Conformal field theories generated by Chern insulators under decoherence or measurement

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We demonstrate that the fidelity between a pure state trivial insulator and the mixed state density matrix of a Chern insulator generated from decoherence or measurement can be mapped to a variety of two-dimensional conformal field theories (CFTs); more specifically, the quantity $\mathcal{Z} = \mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ is mapped to the partition function of the desired CFT, where $\hat{\rho}_\Omega$ and $\hat{\rho}_c^D$ are respectively the density matrix of a pure state trivial insulator and the mixed state density matrix generated from the Chern insulator. For a pure state Chern insulator with Chern number $2N$, the fidelity $\mathcal{Z}$ is mapped to the partition function of the $\mathrm{U}(2N)_1$ CFT; under decoherence or measurement, the Chern insulator density matrix can experience a certain instability, and the "partition function" $\mathcal{Z}$ can flow to other interacting CFTs with smaller central charges. The Rényi relative entropy $\mathcal{F} = -\ln \mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ is mapped to the free energy of the CFT, and we demonstrate that the central charge of the CFT can be extracted from the finite-size scaling of $\mathcal{F}$, analogous to the well-known finite-size scaling of $2d$ CFT.

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I. INTRODUCTION

The $2d$ Chern insulator [1] is the archetypal example of a topological insulator (TI). The Chern insulator is defined as the ground state (hence a pure state) of a two-dimensional tight-binding Hamiltonian of electrons. In reality, any quantum system is exposed to the environment, and experiences a certain level of decoherence through forming entanglement with the ancilla degrees of freedom in the environment. In the last few years TIs or more generally symmetry protected topological (SPT) states in open systems have attracted great interest, and these studies belong to a larger paradigm of classifying and characterizing topological features of density matrices [2–10]. The strongest kind of decoherence would be thermalization, when the system reaches a thermal equilibrium with the environment after a long-time evolution (compared with the microscopic timescales of the system) by interacting and entangling with the ancilla degrees of freedom in the environment. Under this type of long time and massive scale of decoherence, many of the topological features of the Chern insulator (such as its nontrivial edge states) would be lost, and the topological insulators are strictly well defined at zero temperature.

But one can also consider much weaker decoherence caused by exposing the system to the environment for a short amount of time, and the system will form weak entanglements with the ancilla qubits. In this case, after tracing out the ancilla qubits, the pure state topological insulator will still be rendered a mixed density matrix, but in stark contrast with thermalization, it is possible that some of the topological features survive this procedure. This type of weak decoherence is equivalent to the system being "weakly measured" by the environment In the past few years phases and phase transitions that involve quantum measurements have been actively pursued, both theoretically and experimentally [11–35]. The fate of the mixed density matrix after decoherence or measurement strongly depends on the symmetry of the decoherence channel or the measurement, as was pointed out in recent works. It was shown recently that the topological information of a mixed state density matrix can be extracted through "strange correlators," which were originally devised to probe the pure state SPT states and TIs [8,9,36]. In particular, the notion of a type-II strange correlator and a formalism in terms of a "doubled Hilbert space" was developed in order to extract the full anomaly of the mixed density matrix [8].

The notion of the strange correlator is based on the overlap between the SPT state and a trivial state. This overlap is then mapped, under a space-time rotation, to a correlation function on the boundary. Power-law decay of the strange correlator signifies an SPT or TI phase with gapless boundary modes, even while bulk correlation functions remain short-ranged and insensitive to the topological order of the bulk. Motivated by the recent progresses on understanding the topological features of mixed state density matrices, in this work, we investigate the fidelity between a pure state trivial insulator and a mixed state density matrix generated from a Chern insulator through quantum operations that correspond to decoherence and weak measurement, and we will refer to the latter density matrix as a "mixed state Chern insulator." We demonstrate that the fidelity can be mapped to the partition functions of a series of conformal field theories (CFTs), whose central charges depend on the decoherence channel or measurement. We stress that the decoherence or measurements considered in this work are supposed to be "weak"; a strong projective measurement may drive the system into a product state. This includes weak measurement with postselection of measurement outcomes. We also propose a method to extract the central charge of the effective CFT, which is analogous to the finite-size scaling of the actual free energy of ordinary CFTs [37,38]. This analysis enables us to extract the information of the mixed state Chern insulator density matrix without having to know exactly which order parameter to use for the strange correlator.

II. CHERN INSULATOR WITH $\nu = 1$

We are most interested in the following quantities:

$$\mathcal{Z} = \mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}, \quad \mathcal{F} = -\ln \mathcal{Z}. \tag{1}$$

The density matrix of a pure state Chern insulator is $\hat{\rho}_c = |\Psi_c\rangle\langle\Psi_c|$, and $\hat{\rho}_\Omega$ is the density matrix of a trivial insulator. The quantity $\mathcal{Z}$ is the fidelity between a pure state trivial insulator density matrix and the Chern insulator density matrix, which is either a pure state Chern insulator or a mixed state density matrix generated from decoherence or measurement. $\mathcal{F}$ is the Rényi relative entropy. The definition of the fidelity between two general mixed density matrices is more complicated, but it reduces to the simple form of Eq. (1) when one of the density matrices is pure. In our work, it suffices to always keep $\hat{\rho}_\Omega$ the density matrix of the pure state trivial insulator. Later we shall see that $\mathcal{Z}$ is mapped to the partition function of an effective $(2+0)d$ or $(1+1)D$ *nonchiral* conformal field theory (CFT), and $\mathcal{F}$ is mapped to the free energy of the CFT. If we compute the overlap between the wave functions (rather than density matrices) of a pure state Chern insulator and trivial insulator, the overlap would become the partition function of a chiral CFT [39]. Throughout the paper we will always use d to label the spatial dimension, and D to label the space-time dimension.

The quantities $\mathcal{Z}$ and $\mathcal{F}$ encode the "distance" between the two density matrices. We start with a tight-binding Hamiltonian that realizes the Chern insulator with Chern number $\nu = 1$, but we would like to pursue generic physics that is independent of the microscopic details; hence we tune the system close to the topological transition to a trivial insulator. In the case where the system is tuned close to the topological transition, the difference between the Chern insulator and the trivial insulator can be well captured by a single Dirac fermion with a mass (fermion doubling demands there be another Dirac fermion in the momentum space, with presumably a much larger mass), where the sign of the mass determines whether the system is a Chern insulator or a trivial insulator. We also assume that there is sufficient crystal symmetry such as spatial inversion and discrete rotation, which render terms that deviate from a simple Dirac fermion irrelevant in the infrared when the Dirac mass is small. One example of such model is the spinless version of the so-called BHZ model on the square lattice [40], which has multiple discrete symmetries such as charge conjugation, spatial inversion, discrete rotation, etc. These symmetries ensure that close to the topological-trivial transition with band inversion, the system is well captured by a single Dirac fermion at the Gamma point of the Brillouin zone with emergent Lorentz symmetry. The Hamiltonian and the Euclidean space-time Lagrangian of the Dirac fermion read

$$\begin{aligned} H &= \int d^2x\, \mathcal{H} \\ &= \int d^2x\, \psi^\dagger(i\sigma^z\partial_x + i\sigma^x\partial_y + m\sigma^y)\psi, \\ \mathcal{L} &= \psi^\dagger(\partial_\tau + i\sigma^z\partial_x + i\sigma^x\partial_y + m\sigma^y)\psi \\ &= \bar{\psi}(\gamma_0\partial_\tau + \gamma_1\partial_x + \gamma_2\partial_y)\psi + m\bar{\psi}\psi \\ (\gamma_0, \gamma_1, \gamma_2) &= (\sigma^y, -\sigma^x, \sigma^z) \end{aligned} \tag{2}$$

with $\bar{\psi} = \psi^\dagger\gamma_0$.

We would like to consider a mixed state density matrix generated from a Chern insulator after a quantum operation such as weak measurement, with the local quantities measured being invariant under the U(1) charge symmetry. For example, the local quantities being measured could be the current density of the system, and the measurement outcomes are summed over. After this operation, the density matrix of the mixed state Chern insulator can take the following form,

$$\begin{aligned} \hat{\rho}_c^D &= \otimes_{\boldsymbol{x}}\mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c], \\ \mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c] &\sim \hat{\rho}_c + \sum_\mu p_\mu\big(\hat{J}^\mu_{\boldsymbol{x}}\hat{\rho}_c\hat{J}^\mu_{\boldsymbol{x}}\big), \end{aligned} \tag{3}$$

where $\hat{J}^\mu$ is the three-component *Hermitian* current vector:

$$\begin{aligned} \hat{J}^\mu &= (\bar{\psi}\gamma^0\psi, i\bar{\psi}\gamma^1\psi, i\bar{\psi}\gamma^2\psi) \\ &= (\psi^\dagger\psi, -\psi^\dagger\sigma^z\psi, -\psi^\dagger\sigma^x\psi). \end{aligned} \tag{4}$$

The local charge density or current operators are part of the set of "Kraus operators." We would like to to clarify that we do not require the Kraus operators to satisfy the probability-conserving constraint; i.e., we allow a certain level of postselection in our measurement. Since the Kraus operators are invariant under the U(1) symmetry, the mixed density matrix generated will be in the *canonical ensemble* with a fixed number of fermions.

The exact density matrix of a system is often tedious to work with. For a quantum many-body system, we often only care about the long-wavelength behaviors of the system, and this physics can be captured through coarse-graining or renormalization-group techniques. There is a rather convenient formalism that allows us to use these techniques when we study the effects of quantum operations including decoherence or weak measurement. This formalism is based on the well-known fact that the density matrix of the ground state of a system can be generated through path integral in the Euclidean space-time $[\hat{\rho}_0]_{\phi_1(\boldsymbol{x}),\phi_2(\boldsymbol{x})} \sim \lim_{\beta\to\infty}\int D\phi(\boldsymbol{x},\tau)\exp(-\mathcal{S}_0)$, with the temporal boundary condition $\phi(\boldsymbol{x},0) = \phi_1(\boldsymbol{x})$ and $\phi(\boldsymbol{x},\beta) = \phi_2(\boldsymbol{x})$. Here $\mathcal{S}_0$ is the action of the system whose ground state is the desired pure state $|\Psi_0\rangle$. Then under decoherence or weak measurement, the mixed density matrix becomes [29,41] [Fig. 1(a)]

$$\begin{aligned} [\hat{\rho}^D]_{\phi_1(\boldsymbol{x}),\phi_2(\boldsymbol{x})} &\sim \lim_{\beta\to\infty}\int D\phi(\boldsymbol{x},\tau)\exp(-\mathcal{S}_0 - \mathcal{S}^{\mathrm{int}}), \\ \mathcal{S}^{\mathrm{int}} &= \int d\boldsymbol{x}\, \mathcal{L}^{\mathrm{int}}(\phi(\boldsymbol{x},0), \phi(\boldsymbol{x},\beta)); \end{aligned} \tag{5}$$

the boundary condition $\phi(\boldsymbol{x},0) = \phi_1(\boldsymbol{x})$, $\phi(\boldsymbol{x},\beta) = \phi_2(\boldsymbol{x})$ must hold in this path integral. The effect of decoherence is mapped to the interaction $\mathcal{L}^{\mathrm{int}}$ between fields at $\tau = 0$ and $\tau = \beta$.

The symmetry is the most important condition one needs to consider in order to determine the form of $\mathcal{L}^{\mathrm{int}}$. A density matrix can have a *doubled* (or strong) symmetry condition when it is invariant under separate actions from left and right multiplication of the symmetry operation [6–8]. When the measurement only involves quantities that are invariant under the charge-U(1) symmetry, or more technically the Kraus operators that generate the mixed state density matrix are invariant under the U(1) symmetry, $\mathcal{L}^{\mathrm{int}}$ needs to preserve

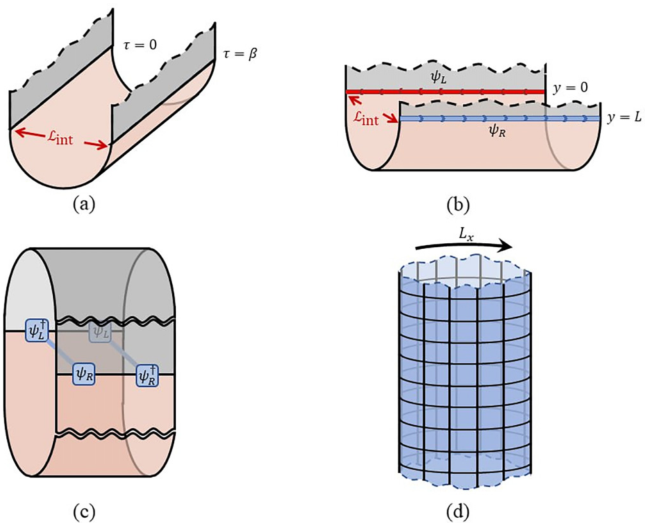

FIG. 1. (a) The fidelity $\mathcal{Z}$ corresponds to path integral in Euclidean space-time, with interfaces at $\tau = 0$ and $\tau = \beta$; the decoherence is mapped to interaction between the two interfaces. (b) Under space-time rotation, the temporal interfaces at $\tau = 0, \beta$ become spatial interfaces $y = 0, L$, and there are chiral edge modes at each interface. (c) Example of type-II strange correlator. (d) The central charge of the effective CFT can be extracted from the finite-size scaling of the Rényi relative entropy $\mathcal{F}$ on a cylindrical geometry.

separate U(1) transformations acting from the left and right side of the density matrix, which translate to the separate U(1) symmetries of the fields at $\tau = 0$ and $\tau = \beta$. In this case $\mathcal{L}^{\rm int}$ should include the following term:

$$\mathcal{L}^{\rm int} \sim \sum_{\mu} -g_\mu J_\mu(\boldsymbol{x}, 0) J_\mu(\boldsymbol{x}, \beta), \tag{6}$$

where $g_\mu \sim p_\mu$ for small p_μ. The charge current operator $\boldsymbol{J} = (J_x, J_y)$ is odd under spatial inversion, and all components of J_μ are odd under charge conjugation. In addition to the doubled U(1) symmetry, we demand the density matrix preserve the diagonal (or weak) charge conjugation and spatial inversion symmetry. This condition precludes terms linear with current operators $J_\mu(\boldsymbol{x}, 0)$, $J_\mu(\boldsymbol{x}, \beta)$ in $\mathcal{L}^{\rm int}$.

Hence the quantity $\mathcal{Z}$ can be computed through a path integral of a Chern insulator action sharing temporal interfaces with a trivial insulator at $\tau = 0, \beta$, and this computation can be approximated by a path integral with the following action on the $(2+0)d$ temporal interfaces $\tau = 0, \beta$,

$$\mathcal{S} = \int d^2x\, \bar{\psi}(\gamma_1 \partial_1 + \gamma_2 \partial_2)\psi + \mathcal{L}^{\rm int}, \tag{7}$$

as there are low-energy modes localized at the temporal interfaces. The low-energy modes at the temporal interface are eigenstates of σ^y. It is much more physically intuitive for us to perform a space-time rotation in the (τ, y) plane, and position the interface at $y = 0, L$, and the modes localized at the spatial interface would be eigenstates of σ^z. The action that describes the low-energy states at both boundaries is

$$\begin{aligned} \mathcal{S}_{1d} &= \int d\tau dx\, \mathcal{L}_{1d} = \int d\tau dx\, \psi^\dagger(\partial_\tau + i\sigma^z\partial_x)\psi \\ &= \int d\tau dx\, \bar{\psi}(\gamma_0\partial_\tau + \gamma_1\partial_x)\psi. \end{aligned} \tag{8}$$

This action corresponds to a $1d$ Hamiltonian of Dirac fermion, whose bosonized form is

$$H_0 = \int dx\, \mathcal{H}_0(x), \quad \mathcal{H}_0(x) = \frac{K}{2\pi}(\nabla_x\theta)^2 + \frac{1}{2\pi K}(\nabla_x\phi)^2. \tag{9}$$

For free fermions without any interaction, $K = 1$. Here we stress that Eq. (8) and its bosonized form already includes physics at the two opposite spatial interfaces, which correspond to the temporal interfaces $\tau = 0$ and β combined before the space-time rotation.

After the space-time rotation, $\mathcal{L}^{\rm int}$ would take the form of interaction between currents on the spatial interfaces $y = 0$ and $y = L$:

$$\mathcal{L}^{\rm int} \sim \sum_{\mu} -g_\mu J'_\mu(x, 0) J'_\mu(x, L), \tag{10}$$

where

$$\begin{aligned} J'_\mu &= (\bar{\psi}\gamma_2\psi, i\bar{\psi}\gamma_1\psi, i\bar{\psi}\gamma_0\psi) \\ &= (-i\psi^\dagger\sigma^x\psi, -\psi^\dagger\sigma^z\psi, i\psi^\dagger\psi). \end{aligned} \tag{11}$$

The g_0 term does not project to an obvious nontrivial operator at the spatial boundary after space-time rotation; g_1 and g_2 have very similar effect, as they both project to the following term at the spatial boundary [Fig. 1(b)]:

$$\mathcal{L}^{\rm int} \sim \mathcal{H}^{\rm int} = g\rho_L(x)\rho_R(x), \quad g \sim g_1 + g_2; \tag{12}$$

hence the effect of the quantum operation is now mapped to a *repulsive* interaction between the left and right moving charge densities in the effective $(1+1)D$ system corresponding to the Hamiltonian density $\mathcal{H}_0 + \mathcal{H}^{\rm int}$.

This interaction would renormalize the Luttinger parameter K, i.e., $K = \sqrt{(2\pi - g)/(2\pi + g)}$, but it will not gap-out the boundary states. This is because the "doubled symmetry" condition demands that the effective action after space-time rotation be invariant under independent $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ symmetries which act on the left and right moving modes of $\mathcal{H}_0$, and it is well known that a system with these symmetries has a perturbative 't Hooft anomaly and cannot be gapped.

III. CHERN INSULATOR WITH $\nu = 2$

We now consider two copies of Chern insulators described by the BHZ model, both with Chern number $\nu = 1$; topologically this is also equivalent to a Chern insulator with Chern number $\nu = 2$. After the quantum operation which preserves the charge number, the density matrix of the Chern insulator becomes

$$\hat{\rho}_c^D = \otimes_{\boldsymbol{x}} \mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c],$$
$$\mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c] \sim \hat{\rho}_c + \sum_{i=1,2} \sum_{\mu} p_{i,\mu} \big(\hat{J}^{\mu}_{i,\boldsymbol{x}} \hat{\rho}_c \hat{J}^{\mu}_{i,\boldsymbol{x}}\big) + p_e \big(\hat{O}_{\boldsymbol{x}} \hat{\rho}_c \hat{O}^{\dagger}_{\boldsymbol{x}} + \hat{O}^{\dagger}_{\boldsymbol{x}} \hat{\rho}_c \hat{O}_{\boldsymbol{x}}\big),$$
$$\hat{O}_{\boldsymbol{x}} = c^{\dagger}_{1,\boldsymbol{x}} \sigma^z c_{2,\boldsymbol{x}}, \tag{13}$$

with small $p_{i,\mu}$ and p_e. The subscript $i = 1, 2$ labels the two copies of the Chern insulators. In addition to the current operators of the system, the quantum operation now involves the exciton operator $\hat{O}$, which is also invariant under the total charge U(1) symmetry. Our goal is to explore the possible effective CFTs that can be realized through $\mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$; hence a specific form of exciton operator was chosen in Eq. (13), which ensures that the realized CFT is unitary.

Following the procedure in the previous section, after a space-time rotation in the (y, τ) plane, the quantity $\mathcal{Z} = \mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ can be mapped to the following path integral of $(1+1)D$ interacting nonchiral fermions:

$$\mathcal{Z} \sim \int D[\psi_i] D[\psi_i^{\dagger}] \exp\left(-\int d\tau dx\, \mathcal{L}_{1d} + \mathcal{L}^{\mathrm{int}}\right)$$
$$\mathcal{L}^{\mathrm{int}} = \sum_{i=1,2} g_i \rho_{i,L} \rho_{i,R} + g_e (\psi^{\dagger}_{1,L} \psi_{2,L} \psi^{\dagger}_{2,R} \psi_{1,R} + \mathrm{H.c.}), \tag{14}$$

where $\mathcal{L}_{1d}$ is the Lagrangian of two flavors of free $(1+1)D$ Dirac fermions. We can turn the path integral problem Eq. (14) into a Hamiltonian formalism in $1d$:

$$H = \int dx \sum_{i=1,2} \frac{K_i}{2\pi} (\nabla_x \theta_i) + \frac{1}{2\pi K_i} (\nabla_x \phi_i)^2 + u \cos(2\phi_1 - 2\phi_2). \tag{15}$$

Here u is proportional to $g_e \sim p_e$, and $K_i = \sqrt{(2\pi - g_i)/(2\pi + g_i)}$ with $g_i \sim p_{i,1} + p_{i,2} > 0$. Here $\exp(i2\phi_i) \sim \psi^{\dagger}_{i,L} \psi_{i,R}$ corresponds to the backscattering terms, and $\exp(i2\theta_i) \sim \psi_{i,L} \psi_{i,R}$ corresponds to the Cooper pairs.

For the most natural choice of parameters where $K_i < 1$, the vertex operator $u \cos(2\phi_1 - 2\phi_2)$ is *relevant*. A relevant u will gap-out the channel $\theta_- \sim \theta_1 - \theta_2$ and $\phi_- \sim \phi_1 - \phi_2$, while leaving the channel $\theta_+ \sim \theta_1 + \theta_2$ and $\phi_+ \sim \phi_1 + \phi_2$ gapless and algebraic. All the fermions will acquire a gap, for example

$$\psi_{1,L} \sim e^{i\theta_1 + i\phi_1} \sim e^{i\frac{1}{2}(\theta_+ + \theta_-) + i\frac{1}{2}(\phi_+ + \phi_-)}, \tag{16}$$

as the short-range correlation of the ϕ_- channel will render all the fermion operators short-ranged. But some composite operators will acquire quasi-long-range or power-law correlation, such as the following four-body operators:

$$\psi_{1,L} \psi_{1,R} \psi_{2,L} \psi_{2,R} \sim e^{i2\theta_+}. \tag{17}$$

The correlation function $\langle e^{i2\theta_+(0,0)} e^{-i2\theta_+(0,x)} \rangle$ of the $(1+1)D$ CFT is actually a "type-II strange correlator" that was proposed in Ref. [8] in the $(2+0)d$ space before the space-time rotation:

$$\big\langle e^{i2\theta_+(0,0)} e^{-i2\theta_+(\tau,x)} \big\rangle \sim \mathrm{tr}\big\{\hat{\rho}_c^D \hat{\Delta}(0) \hat{\Delta}^{\dagger}(\boldsymbol{x}) \hat{\rho}_\Omega \hat{\Delta}(0) \hat{\Delta}^{\dagger}(\boldsymbol{x})\big\}, \tag{18}$$

where $\hat{\Delta}(x)$ is a Cooper pair operator $\hat{\Delta}(\boldsymbol{x}) \sim c_{1,\alpha}(\boldsymbol{x}) c_{2,\alpha}(\boldsymbol{x})$ with the same Dirac index $\alpha = 1, 2$, and $\boldsymbol{x} = (x, y)$ labels the $2d$ spatial coordinate before the space-time rotation.

Hence even under weak measurement, the Chern insulator experiences a certain *instability* in the sense that the CFT whose partition function corresponds to $\mathrm{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$ can have its central charge reduced from $c = 2$ to $c = 1$ through weak measurement. The change of the central charge can be extracted through the finite-size scaling of $\mathcal{F} = -\ln \mathcal{Z}$, which we will discuss later in this paper.

IV. CHERN INSULATOR WITH $\nu = 2N$

Now we consider $\nu = 2N$ copies of the Chern insulator. The system can have a large U$(2N)$ flavor symmetry. There could be many possible choices of the quantum operations that generate a mixed state. As an example, we choose the following quantum operation that acts on the Chern insulator density matrix in the following way:

$$\mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c] \sim \hat{\rho}_c + \sum_a \sum_{\mu} p_{a,\mu} \big(\hat{J}^{a,\mu}_{\boldsymbol{x}} \hat{\rho}_c \hat{J}^{a,\mu}_{\boldsymbol{x}}\big),$$
$$\hat{J}^{a,\mu}_{\boldsymbol{x}} = (\bar{\psi}\gamma^0\tau^a\psi, i\bar{\psi}\gamma^1\tau^a\psi, i\bar{\psi}\gamma^2\tau^a\psi); \tag{19}$$

$\hat{J}^{a,\mu}_{\boldsymbol{x}}$ with $a = 1, 2, 3$ is the current operator for the SU(2) subgroup of the U$(2N)$ flavor symmetry. This quantum operation corresponds to the physics that the currents of the SU(2) flavor symmetry are weakly measured. We would like to keep a *diagonal* SU(2) flavor symmetry; hence $p_{a,\mu}$ does not depend on the SU(2) index a.

Following the same procedure as before, after space-time rotation we arrive at the following effective Hamiltonian at the $(1+1)D$ space-time:

$$\mathcal{H}(x) = \mathcal{H}_0(x) + \sum_a g J^a_L(x) \cdot J^a_R(x). \tag{20}$$

Since the quantum operation breaks the doubled SU(2) symmetry down to the diagonal SU(2), the $\sum_a g J^a_L(x) \cdot J^a_R(x)$ term is allowed, and most naturally $g > 0$. This interaction is *marginally relevant*, and its effect is to gap-out the SU(2) sector of the CFT. The original U$(2N)_1$ CFT has the following decomposition [42]:

$$\mathrm{U}(2N)_1 \simeq \mathrm{SU}(N)_2 \oplus \mathrm{U}(1)_{2N} \oplus \mathrm{SU}(2)_N, \text{ or}$$
$$\mathrm{U}(2N) \simeq \mathrm{O}(4N)_1 \simeq \mathrm{Sp}(N)_1 \oplus \mathrm{SU}(2)_N. \tag{21}$$

Since the SU$(2)_N$ sector of the CFT is gapped out, eventually the central charge is reduced from $c = 2N$ to

$$c = \frac{N(2N+1)}{N+2}. \tag{22}$$

Other CFTs with a coset construction can also be engineered through different types of quantum operations.

V. CENTRAL CHARGE OF THE EFFECTIVE CFT

In the previous sections we have shown that under weak measurement the effective CFT with partition function $\mathcal{Z} = \mathrm{tr}\{\hat{\rho}_c \hat{\rho}_\Omega\}$ can become instable and flow to CFTs with smaller central charges. In this section we propose that, if we design our tight-binding Hamiltonian on a cylinder with finite circumference L along the x direction, while infinite length along y [Fig. 1(d)], the "free energy" $\mathcal{F} = -\ln \mathcal{Z}$ per unit length along the y direction will have the following finite-size scaling,

$$\frac{\mathcal{F}}{L_y} = f_0 L - \frac{\pi c}{6L} + \mathcal{O}\left(\frac{1}{L^2}\right), \tag{23}$$

where c is the central charge. This equation is directly inspired by the well-known formula derived in Refs. [37,38] for the actual $2d$ CFTs.

In the following we compute $\mathcal{F}$ for the pure state Chern insulator, and demonstrate that its finite size-scaling does encode the correct central charge. In Eq. (2), for the Chern insulator we take its mass to be $m > 0$ while for the trivial insulator we take its mass to be $-m$. The Hamiltonian densities are

$$\mathcal{H}_c \sim \psi^\dagger(k)(-k_x\sigma^z - k_y\sigma^x + m\sigma^y)\psi(k), \tag{24}$$

$$\mathcal{H}_\Omega \sim \psi^\dagger(k)(-k_x\sigma^z - k_y\sigma^x - m\sigma^y)\psi(k). \tag{25}$$

The Bloch Hamiltonians can be diagonalized in the momentum basis labeled by

$$k_y \in \mathbb{R},\ k_x = \frac{2\pi(n-1/2)}{L}, \quad n \in \mathbb{Z}. \tag{26}$$

Here we have chosen an antiperiodic boundary condition for the fermions along the x direction, as one can show that the ground state wave function for the Chern insulator with an antiperiodic boundary condition has a lower energy compared with that with a periodic boundary condition.

The fidelity $\mathcal{Z}$ becomes a product of the overlap between Bloch wave functions at each momentum:

$$\mathcal{Z} = |\langle\Psi_\Omega|\Psi_c\rangle|^2 = \prod_k b(k),$$
$$b(k) = \frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + m^2}. \tag{27}$$

The "free energy" $\mathcal{F}$ per unit length in the y direction is

$$\frac{\mathcal{F}}{L_y} = -\frac{\ln \mathcal{Z}}{L_y} = -\frac{1}{2\pi}\sum_{k_x}\int dk_y \ln[b(k_x, k_y)]. \tag{28}$$

The integral in k_y can be performed explicitly, and yields

$$\frac{\mathcal{F}}{L_y} = \frac{1}{2\pi}\sum_{k_x}\left(-2\pi|k_x| + 2\pi\sqrt{k_x^2+m^2}\right). \tag{29}$$

We examine the term $\sum_{k_x} -|k_x|$ first:

$$\sum_{k_x} -|k_x| = \sum_{n=-\infty}^{\infty} -\frac{2\pi}{L}|n-1/2| = \sum_{n=1}^{\infty} -\frac{2\pi}{L}(2n-1). \tag{30}$$

In order to perform the sum of all odd integers, we need to use the Zeta function regularization:

$$\sum_{n=1}^{\infty}(2n-1)^{-s} = (1-2^{-s})\zeta(s). \tag{31}$$

Plugging in $s = -1$ would give us $\sum_n (2n-1) = 1/12$. Hence eventually we obtain the following result:

$$\sum_{k_x} -|k_x| = -\frac{\pi c}{6L}, \tag{32}$$

where $c = 1$. This result is consistent with the well-known formula of finite-size scaling of $2d$ CFTs given in Refs. [37,38].

However, there is also correction from finite mass m which is attained by analyzing the second term of Eq. (29). This term $\sum_{k_x}\sqrt{k_x^2+m^2}$ would physically correspond to the Casmir effect of a massive particle in $1d$. Intuitively a massive particle would not lead to any Casmir effect with large mass, but we will test this intuition as follows. This sum can be regularized using a special form of the Abel-Plana formula for half-integer sums [43]:

$$\sum_{n=0}^{\infty} f_{n+\frac{1}{2}} = \int_0^\infty dt\, f(t) - i\int_0^\infty dt\frac{[f(it) - f(-it)]}{1+e^{2\pi t}}. \tag{33}$$

Only the second integral contains a finite piece independent of the regularization scheme, and for a large $|m|$ the integral can be approximated as $-\sqrt{\frac{2|m|}{\pi L}}e^{-|m|L}$. Therefore, for large $|m|$, the quantity $\mathcal{F}/L_y$ with antiperiodic boundary condition reads

$$\frac{\mathcal{F}_{\text{APBC}}}{L_y} \sim -\frac{\pi c}{6L} + \sqrt{\frac{2|m|}{\pi L}}e^{-|m|L}, \tag{34}$$

with $c = 1$. As expected, the correction to $\mathcal{F}$ arising from the second term of Eq. (29) decays rapidly in the large mass limit and one recovers the exact CFT scaling.

For completeness, one can also compute the sum Eq. (29) for the more energetic case of periodic boundary conditions where the momentum in the x direction now takes values of $k_x \in \frac{2\pi}{L}\mathbb{Z}$, which would yield the result

$$\frac{\mathcal{F}_{\text{PBC}}}{L_y} \sim \frac{\pi}{3L} - \sqrt{\frac{2|m|}{\pi L}}e^{-|m|L}. \tag{35}$$

One can also use the general formula Eq. (33) to evaluate the first sum of Eq. (29). The first integral in Eq. (33) diverges in the UV, and it is proportional to L; the second integral leads to the desired result of $-\pi/(6L)$.

VI. CHERN INSULATOR WITH $\nu = 4$

If we start with the Chern insulator with $\nu = 4$, the physics should be qualitatively similar to the case with other Chern numbers, as long as the quantum operation only performs measurements on quantities that preserve the U(1) charge symmetry. The U(1) charge symmetry will always become a doubled symmetry in the density matrix formalism, and map to the $U(1)_L$ and $U(1)_R$ symmetry after the space-time rotation. Then the perturbative 't Hooft anomaly will exclude the possibility of a gapped "partition function" $\mathcal{Z} \sim \text{tr}\{\hat{\rho}_c^D \hat{\rho}_\Omega\}$.

Let us now break the doubled U(1) symmetry down to a diagonal U(1) symmetry, meaning we allow the quantum operation to measure all bosonic operators, such as all fermion bilinear operators including Cooper pairs. Effectively the $1d$ system after space-time rotation has four left-moving and four right-moving complex fermions, and there is only one global (diagonal) U(1) charge symmetry, when there is no postselection that breaks the diagonal U(1) symmetry. There is also a separate fermion parity of left and right chiral fermions: $Z_2^L \times Z_2^R$. The effect of the weak measurement is mapped to interacting terms of the $1d$ system. According to the classification of interacting TIs, four copies of $(1+1)D$ nonchiral complex fermions can be gapped out by four fermion interactions without breaking any symmetry or ground state degeneracy [44–47]. This mechanism also generalizes to other models and higher dimensions, and is in general referred to as symmetric mass generation (SMG) [48–61]. But the interaction terms in our case have a more restrictive form, and need to be generated by weak measurement. We have not found the corresponding Kraus operators that can generate the exact form of the interactions that have been definitely proven to cause SMG for the system. We leave the possibility of SMG caused solely by decoherence or weak measurement as an open question for future study.

By considering proper Kraus operators and their Hermitian conjugates that respect fermion parity, one can indeed gap out the CFT described by $\mathcal{Z}$ and $\mathcal{F}$. However we note here that the $(1+1)D$ system so obtained from space-time rotation is not a completely trivial state; instead it would spontaneously break the $Z_2^L \times Z_2^R$ symmetry down to a diagonal fermion parity. Let us consider the following quantum operation as an example:

$$\mathcal{E}_{\boldsymbol{x}}[\hat{\rho}_c] \sim \hat{\rho}_c + \sum_{i=1}^{4} p_i\big(\hat{O}_{i,\boldsymbol{x}}\hat{\rho}_c\hat{O}^\dagger_{i,\boldsymbol{x},} + \hat{O}^\dagger_{i,\boldsymbol{x}}\hat{\rho}_c\hat{O}_{i,\boldsymbol{x}}\big),$$
$$\hat{O}_{1,\boldsymbol{x}} = c^\dagger_{1,\boldsymbol{x}}\sigma^z(c^\dagger_{2,\boldsymbol{x}})^T \quad \text{and } \hat{O}_{i\neq 1,\boldsymbol{x}} = c^\dagger_{1,\boldsymbol{x}}\sigma^z c_{i,\boldsymbol{x}}. \tag{36}$$

Following our previous procedure, the fidelity is mapped to a path integral problem of four flavors of $(1+1)D$ Dirac fermions with the following interactions:

$$\mathcal{L}^{\text{int}} = \sum_{i\neq 1} g_i(\psi^\dagger_{1,L}\psi_{i,L}\psi^\dagger_{i,R}\psi_{1,R} + \text{H.c.})$$
$$- g_1(\psi^\dagger_{1,L}\psi^\dagger_{2,L}\psi_{2,R}\psi_{1,R} + \text{H.c.}). \tag{37}$$

These terms after Abelian bosonization become

$$\mathcal{L}^{\text{int}} = \sum_{i\neq 1} u_i\cos(2\phi_1 - 2\phi_i) - u_1\cos(2\phi_1 + 2\phi_2)$$
$$= \sum_{i\neq 1} u_i\cos\left(\Lambda_i^T\mathcal{K}\Phi\right) - u_1\cos\left(\Lambda_1^T\mathcal{K}\Phi\right). \tag{38}$$

The vector Φ is $\Phi = (\phi_{L,1},\ldots,\phi_{L,4};\phi_{R,1},\ldots\phi_{R,4})^T$ where $\phi_{L,i} = \theta_i - \phi_i$ and $\phi_{R,i} = \theta_i + \phi_i$, and the K matrix is $\mathcal{K} = \text{diag}(-1,\ldots,-1;1,\ldots,1)$. In this basis, one can verify that the four Λ-vectors take the following form:

$$\Lambda_1 = (1,-1,0,0;1,-1,0,0)^T,$$
$$\Lambda_2 = (1,0,-1,0;1,0,-1,0)^T,$$
$$\Lambda_3 = (1,0,0,-1;1,0,0,-1)^T,$$
$$\Lambda_4 = (1,1,0,0;1,1,0,0)^T. \tag{39}$$

We will outline the conditions that these vectors satisfy and show that they drive the theory into a Z_2 SSB state. A more complete discussion of these conditions and the Λ-vector formalism can be found in Refs. [47,62].

In order for these vertex operators to completely gap the system, the Λ vectors must be four linearly independent vectors that satisfy the so-called Haldane null-vector condition: $\Lambda_i^T\mathcal{K}\Lambda_j = 0$ for all i, j. This condition ensures that the fields can be rotated to a new basis where each vertex operator contains only one single field instead of a linear combination, and this condition is indeed met by the four vectors in Eq. (39). However, if these terms can indeed trivially gap out the $(1+1)D$ system without any degeneracy (i.e., SMG), it is necessary that the determinants of all possible 4×4 minors of the matrix with rows of Λ_i^T do not share a common factor larger than 1. One can verify that this condition is not met by the four interaction terms generated by Kraus operators chosen above, and our choice of Kraus operators must drive the boundary into a state that spontaneously breaks the fermion parity $Z_2^L \times Z_2^R$ down to its diagonal. As a consequence, the following fermion bilinears in the effective $(1+1)D$ theory acquire a nonzero expectation value,

$$\langle\psi^\dagger_{L,i}\psi_{R,i}\rangle \neq 0. \tag{40}$$

The correlation of these fermion bilinears is the following type-II strange correlator first introduced in Ref. [8] [Fig. 1(c)]:

$$\langle\psi^\dagger_{L,i}\psi_{R,i}(0,0)\psi^\dagger_{R,i}\psi_{L,i}(\tau,x)\rangle$$
$$\sim C^{\text{II}}(\boldsymbol{x}) = \frac{\text{tr}\big[c^\dagger_{i,\alpha}(0)c_{i,\alpha}(\boldsymbol{x})\hat{\rho}_c^D c^\dagger_{i,\alpha}(\boldsymbol{x})c_{i,\alpha}(0)\hat{\rho}_\Omega\big]}{\text{tr}(\hat{\rho}_c\hat{\rho}_\Omega)}, \tag{41}$$

where $i = 1,2,3,4$ and α labels the Dirac index. $\boldsymbol{x} = (x,y)$ labels the $2d$ spatial coordinate before the space-time rotation.

VII. SUMMARY AND DISCUSSION

In this work, we investigate the fidelity between the mixed state Chern insulator, i.e., the mixed state density matrix generated from a Chern insulator under quantum operations such as decoherence or measurement, and a pure state trivial insulator. We demonstrate that the fidelity is mapped to the partition function of a $(2+0)d$ or $(1+1)D$ CFT, and the Rényi relative entropy is mapped to the free energy of the CFT. This quantum operation can lead to a variety of CFTs with different central charges. We also devised a procedure to extract the central charges, without having to know the details of the order parameters that would lead to nontrivial strange correlators.

The Hermiticity of the density matrix demands that, after space-time rotation, the $(1+1)D$ theory is invariant under antiunitary transformation $\psi_L \to \psi_R^\dagger$, $\psi_R \to \psi_L^\dagger$, and $i \to -i$. This alone does not guarantee that the effective $(1+1)D$ CFT be unitary. Indeed, the strange correlator has been utilized as a tool to engineer nonunitary CFTs [9,63,64]. In our current work, the symmetries of the system and the operations ensure that at least there are no *relevant* non-Hermitian terms in the effective $(1+1)D$ theory.

As we have seen in this work, the effect of quantum operation is mapped to interaction terms in the fidelity, which corresponds to the partition function of the $(1+1)D$ CFT. The same physical picture also applies to all other free fermion topological insulators and topological superconductors (TSCs). But as we have mentioned, the quantum operation only realizes certain restricted forms of interactions. We leave a complete discussion of TIs and TSCs under quantum operations for a future study.

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