Case of thermodynamic failure in the Ginzburg-Landau approach to fluctuation superconductivity

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The Ginzburg-Landau approach postulates an energy density, together with an interpretation for the supercurrent, and invokes Ohm's law. We consider quasi-one-dimensional nonuniform superconducting loops, either smooth or piecewise uniform, that enclose a magnetic flux, above the critical temperature. We evaluate the averages of the current and of the power per unit length delivered by the electric field due to thermal fluctuations. We consider three averages: the canonical ensemble average, the canonical ensemble in the reciprocal space, and the time average using a time-dependent model. All the evaluations imply that heat is imparted to part of the loop and removed in the other part, despite the assumption that the loop is at uniform temperature.

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I. INTRODUCTION

The second law of thermodynamics asserts that within a system at uniform temperature there is no heat flux on average, no matter what its Hamiltonian is.

Here we examine the case of a nonuniform onedimensional (1D) superconducting loop that encloses a noninteger number of magnetic flux quanta, slightly above its critical temperature T_c (a situation beyond the realms considered in Refs. [1] or [2]). The present study was motivated by Ref. [3].

The questions which we intend to answer will be spelled out in the following two sections; in Secs. IV–VI we develop and apply three different techniques for the evaluation of the quantities of interest, and in Sec. VII we summarize our results.

II. FORMULATION OF THE PROBLEM

A. Statistical average

If q_1, \ldots, q_f are the coordinates of the phase space that describe the microstate of a system, $F(q_1, \ldots, q_f)$ is the energy of this system, and every part of the system is in equilibrium with a heat bath at temperature T, then the average value of any quantity $Q(q_1, \ldots, q_f)$ is predicted to be

$$\langle Q \rangle = \int Q(q_1, \dots, q_f) e^{-F/k_B T} dq_1, \dots, dq_f / \int e^{-F/k_B T} dq_1, \dots, dq_f,$$
(1)

where k_B is Boltzmann's constant and the integrals extend over the entire phase space.

B. Energy of the system

The system we consider consists of the electrons that are able to flow around a superconducting loop. For the sake of definiteness, let us also describe its environment: We may regard it as composed of four parts, i.e., the lattice of ions that form the loop, the fluid in which the loop is immersed, and the black-body electromagnetic fields within the loop and in the fluid.

For a situation in which the temperature and the applied magnetic field are controlled, the static Ginzburg-Landau (GL) formalism invokes a complex field ψ , known as the "order parameter," and the magnetic potential **A**, which describe the microstate of a superconductor and play the role of the coordinates q_1, \ldots, q_f in Sec. II **A**. The static GL formalism attributes to superconductivity the free energy density [4] $\alpha |\psi|^2 + \beta |\psi|^4/2 + (\hbar^2/2m)|(i\nabla - 2\pi \mathbf{A}/\Phi_0)\psi|^2$, where α and β depend on the superconducting material, *m* is the mass of a Cooper pair, $\Phi_0 = \pi \hbar c/e$ is the quantum of magnetic flux, and *e* is the absolute value of the electron charge. α is an increasing function of temperature and vanishes for $T = T_c$.

The total free energy of the system is the sum of the integral of the superconducting free energy density over the superconducting sample, the integral of the magnetic energy density over the entire space, and a term that does not depend on ψ or **A**. Ignoring fluctuations, the equilibrium values of ψ and **A** are those that minimize the total free energy and are compatible with the constraints of the experimental setup [5,6]. Since the entropy is not assumed to depend on ψ or **A**, if the temperature is kept fixed, the difference between the free energy and the energy becomes an irrelevant additive constant.

The GL free energy can be obtained from microscopic theory as an expansion, assuming that the order parameter is small. The static GL formalism is intended to be a minimal model that retains only those terms that are essential. If $\alpha < 0$, then the quartic term $\beta |\psi|^4/2$ has to be kept in order to obtain a free energy that is bounded from below. However, we will deal with a case in which $\alpha > 0$ and therefore the quartic term will be dropped, except for Sec. VID.

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We consider a superconducting loop of perimeter *L*, in which the linear dimensions of the cross section are much shorter than *L* and than the typical distances over which ψ and **A** vary; in this case the system becomes one dimensional, and the position is described by the arclength $x, 0 \le x \le L$.

Although we consider a situation above the critical temperature, the order parameter does not vanish because of thermal fluctuations. Neglecting terms of order $O(|\psi|^4)$, the GL energy of the system can be written as

$$F = \int_0^L \left[\alpha |\psi|^2 + \frac{\hbar^2}{2m} \left| \left(i \frac{d}{dx} - \frac{2\pi A}{\Phi_0} \right) \psi \right|^2 \right] w(x) dx, \quad (2)$$

where A is the tangential component of A and w(x) is the cross section of the superconducting wire.

We will assume that the magnetic self-inductance of the loop is negligible, as could be the case for a wide thin ribbon that surrounds a region with flattened cross section. As a consequence, the total magnetic flux will be just the applied flux. Consistently, we will assume that the contribution of the current (and therefore of ψ and A) to the magnetic Gibbs free energy is negligible in comparison to F.

The magnetic potential can be eliminated from (2) by defining

$$\tilde{\psi}(x) = \exp\left[\frac{2\pi i}{\Phi_0} \int_{x_1}^x A(x') dx'\right] \psi(x), \tag{3}$$

where x_1 is arbitrary and x can be extended beyond the range $0 \le x \le L$ by identifying x + L with x. Using (3), (2) becomes

$$F = \oint [\alpha |\tilde{\psi}|^2 + (\hbar^2/2m) |d\tilde{\psi}/dx|^2] w(x) dx.$$
 (4)

It follows from (3) that $\tilde{\psi}(x+L) = \tilde{\psi}(x) \exp(2\pi i \Phi/\Phi_0)$, with $\Phi = \oint A(x)dx$, so that $\tilde{\psi}$ is multivalued unless Φ/Φ_0 is an integer.

Although the microstates of the system are described by two fields, ψ and A, the accepted averaging procedure [7–11] includes only ψ in the phase space.

Let us now fix some of the parameters in the system to be considered. The length *L* of the loop, its electrical conductivity σ , its resistance *R* (regarding the loop as an open circuit), and the temperature *T* will be fixed. Unless explicitly stated, we will also fix $\alpha = \hbar^2/2mL^2$, so that the coherence length (the length over which ψ changes significantly) will equal the perimeter of the loop. As mentioned above, the total flux will be the applied flux, which we will fix as $\Phi = \Phi_0/4$, so that $\tilde{\psi}(x + L) = i\tilde{\psi}(x)$. These values define a typical cross section w_0 and a dimensionless quantity γ :

$$w_0 = L/\sigma R, \quad \gamma = \hbar/e^2 R.$$
 (5)

Several possibilities for the cross section w(x) will be considered. A smooth profile will be

$$w(x) = (2w_0/\sqrt{3})[1 - \cos(2\pi x/L)/2];$$
(6)

a family of discrete profiles will be described in the following section.

C. Discrete profiles

We divide the loop into *N* segments of length L/N, centered at $x = x_k := (\lambda + k - 1)L/N$ with $0 \le \lambda < 1$ and k = 1, ..., N. The cross section of segment *k* will be

$$w_k = (w_0/N)[1 - \cos(2\pi x_k/L)/2] \sum_{j=1}^{N} [1 - \cos(2\pi x_j/L)/2]^{-1}.$$
(7)

In the limit of large N, this profile becomes the smooth profile (6).

Let us now build a model for the energy of this system, motivated by the GL energy (4). Instead of the field $\tilde{\psi}(x)$, we assign the value $\tilde{\psi}_k$ to the entire segment k; the derivative $d\tilde{\psi}/dx$ is replaced by the finite difference $N(\tilde{\psi}_{k+1} - \tilde{\psi}_k)/L$, and we define

$$F_{N,\lambda} := \sum_{k=1}^{N} \left\{ \frac{\alpha L w_k}{N} |\tilde{\psi}_k|^2 + \frac{\hbar^2 N w_{k+}}{2mL} |\tilde{\psi}_{k+1} - \tilde{\psi}_k|^2 \right\}, \quad (8)$$

where w_{k_+} is some average between w_k and w_{k+1} , and $\bar{\psi}_{N+1} = \exp(2\pi i\Phi/\Phi_0)\tilde{\psi}_1$. We have studied two possibilities for w_{k_+} : $w_{k_+} = (w_k + w_{k+1})/2$ and $w_{k_+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$.

We still require discretizations for ψ and A. We assign the magnetic potential A_k^+ (A_k^-) to the half segment $x_k < x < x_k + L/2N$ ($x_k > x > x_k - L/2N$) and define

$$\psi_{1} := \psi_{1};$$

$$\tilde{\psi}_{k} := \exp\left[\frac{\pi i L}{N\Phi_{0}} \left(A_{1}^{+} + A_{k}^{-} + \sum_{j=2}^{k-1} (A_{j}^{-} + A_{j}^{+})\right)\right] \psi_{k}$$
for $2 \leq k \leq N.$
(9)

In terms of ψ and A, the energy is

$$F_{N,\lambda} = \sum_{k=1}^{N} \left\{ \frac{\alpha L w_{k}}{N} |\psi_{k}|^{2} + \frac{\hbar^{2} N w_{k_{+}}}{2mL} (|\psi_{k}|^{2} + |\psi_{k+1}|^{2} - 2 \operatorname{Re}[\psi_{k+1}\psi_{k}^{*}e^{(\pi i L/N\Phi_{0})(A_{k}^{+}+A_{k+1}^{-})}]) \right\}.$$
 (10)

Although the systems described here approach the smooth loop only in the limit of large N, any of them, for any N, λ , and $\{w_{k_+}\}$, can be regarded as a self-consistent model that should obey the laws of thermodynamics.

III. QUANTITIES OF INTEREST

Unlike "classical" equilibrium thermodynamics, statistical mechanics can deal not only with state functions, but also with transport quantities such as the total current and heat transport. Expressions for these quantities will be obtained in this section.

In the case of the smooth loop, the supercurrent density is given by [4]

$$J_{S}(x) = -\frac{2e\hbar}{m} \operatorname{Im} \left[\psi^{*} \left(\frac{d}{dx} + \frac{2\pi i}{\Phi_{0}} A \right) \psi \right]$$
$$= -\frac{2e\hbar}{m} \operatorname{Im} \left[\tilde{\psi}^{*} \frac{d\tilde{\psi}}{dx} \right].$$
(11)

TABLE I. Average current in units of ek_BT/\hbar . The "Method" column directs the reader to the section where the evaluation method is described. Here, a, arithmetic mean $w_{k_+} = (w_k + w_{k+1})/2$; h, harmonic mean $w_{k_+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$. The pairs of numbers in the column headings stand for N and λ . Unless explicitly mentioned, $\alpha = \hbar^2/2mL^2$ and $\beta = 0$ in all figures and tables.

Method	w_{k_+}	3,0	3,0.5	4,0	4,0.25	5,0	10,0	15,0
Sec. IV	а	-1.291	-1.286	-1.271	-1.270	-1.259	-1.238	-1.234
Sec. IV	h	-1.215	-1.252	-1.227	-1.232	-1.230	-1.230	-1.230
Sec. VI	h	-1.214	-1.251	-1.224	-1.228	-1.229	-1.225	-1.228

Although the expression for J_S looks the same as the expected current density of a charged particle described by the Schrödinger equation, we should note that in the Schrödinger case this expression follows from charge conservation, whereas in the GL case it follows from variation of the energy functional with respect to A(x). Similarly, in the discrete case, in the half segment where the magnetic potential is A_k^{\pm} , the supercurrent density is given by $J_{Sk}^{\pm} = -(2Nc/Lw_k)\partial F_{N,\lambda}/\partial A_k^{\pm}$, where the meaning of the partial derivative is that the magnetic potential in the other regions of the loop, and also $\{\psi_j\}$, are kept constant. Using (10) and (9), we obtain

$$J_{Sk}^{+} = -(2Ne\hbar w_{k+}/mLw_{k})\text{Im}[\tilde{\psi}_{k}^{*}\tilde{\psi}_{k+1}];$$

$$J_{Sk}^{-} = -(2Ne\hbar w_{(k-1)+}/mLw_{k})\text{Im}[\tilde{\psi}_{k-1}^{*}\tilde{\psi}_{k}].$$
(12)

We denote by *I* the total current around the loop. As usual, we assume electroneutrality, so that *I* does not depend on *x*. By Ohm's law, the electric field is $E(x) = [I/w(x) - J_S(x)]/\sigma$ in the smooth case and

$$E_k^{\pm} = (I/w_k - J_{Sk}^{\pm})/\sigma \tag{13}$$

in the discrete case. Since the magnetic flux through the loop remains constant, the circulation of the electric field vanishes, and the current is

$$I = \oint J_S dx \Big/ \oint w^{-1} dx \tag{14}$$

in the smooth case and, noting that $\sum_{k=1}^{N} w_k^{-1} = N/w_0$,

$$I = \frac{w_0}{2N} \sum_{k=1}^{N} (J_{Sk}^- + J_{Sk}^+)$$
(15)

in the discrete case.

The power per unit length delivered by the electric field to the flowing charges is

$$P(x) = IE(x). \tag{16}$$

If this power is not taken somewhere else by the supercurrent, it has to be passed as heat to the environment. We note that, since I is uniform and E(x) is conservative, the total power delivered by the electric field is zero.

For the sake of definiteness, let us focus our attention on a segment of the lattice of ions close to the position x. If a fraction κ_1 of the work performed by the electric field originates from this segment, and a fraction κ_2 of the heat released by the flowing charges is passed to this segment, then the internal energy of the segment per unit length and time will increase by $(\kappa_2 - \kappa_1)P$. This energy cannot accumulate at the segment, and since the environment is at uniform temperature, implying that there will be no heat flow on average, the average of

 $(\kappa_2 - \kappa_1)P$ has to vanish. κ_1 and κ_2 are not expected to equal one another, because they involve different mechanisms; for example, the electric field at the segment is influenced by distant segments [12], and this field performs work also on Cooper pairs. Therefore the thermodynamic requirement is that the average of P(x) has to vanish at every position along the loop.

Since the expression (4) for the energy penalizes the gradient of $\tilde{\psi}$, at first glance one might think that this gradient vanishes on the average, leading to $\langle I \rangle = 0$, but the gradient cannot vanish because generically $\tilde{\psi}(x + L) \neq \tilde{\psi}(x)$. If the energy density in (4) were not weighed by w(x), then, by symmetry, the average of this gradient would have uniform size. This uniform situation was considered in Ref. [3], and in this case $\langle P \rangle = 0$. In the present situation, the smaller w(x)is, the larger are the fluctuations and the resistance per length near x, and we should anticipate more complex behavior.

IV. EVALUATION OF AVERAGES FOR DISCRETE SYSTEMS

The sets $\{\psi_k\}$ and $\{\tilde{\psi}_k\}$ cover the same phase space, so that it makes no difference over which of them we average. For further simplification, we diagonalize (8) numerically, taking the form

$$F_{N,\lambda} = \sum_{k=1}^{N} f_k |\varphi_k|^2.$$
 (17)

The passage from $\{\bar{\psi}_k\}$ to $\{\varphi_k\}$ is a rotation in the phase space, and any of them can be used for averaging.

The electric field in (13) and the current in (15) are quadratic expressions. $\langle \varphi_j^* \varphi_i \rangle$ vanishes for $j \neq i$, and $\langle \varphi_k^* \varphi_k \rangle = k_B T / f_k$. We found $\langle E_k^{\pm} \rangle = 0$ in all the cases considered, and the values of the average currents obtained in this way are shown in the first two rows of Table I. Note that the current can be expressed solely in terms of $ek_B T / \hbar$ because α has been fixed.

The average power per unit length, $\langle P_k^{\pm} \rangle$, is obtained from Eq. (16). IE_k^{\pm} is a long linear combination of components of the form $\varphi_i^* \varphi_j \varphi_\ell^* \varphi_m$. Most of these components give no contribution to the average, and the only contributions are those of the forms $\langle (\varphi_i^* \varphi_i)^2 \rangle = 2(k_B T/f_i)^2$ and $\langle \varphi_i^* \varphi_i \varphi_j^* \varphi_j \rangle =$ $(k_B T)^2/f_i f_j$ for $i \neq j$. Representative results obtained for small values of N are shown in Fig. 1. From the second law of thermodynamics and the fact that the temperature is uniform, we would expect no local energy flow from the system to the heat bath on average, i.e., $\langle P_k^{\pm} \rangle$ should vanish identically (as would be the case if I and E_k^{\pm} were uncorrelated). Instead, we see in Fig. 1 that, with the exception of the case



FIG. 1. Average power passed to the flowing charges per unit length [Eq. (16)], as a function of position, evaluated as described in Sec. IV, for small values of *N*. The icons at the top of the figure depict the cross section of the system considered, with the position where x = 0 and x = L join at the bottom of the icon. Solid lines stand for $w_{k_+} = (w_k + w_{k+1})/2$, and dashed lines stand for $w_{k_+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$. Red, N = 3, $\lambda = 0$; black, N = 3, $\lambda =$ 0.5; blue, N = 4, $\lambda = 0.25$. γ is defined in Eq. (5).

 $(N, \lambda) = (3, 0)$ with arithmetic mean, the flowing charges receive power from the electric field at the thinnest part of the loop and lose it at the broadest part.

We would like to know whether this disagreement with thermodynamics is a feature of discretization or a property of the GL approach. With this purpose, we investigate the behavior of $\langle P_k^{\pm} \rangle$ as N increases and the smooth limit is approached. Judging by the N dependence of $\langle I \rangle$, we expect a faster convergence when $\{w_{k_+}\}$ are taken as harmonic means. (The values of $\langle I \rangle$ for N = 100 and N = 120 coincide within seven significant figures). The results are shown in Fig. 2. Surprisingly, the effect reverses at intermediate values of N, and as the smooth limit is approached, the flowing charges



FIG. 2. Average power per unit length delivered by the electric field to the flowing charges at the thinnest (blue) and at the broadest (orange) segment of the loop, as functions of *N*. In all cases, *N* is an integer multiple of 4, $w_{k+} = 2(w_k^{-1} + w_{k+1}^{-1})^{-1}$, and $\lambda = 0$. When passing from N = 12 to N = 16, the signs of $\langle P(0) \rangle$ and $\langle P(L/2) \rangle$ reverse.



FIG. 3. The colored segments show the average power per unit length as a function of position, obtained as described in Sec. IV, for N = 30. The purple lines were evaluated taking w_{k_+} as an arithmetic mean, and the other lines were evaluated taking w_{k_+} as a harmonic mean. Purple, $\lambda = 0$; red, $\lambda = 0$; blue, $\lambda = 0.25$; green, $\lambda = 0.5$. The pink segments at x = 0, x = L/2, and x = L mark the limit $N \to \infty$, obtained from extrapolation in Fig. 2. The black curve and the pink circles were obtained as explained in Sec. V, with M = 15 for the curve and M = 50 for the circles.

receive power at the broadest part of the loop and lose it at the thinnest part.

The strong N dependence of $\langle P(x) \rangle$, which can even reverse its sign, is related to the coherence length. If we diminish the coherence length by a factor of 2, by setting $\alpha = 2\hbar^2/mL^2$, then the crossover found in Fig. 2 occurs between N = 4and N = 8. In the opposite direction, if the coherence length increases by a factor of 2, then the crossover occurs between N = 36 and N = 40 for $\langle P(L/2) \rangle$, and between N = 48 and N = 52 for $\langle P(0) \rangle$. In both cases, the larger the coherence length, the larger the tendency for power capture by the field at the broad part of the loop and power delivery by the field at the thin part. The N dependence of the sign of $\langle P \rangle$ can be therefore interpreted as an effective increase of the coherence length due to the requirement of a uniform value of $\tilde{\psi}(x)$ along each entire segment; the smaller N is, the longer the segments and the larger the effective coherence length. The impact of the coherence length will be examined again, for the case of a smooth loop, in Sec. V.

Figure 3 enables us to appraise the influence of discretization on the power delivered, for a moderately large value of N. All the short segments that delineate a bell shape were obtained for N = 30, but using different averages for w_{k_+} or different orientations λ . The fact that all the results nearly coalesce indicates that discretization has no qualitative influence and suggests that, if $\langle P(x) \rangle$ does not vanish for moderately large N, it will also not vanish for the smooth loop.

V. AVERAGING IN THE RECIPROCAL SPACE

In the case of a smooth profile we can write $\tilde{\psi}$ as a modified Fourier series,

$$\tilde{\psi}(x) = \sum b_n e^{2\pi i (n+1/4)x/L}.$$
 (18)

This sum extends over all integers, but we will take $|n| \leq M$, where *M* will be increased until acceptable convergence is obtained.

From (11) and (18),

$$J_{S}(x) = -\frac{4\pi e\hbar}{mL} \operatorname{Re}\left[\sum_{-M \leqslant \ell, n \leqslant M} b_{\ell}^{*} b_{n} \left(n + \frac{1}{4}\right) e^{2\pi i (n-\ell)x/L}\right];$$
(19)

from (14) and (19), using (6),

$$I = -\frac{4\pi e\hbar w_0}{mL} \sum_{-M \leqslant n \leqslant M} \left(n + \frac{1}{4}\right) b_n^* b_n;$$
(20)

and from (4), (6), and (18)

$$F = \frac{2w_0L}{\sqrt{3}} \Biggl\{ \sum_{n=-M}^{M} \Biggl[\alpha + \frac{(2\pi\hbar)^2}{2mL^2} \Biggl(n + \frac{1}{4} \Biggr)^2 \Biggr] b_n^* b_n - \frac{1}{4} \sum_{n=-M}^{M-1} \Biggl[\alpha + \frac{(2\pi\hbar)^2}{2mL^2} \Biggl(n + \frac{1}{4} \Biggr) \Biggl(n + \frac{5}{4} \Biggr) \Biggr] \times (b_{n+1}^* b_n + b_{n+1} b_n^*) \Biggr\}.$$
 (21)

We can now proceed as in Sec. IV. We diagonalize (21), so that it takes the form $F = \sum_{n=-M}^{M} f_n |c_n|^2$. It follows that the only quadratic terms with nonzero average are $\langle c_j^* c_j \rangle = k_B T / f_j$, and the only quartic terms that contribute are those of the forms $\langle (c_i^* c_i)^2 \rangle = 2(k_B T / f_i)^2$ and $\langle c_i^* c_i c_j^* c_j \rangle = (k_B T)^2 / f_i f_j$ for $i \neq j$. Finally, the average of the power per length in (16) is evaluated as

$$\langle P(x)\rangle = \frac{\hbar w_0 \langle I[I - J_S(x)w(x)]\rangle}{\gamma e^2 L w(x)}.$$
 (22)

The values of the average current obtained in this way converge slowly for increasing *M*. For M = 5, 10, 15, and 20 we obtain $\hbar \langle I \rangle / ek_B T = -1.289$, -1.261, -1.251, and -1.246, respectively.

The values of $\langle P(x) \rangle$ obtained from (22) for M = 15 are shown by the black curve in Fig. 3. This curve is not expected to coincide with the results obtained in Sec. IV, because the limits $M, N \rightarrow \infty$ have not been reached. Nevertheless, this figure strongly suggests that all the results would coincide in this limit and rebuts the possibility that $\langle P(x) \rangle \equiv 0$.

The analysis of the smooth loop enables us to disentangle the influence of the coherence length from that of the segments' length found in Sec. IV. Figure 4 shows the delivered powers as functions of position for a wide range of values of α . As expected, these powers vanish for coherence length much shorter than L and saturate for coherence length much larger than L. As in Sec. IV, the electric field delivers energy to the flowing charges in the thin (broad) part of the loop when the coherence length is sufficiently large (short), and the crossover occurs for $\alpha \sim 0.05\hbar^2/mL^2$. The largest energy transfers are found for a coherence length that is about a third of the perimeter of the loop.

The evidence of this and the previous section leads us to conclude that the GL approach does not comply with



FIG. 4. Powers delivered per unit length, as functions of position, for a smooth loop. The values of $\alpha m L^2/\hbar^2$ are shown next to each line. Since $\langle P(L-x) \rangle = \langle P(x) \rangle$, only the range $0 \le x \le L/2$ is shown.

thermodynamics, for either a discrete or a smooth nonuniform loop. We might think of several reservations to this conclusion: (i) Johnson noise has to be added to Eq. (13); (ii) the power that the flowing charges release should not be interpreted as energy transferred to the heat bath, but rather as potential energy that is converted to kinetic energy of the moving charges and then transported as an energy supercurrent; (iii) the magnetic potential A should be included in the phase space; (iv) besides the work that the electric field can perform on the charges, our system can also exchange energy by breaking Cooper pairs [i.e., by changing the value of $|\psi(x)|$]; and (v) the currents shown in Table I do not become negligible in the limit $w_0 \rightarrow 0$. Moreover, ψ has to diverge in the limit of small cross section, and therefore the quartic term in the free energy density cannot be neglected. In order to address these reservations we require a model for time evolution.

VI. TIME-DEPENDENT MODEL

In this section we discretize time into steps Δt that are sufficiently small to justify regarding the changes in ψ and A during a step as infinitesimal.

A. Johnson noise and evolution equations

During a step Δt , the Johnson noise adds to the electric field in Eq. (13) an additional term $\eta_{k,\pm}^A \sqrt{4k_BTN/\sigma w_kL\Delta t}$, where $\eta_{k,\pm}^A$ and $\eta_{k,-}^A$ stand for random variables with zero average, variance 1, and normal distribution. Choosing a gauge such that $cE_k^{\pm} = -\Delta A_k^{\pm}/\Delta t$, where $\Delta A_k^{\pm}(t) = A_k^{\pm}(t + \Delta t) - A_k^{\pm}(t)$ and using (5), Eq. (13) becomes an evolution equation for $A_k^{\pm}(t)$:

$$\Delta A_k^{\pm} = -(cw_0\hbar/\gamma e^2 L)(I/w_k - J_{Sk}^{\pm})\Delta t$$
$$-(2c/eL)\eta_{k+}^A\sqrt{k_BT\hbar Nw_0\Delta t/\gamma w_k}. \tag{23}$$

Equation (23) is implied by the time-dependent Ginzburg-Landau (TDGL) model, which has Ohm's law built into it. The TDGL model is the simplest dynamical model that converges to the GL model when equilibrium is attained. Ignoring fluctuations, for a system in which there are no supercurrents in equilibrium (provided that fluctuations are ignored), and taking a gauge with no electrical potential, TDGL states that $d\psi/dt \propto -\delta F/\delta \psi^*$ and $dA/dt \propto -\delta F/\delta A$ [5,13] or, in discretized form, $\Delta \psi_k/\Delta t \propto -w_k^{-1}\partial F_{N,\lambda}/\partial \psi_k^*$ and $\Delta A_k^{\pm}/\Delta t \propto -w_k^{-1}\partial F_{N,\lambda}/\partial A_k^{\pm} \propto J_{Sk}^{\pm}$. From the equation $\Delta A_k^{\pm}/\Delta t \propto J_{Sk}^{\pm}$, subjected to the constriction $\sum_{k=1}^N (A_k^- + A_k^+) = \text{const}$, and adjusting the constant of proportionality to the conductivity of the loop, we recover Eq. (23) without the stochastic term. The last term is a Langevin term, the variance of which is determined by the fluctuation-dissipation theorem [14].

Similarly, from the equation $\Delta \psi_k / \Delta t \propto -w_k^{-1} \partial F_{N,\lambda} / \partial \psi_k^*$,

$$\frac{\Delta\psi_{k}}{\Delta t} = -\frac{\Gamma\alpha}{\hbar}\psi_{k} - \frac{N^{2}\hbar\Gamma}{2mL^{2}w_{k}} \Big[w_{k+} \big(\psi_{k} - \psi_{k+1}e^{(\pi iL/N\Phi_{0})(A_{k}^{+}+A_{k+1}^{-})}\big) + w_{(k-1)+} \big(\psi_{k} - \psi_{k-1}e^{-(\pi iL/N\Phi_{0})(A_{k-1}^{+}+A_{k}^{-})}\big)\Big] + \eta_{k}^{\psi} \sqrt{\frac{N\Gamma k_{B}T}{\hbar L w_{k} \Delta t}},$$
(24)

where Γ is a dimensionless material constant and both the real and the imaginary parts of η_k^{ψ} are random variables with zero average, variance 1, and normal distribution. Using (9) and assuming that Δt , ΔA_j^{\pm} , and $\Delta \psi_k$ are sufficiently small, (24) gives the evolution of $\tilde{\psi}_k$:

$$\tilde{\psi}_{k}(t+\Delta t) = \left\{ \left(1 - \frac{\Gamma \alpha \Delta t}{\hbar}\right) \tilde{\psi}_{k}(t) + \frac{N^{2} \hbar \Gamma \Delta t}{2mL^{2} w_{k}} \left[w_{(k-1)_{+}} \tilde{\psi}_{k-1}(t) - (w_{(k-1)_{+}} + w_{k_{+}}) \tilde{\psi}_{k}(t) + w_{k_{+}} \tilde{\psi}_{k+1}(t)\right] + \eta_{k}^{\psi} \sqrt{\frac{N \Gamma k_{B} T \Delta t}{\hbar L w_{k}}} \right\} \operatorname{prod}_{k},$$
(25)

with

$$\operatorname{prod}_{1} = 1; \ \operatorname{prod}_{j+1} = \operatorname{prod}_{j} \exp[iLe(\Delta A_{j}^{+} + \Delta A_{j+1}^{-})/Nc\hbar].$$
(26)

The TDGL model is valid for gapless superconductivity. This is the situation that we are considering, since $\alpha > 0$. At any rate, in this paper we are concerned only with the question of compatibility of the TDGL model with thermodynamics, and not with other possible limitations. We point out that, since we are considering uniform temperature, we do not have to deal with heat diffusion, as in Eq. (4) of Ref. [15].

We note that γ and Δt enter (23) only through their ratio; likewise, Γ and Δt enter (25) only through their product. Therefore, changing γ and Γ while keeping their product unchanged leads to the same evolution, although at a different rate. Anyway, in this paper we are interested only in equilibrium values, which should not be affected by the choices of γ and Γ .

Unlike the procedure in Sec. IV, which did not include the magnetic potential in the phase space, in the TDGL model the fields A and ψ stand on the same footing.

B. Energy current and pair creation energy

We may decompose $F_{N,\lambda}$ in (8) into a sum of terms F_k and F_{k_+} , where F_k is the term that contains $|\tilde{\psi}_k|^2$ and F_{k_+} contains $|\tilde{\psi}_{k+1} - \tilde{\psi}_k|^2$. We may regard F_k as located in segment k and F_{k_+} as located at the boundary between k and k + 1.

A change $\Delta \tilde{\psi}_k$ in $\tilde{\psi}_k$ leads to changes ΔF_k , ΔF_{k_+} , and $\Delta F_{(k-1)_+}$ in F_k , F_{k_+} , and $F_{(k-1)_+}$. The time average of Re $[\tilde{\psi}_k^* \Delta \tilde{\psi}_k]$ vanishes, and accordingly we take only cross terms into account. $\Delta F_{k_+} + \Delta F_{(k-1)_+}$ may be interpreted as the change in the energy in segment *k* due to pair production, whereas $I_{Ek}\Delta t = (\Delta F_{k_+} - \Delta F_{(k-1)_+})/2$ may be interpreted as the energy transported by the supercurrent from the negative boundary to the positive boundary of segment k. From (8) we obtain

$$I_{Ek}\Delta t = (\hbar^2 N/2mL) \operatorname{Re}[\Delta \tilde{\psi}_k (w_{(k-1)_+} \tilde{\psi}_{k-1} - w_{k_+} \tilde{\psi}_{k+1})^*].$$
(27)

Equation (27) can be identified as a discretized version of Eq. (D) in Ref. [13], adapted to the case of a nonuniform wire.

C. Results for sections VIA and VIB

The averages in this section are not obtained from Eq. (1), but are rather time averages. We took $\gamma \Gamma = 1$, and for evaluation of the average current, we took $\Delta t = 2 \times 10^{-4} \gamma \hbar/k_B T$ and followed the evolution equations (23) and (25) during 2×10^{10} steps for N < 10 (10^{10} steps for $N \ge 10$). The initial value of $\tilde{\psi}$ was zero, and the 10^7 initial steps were not included in the average. Our results for the current are shown in the last row in Table I.

Comparison of the last two rows in Table I shows that these two apparently independent procedures lead to the same results, but the time averages have a random uncertainty, slightly larger than $10^{-3} ek_B T/\hbar$, which can be attributed to the fact that the averaging spanned time was not infinite. There is also a systematic discrepancy of a similar size, which can be attributed to the fact that Δt was not infinitesimal.

Evaluation of $\langle I_E \rangle$ and $\langle P \rangle$ required longer periods of time and was limited to small values of N. For N = 5, the evolution of $\tilde{\psi}$ and A was followed during 1.6×10^{11} steps of $\Delta t = 2 \times 10^{-3} \gamma \hbar/k_B T$ and, for N = 8, during 10^{11} steps of $\Delta t = 10^{-3} \gamma \hbar/k_B T$. For every k, $\langle \Delta F_{k_+} \rangle / \Delta t$ vanished within its statistical uncertainty, which was of the order of



FIG. 5. Powers per unit length delivered by the electric field to the flowing charges, as functions of position. The horizontal lines were evaluated using Eq. (1), and the vertical lines are time averages, obtained as explained in Sec. VI. The vertical lines are centered at the time averages, and their half-lengths equal the standard deviation divided by the square root of the number of steps. The horizontal lines are free of statistical uncertainty. Black, N = 8, $\lambda = 0.5$; red, N = 5, $\lambda = 0.1$. w_{k_+} was taken as the harmonic mean of w_k and w_{k+1} .

 $10^{-4} (k_B T)^2 / \gamma \hbar$, and therefore the energy supercurrent and the pair production energy had no significant influence on the transferred powers.

Figure 5 compares the local delivered powers found in this section with those obtained using Eq. (1). Within the expected statistical uncertainty, the results obtained by the two procedures agree with each other in all the cases, and in most cases they are convincingly different from zero.

Figure 5 shows that Johnson noise does not clear away the local power transfer. Moreover, if we use (13) instead of (23), we obtain different results for $\langle I \rangle$, $\langle I_E \rangle$, and $\langle P \rangle$, indicating that this noise can actually be regarded as a source of the power transfer distribution.

D. Influence of the quartic term

When fluctuations are present, any coherent part of the system acquires energies of the order of k_BT . The smaller w_0 is, the smaller is the volume of these parts and the larger are the energy densities and the values that $|\psi|$ has to reach. However, the GL formalism relies on the premise that the energy density can be expanded in powers of $|\psi|$. Therefore the situation we are considering is such that w_0 is sufficiently small to permit a 1D treatment but sufficiently large to permit invoking the GL formalism.

In this section we want to examine whether the nonzero powers that we have found are an artifact of the quadratic form of the energy density that we have kept, and will be wiped out if the quartic term is not neglected. Including this term is straightforward: All we have to do is replace α with $\alpha + \beta |\tilde{\psi}|^2$ in Eqs. (24) and (25).

Table II displays the results that we have obtained for several values of α and β , for a particular discrete loop and for the particular temperature $T = \hbar^2/k_B m L^2$. We present only one "power per length," which was evaluated in the four half

TABLE II. Current and power per length for the profile N = 5, $\lambda = 0.1$, for temperature $T_0 = \hbar^2/k_BmL^2$. α is in units of \hbar^2/mL^2 , β is in units of \hbar^2w_0/mL , and the power per length is in units of $(k_BT_0)^2/\gamma \hbar L$. The power per length was evaluated in the region $-0.18 \le x/L \le 0.22$ and has a statistical uncertainty of ~ 0.004 . The global correlation is defined as $\prod_{j=1}^{N} [|\langle \tilde{\psi}_j^* \exp(-2\pi i \Phi/N \Phi_0) \tilde{\psi}_{j+1} \rangle]/\langle |\tilde{\psi}_j \tilde{\psi}_{j+1} \rangle].$

α	β	$\langle I \rangle$	Power per length	Global correlation
-0.5	1	-1.076	0.133	0.392
0	1	-0.830	0.095	0.331
0	2	-0.571	0.057	0.251
0.005	0	-1.971	0.297	0.548
0.5	0	-1.222	0.153	0.426
0.5	0.5	-0.827	0.087	0.331
0.5	1	-0.652	0.061	0.279
0.5	2	-0.471	0.038	0.216
1	0	-0.846	0.083	0.338

segments in the region $(\lambda - 1)L/N \le x \le (\lambda + 1)L/N$, at the thin part of the loop.

The results in Table II indicate that the power per length is not determined by the relative size of the quartic term in comparison to the quadratic term. It appears that the power per length is determined by the "global correlation," which we define as $\prod_{j=1}^{N} [|\langle \tilde{\psi}_{j}^{*} \exp(-2\pi i \Phi/N\Phi_{0})\tilde{\psi}_{j+1}\rangle]/\langle|\tilde{\psi}_{j}\tilde{\psi}_{j+1}|\rangle].$ We conclude that though a large quartic term diminishes the power transfers, it does not eliminate them.

VII. SUMMARY AND EXPECTATIONS

Based on the Ginzburg-Landau model for superconductivity, we have evaluated the average local power absorbed (or released) from the fluctuating electric field by the circulating current, in a family of systems that represent the flowing charges in a loop with nonuniform cross section threaded by a magnetic flux. The evaluation was performed following three apparently independent procedures.

Our results are independent of the followed procedure and violate the requirement that in a system in equilibrium with a heat bath at uniform temperature there should be no heat flow.

The present study analyzes a gedanken experiment: We have dealt with a question of principle, and not with technical feasibility. Let us now estimate the performance that the Ginzburg-Landau approach predicts for an aluminium loop of perimeter $L = 10^{-6}$ m, mean cross section $w_0 = 10^{-14}$ m², and residual resistivity $10^{-8} \Omega$ m. We first check whether the requirements of small coefficients α and β are met. The value of α can in principle be chosen by controlling the temperature; the values of β , according to Table II, should not be much larger than $\hbar^2 w_0/mL \sim 6 \times 10^{-47}$ kg m⁵ s⁻², whereas the value of β for aluminium is smaller by seven orders of magnitude, indicating that also superconducting materials with stronger coupling would be permissible. The predicted average current is of the order of $ek_BT_c/\hbar \sim 10^{-8}$ A, and the power per length is of the order of $10^{-1} (ek_B T_c)^2 / \hbar^2 \sigma w_0 \sim$ $10^{-10} \text{ W m}^{-1}$

Conceivably, the disagreement between the GL approach and the experimental measurements [16] for small

pair-breaking interaction is related to the thermodynamic failure found here. This disagreement can be amended by addition of the terms found by Maki [17] and Thompson [18]. Thus far we have not investigated whether the addition of correction terms to the GL formalism could lead to compliance with thermodynamics.

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