Eliashberg-like theory for Dynes superconductors

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(Received 6 October 2023; accepted 21 December 2023; published 10 January 2024)

The existing literature on the influence of impurities on superconductivity follows two different approaches. In the first one, a simple BCS-like instantaneous electron-electron interaction is considered, and special care is taken to treat the scattering on impurities as accurately as possible. On the other hand, the second approach starts with more realistic phonon-induced retarded electron-electron interactions, but treats the impurities only within the simplest Born approximation. Here we develop a theory combining the strengths of both of these approaches. This allows us to describe the recently introduced Dynes superconductors within an Eliashberg-like theory. One of the results of the developed theory is that the Dynes scattering parameter may depend on temperature, in qualitative agreement with recent experiments.

DOI: 10.1103/PhysRevB.109.014507

I. INTRODUCTION

The relation between the BCS model and real-life superconductors is the same as that between the free Fermi-gas model and real-life normal metals. As emphasized by Anderson [1], both models are believed to exhibit the correct symmetries of the superconducting and normal-metallic states, respectively. However, in order to provide finite values of certain physical quantities (e.g., resistivity in the normal state), both models have to take into account that the quasiparticle lifetime has a finite value.

As is well known, finite lifetime can be caused by elastic scattering on disorder and/or by inelastic scattering on dynamic excitations of the system. Let us start by considering the case of disorder scattering. In normal metals, in many cases such processes can be taken into account by a simple substitution of the energy ω by $\omega + i\Gamma$, where Γ is the inverse lifetime [2].

On the other hand, in superconductors it is essential to distinguish between two different types of scattering processes, depending on their action on the Cooper pairs. In order to simplify the discussion, in this paper we will consider single-band isotropic superconductors. If the scattering does not affect the Cooper pairs, for instance, if it is due to presence of finite time-reversal invariant potential disorder, it is called pair-conserving and the rate of such scattering will be denoted as Γ^{S}. In the opposite case, i.e., when the scattering does affect the Cooper pairs, as in the presence of finite time-reversal breaking magnetic disorder, it is called pair-breaking, with the corresponding rate to be denoted Γ^{M}. It has been known for a long time that, due to Anderson's theorem [3], weak pair-conserving scattering does not affect the thermodynamic properties of superconductors, while pair-breaking scattering does strongly disturb the superconducting state. Therefore, obviously, a superconductor has to be characterized by at least two lifetimes, one for each type of processes [2].

However, if pair-breaking scattering is present, no closed-form expressions for the electron propagator can be found at the simplest level of the self-consistent Born approximation [2]. Therefore, it came as a big surprise that, when disorder was described within the much more sophisticated coherent potential approximation (CPA) [4], such closed-form expressions featuring two lifetimes could in fact be found under only some mild assumptions about the distribution of the disorder [5]. The resulting description of two-lifetime superconductors has been dubbed the theory of Dynes superconductors, because, *inter alia*, it does explain the frequently observed phenomenological Dynes formula [6] for the tunneling density of states $N(\omega)$. The Dynes formula describes $N(\omega)$ by two parameters: the gap magnitude Δ and the Dynes parameter Γ. In [5] it has been shown that, within the theory of Dynes superconductors, Γ is equal to the pair-breaking scattering rate Γ^{M}, and therefore it does not depend on temperature.

On the other hand, due to Migdal's theorem, it is widely believed that inelastic scattering of electrons on phonons and the resulting effective phonon-mediated electron-electron interactions are well described by the self-consistent Born approximation, resulting in the well-known Eliashberg theory [2]. A careful analysis of the theory shows that, at a finite temperature T, the low-energy region of the density of states $N(\omega)$ is also described by the Dynes formula [7]. However, in this case, the Dynes parameter Γ does depend on temperature. As a matter of fact, the temperature dependence of Γ is very strong in this case: at $T = 0$, Γ vanishes, and with increasing temperature it grows, until reaching the normal-state rate of scattering on the phonons at T_c.

This means that there exist two independent explanations of the Dynes formula, taking into account either elastic or inelastic scattering. Unfortunately, the existing explanations make use of very different formalisms: CPA for elastic scattering [5], and self-consistent Born approximation in the inelastic case [7].

The goal of this paper is to develop a unified theory, to be called E-CPA, with the following two properties: (i) When the retarded phonon-mediated electron-electron interaction is replaced by the instantaneous BCS model interaction, the theory reduces to the theory for Dynes superconductors [5].

(ii) When elastic scattering is switched off, the theory reduces to the usual Eliashberg theory [2]. This goal will be solved in Sec. II.

In the rest of this paper we pursue the predictions of the E-CPA theory. In Sec. III we study the thermodynamic aspects of the theory. In particular, we ask whether the magnitude of the critical pair breaking (which leads to a complete destruction of superconductivity) bears some information on the pairing mechanism. Next, motivated by the recent Ref. [8] where a very strong temperature dependence of the Dynes parameter Γ was reported in tunneling experiments, in Sec. IV we study the temperature dependence of the tunneling density of states within E-CPA.

II. THE E-CPA EQUATIONS

We consider a single band of electrons with bare energy $\varepsilon_{\mathbf{k}}$. The standard 2×2 Nambu-Gor'kov Green's function of the clean noninteracting problem in imaginary time is given by $\hat{G}^{-1}_{0;n\mathbf{k}} = i\omega_n\tau_0 - \varepsilon_{\mathbf{k}}\tau_3$, where ω_n is the Matsubara frequency, and τ_i with $i=0,\ldots,3$ are the unit matrix and the Pauli matrices.

The electrons are supposed to be subject to scattering on random spatially uncorrelated pair-conserving and pair-breaking fields U and V with distribution functions $P(U)$ and $P_{\mathrm{M}}(V)$, respectively. At the lattice site l, the local impurity potential therefore reads

$$\hat{W}_l = U_l\tau_3 + V_l\tau_0. \tag{1}$$

Physically, U_l can be thought of as a random scalar potential at site l, whereas V_l is a magnetic field with a random magnitude and a fixed direction. Let us note in passing that, following the discussion in [9], the direction of the magnetic field can be taken as fluctuating as well, with no change of the subsequent considerations. Since in that case we would need to work with 4×4 matrices, in what follows we shall deal instead with the simpler case with 2×2 matrices.

The averaged full Green's function $\hat{G}_{n\mathbf{k}}$ is given by the Dyson equation

$$\hat{G}^{-1}_{n\mathbf{k}} = \hat{G}^{-1}_{0;n\mathbf{k}} - \hat{\Sigma}_n, \tag{2}$$

where we have assumed, as usual in the case of an isotropic system with a featureless density of states, that the averaged electron self-energy $\hat{\Sigma}_n$ depends only on the frequency ω_n and not on the momentum $\mathbf{k}$. Making use of $\hat{G}_{n\mathbf{k}}$, it is useful to define also the averaged local Green's function,

$$\hat{\mathcal{G}}_n = \frac{1}{\mathcal{N}}\sum_{\mathbf{k}}\hat{G}_{n\mathbf{k}} = N_0\int d\varepsilon_{\mathbf{k}}\hat{G}_{n\mathbf{k}}, \tag{3}$$

where N_0 is the normal-state density of states at the Fermi level.

Furthermore we will assume that the electrons interact also with the phonons. The contribution of this interaction to the electron self-energy, when evaluated within the self-consistent Born approximation, is given by [2]

$$\hat{\Sigma}^{\mathrm{ph}}_n = T\sum_m D_{n-m}\tau_3\hat{\mathcal{G}}_m\tau_3, \tag{4}$$

where D_{n-m} is the phonon-induced electron-electron interaction with energy transfer $\omega_n-\omega_m$, appropriately averaged over the Fermi surface, as usual.

This means that, at site l, the electrons are acted upon by the total fluctuating potential $\hat{W}_l + \hat{\Sigma}^{\mathrm{ph}}_n$ due to both, disorder and phonons. The goal of the theory is to choose the averaged total self-energy of the electron $\hat{\Sigma}_n$ in such a way that it best represents the action of the total fluctuating potential.

We solve this task within the standard single-site CPA [4], as recently reformulated in [10]. In complete analogy with Ref. [10], we require that the averaged self-energy $\hat{\Sigma}_n$ has to satisfy the following elegant and intuitively appealing equation:

$$\hat{\mathcal{G}}_n = \left\langle\left(\hat{\mathcal{G}}_n^{-1} - \hat{W} - \hat{\Sigma}^{\mathrm{ph}}_n + \hat{\Sigma}_n\right)^{-1}\right\rangle, \tag{5}$$

where the angular brackets represent taking an average with respect to the disorder distribution functions $P(U)$ and $P_{\mathrm{M}}(V)$. In what follows, we will assume that both distribution functions $P(U)$ and $P_{\mathrm{M}}(V)$ are even.

Equations (2), (3), (4), and (5), to be called E-CPA equations in what follows, form a closed set of equations for the unknown functions of the Matsubara frequency $\hat{\Sigma}_n$ and $\hat{\mathcal{G}}_n$.

By considering the matrix structure of the E-CPA equations one can show that the averaged self-energy $\hat{\Sigma}_n$ is a linear combination of the matrices τ_0 and τ_1. Thus $\hat{\Sigma}_n$ can be parameterized by just two functions, the wave-function renormalization Z^{tot}_n and the gap function Δ_n, as usual in the Eliashberg theory:

$$\hat{\Sigma}_n \equiv \hat{\Sigma}^{\mathrm{tot}}_n = i\omega_n\left(1-Z^{\mathrm{tot}}_n\right)\tau_0 + Z^{\mathrm{tot}}_n\Delta_n\tau_1. \tag{6}$$

Neglecting the possibility of odd-frequency pairing, one can show that Z^{tot}_n and Δ_n are real and even functions of ω_n. It is also worth pointing out that the functional form for $\hat{\Sigma}_n$ implies that the local Green's function reads

$$\hat{\mathcal{G}}_n = -\pi N_0\frac{i\omega_n\tau_0 + \Delta_n\tau_1}{\sqrt{\omega_n^2+\Delta_n^2}}. \tag{7}$$

In what follows we shall prove that the E-CPA equations do satisfy the conditions (i) and (ii) mentioned in the Introduction. In doing so, we will therefore show that the E-CPA equations do correspond to the generalized theory we were looking for.

In order to prove (i), let us assume that the electron-electron interaction does not depend on frequency, $D_{n-m}=D$, up to a finite cutoff Ω. Plugging Eq. (7) into Eq. (4) one finds, since Δ_n is an even function of frequency, that the phonon self-energy $\hat{\Sigma}^{\mathrm{ph}}_n = \Delta\tau_1$. Here the magnitude of the frequency-independent gap Δ is given by a BCS-like gap equation

$$\Delta = g\pi T\sum_{\omega_n=-\Omega}^{\Omega}\frac{\Delta_n}{\sqrt{\omega_n^2+\Delta_n^2}}$$

with the coupling constant $g=N_0D$. One notes readily that, with these identifications, our Eq. (5) reduces precisely to the CPA equation (5) of [10], while our gap equation coincides with Eq. (6) in [10]. Thus, if the electron-electron interaction is frequency-independent (instantaneous), the E-CPA theory does in fact reduce to the CPA treatment of disorder scattering which was utilized in previous works on Dynes superconductors [5,10].

On the other hand, in order to prove (ii), let us assume that disorder is absent, $W_l = 0$. Plugging this assumption into our Eq. (5), one finds readily that the total self-energy $\hat{\Sigma}_n$ is simply equal to $\hat{\Sigma}_n^{\rm ph}$. In other words, in the absence of disorder the E-CPA theory does reproduce the Eliashberg theory, as was to be shown.

To summarize, the E-CPA theory provides the necessary framework which unifies the Eliashberg theory with the CPA treatment of disorder scattering. In Appendix A we show that the E-CPA equations can be derived variationally from an appropriately chosen Luttinger-Ward-type functional.

Dynes superconductors within E-CPA theory

The actual form of the solution for $Z_n^{\rm tot}$ and Δ_n depends on the electron-electron interaction D_{n-m}, as well as on the distribution functions $P(U)$ and $P_{\rm M}(V)$. In general, further progress is possible only numerically.

In order to proceed, in what follows we will assume that the distribution function $P_{\rm M}(V)$ for magnetic fields is a Lorentzian centered around $V = 0$ with a width of $\Gamma^{\rm M}$. On the other hand, no further assumptions are needed for the distribution function $P(U)$ for scalar potentials, except for requiring that it is even. These requirements are precisely the same as those imposed in the theory of Dynes superconductors at the BCS level [5,10].

As shown in Appendix B, with this special choice of the distribution functions, describing what we call Dynes superconductors, the disorder averaging in (5) can be performed analytically. Plugging the ansatz (6) into the E-CPA equations, after some work we find that the gap function Δ_n and an auxiliary function Z_n, which corresponds to the wave-function renormalization of a hypothetical system without scalar disorder, satisfy the coupled set of equations

$$Z_n = 1 + \frac{\Gamma^{\rm M}}{|\omega_n|} + \frac{\pi T}{\omega_n}\sum_m g_{n-m}\frac{\omega_m}{\sqrt{\omega_m^2+\Delta_m^2}},$$
$$Z_n\Delta_n = \pi T\sum_m g_{n-m}\frac{\Delta_m}{\sqrt{\omega_m^2+\Delta_m^2}}, \tag{8}$$

where we have introduced a dimensionless electron-electron interaction $g_{n-m} = N_0 D_{n-m}$. In the presence of finite elastic pair-conserving scattering, the (physical) wave-function renormalization $Z_n^{\rm tot}$ is given by

$$Z_n^{\rm tot} = Z_n + \frac{\Gamma^{\rm S}}{\sqrt{\omega_n^2+\Delta_n^2}}, \tag{9}$$

where the scattering rate $\Gamma^{\rm S}$ depends on the distribution function $P(U)$. It can be determined from the self-consistent equation

$$\frac{1}{1-\pi N_0\Gamma^{\rm S}} = \int_{-\infty}^{\infty}\frac{P(U)\,dU}{(1-\pi N_0\Gamma^{\rm S})^2+(\pi N_0 U)^2}. \tag{10}$$

For the sake of completeness, we note that the above results hold provided the inequality $\Gamma^{\rm S} \lesssim 1/(\pi N_0)$ is satisfied. This condition requires that the disorder is not too large, but admits in fact quite strong disorder scattering.

At this point, several remarks should be made. First, the gap function Δ_n is not influenced by elastic pair-conserving scattering, since it is determined by the closed set of equations (8) where $\Gamma^{\rm S}$ does not enter. This is of course fully consistent with Anderson's theorem [3].

Second, once the closed Eqs. (8) have been solved, the physical wave-function renormalization $Z_n^{\rm tot}$ can be determined easily, without the need to solve a self-consistent problem. Thus, the presence of a finite scattering rate $\Gamma^{\rm S}$ introduces only trivial changes (which are visible, e.g., in the spectral functions). Therefore, in what follows we will discuss only the case $\Gamma^{\rm S} = 0$.

Third, when comparing Eqs. (8) with the standard Eliashberg equations, one notes that the only difference is that, in the present case, a term proportional to $\Gamma^{\rm M}$ enters the equation for Z_n. This can be viewed as yet another confirmation of the property (2) of the E-CPA equations.

III. DESTRUCTION OF SUPERCONDUCTING ORDER BY $\Gamma^{\rm M}$

It is well known that elastic pair-breaking scattering, gauged in the present paper by the parameter $\Gamma^{\rm M}$, leads to a decrease of the superconducting critical temperature, T_c. Ultimately, when $\Gamma^{\rm M}$ becomes larger than a certain critical value, to be called $\Gamma_c^{\rm M}$, the critical temperature vanishes completely.

Previous BCS-like work on the Dynes superconductors has found that the critical pair-breaking rate, when measured in units of the critical temperature T_{c0} of a system in the absence of pair-breaking disorder, is given by $\Gamma_c^{\rm M}/T_{c0} \approx 0.88$ in the limit of weak coupling [5]. The question we would like to answer in this section is how the ratio $\Gamma_c^{\rm M}/T_{c0}$ changes in a full-fledged E-CPA description of a Dynes superconductor. In particular, can one draw conclusions about the effective electron-electron interaction g_{n-m} from the measured value of $\Gamma_c^{\rm M}/T_{c0}$?

As is customary in the literature, instead of the dimensionless electron-electron interaction g_{n-m}, we introduce the interaction function $\alpha^2F(y)$ defined by

$$g_{n-m} = \int_{-\infty}^{\infty}\frac{\alpha^2F(y)\,dy}{i\omega_n - i\omega_m + y}. \tag{11}$$

Note that $\alpha^2F(y)$ is an odd function of its argument, $\alpha^2F(-y) = -\alpha^2F(y)$. In terms of the function $\alpha^2F(y)$, it is customary to define the dimensionless coupling constant $\lambda = 2\int_0^\infty dy\alpha^2F(y)/y$.

In the numerical examples to be studied in later sections, we will study the following set of model interaction functions $\alpha^2F(y)$:

$$\alpha^2F(y) = \frac{\beta\lambda}{2}\,\mathrm{sgn}(y)\left|\frac{y}{\Omega}\right|^\beta, \quad |y| < \Omega. \tag{12}$$

For $|y| > \Omega$, the interaction functions $\alpha^2F(y)$ are taken to be zero. Note that the normalization in Eq. (12) is chosen so that the dimensionless coupling constant equals λ. Thus, interaction functions $\alpha^2F(y)$ described by Eq. (12) are fully characterized by three parameters: the energy scale Ω (to be used as the unit of energy in what follows) and two dimensionless parameters, the exponent β and the coupling constant λ.

Different functions $\alpha^2F(y)$ correspond to different effective electron-electron interactions. For instance, a three-dimensional system of electrons coupled to acoustic phonons

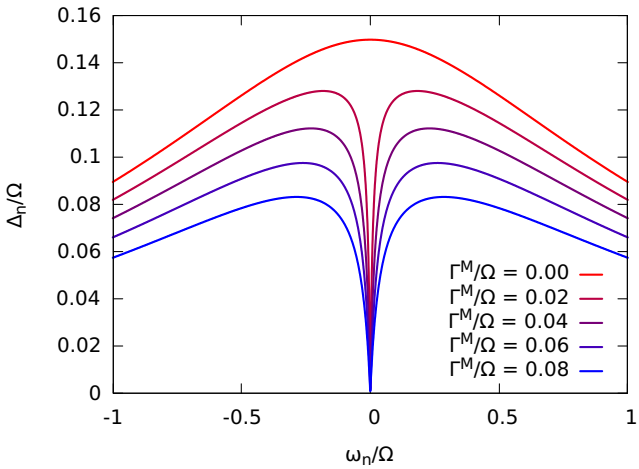

FIG. 1. Matsubara gap function Δ_n for a superconductor described by the model interaction function (12) with $\beta = 2$ and $\lambda = 1$ at temperature $T/\Omega = 10^{-4}$. The indicated pair-breaking rates Γ^{M} correspond to top to bottom lines.

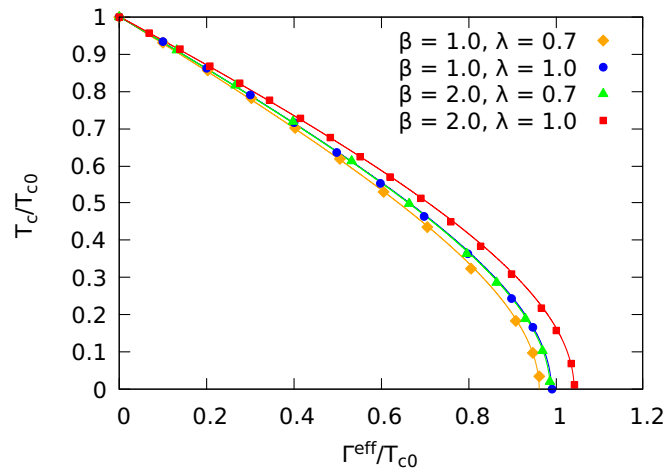

FIG. 2. Critical temperature $\theta = T_c/T_{c0}$ as a function of $\gamma = \Gamma^{\mathrm{eff}}/T_{c0}$ for superconductors described by Eq. (12) with several choices of the parameters β and λ (symbols). Guides to the eye of the form $\theta = \sqrt{(1-a\gamma)(1-b\gamma)}$ are also shown.

is described by a model with $\beta = 2$. In this case, Ω is the Debye energy of the phonons. Smaller values of β correspond to systems with increasing importance of electron-boson interactions in the limit of low energies. For instance, as will be shown later, the model with $\beta = 1$ generates a standard Fermi-liquid-like behavior in the normal state, while models with $\beta < 1$ lead to a more anomalous behavior.

To begin, in Fig. 1 we present the Matsubara gap function Δ_n calculated from Eqs. (8) using the model interaction function (12) and several values of the pair-breaking rate Γ^{M}. The parameters λ, β and the temperature T are kept fixed. With increasing Γ^{M}, one can observe that the overall gap size decreases. At the same time, in the low-energy limit we find that $\Delta_n \propto |\omega_n|$, with the constant of proportionality decreasing with increasing Γ^{M}.

In Fig. 2 we show how the critical temperature T_c (measured in units of T_{c0}) of a superconductor described by Eqs. (8) decreases with increasing elastic pair breaking. Since, as shown in the next section, the observable pair-breaking scattering is roughly given by $\Gamma^{\mathrm{eff}} = \Gamma^{\mathrm{M}}/(1+\lambda)$, the magnitude of pair breaking is characterized by $\Gamma^{\mathrm{eff}}/T_{c0}$. Note that for a Dynes superconductor with a BCS-like interaction, Γ^{eff} is simply equal to Γ^{M}, because a frequency-independent interaction does not generate mass enhancement.

Figure 2 shows that the overall shape of the T_c/T_{c0} vs $\Gamma^{\mathrm{eff}}/T_{c0}$ curves is qualitatively similar for all studied interaction functions (12). The main difference between different choices of $\alpha^2F(y)$ concerns the value of the critical pair-breaking rate $\Gamma_c^{\mathrm{eff}}/T_{c0}$. For Dynes superconductors with a BCS-like interaction it has been found that $\Gamma_c^{\mathrm{eff}}/T_{c0} \approx 0.88$, while Fig. 2 shows that larger values are obtained within E-CPA theory. We find that $\Gamma_c^{\mathrm{eff}}/T_{c0}$ increases with both, increasing coupling constant λ and increasing exponent β. However, in order to determine the individual values of λ and β, independent information is needed. Such information is provided, e.g., by the magnitude of the gap-to-T_c ratio in the absence of elastic pair-breaking, which is accessible via the standard Eliashberg theory for $\Gamma^{\mathrm{M}} = 0$.

IV. TUNNELING DENSITY OF STATES

Tunneling experiments with their very high energy resolution and wide accessibility are a popular tool in superconductivity research. When studying the differential conductance between a disordered superconductor and an electrode in the normal state, it has become customary (see, e.g., [8,11,12]) to fit the conductance vs voltage curve by the thermally smeared Dynes formula [6].

Very good fits are often obtained with a temperature-independent pair-breaking parameter Γ; for a couple of examples see the references in Ref. [5]. However, in some cases the extracted parameter Γ has been found to exhibit a slight increase with increasing temperature; see, e.g., Ref. [12]. Moreover, in a recent paper by Boschker *et al.*, a very large increase of Γ has been reported [8].

The goals of this section are twofold. First, we want to determine a procedure for extracting the Dynes parameters Δ and Γ from the results of the E-CPA analysis. Second, we want to estimate the temperature dependence of Γ, taking for the interaction function (12) with several choices of the parameters λ and β.

Let us start by observing that, within E-CPA, the tunneling density of states $N(\omega)$ in the superconducting state is given by

$$\frac{N(\omega)}{N_0} = \mathrm{Re}\left[\frac{\omega}{\sqrt{\omega^2 - \Delta^2(\omega)}}\right], \tag{13}$$

exactly as in the standard Eliashberg theory. The density of states is seen to depend on the gap function. However, instead of the Matsubara gap function Δ_n, one has to use the gap function on the real axis, $\Delta(\omega)$. Therefore, it is necessary to perform the analytic continuation from the imaginary to the real axis.

A. Analytical continuation to the real axis

Let us start by fixing the vocabulary. Complex frequencies in the upper half-plane will be denoted z, and under $X(z)$ we will understand an analytic continuation of the function X from the values $X_n \equiv X(i\omega_n)$ in the Matsubara points $i\omega_n$ on the imaginary axis to the whole upper half of the complex plane. On the other hand, $X(\omega)$ will be reserved for the values of $X(z)$ infinitesimally above the real axis, i.e., for $z = \omega + i0$ with real ω. In general, X is a complex number with real and imaginary parts to be denoted as X' and X'', respectively, i.e., $X = X' + iX''$.

Making use of the Eliashberg functions $Z(z)$ and $\Delta(z)$ which derive from Z_n and Δ_n, respectively, let us introduce two other functions: $\tilde{\omega}(z) = Z(z)z$ and $\Phi(z) = Z(z)\Delta(z)$. In terms of these functions, the averaged self-energy can be written for $\Gamma^{\rm S} = 0$ (i.e., in the absence of elastic pair-conserving scattering) as $\hat{\Sigma}(z) = (z - \tilde{\omega}(z))\tau_0 + \Phi(z)\tau_1$. Thus, the real-axis solution for the Green's function is completely specified by two complex functions of frequency, $\tilde{\omega}(\omega)$ and $\Phi(\omega)$.

The problem of analytic continuation to the real axis is known to be notoriously difficult. Here it will be circumvented by a trick developed in Ref. [13]. As observed in that paper, once the imaginary-axis solutions Z_n and Δ_n of Eqs. (8) have been found, the functions $\tilde{\omega}(\omega)$ and $\Phi(\omega)$ can be determined from a coupled set of equations, which in the present case read

$$\begin{aligned}\tilde{\omega}(\omega) &= \omega + i\Gamma^{\rm M} + 2\pi T \sum_{\omega_n>0} g''(\omega + i\omega_n)\frac{\omega_n}{\sqrt{\omega_n^2 + \Delta_n^2}} \\ &\quad + i\pi \int_{-\infty}^{\infty} \alpha^2 F(y)[f(\omega + y) + b(y)]n(\omega + y)\, dy, \\ \Phi(\omega) &= 2\pi T \sum_{\omega_n>0} g'(\omega + i\omega_n)\frac{\Delta_n}{\sqrt{\omega_n^2 + \Delta_n^2}} \\ &\quad + i\pi \int_{-\infty}^{\infty} \alpha^2 F(y)[f(\omega + y) + b(y)]p(\omega + y)\, dy.\end{aligned} \tag{14}$$

Here the function $g(z)$ represents the analytic continuation of the dimensionless electron-electron interaction from $g_{n-m} \equiv g(i\omega_n - i\omega_m)$ to the complex plane, $g(z) = \int_{-\infty}^{\infty} dy \alpha^2 F(y)/(z + y)$. Moreover, we have introduced the following auxiliary functions:

$$n(z) = \frac{\tilde{\omega}(z)}{\sqrt{\tilde{\omega}^2(z) - \Phi^2(z)}}, \quad p(z) = \frac{\Phi(z)}{\sqrt{\tilde{\omega}^2(z) - \Phi^2(z)}}.$$

It should be noted that Eqs. (14), which mix the Matsubara and real-axis formulations, are much easier to solve numerically than the corresponding integral equations written directly on the real axis. A simplified derivation of Eqs. (14) is presented in Appendix C.

The utility of Eqs. (14) is demonstrated in Fig. 3. There we compare the gap function Δ_n at discrete Matsubara frequencies ω_n, obtained by the solution of Eqs. (8), with the result of analytic continuation. Note that, when looking just at the Matsubara points, there exists no indication that the function $\Delta(z)$ should vanish when $z \to 0$. Nevertheless, making use of Eqs. (14), this nontrivial behavior (which will turn out to be crucial in the discussion which follows) is correctly reproduced.

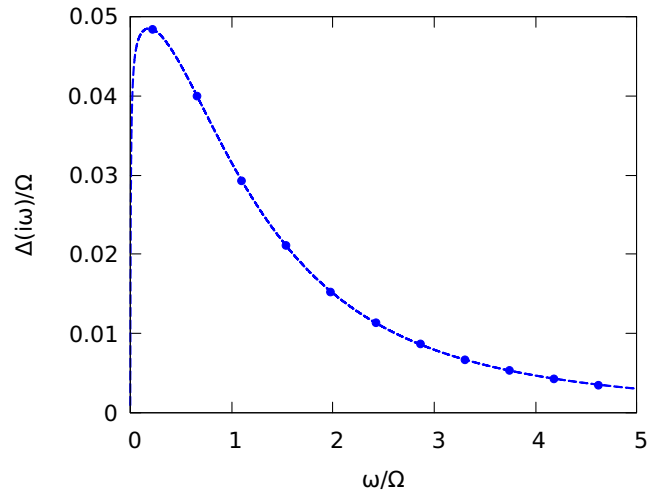

FIG. 3. The dots denote the values of the gap function Δ_n in the Matsubara frequencies $\omega_n = (2n + 1)\pi T$ for a superconductor described by Eq. (12) with $\beta = 2$, $\lambda = 1$, and $\Gamma^{\rm M} = 0$ at temperature $T/\Omega = 0.07$. The broken line is the plot of the analytic continuation $\Delta(i\omega)$ to the whole imaginary axis, obtained from the solutions of Eqs. (14).

B. $N(\omega)$ in the low-energy limit

The Dynes formula is meant to describe $N(\omega)$ in the limit of low energy, thus from Eq. (13) it follows that we need an approximation for $\Delta(\omega)$ in that limit. To this end, let us first observe that $\Delta(\omega) = \omega\phi(\omega)/\tilde{\omega}(\omega)$. Within the Eliashberg theory, it was first recognized by Karakozov *et al.* that at low energies one can use the low-energy expansions $\phi(\omega) \approx \phi_0$ and $\tilde{\omega}(\omega) \approx R\omega + iS$, where ϕ_0, R and S are (temperature-dependent) constants [14]. A crucial observation was that S is finite at any finite temperature, due to scattering on real thermal phonons. From here it follows that

$$\Delta(\omega) = \frac{\omega\phi_0}{R\omega + iS} \equiv \frac{\omega\Delta}{\omega + i\Gamma}, \tag{15}$$

where we have introduced $\Delta = \phi_0/R$ and $\Gamma = S/R$. But it is well known that, once the gap function $\Delta(\omega)$ is given by Eq. (15), the density of states is described by the Dynes formula [5].

It turns out that also within the E-CPA theory, Eq. (15) is still valid. Thus, in order to determine the temperature dependence of the pair-breaking parameter Γ, one needs to calculate the parameters R and S.

Let us first note that the E-CPA equations on the real axis differ from the standard Eliashberg equations only by the presence of the term $i\Gamma^{\rm M}$ in the equation for $\tilde{\omega}(\omega)$. Therefore, closely following the discussion in [14], one finds that the scattering parameter S is given by

$$S(T) = \Gamma^{\rm M} + 2\pi \int_0^{\infty} dx \frac{n'(x)\alpha^2 F(x)}{\sinh(x/T)}. \tag{16}$$

In passing, let us note that from Eq. (16) it follows that the inelastic part of the scattering rate S in the normal state

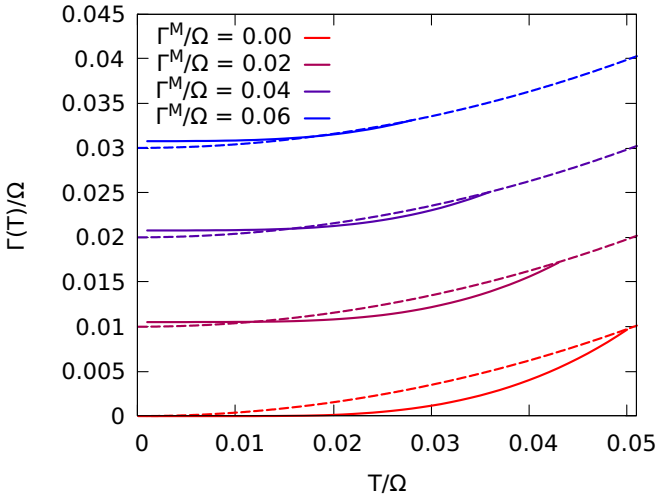

FIG. 4. Temperature dependence of the Dynes parameter Γ for a superconductor described by Eq. (12) with $\beta = 1$ and $\lambda = 1$. The indicated pair-breaking rates Γ^{M} correspond to bottom to top lines. The dashed lines show the values of Γ in the hypothetical normal state.

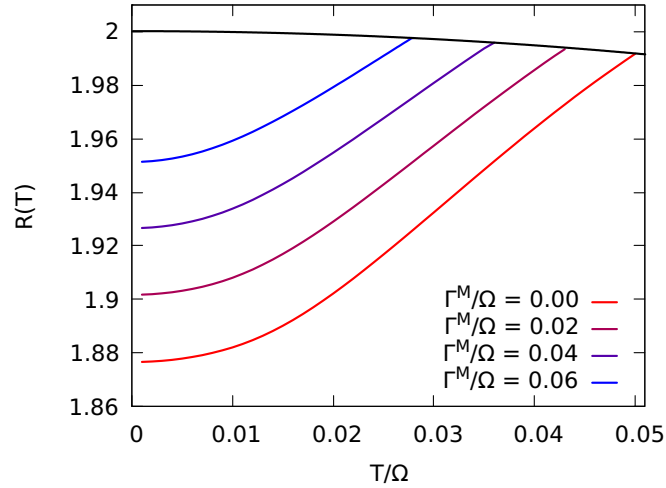

FIG. 5. Temperature dependence of the mass-renormalization parameter R for the same parameters as used in Fig. 4 (bottom to top lines). The topmost black line corresponds to the hypothetical normal state.

scales as $\propto T^{\beta+1}$. In particular, as mentioned in the previous section, the model interaction function Eq. (12) generates a Fermi-liquid behavior in the normal state, if for the exponent we take $\beta = 1$.

The mass-renormalization parameter R is given by a less transparent expression. However, at temperature $T = 0$, the formula for R simplifies to

$$R(0) = 1 + 2\int_0^\infty dy\alpha^2F(y)\int_0^\infty dx\frac{n'(x)}{(x+y)^2}. \qquad (17)$$

In the hypothetical normal state at $T = 0$, when $n'(x) = 1$, this formula reduces to the well-known result $R(0) = 1 + \lambda$, where λ is the dimensionless coupling constant.

In order to compare our theory with experimental results, let us introduce the total change $\delta\Gamma \equiv \Gamma(T_c) - \Gamma(0)$ of the Dynes parameter $\Gamma(T)$ between $T = 0$ and $T = T_c$. When evaluating $\delta\Gamma$, let us first neglect the temperature dependence of R and assume that $R(T) \approx 1 + \lambda$. Since $\Gamma(T_c) = \Gamma_n(T_c)$, in that case we obtain the estimate

$$\frac{\delta\Gamma}{T_c} \approx \frac{\pi\lambda I_\beta}{1+\lambda}\left(\frac{T_c}{\Omega}\right)^\beta, \qquad (18)$$

where we have introduced a β-dependent parameter $I_\beta = \beta\int_0^\infty dt t^\beta/\sinh(t)$. One observes easily that, in usual situations, $\delta\Gamma/T_c$ is small, because the fraction T_c/Ω is usually very small. A non-negligible temperature dependence of $\Gamma(T)$ is therefore possible only in special situations: either the fraction T_c/Ω should not be very small, i.e., the coupling constant λ should be large, or the exponent β should be anomalously low, or both.

Next we study how does the presence of a finite elastic pair-breaking rate Γ^{M} influence the temperature dependence of the Dynes parameter Γ. In Fig. 4 we plot the numerical results for $\Gamma(T)$ obtained for several values of Γ^{M}. As was to be expected, a finite value of Γ^{M} increases the overall magnitude of Γ.

In Fig. 4 we also plot the expected values of $\Gamma_n(T)$ in the hypothetical normal state. The low-temperature limit $\Gamma_n(T)$ is readily seen to be given by $\Gamma^{\mathrm{M}}/(1+\lambda)$. However, note that if the elastic pair-breaking rate Γ^{M} is finite, the Dynes parameter $\Gamma(0)$ in the true superconducting state is not equal to $\Gamma^{\mathrm{M}}/(1+\lambda)$, but it is larger. As shown in Fig. 5, this effect is caused by the fact that, in the superconducting state, the parameter $R(T)$ differs from its normal-state value. This difference decreases with increasing Γ^{M}.

This means that, if the elastic pair-breaking rate Γ^{M} is finite, the overall temperature dependence of the Dynes parameter

$$\delta\Gamma = [\Gamma_n(T_c) - \Gamma_n(0)] - [\Gamma(0) - \Gamma_n(0)]$$

decreases for two reasons. First, a finite value of Γ^{M} reduces T_c, and, therefore, also the inelastic term $\Gamma_n(T_c) - \Gamma_n(0)$. Second, a finite Γ^{M} leads at the same time to an increase of the elastic term $\Gamma(0) - \Gamma_n(0)$.

Summarizing, in ordinary situations (i.e., for $\beta > 1$ and $\lambda \lesssim 1$), the parameter $\delta\Gamma/T_c$ is small already for $\Gamma^{\mathrm{M}} = 0$, and finite values of Γ^{M} make it even smaller. This conclusion is in qualitative agreement with most experiments [11,12].

Before concluding, let us discuss whether also the anomalous results of Ref. [8] can be explained within the present E-CPA theory. A successful description should explain two observations: at $T = 0$, the ratio $\Gamma(0)/\Delta(0)$ is approximately equal to 0.5, and, at the same time, the temperature increase $\delta\Gamma$ is comparable to $\Delta(0)$. Note that a large value of $\Gamma(0)$ requires that Γ^{M} should also be large. Moreover, a large value of $\delta\Gamma$ might obtain only if β is small and λ is large.

In Fig. 6 we plot the temperature dependence of the Dynes parameters Δ and Γ for a superconductor described by Eq. (12) for several moderately anomalous choices of parameters β and λ. In order to obtain $\Gamma(0)/\Delta(0) \approx 0.5$, we have made the following choices of the pair-breaking parameter Γ^{M}: $\Gamma^{\mathrm{M}}/\Omega = 0.115$ for $\beta = 0.5$ and $\lambda = 2$, $\Gamma^{\mathrm{M}}/\Omega = 0.0315$ for $\beta = 0.5$ and $\lambda = 1$, and $\Gamma^{\mathrm{M}}/\Omega = 0.06$ for $\beta = 1$ and

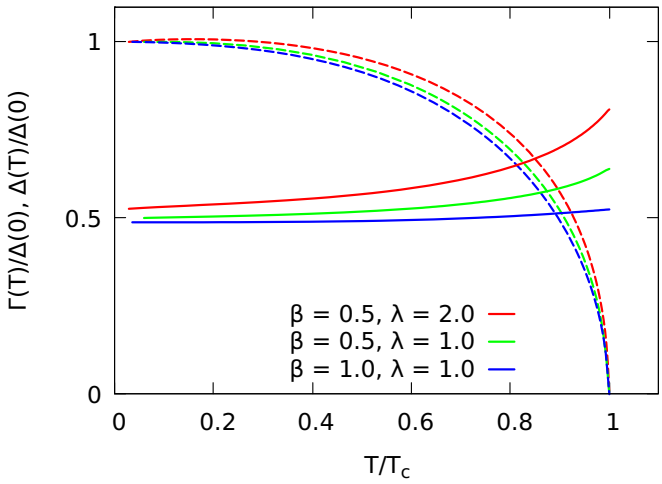

FIG. 6. Temperature dependence of the Dynes parameters Δ (dashed lines) and Γ (solid lines) for superconductors described by Eq. (12) with several moderately anomalous choices of β and λ (the indicated parameters correspond to top to bottom lines). In all cases, the pair-breaking scattering rate Γ^{M} is chosen so that $\Gamma(T)/\Delta(0) \approx 0.5$.

$\lambda = 1$. One observes that, in agreement with expectations, the overall temperature dependence of the Dynes parameter $\delta\Gamma$, measured in units of the zero-temperature gap $\Delta(0)$, grows with increasing interaction strength λ and decreasing exponent β. However, even in the most anomalous case we have studied, $\beta = 0.5$ and $\lambda = 2$, the value of $\delta\Gamma/\Delta(0) \approx 0.3$ does not nearly come close to the experimentally observed value $\delta\Gamma/\Delta(0) \gtrsim 1$ [8].

V. CONCLUSIONS

In this paper we have developed what we call the E-CPA theory of the superconducting state. The theory combines two successful microscopic descriptions of superconductors: an Eliashberg-like treatment of boson-mediated electron-electron interactions, and a coherent-potential-approximation-based description of elastic scattering.

For a Lorentzian distribution of pair-breaking disorder, the theory reduces to Eqs. (8). These differ from the standard Eliashberg equations only by the presence of a new term $\Gamma^{\mathrm{M}}/|\omega_n|$ in the equation for the wave-function renormalization Z_n, where Γ^{M} is the rate of elastic pair-breaking scattering.

We have shown that, if the boson-mediated electron-electron interaction is replaced by an instantaneous interaction, Eqs. (8) reduce to the recently developed theory for Dynes superconductors [5,10].

In agreement with Anderson's theorem [3], we have also shown that pair-conserving scattering on disorder, even if present, does not influence the form of the Eliashberg-like Eqs. (8), although it does influence the electron spectral function via Eq. (9).

Within E-CPA theory, we can ask several questions about Dynes superconductors with retarded electron-electron interactions. In this work, we have presented a study of simple model interaction functions with anomalous low-energy behavior, but it is possible to apply the theory also to more realistic interaction functions which are determined by first-principles calculations. Such calculations take into account the coupling of electrons to all relevant modes from low to high energy. It will be interesting to see the new features which such realistic interaction functions imply.

We have applied the theory to two specific problems.

The first question we have asked is: how does the critical value of the pair-breaking rate Γ_c^{M} (leading to a complete suppression of superconductivity) depend on the pairing glue? Can one draw conclusions from the knowledge of Γ_c^{M} about the pairing glue? To this end, we have studied Γ_c^{M} for superconductors with model interaction functions (12) for several choices of coupling constants λ and exponents β.

We have found that Γ_c^{M}, when measured in units of the critical temperature in the absence of disorder T_{c0}, does depend on λ and β. Of course, if only the value of $\Gamma_c^{\mathrm{M}}/T_{c0}$ is known, it is not possible to conclude about the individual values of λ and β. To this end, independent information is required.

The second problem we have studied dealt with predictions of E-CPA theory for the tunneling density of states $N(\omega)$. Experimentally, it is well known that, quite often, $N(\omega)$ is given by the Dynes formula with a phenomenological pair-breaking parameter Γ. Usually, Γ is not temperature-dependent [11]. However, recently both weak [12] and strong [8] T dependence of $\Gamma(T)$ has been observed. The question was: can such T dependencies be explained within E-CPA theory?

We have found that, generically, our E-CPA formalism does predict a T-dependent Γ. A rough estimate of $\delta\Gamma = \Gamma(T_c) - \Gamma(0)$ is given by Eq. (18), which shows that $\delta\Gamma$ increases with growing interaction strength λ and decreasing exponent β. For standard values of λ and β, E-CPA theory is qualitatively consistent with no or weak T dependence of Γ, in agreement with most experiments [11,12]. However, even when taking the very anomalous values $\lambda = 2$ and $\beta = 0.5$, we were not able to reproduce the large value of $\delta\Gamma$ observed in [8].

We conclude that the very strong temperature dependence of the Dynes parameter $\Gamma(T)$ observed in [8] is puzzling. Within the one-band scenario considered in this work such T dependence would require an extremely anomalous pairing glue with very small exponent β and/or unphysically large λ [15]. However, preliminary work suggests that when β decreases and/or λ increases, the Dynes formula gives progressively worse and worse description of the true density of states $N(\omega)$.

Could it be that the actual experimental $N(\omega)$ is not given by the Dynes formula? Since the quantity which is actually measured, namely, the differential conductance dI/dV, is proportional to the thermally smeared density of states, one would need to "unsmear" the data for dI/dV in order to determine the true $N(\omega)$. Obviously, it is most difficult to carry out this "unsmearing" procedure close to T_c. But, unfortunately, it is precisely close to T_c where also the reported T dependence of $\Gamma(T)$ is strongest.

ACKNOWLEDGMENTS

This work was supported by the Slovak Research and Development Agency under Contract No. APVV-19-0371 and by Comenius University under Contract No. UK/399/2023.

APPENDIX A: DERIVATION OF E-CPA EQUATIONS

In this Appendix we present a variational derivation of the E-CPA equations. We start by defining a Luttinger-Ward-type free-energy functional $\mathcal{F}$, which is analogous to that introduced in the study of thermodynamic properties of Dynes superconductors in [10]:

$$\begin{aligned}\mathcal{F} = &-\frac{T}{\mathcal{N}}\sum_{n\mathbf{k}}\mathrm{Tr}\ln\hat{G}^{-1}_{n\mathbf{k}}(\hat{\Sigma}) + T\sum_{n}\mathrm{Tr}\ln\hat{\mathcal{G}}^{-1}_{n}\\ &- T\sum_{n}\left\langle\mathrm{Tr}\ln\left(\hat{\mathcal{G}}^{-1}_{n}-\hat{W}-\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}})+\hat{\Sigma}_{n}\right)\right\rangle\\ &-\frac{T}{2}\sum_{n}\mathrm{Tr}\,\hat{\mathrm{G}}_{n}\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}}),\end{aligned}\tag{A1}$$

Here we have assumed that $\mathcal{F}=\mathcal{F}[\hat{\Sigma}_n,\hat{\mathcal{G}}^{-1}_n,\hat{\mathrm{G}}_n]$, i.e., the free energy $\mathcal{F}$ is a functional of three objects: self-energy $\hat{\Sigma}_n$, local Green's function $\hat{\mathcal{G}}^{-1}_n$, and the auxiliary variable $\hat{\mathrm{G}}_n$, which plays the role of the variable Δ in the functional $\mathcal{F}=\mathcal{F}[\hat{\Sigma}_n,\hat{\mathcal{G}}^{-1}_n,\Delta]$ introduced in [10].

In order to simplify the appearance of Eq. (A1), we have introduced the symbol $\hat{G}^{-1}_{n\mathbf{k}}(\hat{\Sigma})$, which is related to the self-energy $\hat{\Sigma}_n$ by the Dyson equation (2). Similarly, the symbol $\hat{\Sigma}^{\mathrm{ph}}_n(\hat{\mathrm{G}})$ is an abbreviation for

$$\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}}) = T\sum_{m}D_{n-m}\tau_3\hat{\mathrm{G}}_m\tau_3.$$

Having defined the functional, we can search for the equations of motion which it implies. Minimization with respect to $(\hat{\Sigma}_n)_{ij}$ gives

$$\frac{1}{\mathcal{N}}\sum_{\mathbf{k}}\hat{G}_{n\mathbf{k}}(\hat{\Sigma}) = \left\langle\left(\hat{\mathcal{G}}^{-1}_{n}-\hat{W}-\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}})+\hat{\Sigma}_{n}\right)^{-1}\right\rangle.\tag{A2}$$

On the other hand, requiring $\mathcal{F}$ to be stationary with respect to $(\hat{\mathrm{G}}_n)_{ij}$ yields in turn

$$\hat{\mathrm{G}}_{n} = \left\langle\left(\hat{\mathcal{G}}^{-1}_{n}-\hat{W}-\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}})+\hat{\Sigma}_{n}\right)^{-1}\right\rangle.\tag{A3}$$

Finally, optimization with respect to $(\hat{\mathcal{G}}^{-1}_n)_{ij}$ yields

$$\hat{\mathcal{G}}_{n} = \left\langle\left(\hat{\mathcal{G}}^{-1}_{n}-\hat{W}-\hat{\Sigma}^{\mathrm{ph}}_{n}(\hat{\mathrm{G}})+\hat{\Sigma}_{n}\right)^{-1}\right\rangle.\tag{A4}$$

Comparing the above three equations of motion, we find that $\hat{\mathrm{G}}_n=\hat{\mathcal{G}}_n$. In other words, the auxiliary variable $\hat{\mathrm{G}}_n$ can be replaced by the local Green's function $\hat{\mathcal{G}}_n$. Moreover, we recover the definition (7) as well as the CPA equation (5). This completes the variational derivation of the E-CPA equations.

APPENDIX B: DERIVATION OF EQS. (8)

To begin, let us rewrite the expression for the total self-energy (6) in a simpler form, $\hat{\Sigma}^{\mathrm{tot}}_n = -i\Gamma^{\mathrm{tot}}_n\tau_0 + \Phi^{\mathrm{tot}}_n\tau_1$. Similarly, let us parametrize the phonon self-energy Eq. (4) as $\hat{\Sigma}^{\mathrm{ph}}_n = -i\Gamma^{\mathrm{ph}}_n\tau_0 + \Phi^{\mathrm{ph}}_n\tau_1$. Let us also introduce the notation $x_n = \Phi^{\mathrm{tot}}_n/\sqrt{(\omega_n+\Gamma^{\mathrm{tot}}_n)^2+(\Phi^{\mathrm{tot}}_n)^2}$ and $y_n = (\omega_n+\Gamma^{\mathrm{tot}}_n)/\sqrt{(\omega_n+\Gamma^{\mathrm{tot}}_n)^2+(\Phi^{\mathrm{tot}}_n)^2}$.

With these notations, Eq. (5) can be written as

$$\hat{\mathcal{G}}_n = \langle(i(A_n+iV)\tau_0 - B_n\tau_1 + U\tau_3)^{-1}\rangle_{U,V},\tag{B1}$$

where we have introduced

$$A_n = \frac{y_n}{\pi N_0}+\Gamma^{\mathrm{ph}}_n-\Gamma^{\mathrm{tot}}_n,\quad B_n = \frac{x_n}{\pi N_0}+\Phi^{\mathrm{ph}}_n-\Phi^{\mathrm{tot}}_n.$$

Taking the inverse of the matrix on the right-hand side of Eq. (B1) and performing averaging with respect to V, we find

$$\begin{aligned}\hat{\mathcal{G}}_n &= -\left\langle\frac{i(A_n+iV)\tau_0+B_n\tau_1}{(A_n+iV)^2+B_n^2+U^2}\right\rangle_{U,V}\\ &= -K_n\left[i\left(A_n+\Gamma^{\mathrm{M}}s_n\right)\tau_0+B_n\tau_1\right],\end{aligned}\tag{B2}$$

where in the second step we have used the notation $s_n=\mathrm{sgn}\,A_n$ and we have also introduced

$$K_n = \left\langle\frac{1}{(A_n+\Gamma^{\mathrm{M}}s_n)^2+B_n^2+U^2}\right\rangle_U.\tag{B3}$$

Comparing the coefficients in front of τ_0 and τ_1 on both sides of Eq. (B2), we obtain the equations

$$\frac{\pi N_0}{K_n}y_n = A_n+\Gamma^{\mathrm{M}}s_n,\quad \frac{\pi N_0}{K_n}x_n = B_n.\tag{B4}$$

Adding the squares of the two equations in Eq. (B4) we find that $(A_n+\Gamma^{\mathrm{M}}s_n)^2+B_n^2=(\pi N_0K_n^{-1})^2$. Plugging this identity into Eq. (B3), we observe that K_n is given by the self-consistent equation

$$K_n = \left\langle\frac{1}{\left(\pi N_0K_n^{-1}\right)^2+U^2}\right\rangle_U.\tag{B5}$$

From here it follows that K_n does not depend on frequency, $K_n=K$. If we introduce $\Gamma^{\mathrm{S}}=(\pi N_0)^{-1}-\pi N_0/K$, one checks easily that Eq. (B5) reproduces the definition (10) of Γ^{S} from the main text.

Making use of the explicit form of A_n and B_n in Eqs. (B4), these expressions simplify to

$$\Gamma^{\mathrm{S}}x_n = \Phi^{\mathrm{tot}}_n-\Phi_n,\quad \Gamma^{\mathrm{S}}y_n = \Gamma^{\mathrm{tot}}_n-\Gamma_n,\tag{B6}$$

where we have defined $\Phi_n\equiv\Phi^{\mathrm{ph}}_n$ and $\Gamma_n\equiv\Gamma^{\mathrm{ph}}_n+\Gamma^{\mathrm{M}}s_n$. From here it follows that

$$y_n/x_n = \left(\Gamma^{\mathrm{tot}}_n-\Gamma_n\right)/\left(\Phi^{\mathrm{tot}}_n-\Phi_n\right).$$

But, on the other hand, by definition we have $y_n/x_n=(\omega_n+\Gamma^{\mathrm{tot}}_n)/\Phi^{\mathrm{tot}}_n$. Comparing these two expressions for y_n/x_n, we find $(\omega_n+\Gamma^{\mathrm{tot}}_n)/\Phi^{\mathrm{tot}}_n=(\omega_n+\Gamma_n)/\Phi_n$. But this means that we can express x_n and y_n in terms of Φ_n and Γ_n, instead of Φ^{tot}_n and Γ^{tot}_n:

$$x_n = \Phi_n/\mathcal{D}_n,\quad y_n = (\omega_n+\Gamma_n)/\mathcal{D}_n,$$

where $\mathcal{D}_n=\sqrt{(\omega_n+\Gamma_n)^2+\Phi_n^2}$. Making use of these expressions in Eq. (B6), we find

$$\begin{aligned}\Phi^{\mathrm{tot}}_n &= (1+\Gamma^{\mathrm{S}}/\mathcal{D}_n)\Phi_n,\\ \omega_n+\Gamma^{\mathrm{tot}}_n &= (1+\Gamma^{\mathrm{S}}/\mathcal{D}_n)(\omega_n+\Gamma_n).\end{aligned}$$

Introducing next the parametrizations $\Phi^{\mathrm{tot}}_n = Z^{\mathrm{tot}}_n\Delta^{\mathrm{tot}}_n$ and $\omega_n+\Gamma^{\mathrm{tot}}_n = Z^{\mathrm{tot}}_n\omega_n$, and similarly $\Phi_n = Z_n\Delta_n$ and

$\omega_n + \Gamma_n = Z_n\omega_n$, we find that the relation between $Z_n^{\rm tot}$ and Z_n is given by Eq. (9), and that $\Delta_n^{\rm tot} = \Delta_n$, i.e., that $\Delta_n^{\rm tot}$ does not depend on the pair-conserving scattering rate $\Gamma^{\rm S}$.

To finish the derivation, one just has to note that the phonon self-energy $\hat{\Sigma}_n^{\rm ph}$ is given by standard Eliashberg-like expressions. Moreover, since $\Phi_n = \Phi_n^{\rm ph}$ and $\Gamma_n = \Gamma_n^{\rm ph} + \Gamma^{\rm M} s_n$, the only difference between E-CPA and the standard expressions comes from the presence of the term $\Gamma^{\rm M} s_n$ in Γ_n. If one observes that s_n is the sign of ω_n, one finally arrives at Eqs. (8).

APPENDIX C: DERIVATION OF EQS. (14)

The E-CPA equations (8) can be written down as a single equation for the matrix self-energy $\hat{\Sigma}_n = -i\Gamma_n\tau_0 + \Phi_n\tau_1$ defined in Appendix B,

$$\hat{\Sigma}_n = -i\Gamma^{\rm M}\,\mathrm{sgn}\,\omega_n\tau_0 + \frac{T}{N_0}\sum_m g_{n-m}\tau_3\hat{\mathcal{G}}_m\tau_3. \tag{C1}$$

In the upper half-plane we have $\mathrm{sgn}\,\omega_n = 1$, and therefore the analytic continuation of the first term in $\hat{\Sigma}_n$ is simply $-i\Gamma^{\rm M}\tau_0$. The second term can be written as

$$\frac{1}{N_0}\int_{-\infty}^{\infty} dy\alpha^2F(y)\tau_3\hat{H}_n(y)\tau_3, \tag{C2}$$

where we have introduced the following function of the Matsubara frequency:

$$\hat{H}_n(y) \equiv T\sum_m \frac{\hat{\mathcal{G}}(i\omega_m)}{i\omega_n - i\omega_m + y}. \tag{C3}$$

Our goal is to find an analytic continuation of $\hat{H}_n(y)$ to the whole upper half of the complex plane, $\hat{H}(z,y)$. The solution is

$$\hat{H}(z,y) = T\sum_m \frac{\hat{\mathcal{G}}(i\omega_m)}{z - i\omega_m + y} + \hat{A}(z,y). \tag{C4}$$

Here we have defined

$$\hat{A}(z,y) = \hat{\mathcal{G}}(z+y)[f(z+y) + b(y)], \tag{C5}$$

where $f(w) = 1/(e^{w/T}+1)$ and $b(w) = 1/(e^{w/T}-1)$ are the Fermi and Bose functions of the complex argument w.

In order to prove that Eq. (C4) does solve the problem, we have to show (i) that $H(i\omega_n, y) = H_n(y)$ and (ii) that $H(z,y)$ is analytic in the upper half-plane of z.

As regards (i), using Eq. (C4) we find $H(i\omega_n, y) = H_n(y) + \hat{A}(i\omega_n, y)$. But, since for fermionic frequencies we have $f(i\omega_n + y) + b(y) = 0$, also $\hat{A}(i\omega_n, y) = 0$ and the condition (i) is satisfied.

As regards (ii), since the function $\hat{\mathcal{G}}(z)$ is assumed to be analytic in the upper half-plane, the function $\hat{H}(z,y)$ might have poles only at some of the points $z = i\omega_m - y$. Let us therefore assume that $z = i\omega_m - y + u$, where u is an infinitesimal complex number. Then the possibly singular contribution to $\hat{H}(z,y)$ is given by

$$\hat{H}(z,y) \approx \frac{T}{u}[\hat{\mathcal{G}}(i\omega_m) - \hat{\mathcal{G}}(i\omega_m + u)].$$

But, since $\hat{\mathcal{G}}(z)$ is analytic at $z = i\omega_m$, the residue of $\hat{H}(z,y)$ at $z = i\omega_m - y$ vanishes. This finishes the proof.

Finally, plugging Eqs. (C4) and (C5) into Eq. (C2), making use of the fact that $\hat{\mathcal{G}}(z) = -i\pi N_0[n(z)\tau_0 + p(z)\tau_1]$, and requiring that $z = \omega + i0$, we arrive at Eq. (14).

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[15] The multiband explanation proposed in [8] has different problems of its own: In this picture, the Dynes density of states in the studied superconducting band 1 arises from internal proximity coupling to a nonsuperconducting band 2. However, this explanation works only if the density of states at the chemical potential is much larger in band 2 than in band 1. In order to explain that tunneling experiments are sensitive only to band 1, one has to furthermore assume that the matrix element for coupling between the microscope and band 2 is much smaller than for band 1. Even if we accept this assumption, one still has to explain why the interband coupling Γ_{12} strongly depends on temperature.