Lindblad system. We introduce a model with a local Hamiltonian and local jump operators in the bulk, which break the standard U(1) symmetry of the Hamiltonian. Nevertheless, we find hidden strong symmetries in the form of quasilocal charges: extensive operators with a quasilocal operator density. Previously, such operators were treated in the context of the generalized Gibbs ensemble [14,15], but our work uncovers quasilocal charges in a Lindblad system with local driving in the bulk.

We also find explicit and exact formulas for the degenerate NESS in our model: we present them as matrix-product operators (MPOs) with fixed bond dimension. We argue that they are analogous to the quantum many-body scars known from Hermitian systems [9,16]. We also consider time evolution from selected initial states and rigorously compute the steady-state values of selected observables, thereby proving that the system retains memory of the initial state. Furthermore, we show that in the infinite volume limit, the emerging steady states can also be described by a Gibbs ensemble constructed from the hidden quasilocal charge.

II. LINDBLAD SYSTEMS

We consider the dynamics of a quantum spin-1/2 chain in contact with its environment. If the environment is Markovian, the time evolution of the density matrix ρ of the system can be described by the Lindblad equation, which reads

$$\dot{\rho} = i[\rho, H] + \sum_a u_a \left[\ell_a \rho \ell_a^\dagger - \frac{1}{2}\{\ell_a^\dagger \ell_a, \rho\} \right], \tag{1}$$

and, equivalently, in the superoperator formalism $\dot{\rho} = \mathcal{L}\rho$, where $\mathcal{L}$ is the so-called Lindblad superoperator [17,18].

Here, H is the Hamiltonian of the system and ℓ_a are the jump operators, which describe processes mediated by the environment. The parameters $u_a \in \mathbb{R}^+$ are coupling constants and the index a labels the various jump operators.

We are interested in models where the jump operators are localized in real space and the system is translationally invariant. Furthermore, we consider periodic boundary conditions and one family of jump operators in the bulk. In such a case, $u_a \equiv U$ with a uniform coupling U and $\ell_a \equiv \ell(j)$ is a fixed short-range operator localized around the site j.

A. Symmetries and NESS

In a Lindblad system, the nonequilibrium steady states (NESS) are the density matrices ρ which emerge in the long-time limit, and they satisfy $\mathcal{L}\rho = 0$. In a generic Lindblad system without symmetries, there is a unique NESS, but counterexamples are also known [5,12]. In such exceptional cases, the system preserves memory of the initial state because different initial density matrices evolve to different NESS in the long-time limit. One of the possible ways to have nonunique NESS is to have conservation laws in the model because different initial mean values of the conserved quantity necessarily lead to multiple NESS.

Conservation laws are typically associated with symmetries. In Hermitian quantum mechanics, symmetries are represented by linear operators which commute with the Hamiltonian, and every symmetry automatically leads to a conservation law for an observable quantity. The situation is very different in the non-Hermitian setting of the Lindblad equation [11]. In these systems, a symmetry operation might or might not lead to a conserved quantity, and not all conserved quantities originate in symmetries.

However, there is a direct connection in the case of a "strong symmetry." We say that an operator Q is a strong symmetry of a Lindblad system if Q commutes with the Hamiltonian H and all jump operators individually. In this case, $\mathcal{L}^\dagger Q = \mathcal{L}Q = 0$ and thus Q is also a NESS. Of special interest are those strong symmetries which are represented by *extensive operators*, i.e., $Q = \sum_j q(j)$, where $q(j)$ is the operator density of the conserved charge.

B. The Hubbard Lindbladian

An example for a Lindblad system with such a strong symmetry was considered in [19]. Using the notation X_j, Y_j, Z_j for the Pauli matrices acting on site j of the spin chain, we can write the Hamiltonian and the jump operators of the model of [19] as

$$H = \sum_j X_j X_{j+1} + Y_j Y_{j+1}, \quad \ell(j) = Z_j. \tag{2}$$

The system is homogeneous with a global coupling constant U.

Here the Hamiltonian describes the so-called *XX* model, while the jump operators describe local dephasing effects. Substituting (2) into (1), the resulting Lindblad superoperator can be seen as the Hubbard model with imaginary coupling constant [19], which implies that the superoperator is Yang-Baxter integrable, and the Lindblad superoperator can be diagonalized using the Bethe ansatz technique.

This model has an extensive strong symmetry given by

$$Q_0 = \sum_j Z_j, \tag{3}$$

which is the global magnetization. Accordingly, in this model, the NESS is not unique and, in a finite volume L, the null space of the superoperator $\mathcal{L}$ is $L+1$ dimensional. Representative NESS can be chosen as the $L+1$ projectors P_N to the different sectors of the Hilbert space with a given total magnetization N. Alternatively, an overcomplete basis for the null space can be chosen as

$$\rho(\alpha) \sim e^{\alpha Q_0} = \prod_j e^{\alpha Z_j}, \quad \alpha \in \mathbb{R}. \tag{4}$$

These density matrices are linear combinations of P_N. They are product operators in real space: their operator space entanglement is zero.

III. PRESENTATION OF THE MODEL

We consider a deformation of the model given by (2). In our case, the Hamiltonian is

$$H = \sum_j X_j Y_{j+1} - Y_j X_{j+1}, \tag{5}$$

which is known as the Dzyaloshinskii-Moriya interaction term. It can be related to the *XX* Hamiltonian (2) by applying a homogeneous twist along the chain [20]. We have a global coupling constant U, and the jump operators are given by

$$\ell(j) = \frac{1}{1+\gamma^2}[Z_{j+1} + \gamma(X_j + X_{j+2})X_{j+1} - \gamma^2 X_j Z_{j+1} X_{j+2}], \tag{6}$$

where $\gamma \in \mathbb{R}$ is seen as a deformation parameter, such that $\gamma = 0$ describes the original model (2) treated in [19]. The jump operator (6) acts nontrivially on three neighboring sites and simple computation shows that it satisfies the special relations

$$[\ell(j)]^\dagger = \ell(j), \quad [\ell(j)]^2 = 1, \tag{7}$$

neighboring jump operators do not commute, but $[\ell(j), \ell(k)] = 0$ if $|j-k| \geqslant 2$.

For simplicity, we consider the regime $0 < \gamma < 1$ throughout this paper. Other regimes can be treated by special similarity and duality transformations. Furthermore, the points $\gamma = \pm 1$ require special care due to extra U(1) charges, which enlarge the null space of the Lindbladian. The other regimes and the special points deserve a separate study.

The model can also be formulated in terms of fermion operators, following the usual Jordan-Wigner transformation [21]. Introducing the Majorana operators $\psi_{2j-1} = X_j \prod_{l<j} Z_l$, $\psi_{2j} = Y_j \prod_{l<j} Z_l$, which satisfy $\{\psi_a, \psi_b\} = 2\delta_{a,b}$, we have

$$H = \sum_k \psi_{k-1}\psi_{k+1}, \tag{8}$$

where the sum is now over twice the number of sites of the original spin model. Considering the spin chain defined on L sites with periodic boundary conditions translates, in the Majorana language, into $\psi_{L+k} = \mathcal{Z}\psi_k$, where $\mathcal{Z} \equiv (-1)^F \equiv \prod_j Z_j$ is the fermion number parity. The jump operators take the form

$$\ell(j) = \frac{i}{1+\gamma^2}(\psi_{2j+2} - \gamma\psi_{2j})(\psi_{2j+1} - \gamma\psi_{2j+3}). \tag{9}$$

The jump operators break the U(1) symmetry of the original model: they induce particle creation and annihilation, but due to conservation of $\mathcal{Z}$, the creation and annihilation happen in pairs.

While the Hamiltonian (8) is bilinear in terms of the Majorana operators and can therefore be diagonalized using free-fermion techniques [21], the jump operators (9) introduce quartic terms in the Lindblad equation (1), and our model is therefore truly interacting.

A. Integrability properties

The work in Ref. [19] initiated the study of integrable Lindbladians: these are models where the superoperator is one of the conserved charges on an integrable model and it originates from solutions of the Yang-Baxter equation. Recently, a systematic search was initiated to find integrable Lindbladians by using the boost operator [22] (see also [23]), and the present model was discovered with the same method. We refer to Appendix A and Ref. [24] for a review of the method. The model given by (2) can be related to the Hubbard model, whereas our Lindblad superoperator is related to the deformation of the Hubbard model treated in the recent work [25]. Therefore, our model is also Yang-Baxter integrable. Interestingly, the derivations below do not make use of this property. They will, however, make use of the "superintegrability" property of the Hamiltonian (5), namely, the fact that it allows for non-Abelian families of conserved charges, which commute with H but not necessarily with one another [20,26,27] (see Appendix B 1 for a detailed discussion).

B. Construction of the NESS space

We find that our Lindbladian possesses a null space which is $L+1$ dimensional in a finite volume L. The existence of the degenerate NESS is explained by an unexpected strong symmetry in the system. This symmetry and the associated conserved charge are obtained from the original Q_0 (3) of the undeformed model via a nonlocal transformation, which is performed by a matrix-product operator (MPO).

More specifically, let us define the MPO $T(\gamma)$ as

$$T(\gamma) = \mathrm{Tr}_{\mathcal{A}}[A_L(\gamma)A_{L-1}(\gamma)\dots A_1(\gamma)]. \quad (10)$$

Here, $\mathcal{A}$ is a two-dimensional ancillary space, and the tensor $A(\gamma)$ is written with respect to this space as

$$A_j(\gamma) = \frac{1}{2}\begin{pmatrix} g^- + g^+ Z_j & g^+ X_j - i g^- Y_j \\ g^- X_j + i g^+ Y_j & g^+ - g^- Z_j \end{pmatrix}, \quad (11)$$

where $g^\pm = \sqrt{1\pm\gamma}$. The operators $T(\gamma)$ form a mutually commuting family, namely, $[T(\gamma), T(\gamma')] = 0$: In Appendix B 2, we show that they can be recast as a series expansion in powers of γ, whose coefficients are expressed in terms of a family of mutually commuting conserved charges of H. It will be crucial for the following to note that while these charges [and hence the operators $T(\gamma)$] commute with one another, they do not commute with the U(1) charge Q_0, a sign of the superintegrability of the Hamiltonian H. We further show that the operators $T(\gamma)$ and $T(\gamma)^\dagger$ obey the property

$$T(\gamma)T(\gamma)^\dagger = T(\gamma)^\dagger T(\gamma) = 1 + \gamma^L \mathcal{Z}. \quad (12)$$

Hence, in the $L\to\infty$ limit, they become the inverse of each other.

The operators $T(\gamma)$ can be used to relate our model to another family of Lindbladians, which have the original U(1) charge Q_0 as a strong symmetry. In Appendix B 3, we prove the following identity:

$$T(\gamma)\ell(j)T(\gamma)^\dagger = (1+\gamma^L\mathcal{Z})\tilde{\ell}(j), \quad (13)$$

or, equivalently,

$$T(\gamma)\ell(j)T(\gamma)^{-1} = \tilde{\ell}(j), \quad (14)$$

where

$$\tilde{\ell}(j) \equiv \frac{1}{1+\gamma^2}[Z_{j+2} + \gamma(X_{j+1}X_{j+2} + Y_{j+1}Y_{j+2}) + \gamma^2 Z_{j+1}]. \quad (15)$$

The modified jump operators $\tilde{\ell}(j)$ act nontrivially on two neighboring sites and all square to one and commute with the global charge $Q_0 = \sum_j Z_j$. Since, furthermore, $T(\gamma)HT(\gamma)^{-1} = H$, we can readily conclude that all powers of Q_0 or, equivalently, all exponentials of the form $e^{\alpha Q_0}$ are (unnormalized) NESS of the Lindbladian defined from the Hamiltonian H and the jump operators $\tilde{\ell}(j)$. These form a basis for a $(L+1)$-dimensional space, including the identity.

Conversely, we now define the deformation of Q_0 as

$$Q_\gamma = T(\gamma)^\dagger Q_0 T(\gamma), \quad (16)$$

which, in the $L\to\infty$ limit, corresponds to a conjugation relation. This conjugation can be understood as a quasilocal deformation of Q_0, involving the non-Abelian conserved charges of the Hamiltonian (5). Q_γ remains an extensive operator, but its operator density $q_\gamma(j) = T(\gamma)^\dagger Z_j T(\gamma)$ becomes quasilocal; details are given in Appendix B 2.

From the preceding discussion, it is clear that the operator Q_γ is a strong symmetry of the Lindbladian: it commutes with the Hamiltonian (5) and also with the jump operators (6). This implies that it is a conserved charge for the Lindbladian time evolution. More generally, the matrices

$$\rho_\gamma(\alpha) = T(\gamma)^\dagger e^{\alpha Q_0} T(\gamma) = T(\gamma)^\dagger \left[\prod_j e^{\alpha Z_j}\right] T(\gamma) \quad (17)$$

are (unnormalized) density matrices: they are Hermitian and positive definite. They are also strong symmetries. It follows that the matrices $\rho_\gamma(\alpha)$, $\alpha\in\mathbb{R}$ are NESS of the Lindbladian with fixed deformation parameter γ and arbitrary coupling strength U. Alternatively, we could consider the density matrices $\tilde{\rho}_\gamma(\alpha) = T(\gamma)^{-1}e^{\alpha Q_0}T(\gamma)$, which coincide with (17) up to corrections of the order of γ^L.

The operators $\rho_\gamma(\alpha)$, $\alpha\in\mathbb{R}$ form an overcomplete basis for the null space of the Lindbladian, which has dimension $L+1$ in a finite volume L. This can be proven by expanding $\rho_\gamma(\alpha)$ into a power series in α: this produces the powers of Q_γ (up to corrections of the order γ^L), which (together with the identity) span a space of dimension $L+1$.

Steady states in MPO form have been found earlier in multiple instances in the literature (for systems with boundary driving, see, for example, [12,28–30]). Our results are unique because we treat a system locally driven in the bulk, and the bond dimension of the MPO is a fixed small number.

It is also worth emphasizing that the transformation (14) maps between two *local* families of jump operators, $\ell(j)$ and $\tilde{\ell}(j)$, where, as mentioned, $\ell(j)$ is acting nontrivially on three neighboring site of the spin chain, while $\tilde{\ell}(j)$ on two adjacent sites. Such a property is highly nontrivial and not true for generic operators: for instance, the local densities Z_j of the charge Q_0 are mapped onto *quasilocal* densities $q_\gamma(j)$.

IV. FRUSTRATION-FREE PROPERTY AND LINDBLADIAN SCARS

The density matrices $\rho_\gamma(\alpha)$ can be written as an MPO with bond dimension 4. Therefore, their operator space entanglement satisfies an area law. Interestingly, the $\rho_\gamma(\alpha)$ are related to frustration-free Hamiltonians.

To see this, we define an auxiliary Hermitian superoperator M, which acts on any ρ as

$$M\rho = \sum_j \ell(j)\rho\ell^\dagger(j). \tag{18}$$

In our case, the strong symmetry and the relations (7) imply that $\rho_\gamma(\alpha)$ are eigenvectors of M with eigenvalue L, and that is the maximal possible eigenvalue of M. By definition, this means that the superoperator M is frustration free.

A related model with the frustration-free property was investigated in [31] (see also [32,33]). Their Hamiltonian acts on the spin-1/2 Hilbert space and it can be written as

$$K = \sum_j \ell(j). \tag{19}$$

It has two extremal states $|\Psi_\pm\rangle$ satisfying the frustration-free condition, $\ell(j)|\Psi_\pm\rangle = \pm|\Psi_\pm\rangle$. It follows that the density matrices $\rho_\pm = |\Psi_\pm\rangle\langle\Psi_\pm|$ are frustration-free eigenstates of M. Furthermore, they are NESS for our Lindbladian and they are reproduced by $\rho_\gamma(\alpha)$ in the $\alpha \to \pm\infty$ limit. Our procedure to obtain the density matrices $\rho_\gamma(\alpha)$ can be seen as a generalization of the methods of [31] to the Lindbladian setting.

After renormalization and shifting by a matrix proportional to the identity, the action of the full superoperator can be written as

$$\tilde{\mathcal{L}}\rho \equiv (U^{-1}\mathcal{L} + L)\rho = M\rho + iU^{-1}[\rho, H]. \tag{20}$$

The superoperator $\tilde{\mathcal{L}}$ becomes Hermitian [34] for $U = iu$, $u \in \mathbb{R}$. In such a case $\rho_\gamma(\alpha)$ are still eigenoperators of $\tilde{\mathcal{L}}$, they have low spatial entanglement, and they are in the middle of the spectrum for a generic real u. Therefore, they can be seen as quantum many-body scars of $\tilde{\mathcal{L}}$ [9,16]. In light of the rescaling (20), we suggest to call the ρ eigenoperators *Lindbladian scars* for our original superoperator $\mathcal{L}$ [35]. In the standard scenario, a scar is a state of the Hilbert space which breaks ergodicity, but in the Lindbladian setting, it is very natural to see the density matrix corresponding to the NESS as a scar in the Hermitian setting.

V. MEAN VALUES

The physical properties of $\rho_\gamma(\alpha)$ can be demonstrated by computing the mean values of local observables in these states, which can be done using standard MPO techniques; see Appendix C. First we compute the mean value of the local operator Z placed at any site j. We find

$$\langle Z_j\rangle = \frac{\mathrm{Tr}[\rho_\gamma(\alpha)Z_j]}{\mathrm{Tr}\rho_\gamma(\alpha)} = \frac{(1-\gamma^2)\tanh(\alpha)}{[\gamma\tanh(\alpha)]^L + 1}. \tag{21}$$

In the large volume limit, this gives

$$\langle Z_j\rangle_{L\to\infty} = (1-\gamma^2)\tanh(\alpha). \tag{22}$$

In the undeformed model ($\gamma = 0$), the mean value is $\tanh(\alpha)$, and thus the transformation (17) decreases the mean value by a factor that depends only on γ.

It is also useful to consider a measure for the breaking of the standard U(1) symmetry. We choose the two-site operator

$$X_jX_{j+1} - Y_jY_{j+1} = 2(\sigma_j^+\sigma_{j+1}^+ + \sigma_j^-\sigma_{j+1}^-), \tag{23}$$

which is sensitive to the creation/annihilation of pairs of particles. For the mean value, we find

$$\frac{[\gamma\tanh(\alpha)]^{L-1} - \gamma(\gamma^2-2)\tanh(\alpha)}{[\gamma\tanh(\alpha)]^L + 1}. \tag{24}$$

The infinite volume limit becomes

$$\langle X_jX_{j+1} - Y_jY_{j+1}\rangle_{L\to\infty} = \gamma(2-\gamma^2)\tanh(\alpha). \tag{25}$$

Having a nonzero mean value for the deformed model is a clear sign of the breaking of the original U(1) symmetry.

VI. DYNAMICS AND GIBBS ENSEMBLE

We consider real-time evolution from selected initial states, focusing on

$$\rho(t=0) = \rho_0(\beta) \equiv \frac{e^{\beta Q_0}}{(2\cosh\beta)^L}. \tag{26}$$

These are steady states of the undeformed model ($\gamma = 0$). They are product operators in real space and, in the limit $\beta \to \pm\infty$, they also include pure states obtained from the reference states with all spins up/down.

Due to the conserved charge Q_γ, we expect that the system has memory: the long-time limit of the observables will depend on the initial state. On the other hand, since the emerging NESS are strong symmetries of the model, they are independent from the coupling U, and therefore we expect that U influences only the speed of convergence towards them. This is confirmed by numerical computation of the real-time dynamics for small volumes, with the results presented in Fig. 1.

It is important to clarify the nature of the emerging steady states. Our Lindblad system has a single extensive conserved charge Q_γ. In analogy with thermalization in an isolated system, we postulate that in large volumes, the emerging steady states can be described by a Gibbs ensemble of the form

$$\rho_{\mathrm{G}} \sim e^{-\lambda Q_\gamma}, \tag{27}$$

such that λ should be determined by the initial mean value,

$$\mathrm{Tr}(\rho_{\mathrm{G}}Q_\gamma) = \mathrm{Tr}(\rho_0 Q_\gamma). \tag{28}$$

For the initial density matrices (26), this computation can be performed easily in the infinite volume limit, yielding $\tanh(\lambda) = -(1-\gamma^2)\tanh(\beta)$. This result can be used to compute mean values of local observables in the Gibbs ensemble. We obtain, for example, the prediction

$$\lim_{t\to\infty}\langle Z_j(t)\rangle = \mathrm{Tr}(\rho_{\mathrm{G}}Z_j) = (1-\gamma^2)^2\tanh(\beta). \tag{29}$$

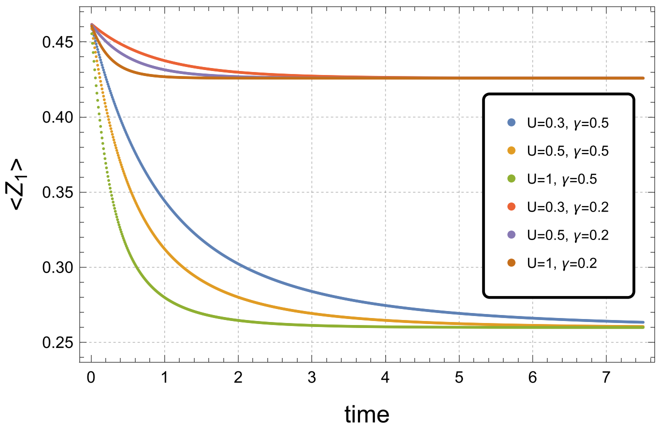

FIG. 1. Time evolution of $\langle Z_1(t)\rangle$ from a selected initial density matrix $\rho_0(\beta)$ (26) with $\beta = 0.5$, in a finite volume $L = 7$. We choose two different deformation parameters γ and three coupling strengths U. It is seen that the asymptotic values depend only on γ and not on U, which influences only the speed of convergence. The asymptotic values agree with those predicted by the exact formula (30); therefore, they also confirm our postulate about the emergence of the Gibbs ensemble.

Remarkably, we also performed an exact finite volume computation to find the asymptotic mean values. Details are given in Appendix C. For the observable Z_j, we find

$$\lim_{t\to\infty}\langle Z_j\rangle = \frac{(\gamma^2-1)^2\tanh\beta(1-2\gamma^L\tanh^{L-2}\beta+\gamma^{2L})}{(1-\gamma^{2L})^2}. \tag{30}$$

These values are confirmed by the numerics at finite L. Furthermore, it is easy to take the large volume limit, and for $0<\gamma<1$, we always recover (29), thus also confirming our postulate about the Gibbs ensemble.

VII. CONCLUSIONS

We demonstrated that a Lindblad system with local jump operators can have quasilocal symmetries, crucially affecting the real-time dynamics. The steady states of our model were obtained from those of the "Hubbard Lindbladian" after a similarity transformation with an MPO. Surprisingly, this similarity transformation is compatible with local jump operators, in particular the jump operators of range three get mapped to the jump operator of range two. Curiously, we did not use the integrability of the Lindbladian, but the superintegrability of the Hamiltonian did play a crucial role. Perhaps integrability plays a hidden role in the derivation of (30).

Additional physical properties of the model, such as the Lindbladian gap, could be computed from a full Bethe ansatz solution, which is not yet available. Also, it would be interesting to consider analogous models with discrete-time evolution [36,37]. This would open up the way towards the experimental realization of our findings.

ACKNOWLEDGMENTS

We are thankful for useful discussions with M. Ljubotina, S. Moudgalya, T. Prosen, M. Serbyn, and T. Rakovszky. M.d.L. was supported by SFI, the Royal Society and the EPSRC with funding under Grants No. UF160578, No. RGF/R1/181011, No. RGF/EA/180167, and No. 18/EPSRC/3590. C.P. was supported by Grant No. RGF/EA/180167.

APPENDIX A: INTEGRABLE LINDBLADIANS

Here we review the key statements about integrable Lindbladians, based on our recent work [22]. Consider a Lindblad equation with one family of jump operators in the bulk,

$$\dot{\rho} \equiv \mathcal{L}\rho = i[\rho, H] + \sum_j \left[\ell_j\rho\ell_j^\dagger - \frac{1}{2}\{\ell_j^\dagger\ell_j, \rho\}\right]. \tag{A1}$$

$\mathcal{L}$ is the Lindblad superoperator, which acts on the Hilbert space of density matrices. We assume that $\ell_j = \ell(j)$ are short-range operators localized at site j, and $H = \sum_j h(j)$ is a local nearest-neighbor interacting Hamiltonian. We will first consider the case when $\ell(j)$ acts on two sites only and we identify it as $\ell_{j,j+1}$.

We perform a standard operator-state correspondence: the Hilbert space of the density matrices is related to the space of a spin ladder of the form $\mathcal{H} = \mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}$, where $\mathcal{H}^{(1,2)}$ correspond to the bra/ket "sides" of the density matrices. Then one can transform the superoperator into an operator that acts on a spin ladder. In this way, we obtain a spin ladder (non-Hermitian) superoperator $\mathcal{L} = \sum_j \mathcal{L}_{j,j+1}$ with operator density

$$\mathcal{L}_{j,j+1} = -ih^{(1)}_{j,j+1} + ih^{(2)*}_{j,j+1} + \ell^{(1)}_{j,j+1}\ell^{(2)*}_{j,j+1} - \tfrac{1}{2}\ell^{(1)\dagger}_{j,j+1}\ell^{(1)}_{j,j+1} - \tfrac{1}{2}\ell^{(2)T}_{j,j+1}\ell^{(2)*}_{j,j+1}. \tag{A2}$$

Here, the superscript T denotes transpose and the asterisk denotes the complex conjugation of each of the components of the matrix. For any operator A, the notation $A^{(1)}$ or $A^{(2)}$ means that the operator acts nontrivially only on $\mathcal{H}^{(1)}$ or $\mathcal{H}^{(2)}$.

Now one can apply techniques of integrability to find and classify those Lindbladians, where the spin ladder operator is integrable. For a nearest-neighbor operator $Q^{(2)} = \sum_j q(j)$, the Reshetikhin condition states that if a model is integrable, then there is an extensive operator $Q^{(3)}$ satisfying

$$[Q^{(3)}, Q^{(2)}] = 0 \tag{A3}$$

in every volume L, such that the operator density of $Q^{(3)}$ is

$$q^{(3)}(j) = [q(j), q(j+1)] + \tilde{q}(j), \tag{A4}$$

where $\tilde{q}(j)$ is a two-site operator.

It is conjectured that if $q(j)$ and $\tilde{q}(j)$ are such that the commutativity relation above holds, then the model is Yang-Baxter integrable: it has a commuting set of transfer matrices and it is associated to a so-called R matrix which is a solution to the Yang-Baxter equations. The above condition is cubic in $q(j)$ and linear in $\tilde{q}(j)$ and it can be used to find and classify integrable models.

In our case, we used this condition to find integrable Lindblad superoperators: we identified $Q^{(2)}$ with the ladder operator $\mathcal{L}$ given by (A2). A number of solutions with local U(1) symmetry were presented in [22]. Our current model was found by continuing the classification started in [22], allowing for the breaking of U(1) symmetry. The model was found first in a two-site version, but it was transformed into the three-site formulation using a duality transformation (for details, see [25]).

In our three-site model, the ladder operator still takes the form (A2), with the only modification that now $\ell(j)$ acts on three sites instead of two. Specializing the model to $\gamma = 0$, we obtain $\ell(j) = Z_j$ and the ladder operator becomes identical to the Hubbard model with an imaginary coupling constant [19].

APPENDIX B: CONSTRUCTION OF THE NESS USING THE CONSERVED CHARGES OF THE HAMILTONIAN

In this Appendix, we show how the deformed Lindbladian considered in the main text as well as the corresponding $L+1$ NESS can be constructed as a deformation of the $\gamma = 0$ case (imaginary coupling Hubbard model) using the conserved charges of the Hamiltonian (5) in the main text.

1. Conserved charges of the Hamiltonian

The Hamiltonian (5) commutes with an extensive set of charges which we label as $[ab]_m$, where a and b can take the label X or Y, and for $m \geqslant 0$,

$$[ab]_m \equiv \sum_{j=1}^{L} a_j \left(\prod_{1\leqslant k<m} Z_{j+k} \right) b_{j+m}, \tag{B1}$$

where X_j, Y_j, and Z_j are the Pauli matrices acting on site j of the spin chain.

In the fermionic formulation of the model, detailed in the main text, the corresponding charges are the set of all possible translationally invariant fermion bilinears.

An extensive set of local charges usually signals integrability, and indeed the Hamiltonian (5) can be related to the well-known integrable *XX* Hamiltonian [see Eq. (2) in the main text] by a homogeneous twist along the chain. It is, in fact, *superintegrable*, as the charges $[ab]_m$ form various families which, in turn, do not commute with one another. For instance, the sets of charges $\{[XY]_m\}$ and $\{[YX]_n\}$ commute with one another, but only the combinations $\{[XY]_m - [YX]_m\}$ commute with the charges $\{[XX]_n\}$ or $\{[YY]_n\}$. We also introduce the charge

$$\mathcal{Z} = \prod_{j=1}^{L} Z_j, \tag{B2}$$

which commutes with the Hamiltonian as well as with all the charges $[ab]_m$.

2. The operator $T(\gamma)$

As usual when dealing with quantum integrable models, families of mutually conserved charges can be generated by a matrix-product operator (MPO) called the transfer matrix. Introduce the following MPO:

$$T(\gamma) = \mathrm{Tr}_{\mathcal{A}}[A_L(\gamma)A_{L-1}(\gamma)\dots A_1(\gamma)], \tag{B3}$$

where the ancillary space $\mathcal{A}$ has dimension 2, and where the matrices $A_j(\gamma)$ are defined as

$$A_j(\gamma) = \begin{pmatrix} \frac{\sqrt{1-\gamma}+\sqrt{1+\gamma}Z_j}{2} & \frac{\sqrt{1+\gamma}X_j - i\sqrt{1-\gamma}Y_j}{2} \\ \frac{\sqrt{1-\gamma}X_j + i\sqrt{1+\gamma}Y_j}{2} & \frac{\sqrt{1+\gamma}-\sqrt{1-\gamma}Z_j}{2} \end{pmatrix}. \tag{B4}$$

The matrices $T(\gamma)$ commute with one another for different γ, as can be traced back to the known integrability properties of the *XX* chain [more precisely, they correspond to transfer matrices based on cyclic representations of the quantum group $U_q(sl_2)$ at $q = i$], and admit the following series expansion around $\gamma = 0$:

$$T(\gamma) = \mathcal{U} \exp[\mathcal{G}(\gamma)], \tag{B5}$$

where $\mathcal{U}$ is the one-site discrete translation operator, and

$$\mathcal{G}(\gamma) = i \sum_{m\geqslant 1} \frac{\gamma^m}{2m} [YX]_m. \tag{B6}$$

We emphasize that the expansion (B6) holds at all orders, even for a system for finite size L, as can be checked by explicitly computing the successive logarithmic derivatives of $T(\gamma)$ at $\gamma = 0$. For $L \to \infty$, the series (B6) defines a quasilocal operator for $|\gamma| < 1$. For finite L, it can be further rearranged using the properties $[YX]_{m+L} = -\mathcal{Z}[YX]_m$ for $m \geqslant 1$, and $[YX]_L = -iL\mathcal{Z}$. A practical expression is

$$\mathcal{G}(\gamma) = \frac{1}{2}\ln(1+\gamma^L \mathcal{Z}) + i \sum_{\substack{m\geqslant 1 \\ m\notin L\mathbb{Z}}} \frac{\gamma^m}{2m}[YX]_m, \tag{B7}$$

which splits between a first term, which is Hermitian, and an anti-Hermitian part. From there, see, in particular,

$$T(\gamma)T(\gamma)^\dagger = 1 + \gamma^L \mathcal{Z}, \tag{B8}$$

or, equivalently,

$$T(\gamma)^{-1} = \frac{1-\gamma^L\mathcal{Z}}{1-\gamma^{2L}} T(\gamma)^\dagger. \tag{B9}$$

The result of Eq. (B8) can be seen directly in the MPO formalism. We can write $T(\gamma)T(\gamma)^\dagger$ as a MPO of bond dimension 4, with ancillary space $\mathcal{A}\otimes\mathcal{A}$, namely,

$$T(\gamma)T(\gamma)^\dagger = \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{A}}[\mathcal{M}_L(\gamma)\mathcal{M}_{L-1}(\gamma)\dots\mathcal{M}_1(\gamma)], \tag{B10}$$

where the $\mathcal{M}_j(\gamma)$ are 4×4 matrices with entries expressed in terms of X_j, Y_j, Z_j. The MPO is invariant under any change of basis performed in the ancillary space. Defining $V = e^{\frac{i\pi}{4} Y\otimes X}$, where X and Y are now Pauli matrices acting in each copy of the ancillary space $\mathcal{A}$, Eq. (B10) can therefore be recovered by replacing the matrices $\mathcal{M}_j(\gamma)$ with $V\mathcal{M}_j(\gamma)V^{-1}$, which take the form

$$V\mathcal{M}_j(\gamma)V^{-1} = \begin{pmatrix} 1 & \sqrt{1-\gamma^2}X_j & -iY_j & -\sqrt{1-\gamma^2}Z_j \\ 0 & \gamma Z_j & 0 & \gamma X_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B11}$$

From the block-diagonal form of (B11), it is clear that after taking the trace in (B10), only the two diagonal terms contribute, which give rise to the two terms in (B8).

3. Construction of the NESS

As we will now see, the transfer matrix $T(\gamma)$ can be used to construct the jump operators and NESS described in the main text. Let us study the transformation of the jump operators $\ell(j)$ [defined in Eq. (2) of the main text] under conjugation by $T(\gamma)$. For this, it will be useful to introduce the following MPOs:

$$T(\gamma)B_jT(\gamma)^\dagger = \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{A}}\big[\mathcal{M}_L(\gamma)\dots\mathcal{M}^B_j(\gamma)\dots\mathcal{M}_1(\gamma)\big], \tag{B12}$$

where $B\in\{X,Y,Z\}$. We find, similarly,

$$V\mathcal{M}^X_j(\gamma)V^{-1} = \begin{pmatrix} 0 & \gamma Z_j & 0 & \gamma X_j \\ 1 & \sqrt{1-\gamma^2}X_j & -iY_j & -\sqrt{1-\gamma^2}Z_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B13}$$

$$V\mathcal{M}^Y_j(\gamma)V^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sqrt{1-\gamma^2} & iX_j & \sqrt{1-\gamma^2}Y_j & -iZ_j \\ \gamma Y_j & 0 & -i\gamma & 0 \end{pmatrix}, \tag{B14}$$

$$V\mathcal{M}^Z_j(\gamma)V^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\gamma Y_j & 0 & \gamma & 0 \\ -\sqrt{1-\gamma^2} & -X_j & i\sqrt{1-\gamma^2}Y_j & Z_j \end{pmatrix}. \tag{B15}$$

It will also be useful, for practical calculations, to introduce $\mathcal{M}^{(\alpha)}_j = \cosh\alpha\mathcal{M}_j + \sinh\alpha\mathcal{M}^Z_j$. After rotation, we have, similarly,

$$V\mathcal{M}^{(\alpha)}_j(\gamma)V^{-1} = \begin{pmatrix} \cosh\alpha & \cosh\alpha\sqrt{1-\gamma^2}X_j & -i\cosh\alpha Y_j & -\cosh\alpha\sqrt{1-\gamma^2}Z_j \\ 0 & \gamma\cosh\alpha Z_j & 0 & \gamma\cosh\alpha X_j \\ i\gamma\sinh\alpha Y_j & 0 & \gamma\sinh\alpha & 0 \\ -\sinh\alpha\sqrt{1-\gamma^2} & -\sinh\alpha X_j & i\sinh\alpha\sqrt{1-\gamma^2}Y_j & \sinh\alpha Z_j \end{pmatrix}. \tag{B16}$$

For any three consecutive sites $j, j+1, j+2$, we then have

$$T(\gamma)\ell(j)T(\gamma)^\dagger = \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{A}}[\mathcal{M}_L(\gamma)\dots\mathcal{M}^\ell_{j,j+1,j+2}(\gamma)\dots\mathcal{M}_1(\gamma)], \tag{B17}$$

where

$$\begin{aligned}\mathcal{M}^\ell_{j,j+1,j+2}(\gamma) \equiv \frac{1}{1+\gamma^2}\big[&\mathcal{M}_{j+2}\mathcal{M}^Z_{j+1}\mathcal{M}_j \\ &+\gamma\big(\mathcal{M}^X_{j+2}\mathcal{M}^X_{j+1}\mathcal{M}_j \\ &+\mathcal{M}_{j+2}\mathcal{M}^X_{j+1}\mathcal{M}^X_j\big) \\ &-\gamma^2\mathcal{M}^X_{j+2}\mathcal{M}^Z_{j+1}\mathcal{M}^X_j\big],\end{aligned} \tag{B18}$$

which can be brought to the following form after rotation in ancillary space:

$$V\mathcal{M}^\ell_{j,j+1,j+2}(\gamma)V^{-1} = \begin{pmatrix} \tilde{\ell}(j) & \dots & \dots & \dots \\ 0 & \gamma^3 Z_jZ_{j+1}Z_{j+2}\tilde{\ell}(j) & 0 & \dots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B19}$$

Here we have defined

$$\tilde{\ell}(j) \equiv \frac{1}{1+\gamma^2}[Z_{j+2} + \gamma(X_{j+1}X_{j+2} + Y_{j+1}Y_{j+2}) + \gamma^2 Z_{j+1}], \tag{B20}$$

and the … denote other combinations of the Pauli matrices, which we will not need to consider. Indeed, from the triangular structure of (B19), we see again that only the two nonzero diagonal entries give a nonzero contribution to the trace (B17). As a result, we find

$$T(\gamma)\ell(j)T(\gamma)^\dagger = (1+\gamma^L\mathcal{Z})\tilde{\ell}(j), \tag{B21}$$

or, equivalently,

$$T(\gamma)\ell(j)T(\gamma)^{-1} = \tilde{\ell}(j). \tag{B22}$$

The modified jump operators $\tilde{\ell}(j)$ all square to one and commute with the global charge $Q_0 = \sum_j Z_j$. Since, furthermore, $T(\gamma)HT(\gamma)^{-1} = H$, we can readily conclude that all powers of Q_0 or, equivalently, all exponentials of the form $e^{\alpha Q_0}$ are (unnormalized) NESS of the Lindbladian defined from the Hamiltonian H and the jump operators $\tilde{\ell}(j)$. These form a basis for a $(L+1)$-dimensional space, including the identity.

Undoing the similarity transformation, this shows that the matrices $T(\gamma)^{-1}e^{\alpha Q_0}T(\gamma)$ are (unnormalized) NESS for the Lindbladian constructed out of the Hamiltonian H and jump operators $\ell(j)$. Since $\mathcal{Z}$ commutes with both the Hamiltonian and jump operators, we can further replace $T(\gamma)^{-1}$ by $T(\gamma)^\dagger$, and conclude that the density matrices (14) in the main text are a family of NESS.

APPENDIX C: MEAN VALUES IN NESS

In this Appendix, we compute mean values of local observables in states of the form $\rho_\gamma(\beta) = T(\gamma)^\dagger e^{\beta Q_0} T(\gamma)$. In particular, we derive Eqs. (21) and (24) of the main text.

1. Expectation values of matrix product operators

We start by computing the following objects:

$$\mathcal{G}(\alpha, \beta) = \mathrm{Tr}[e^{\alpha Q_0} T(\gamma)^\dagger e^{\beta Q_0} T(\gamma)], \tag{C1}$$

$$\widetilde{\mathcal{G}}(\alpha, \beta) = \mathrm{Tr}[e^{\alpha Q_0} T(\gamma)^{-1} e^{\beta Q_0} T(\gamma)], \tag{C2}$$

in terms of which we will see that all quantities of interest can be expressed.

Let us start with $\mathcal{G}(\alpha, \beta)$. Using the MPO formalism above, we can rewrite it as

$$\mathcal{G}(\alpha, \beta) = \mathrm{Tr}_{\mathcal{A}\otimes\mathcal{A}} \prod_{j=L}^{1} \mathrm{tr}_j\big[\mathcal{M}_j^{(\alpha,\beta)}(\gamma)\big], \tag{C3}$$

where $\mathcal{M}_j^{(\alpha,\beta)}(\gamma) \equiv \mathcal{M}_j^{(\alpha)}(\gamma) e^{\beta Z_j}$. After rotation in ancillary space [see Eq. (B16)], we have

$$\mathrm{tr}_j\big(V\mathcal{M}_j^{(\alpha,\beta)}(\gamma)V^{-1}\big) = \begin{pmatrix} 2\cosh\alpha\cosh\beta & 0 & 0 & -2\sqrt{1-\gamma^2}\cosh\alpha\sinh\beta \\ 0 & 2\gamma\cosh\alpha\sinh\beta & 0 & 0 \\ 0 & 0 & 2\gamma\sinh\alpha\cosh\beta & 0 \\ -2\sqrt{1-\gamma^2}\sinh\alpha\cosh\beta & 0 & 0 & 2\sinh\alpha\sinh\beta \end{pmatrix}. \tag{C4}$$

The computation of $\mathcal{G}(\alpha, \beta)$ can be performed by diagonalizing (C4) in ancillary space, leading to

$$\mathcal{G}(\alpha, \beta) = [\lambda_1(\alpha, \beta)]^L + [\lambda_2(\alpha, \beta)]^L + [\lambda_3(\alpha, \beta)]^L + [\lambda_4(\alpha, \beta)]^L, \tag{C5}$$

where

$$\begin{aligned} \lambda_1(\alpha, \beta) &= \cosh(\alpha+\beta) + \sqrt{\cosh^2(\alpha+\beta) - \gamma^2\sinh(2\alpha)\sinh(2\beta)}, \\ \lambda_2(\alpha, \beta) &= \cosh(\alpha+\beta) - \sqrt{\cosh^2(\alpha+\beta) - \gamma^2\sinh(2\alpha)\sinh(2\beta)}, \\ \lambda_3(\alpha, \beta) &= 2\gamma\sinh\alpha\cosh\beta, \\ \lambda_4(\alpha, \beta) &= 2\gamma\cosh\alpha\sinh\beta, \end{aligned} \tag{C6}$$

are the eigenvalues of (C4). For later use, we evaluate the function $\mathcal{G}(\alpha, \beta)$ and its derivatives at particular points,

$$\begin{aligned} \mathcal{G}(0, \beta) &= 2^L[\cosh\beta^L + (\gamma\sinh\beta)^L], \\ \frac{1}{L}\partial_\alpha\mathcal{G}(\alpha, \beta)|_{\alpha=0} &= (1-\gamma^2)2^L\tanh\beta\cosh^L(\beta), \\ \mathcal{G}(-i\pi/2, \beta) &= (-2i)^L(\gamma^L\cosh^L\beta + \sinh^L\beta), \\ \frac{1}{L}\partial_\alpha\mathcal{G}(\alpha, \beta)|_{\alpha=-i\pi/2} &= (1-\gamma^2)(-2i)^L\coth\beta\sinh^L\beta. \end{aligned} \tag{C7}$$

We now turn to $\widetilde{\mathcal{G}}(\alpha, \beta)$. Using (B9), we have

$$\begin{aligned} \widetilde{\mathcal{G}}(\alpha, \beta) &= \mathrm{Tr}\bigg[e^{\alpha Q_0}\frac{1-\gamma^L\mathcal{Z}}{1-\gamma^{2L}}T(\gamma)^\dagger e^{\beta Q_0}T(\gamma)\bigg] \\ &= \frac{\mathcal{G}(\alpha, \beta) - (i\gamma)^L\mathcal{G}(\alpha - i\pi/2, \beta)}{1-\gamma^{2L}}, \end{aligned} \tag{C8}$$

where, in the last equality, we have used the identity $Z_j e^{\alpha Z_j} = i e^{(\alpha - i\pi/2)Z_j}$.

2. Mean values in NESS

Now we compute the mean values of local operators Z_j and $X_jX_{j+1} - Y_jY_{j+1}$. Using translation invariance, we find

$$\begin{aligned} \langle Z_j\rangle_\beta &\equiv \frac{\mathrm{Tr}[Z_jT(\gamma)^\dagger e^{\beta Q_0}T(\gamma)]}{\mathrm{Tr}[T(\gamma)^\dagger e^{\beta Q_0}T(\gamma)]} = \frac{\frac{1}{L}\partial_\alpha\mathcal{G}(\alpha, \beta)|_{\alpha=0}}{\mathcal{G}(0, \beta)} \\ &= \frac{(1-\gamma^2)\tanh\beta}{1+(\gamma\tanh\beta)^L}, \end{aligned} \tag{C9}$$

hence recovering Eq. (21) in main text, after the substitution $\beta \to \alpha$.

The expectation of other local operators such as $X_jX_{j+1} - Y_jY_{j+1}$ could be similarly obtained from suitably defined generating functions, but here we compute directly:

$$\begin{aligned} &\mathrm{Tr}[X_jX_{j+1}T(\gamma)^\dagger e^{\beta Q_0}T(\gamma)] \\ &= \mathrm{Tr}\big\{\mathrm{Tr}_{\mathcal{A}\otimes\mathcal{A}}\big[\mathcal{M}_L^{(0,\beta)}(\gamma)\dots\mathcal{M}_{j+1}^{(X,\beta)}(\gamma)\mathcal{M}_j^{(X,\beta)}(\gamma)\dots \\ &\quad \times \mathcal{M}_1^{(0,\beta)}(\gamma)\big]\big\}, \end{aligned} \tag{C10}$$

where $\mathcal{M}_j^{(X,\beta)} = X_j\mathcal{M}_j^{(0,\beta)} = \mathcal{M}_j^X e^{\beta Z_j}$. A similar formula holds with $X \to Y$.

Using

$$\mathrm{tr}_j\big(V\mathcal{M}_j^{(0,\beta)}V^{-1}\big) = \begin{pmatrix} 2\cosh\beta & 0 & 0 & -2\sqrt{1-\gamma^2}\sinh\beta \\ 0 & 2\gamma\sinh\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C11}$$

$$\mathrm{tr}_j\big(V\mathcal{M}_j^{(X,\beta)}V^{-1}\big) = \begin{pmatrix} 0 & 2\gamma\sinh\beta & 0 & 0 \\ 2\cosh\beta & 0 & 0 & -2\sqrt{1-\gamma^2}\sinh\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C12}$$

$$\mathrm{tr}_j(V\mathcal{M}_j^{(Y,\beta)}V^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i\sqrt{1-\gamma^2}\cosh\beta & 0 & 0 & -2i\sinh\beta \\ 0 & 0 & -2i\gamma\cosh\beta & 0 \end{pmatrix}, \tag{C13}$$

we find

$$\langle X_jX_{j+1}-Y_jY_{j+1}\rangle_\beta = \frac{\mathrm{Tr}[(X_jX_{j+1}-Y_jY_{j+1})T(\gamma)^\dagger e^{\beta Q_0}T(\gamma)]}{\mathrm{Tr}[T(\gamma)^\dagger e^{\beta Q_0}T(\gamma)]} = \frac{\gamma(2-\gamma^2)\tanh(\beta)+[\gamma\tanh(\beta)]^{L-1}}{\gamma^L\tanh^L(\beta)+1}, \tag{C14}$$

which recovers (24) in the main text.

APPENDIX D: LATE-TIME EXPECTATION VALUES—GIBBS ENSEMBLE

Here we consider real-time evolution in the Lindblad system and the emergence of the Gibbs ensemble. We show how to compute the Lagrange multiplier of the Gibbs ensemble, which eventually leads to a prediction for the long-time limit of local observables. The computations in this section are performed in the infinite volume limit. Exact finite volume computations confirming the general statements for specific cases will be provided in the next Appendix.

The conservation of Q_γ implies that if a Gibbs ensemble $\rho_G \sim e^{-\lambda Q_\gamma}$ emerges during time evolution, then it has to satisfy

$$\mathrm{Tr}(\rho_0 Q_\gamma) = \frac{\mathrm{Tr}(e^{-\lambda Q_\gamma}Q_\gamma)}{\mathrm{Tr}e^{-\lambda Q_\gamma}}. \tag{D1}$$

This equation can be used to fix λ using knowledge of the initial state.

Once λ is found, the predictions for the steady-state values of observables can be given by results from the previous Appendix. For example, for the mean value of Z_j, we find

$$\langle Z_j\rangle = \frac{\mathrm{Tr}(e^{-\lambda Q_\gamma}Q_\gamma)}{\mathrm{Tr}e^{-\lambda Q_\gamma}} = -(1-\gamma^2)\tanh(\lambda). \tag{D2}$$

Using the similarity transformation, the right-hand side is actually found to be

$$-\tanh(\lambda). \tag{D3}$$

Furthermore, the left-hand side can be expressed as

$$\mathrm{Tr}[T(\gamma)\rho_0T^\dagger(\gamma)Q_0]. \tag{D4}$$

If the initial density matrix is chosen to be

$$\rho_0 \sim e^{\beta Q_0}, \tag{D5}$$

then, once again, we can use the results from the next Appendix to conclude

$$(1-\gamma^2)\tanh(\beta) = -\tanh(\lambda). \tag{D6}$$

Combining everything, we obtain the prediction for the long-time limit,

$$\langle Z_j\rangle = (1-\gamma^2)^2\tanh(\beta). \tag{D7}$$

APPENDIX E: LATE-TIME EXPECTATION VALUES—EXACT COMPUTATIONS

In this section, we detail the computation of the late-time expectation values of local observables following a quantum quench from initial states given by Eq. (26) in the main text. In particular, we focus on the local observable $\langle Z_j\rangle$.

Let us start by recalling that an overcomplete basis of the $(L+1)$-dimensional space of NESS can be generated by the $\tilde{\rho}_\gamma(\alpha) = T(\gamma)^{-1}e^{\alpha Q_0}T(\gamma)$. Since $Q_0 = \sum_j Z_j$ has $L+1$ distinct eigenvalues of the form $2n-L$ with $n = 0,\dots,L$, we can alternatively define a basis of the space of NESS in terms of the projectors $\widetilde{P}_n = T(\gamma)^{-1}P_nT(\gamma)$, where

$$P_n = \frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L+Q_0}{2}-n)} \tag{E1}$$

is the projector onto the subspace where Q_0 has eigenvalue $2n-L$.

As a consequence, any density matrix in the space of NESS can be decomposed as

$$\rho_{\mathrm{NESS}} = \sum_{n=0}^{L}\frac{\mathrm{Tr}(\rho_{\mathrm{NESS}}\widetilde{P}_n)}{\mathrm{Tr}(\widetilde{P}_n)}\widetilde{P}_n. \tag{E2}$$

Starting from an arbitrary $\rho(t=0)$, we therefore have, at late times,

$$\lim_{t\to\infty} e^{\mathcal{L}t}\rho(t=0) = \sum_{n=0}^{L}\frac{\mathrm{Tr}[\rho(t=0)\widetilde{P}_n]}{\mathrm{Tr}(\widetilde{P}_n)}\widetilde{P}_n \tag{E3}$$

and, therefore, for any observable $\mathcal{O}$,

$$\lim_{t\to\infty}\langle\mathcal{O}\rangle = \sum_{n=0}^{L}\frac{\mathrm{Tr}[\rho(t=0)\widetilde{P}_n]}{\mathrm{Tr}(\widetilde{P}_n)}\mathrm{Tr}(\widetilde{P}_n\mathcal{O}). \tag{E4}$$

All the traces involved in (E4) can be computed using the matrix product operator techniques.

Focusing on the observable Z_j, we need to compute

$$\lim_{t\to\infty}\langle Z_j\rangle = \sum_{n=0}^{L}\frac{\mathrm{Tr}[\rho(t=0)\widetilde{P}_n]}{\mathrm{Tr}(\widetilde{P}_n)}\mathrm{Tr}(\widetilde{P}_nZ_j) \tag{E5}$$

in the initial expression (E5) for $\lim_{t\to\infty}\langle Z_j\rangle$.

The first trace $\mathrm{Tr}(\widetilde{P}_n) = \mathrm{Tr}(P_n)$ can easily be computed without resorting to MPO techniques, as it corresponds to the dimension of the eigenspace of P_n with eigenvalue $2L-n$; however, as a warm-up, we present its computation using the previously computed function $\widetilde{\mathcal{G}}(\alpha,\beta)$. Using the decomposition (E1) of the projector P_n, we have

$$\begin{aligned}\mathrm{Tr}(\widetilde{P}_n) &= \frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\mathrm{Tr}[T(\gamma)^{-1}e^{i\frac{\pi k}{L+1}Q_0}T(\gamma)] \\ &= \frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\widetilde{\mathcal{G}}\left(0,\frac{ik\pi}{L+1}\right) \\ &= \frac{2^L}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\left(\cos\frac{k\pi}{L+1}\right)^L \\ &= \binom{L}{n}. \end{aligned} \tag{E6}$$

The second trace can be similarly evaluated as

$$\begin{aligned}\mathrm{Tr}(\widetilde{P}_n Z_j) &= \frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\frac{1}{L}\partial_\alpha\widetilde{\mathcal{G}}\left(\alpha,\frac{ik\pi}{L+1}\right)|_{\alpha=0} \\ &= i2^L\frac{1-\gamma^2}{1-\gamma^{2L}}\frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L}(\frac{L}{2}-n)}\left[\sin\frac{k\pi}{L+1}\left(\cos\frac{k\pi}{L+1}\right)^{L-1} + (i\gamma)^L\cos\frac{k\pi}{L+1}\left(\sin\frac{k\pi}{L+1}\right)^{L-1}\right] \\ &= -\frac{1-\gamma^2}{1-\gamma^{2L}}[1-(-1)^{L-n}\gamma^L]\frac{L-2n}{n}\binom{L-1}{n-1}. \end{aligned} \tag{E7}$$

We now move to the third trace, $\mathrm{Tr}[\rho(t=0)\widetilde{P}_n]$. Taking the normalized density matrix $\rho(t=0) = e^{\alpha Q_0}/(2\cosh\alpha)^L$,

$$\begin{aligned}\mathrm{Tr}[\rho(t=0)\widetilde{P}_n] &= \frac{1}{(2\cosh\alpha)^L}\frac{1}{L+1}\sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\widetilde{\mathcal{G}}\left(\alpha,\frac{ik\pi}{L+1}\right) \\ &= \frac{1}{(2\cosh\alpha)^L}\frac{1}{L+1}\frac{1}{1-\gamma^{2L}}[\mathcal{F}_n(\alpha)-(i\gamma)^L\mathcal{F}_n(\alpha-i\pi/2)], \end{aligned} \tag{E8}$$

where, in the last line, we have used the expression (C8) of $\widetilde{\mathcal{G}}$ in terms of $\mathcal{G}$, and introduced the functions

$$\mathcal{F}_n(\alpha) \equiv \sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\widetilde{\mathcal{G}}\left(\alpha,\frac{ik\pi}{L+1}\right) \tag{E9}$$

$$= \sum_{k=0}^{L} e^{i\frac{2\pi k}{L+1}(\frac{L}{2}-n)}\left[\lambda_1\left(\alpha,\frac{ik\pi}{L+1}\right)^L + \lambda_2\left(\alpha,\frac{ik\pi}{L+1}\right)^L + \lambda_3\left(\alpha,\frac{ik\pi}{L+1}\right)^L + \lambda_4\left(\alpha,\frac{ik\pi}{L+1}\right)^L\right] \tag{E10}$$

$$\equiv \mathcal{F}_n^{(1)}(\alpha)+\mathcal{F}_n^{(2)}(\alpha)+\mathcal{F}_n^{(3)}(\alpha)+\mathcal{F}_n^{(4)}(\alpha). \tag{E11}$$

Using the expressions (C6) of the eigenvalues λ_i, the contributions $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(4)}$ are easily evaluated. We find

$$\frac{1}{(2\cosh\alpha)^L}\frac{1}{L+1}\frac{1}{1-\gamma^{2L}}\mathcal{F}_n^{(3)}(\alpha) = \frac{1}{2^L}\frac{(\gamma\tanh\alpha)^L}{1-\gamma^{2L}}\binom{L}{n}, \tag{E12}$$

$$\frac{1}{(2\cosh\alpha)^L}\frac{1}{L+1}\frac{1}{1-\gamma^{2L}}\mathcal{F}_n^{(4)}(\alpha) = \frac{1}{2^L}\frac{\gamma^L}{1-\gamma^{2L}}(-1)^{L-n}\binom{L}{n}. \tag{E13}$$

We now move to the contribution $\mathcal{F}^{(1)}+\mathcal{F}^{(2)}$. Using the expression (C6),

$$\lambda_1(\alpha,\beta)^L+\lambda_2(\alpha,\beta)^L = 2\sum_{\substack{j=0\\ j\text{ even}}}^{L}\binom{L}{j}[\cosh(\alpha+\beta)]^{L-j}[\cosh^2(\alpha+\beta)-\gamma^2\sinh(2\alpha)\sinh(2\beta)]^{j/2}. \tag{E14}$$

Hence,

$$\mathcal{F}_n^{(1)}(\alpha)+\mathcal{F}_n^{(2)}(\alpha)=\frac{2}{2^L}\sum_{k=0}^{L}\sum_{\substack{j=0\\ j\text{ even}}}^{L}\binom{L}{j}e^{-\alpha L}e^{\frac{-2ink\pi}{L+1}}\left(1+e^{\frac{2ik\pi}{L+1}}e^{2\alpha}\right)^{L-j}\left[\left(1+e^{\frac{2ik\pi}{L+1}}e^{2\alpha}\right)^2-\gamma^2\left(1-e^{\frac{4ik\pi}{L+1}}\right)\left(1-e^{4\alpha}\right)\right]^{j/2} \tag{E15}$$

$$\begin{aligned}&=\frac{e^{-\alpha L}}{2^{L-1}}\sum_{k=0}^{L}\sum_{l=0}^{L/2}\sum_{a=0}^{L-2l}\sum_{\substack{b_1,b_2\geqslant 0\\ b_1+b_2\leqslant l}}\binom{L}{2l}\binom{L-2l}{a}\binom{l}{b_1,b_2,l-b_1-b_2}e^{\frac{2i(a+b_1+2b_2-n)k\pi}{L+1}}e^{2a\alpha}(2e^{2\alpha})^{b_1}\\&\quad\times\frac{[e^{4\alpha}(1-\gamma^2)+\gamma^2]^{b_2}}{[1-\gamma^2(1-e^{4\alpha})]^{b_1+b_2-l}}\\&=\frac{L+1}{e^{\alpha L}}\frac{2}{2^L}\sum_{l=0}^{L/2}\sum_{\substack{b_1,b_2\geqslant 0\\ b_1+b_2\leqslant l}}\frac{L!\,e^{2n\alpha}[1-\gamma^2(1-e^{4\alpha})]^{l-b_1}2^{b_1}}{(2l)!(n-b_1-2b_2)!(L-2l-n+b_1+2b_2)!}\binom{l}{b_1,b_2,l-b_1-b_2}\\&\quad\times\left[\frac{1-\gamma^2(1-e^{-4\alpha})}{1-\gamma^2(1-e^{4\alpha})}\right]^{b_2}.\end{aligned} \tag{E16}$$

We further expand this expression to write it as a polynomial in γ (the deformation parameter) and we obtain (after proper rearranging of the sum)

$$\frac{(L+1)\,L!\,e^{2\alpha n}}{2^{L-1}e^{\alpha L}}\sum\frac{l!\,2^{l-b_3}(e^{-4\alpha}-1)^t(e^{4\alpha}-1)^{f-t}}{(2l)!\,t!\,(b_2-t)!(l-b_3)!(f-t)!(t-b_2+b_3-f)!(n-2b_2+b_3-l)!(2b_2-b_3-l+L-n)!}\gamma^{2f}, \tag{E17}$$

where we used the shortcut

$$\sum\to\sum_{f=0}^{L/2}\sum_{l=f}^{L/2}\sum_{b_3=f}^{l}\sum_{b_2=0}^{b_3}\sum_{t=0}^{b_2}. \tag{E18}$$

We used the software *Mathematica* 12.3 to further simplify this expression and we obtain

$$\mathcal{F}_n^{(1)}(\alpha)+\mathcal{F}_n^{(2)}(\alpha)=\kappa\sum_{f=0}^{L/2}\frac{(e^{4\alpha}-1)^f\gamma^{2f}(L-f)!}{f!n!\,e^{\alpha(L-2n)}\,2^{L-n}}\,{}_3\tilde{F}_2\left(-f,\frac{1-n}{2},-\frac{n}{2};\frac{L-n-2f+1}{2},\frac{L-n-2f+2}{2};\frac{1}{e^{4\alpha}}\right), \tag{E19}$$

where $\kappa=(L+1)\sqrt{\pi}L$ and ${}_3\tilde{F}_2$ is the hypergeometric function regularized.

Reporting the results (E12), (E13), and (E19) into (E8) [where the terms $\mathcal{F}_n(\alpha-i\pi/2)$ can just be obtained by shifting the argument], we get

$$\begin{aligned}\mathrm{Tr}[\rho(t=0)\tilde{P}_n]=&\frac{1}{2^L}\frac{(-1)^n\gamma^L}{\gamma^{2L}-1}\binom{L}{n}\{[-\gamma\tanh(\alpha)]^L-(-1)^n\tanh^L(\alpha)+(-1)^n\gamma^L-(-1)^L\}\\&+\frac{1}{(2\cosh\alpha)^L}\frac{1}{1-\gamma^{2L}}\sum_{f=0}^{L/2}\frac{\sqrt{\pi}L(1-e^{4\alpha})^f\gamma^{2f}(-1)^{f+n}2^{n-L}(L-f)!e^{-\alpha(L-2n)}[(-1)^{L+1}\gamma^L+(-1)^n]}{f!n!}\\&\times{}_3\tilde{F}_2\left(-f,\frac{1-n}{2},-\frac{n}{2};\frac{L-n-2f+1}{2},\frac{L-n-2f+2}{2};\frac{1}{e^{4\alpha}}\right).\end{aligned} \tag{E20}$$

The three factors (E6), (E7), and (E20) can now be gathered in the initial expression (E5) for $\lim_{t\to\infty}\langle Z_j\rangle$. Performing the sum over n, we find that the contributions coming from $\mathcal{F}_n^{(3)}$ and $\mathcal{F}_n^{(4)}$ vanish. It remains to compute

$$\begin{aligned}\lim_{t\to\infty}\langle Z_j\rangle=&\frac{\sqrt{\pi}(1-\gamma^2)e^{-\alpha L}}{2^L(\gamma^{2L}-1)^2[\sinh(2\alpha)\mathrm{csch}(\alpha)]^L}\sum_{n=0}^{L}\sum_{f=0}^{L/2}\frac{2^n(e^{4\alpha}-1)^f\gamma^{2f}(2n-L)e^{2\alpha n}(L-f)![(-1)^n-(-\gamma)^L]^2}{(f+1)!(n+1)!}\\&\times{}_3\tilde{F}_2\left(-f,\frac{1-n}{2},-\frac{n}{2};\frac{L-n-2f+1}{2},\frac{L-n-2f+2}{2};\frac{1}{e^{4\alpha}}\right).\end{aligned} \tag{E21}$$

This expression looks complicated at first sight, but we will now see that it is equivalent to the expression (30) in the main text. Both expressions contain a prefactor $\frac{1-\gamma^2}{(1-\gamma^{2L})^2}$ which we therefore omit in the following, and compare the remaining polynomials in γ order by order. Starting from (30), the remaining polynomial takes the form

$$(1-\gamma^2)\tanh(\alpha)(1+\gamma^{2L}-2\gamma^L\tanh^{L-2}\alpha), \tag{E22}$$

in particular the exponents of γ that give a contribution different from 0 are $0, 2, L, 2L, L+2, 2L+2$. We shall now demonstrate that all other powers indeed vanish in the polynomial associated with expression (E21). In (E21), the coefficients are $2f, 2f+L, 2f+2L$, so the only nonzero contribution should come from $f = 0, 1, L/2$, the last one only for L even.

By direct computation, we found that to obtain (E22), it is enough to sum the contribution of $f = 0$ and $f = 1$. Let us show that the other terms vanish. First, we consider the contribution of $f = L/2$ in (E21). This is proportional to

$$\sum_{n=0}^{L} \frac{(L-2n)e^{2\alpha n}[(-1)^n - (-\gamma)^L]^2 \sin(\pi n)}{n} \, {}_2F_1\left(-\frac{L}{2}, -\frac{n}{2}; 1-\frac{n}{2}; \frac{1}{e^{4\alpha}}\right). \tag{E23}$$

Since the function ${}_2F_1(-\frac{L}{2}, -\frac{n}{2}; 1-\frac{n}{2}; e^{-4\alpha})$ is finite for n odd, (E23) vanishes due to $\sin(n\pi)$, while for n even,

$$ {}_2F_1\left(-\frac{L}{2}, -\frac{n}{2}; 1-\frac{n}{2}; x\right) = \sum_{k=0}^{L/2} \frac{(-1)^k n \binom{L/2}{k} x^k}{n-2k}, \tag{E24}$$

and if we substitute this into (E23), we get

$$\sum_{k=0}^{L/2}\sum_{n=0}^{L} \frac{(-1)^k (L-2n)\sin(\pi n)\binom{\frac{L}{2}}{k} e^{2\alpha(n-2k)}}{n-2k} = \sum_{n=0}^{L} i^n (L-2n)\binom{\frac{L}{2}}{\frac{n}{2}} = 0, \tag{E25}$$

where, since $\sin(n\pi)$ is zero, we need to only keep the singular term.

It remains to show that all the terms with $f > 1$ do not contribute. Removing the irrelevant terms and considering $e^{-\alpha} = z$, we should prove that the following term vanishes:

$$\sum_{n=0}^{L} \frac{(L-2n)z^{L-2n}}{(n+1)!(1-2f+L-n)!} {}_3F_2\left(-f, \frac{1-n}{2}, -\frac{n}{2}; \frac{L-n-2f+1}{2}, \frac{L-n-2f+2}{2}; z^4\right). \tag{E26}$$

Expanding as a series in z and reshifting the sum over n, we obtain

$$\sum_{m=0}^{\infty}\sum_{n=-2m}^{L-2m} \frac{(-1)^m\binom{f}{m}(L-4m-2n)z^{L-2n}}{(n+1)!\,(1-2f+L-n)!} = \sum_{m=0}^{\infty}\sum_{n=0}^{L} \frac{(-1)^m\binom{f}{m}(L-4m-2n)z^{L-2n}}{(n+1)!\,(1-2f+L-n)!}. \tag{E27}$$

We can now sum over m and we are left with

$$\sum_{n=0}^{L} (L-2n)\, {}_1F_0(-f;;1) + 4f\, {}_1F_0(1-f;;1). \tag{E28}$$

Considering that

$$ {}_1F_0(a;;x) = (1-x)^{-a}, \tag{E29}$$

each of the terms of (E28) is zero for $f > 1$, as stated at the beginning.

To summarize, we proved that (E5) is equivalent to the expression (30) given in the main text.

[1] G. De Palma, A. Serafini, V. Giovannetti, and M. Cramer, Necessity of eigenstate thermalization, Phys. Rev. Lett. **115**, 220401 (2015).

[2] F. Lange, Z. Lenarčič, and A. Rosch, Pumping approximately integrable systems, Nat. Commun. **8**, 15767 (2017).

[3] T. N. Ikeda and M. Sato, General description for nonequilibrium steady states in periodically driven dissipative quantum systems, Sci. Adv. **6**, eabb4019 (2020).

[4] D. Manzano, A short introduction to the Lindblad master equation, AIP Adv. **10**, 025106 (2020).

[5] D. Nigro, On the uniqueness of the steady-state solution of the Lindblad-Gorini-Kossakowski-Sudarshan equation, J. Stat. Mech. (2019) 043202.

[6] V. V. Albert, Lindbladians with multiple steady states: Theory and applications, arXiv:1802.00010.

[7] R. Nandkishore and D. A. Huse, Many body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).

[8] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. (2016) 064002.

[9] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and Hilbert space fragmentation: A review of exact results, Rep. Prog. Phys. **85**, 086501 (2022).

[10] B. Buča and T. Prosen, A note on symmetry reductions of the Lindblad equation: Transport in constrained open spin chains, New J. Phys. **14**, 073007 (2012).

[11] V. V. Albert and L. Jiang, Symmetries and conserved quantities in Lindblad master equations, Phys. Rev. A **89**, 022118 (2014).

[12] E. Ilievski and T. Prosen, Exact steady state manifold of a boundary driven spin-1 Lai-Sutherland chain, Nucl. Phys. B **882**, 485 (2014).

[13] Z. Zhang, J. Tindall, J. Mur-Petit, D. Jaksch, and B. Buča, Stationary state degeneracy of open quantum systems with non-Abelian symmetries, J. Phys. A **53**, 215304 (2020).

[14] E. Ilievski, M. Medenjak, T. Prosen, and L. Zadnik, Quasilocal charges in integrable lattice systems, J. Stat. Mech. (2016) 064008.

[15] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete generalized Gibbs ensembles in an interacting theory, Phys. Rev. Lett. **115**, 157201 (2015).

[16] Z. Papić, Weak ergodicity breaking through the lens of quantum entanglement, in *Entanglement in Spin Chains: From Theory to Quantum Technology Applications*, edited by A. Bayat, S. Bose, and H. Johannesson (Springer International, Cham, 2022), pp. 341–395.

[17] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[18] H.-P. Breuer, F. Petruccione *et al.*, *The Theory of Open Quantum Systems* (Oxford University Press on Demand, Oxford, 2002).

[19] M. V. Medvedyeva, F. H. L. Essler, and T. Prosen, Exact Bethe Ansatz spectrum of a tight-binding chain with dephasing noise, Phys. Rev. Lett. **117**, 137202 (2016).

[20] E. Vernier, E. O'Brien, and P. Fendley, Onsager symmetries in U(1)-invariant clock models, J. Stat. Mech. (2019) 043107.

[21] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. **16**, 407 (1961).

[22] M. de Leeuw, C. Paletta, and B. Pozsgay, Constructing integrable Lindblad superoperators, Phys. Rev. Lett. **126**, 240403 (2021).

[23] A. A. Ziolkowska and F. H. L. Essler, Yang-Baxter integrable Lindblad equations, SciPost Phys. **8**, 044 (2019).

[24] C. Paletta, Yang-Baxter integrable open quantum systems, Ph.D. thesis, Trinity College, Dublin, 2023, arXiv:2312.00064.

[25] M. de Leeuw, C. Paletta, and B. Pozsgay, A range three elliptic deformation of the Hubbard model, SciPost Phys. **15**, 071 (2023).

[26] W. Miller, Jr., S. Post, and P. Winternitz, Classical and quantum superintegrability with applications, J. Phys. A **46**, 423001 (2013).

[27] M. Fagotti, On conservation laws, relaxation and prerelaxation after a quantum quench, J. Stat. Mech. (2014) P03016.

[28] T. Prosen, Open XXZ spin chain: Nonequilibrium steady state and a strict bound on ballistic transport, Phys. Rev. Lett. **106**, 217206 (2011).

[29] T. Prosen, E. Ilievski, and V. Popkov, Exterior integrability: Yang-Baxter form of non-equilibrium steady-state density operator, New J. Phys. **15**, 073051 (2013).

[30] T. Prosen, Exact nonequilibrium steady state of a strongly driven open *XXZ* chain, Phys. Rev. Lett. **107**, 137201 (2011).

[31] N. G. Jones, J. Bibo, B. Jobst, F. Pollmann, A. Smith, and R. Verresen, Skeleton of matrix-product-state-solvable models connecting topological phases of matter, Phys. Rev. Res. **3**, 033265 (2021).

[32] I. Peschel and V. Emery, Calculation of spin correlations in two-dimensional Ising systems from one-dimensional kinetic models, Z. Phys. B **43**, 241 (1981).

[33] J. Wouters, H. Katsura, and D. Schuricht, Interrelations among frustration-free models via Witten's conjugation, SciPost Phys. Core **4**, 027 (2021).

[34] We would like to clarify that we are referring to the Hermiticity property of the action of the superoperator $\tilde{\mathcal{L}}$. This is not the same as Hermiticity or anti-Hermiticity of the commutator $[\rho, H]$.

[35] Closely related notions of a Lindbladian scar appeared in [38,39]; see also [40].

[36] L. Sá, P. Ribeiro, and T. Prosen, Integrable nonunitary open quantum circuits, Phys. Rev. B **103**, 115132 (2021).

[37] L. Su and I. Martin, Integrable nonunitary quantum circuits, Phys. Rev. B **106**, 134312 (2022).

[38] B. Buča, J. Tindall, and D. Jaksch, Nonstationary coherent quantum many-body dynamics through dissipation, Nat. Commun. **10**, 1730 (2019).

[39] H.-R. Wang, D. Yuan, S.-Y. Zhang, Z. Wang, D.-L. Deng, and L. M. Duan, Embedding quantum many-body scars into decoherence-free subspaces, arXiv:2304.08515.

[40] Q. Chen, S. A. Chen, and Z. Zhu, Weak ergodicity breaking in non-Hermitian many-body systems, SciPost Phys. **15**, 052 (2023).