Exactly solvable dynamics and signatures of integrability in an infinite-range many-body Floquet spin system

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We study N qubits having infinite-range Ising interaction and subjected to a periodic pulse of an external magnetic field. We analytically solve the cases of $N = 5$ to 11 qubits, finding its eigensystem, the dynamics of the entanglement for various initial states, and the unitary evolution operator. These quantities shows signatures of quantum integrability. For the general case of $N > 11$ qubits, we provide a conjecture on quantum integrability based on the numerical evidence such as degenerate spectrum, and the exact periodic nature of the time-evolved unitary evolution operator and the entanglement dynamics. Using linear entropy, we show that for the class of initial unentangled state, the entanglement periodically displays maximum and zero values.

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Introduction. Classical and quantum systems with long-range interactions have played an important role in our understanding [1–14]. They have been found to be useful for quantum technology applications such as quantum heat engine [15], quantum computing [16,17], ion traps [18], etc. The generalized Higgs mechanism was used recently to understand such systems [14]. There are several experimental systems where these interactions are present, for example, cold atoms in cavities [19], polar molecules [20], dipolar quantum gases [21,22], and Rydberg atoms [23]. They have been found to be efficient for quantum computing and quantum simulation tasks as they can realize highly entangled states [24–27]. These systems have given rise to new phases of matter [28] and a measurement-induced phase transition [29–31]. Studies have shown that one can view them as short-range interacting systems in higher dimensions [32,33]. Recent work has studied quantum many-body scars in them [34]. Various studies have also addressed entanglement in such systems [35–37]. The propagation of multipartite entanglement and quantum scrambling in them is also addressed [38], which can be measured in experiments [39]. The bound on the scrambling in such all-to-all interaction models is well understood [40–42]. It is known that the semiclassical limit for these systems can be obtained [43,44]. Thus, one can study the effects of the underlying classical correlations on the quantum correlations [38,45–52].

The interaction in the long-range system decays as a power law with distance ($1/r^{\alpha}$), where α characterizes the given system. For van der Waals interactions in Rydberg atoms, $\alpha = 6$, for polar molecules and magnetic atoms (dipole-dipole interaction), $\alpha = 3$, for monopole-dipole, $\alpha = 2$, and for monopole-monopole or Coulomb-like, $\alpha = 1$, whereas for atoms coupled to cavities, $\alpha = 0$ [20,25,26,53–58]. The case $\alpha = 0$ corresponds to the class of infinite-range or all-to-all interaction physical systems. This kind of interaction is very useful for the construction of robust and high-fidelity geometric quantum logic gates [59].

In this work, we consider a system of qubits in magnetic field with periodic application (or kicking) of uniform infinite-range Ising interaction ($\alpha = 0$). Thus, the energy is not conserved due to periodic kicking. In such models, the Hamiltonian can also be reduced to total spin operators [38,45–47,60–65]. In fact, there is a one-to-one correspondence between all-to-all Ising interacting qubits and total spin operators. It is used depending on the problem under consideration. Specifically, the qubit representation is useful for studying quantum correlations, whereas the total spin operators help in understanding the semiclassical limit. Such Hamiltonians with and without energy conservation are studied from the direction of integrability-chaos transition in Refs. [66] and [67], respectively.

In this paper, we study a model consisting of N qubits kept in an external magnetic field and subjected to periodic global pulses of uniform infinite-range Ising interaction. We show that the system shows quantum integrability [68–71]. Its signatures can be seen in spectral statistics being Poissonian or degenerate spectrum or level crossings, exact periodicity of the time-evolution operator, etc. [68,69,71–76]. It should be noted that Refs. [68,69,71] consider a parameter-dependent family of quantum integrable Hamiltonians, whereas we get integrability only at a special case. And we expect these general signatures to remain intact for our case too. Our model has a connection to the nearest-neighbor Ising interaction model and a special case of the quantum chaotic kicked top (QKT) (to be discussed below). The QKT has been implemented in various experimental beds for values of N up to six [45,77,78]. For larger values of N of the order of 100, the use of ion traps [8,10] has been proposed in Ref. [79]. Once our model is mapped to the QKT, it is shown to display integrability *only* up to four qubits [63], whereas for large N, it is not understood whether or not the integrability persists.

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In our present work, we first analytically solve the model for the cases of five to 11 qubits. We analytically obtain the eigenvalues, eigenvectors, and entanglement and operator dynamics exactly. We show the time-periodic nature of both the entanglement and the operator itself.

This nature is in accordance with the results from Refs. [73,80–82], where the quantum integrable kicked Ising model for a range of parameters is considered [83,84]. This is an important observation as our model falls in the same category of integrability class as that of these models [refer to Eq. (3)] [85]. Thus, we can use these signatures to quantify integrability in our model too.

In Refs. [73,80–82], using numerical and analytical studies, it is found that the integrable periodically kicked spin chains with Ising interaction show a time-periodic nature of the entanglement for various initial states. In Ref. [82], it is shown that the integrable systems display the time-periodic nature of the entangling power of the Floquet operator, whereas in Ref. [73], the time-periodic nature of the Floquet operator itself is observed, which implies its time-periodic entangling power. In our work, we show a result for the general value of N, analytically for five to 11 qubits and numerically for $N > 11$, that the Floquet operator itself is time periodic, implying the time-periodic nature of the entangling power. Another signature of integrability can be observed from the degeneracy in the spectra of the Hamiltonian [68,71]. It means the system eigenvalues are lacking the repulsion among themselves. Similarly, in Ref. [73], a highly degenerate spectra is observed in the periodically driven Ising system which is also integrable. Thus, we can conclude that a system with Ising-like interaction shows quantum integrability if its spectra are degenerate; the entanglement and the evolution operator dynamics are time periodic.

For the general case of $N > 11$ qubits, we provide numerical evidence of the integrability using the degeneracy in the spectra, the periodic nature of entanglement dynamics, as well as the corresponding Floquet operator itself.

Model. The model Hamiltonian is given as follows:

$$H(t) = \sum_{l<l'=1}^{N} \sigma_l^z \sigma_{l'}^z + \sum_{n=-\infty}^{\infty} \delta(n - t/\tau) \sum_{l=1}^{N} \sigma_l^y, \qquad (1)$$

where τ is the period with which the magnetic field along the y axis is periodically applied (second term). The strength of the field is set equal to that of the Ising interaction (first term). In our model, the Ising interaction is uniform and all to all. The model is also permutation symmetric under the exchange of spins and thus the Hilbert space dimension is $N + 1$. Its special case, the one with only nearest-neighbor (NN) interaction, has been extensively studied [73,80,81,86–88]. In Ref. [81], $\tau = \pi/4$ is shown to generate nonlocal Bell pairs (maximum entanglement between two qubits) and multiqubit entanglement. Here, we also restrict ourselves to the same τ. In fact, we find that the model is quantum integrable *only* for $\tau = \pi/4$, whereas for other values, it is not (discussed below). Scrambling in models similar to ours has been studied in Refs. [40–42,89–91]; the one in Ref. [41], which is studied from the perspective of holographic duality, is very similar to ours. All these models are energy conserving ones, whereas ours is not.

The Hamiltonian in Eq. (1) also has a connection with the model of quantum chaotic kicked top (QKT) [44,92,93]. Its Hamiltonian is given by

$$H_{QKT}(t) = \frac{p}{\tau'} J_y + \frac{k}{2j} J_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau'). \qquad (2)$$

It is a time-dependent Hamiltonian where the first term represents a rotation and the second one is the torsion applied at periodic δ kicks. Here, $J_{x,y,z}$ are components of the angular momentum operator $\mathbf{J}$. For a given spin of the top j, the top can be decomposed in $N = 2j$ spin-half particles [60,61]. Here, the time between periodic kicks is τ', p measures rotation about the y axis, and k controls the degree of chaos in the classical limit. It can be seen that $H_{QKT}(t)$ has total spin operators. Various entanglement content has been studied in this model in recent times [45–49,52,62,63,78,92–99]. Particularly, it is important to note that for a given initial quantum state, the quantum correlations are periodic in k with period $2j\pi$ [93].

For the special case of parameters $p = \pi/2$, $k = j\pi (= N\pi/2)$, $\tau' = 1$ and using many-qubit transformation $J_{x,y,z} = \sum_{l=1}^{2j} \sigma_l^{x,y,z}/2$, where $\sigma_l^{x,y,z}$ are the standard Pauli matrices, it can be shown that $H_{QKT}(t)$ has a close resemblance or mapping with $H(t)$ from Eq. (1). Due to this mapping, our model can also be shown to have a connection with the integrable Lipkin-Meshkov-Glick (LMG) model [100]. The Floquet operator corresponding to Eq. (1) for $\tau = \pi/4$ is as follows:

$$\mathcal{U} = \exp\left(-i\frac{\pi}{4} \sum_{l<l'=1}^{N} \sigma_l^z \sigma_{l'}^z\right) \exp\left(-i\frac{\pi}{4} \sum_{l=1}^{N} \sigma_l^y\right). \qquad (3)$$

It gives the evolution of states just before a kick to just before the next one. In Ref. [63], a small number of qubits ($N = 3$ and 4) is extensively studied for these parameters.

In this work, our initial states are the standard SU(2) coherent states on the unit sphere, with spherical coordinates (θ_0, ϕ_0) and given by [101,102]

$$|\theta_0, \phi_0\rangle = \otimes^N [\cos(\theta_0/2)|0\rangle + e^{-i\phi_0} \sin(\theta_0/2)|1\rangle]. \qquad (4)$$

These are then evolved using the Floquet operator $\mathcal{U}$, and their entanglement dynamics as a function of time is studied. For this study, we use linear entropy [103] and concurrence [104,105] as measures of entanglement. For the analytical solution, we use the following standard basis permutation symmetric space by generalizing the one given for three and four qubits in Ref. [63]. It is based on the commutation relation $[\mathcal{U}, \otimes_{l=1}^{N} \sigma_l^y] = 0$, proved in the Supplemental Material [106]. Thus, the general basis for the odd number of qubits is

$$|\phi_q^{\pm}\rangle = \frac{1}{\sqrt{2}} \big(|w_q\rangle \pm i^{(N-2q)} \, |\overline{w_q}\rangle \big), 0 \leqslant q \leqslant \frac{N-1}{2}. \qquad (5)$$

For the even case,

$$|\phi_r^{\pm}\rangle = \frac{1}{\sqrt{2}} \big[\, |w_r\rangle \pm (-1)^{(N/2-r)} \, |\overline{w_r}\rangle \, \big], \quad 0 \leqslant r \leqslant N/2 - 1$$

$$\text{and} \quad |\phi_{N/2}^{+}\rangle = \left(1\Big/\sqrt{\binom{N}{N/2}}\right) \sum_{\mathcal{P}} \big(\otimes^{N/2} |0\rangle \otimes^{N/2} |1\rangle \big)\mathcal{P}, \qquad (6)$$

where $|w_q\rangle = \left(1/\sqrt{\binom{N}{q}}\right) \sum_{\mathcal{P}} (\otimes^q |1\rangle \otimes^{(N-q)} |0\rangle)_{\mathcal{P}}$ and $|\overline{w_q}\rangle = \left(1/\sqrt{\binom{N}{q}}\right) \sum_{\mathcal{P}} (\otimes^q |0\rangle \otimes^{(N-q)} |1\rangle)_{\mathcal{P}}$, with both being definite particle states [107]. The $\sum_{\mathcal{P}}$ denotes the sum over all possible permutations. These basis states are parity symmetric and follow $\otimes_{l=1}^{N} \sigma_l^y |\phi_j^\pm\rangle = \pm |\phi_j^\pm\rangle$. In this basis, $\mathcal{U}$ becomes block diagonal, simplifying further analysis. We derive entanglement dynamics for the coherent states $|\theta_0 = 0, \phi_0 = 0\rangle$ and $|\theta_0 = \pi/2, \phi_0 = -\pi/2\rangle$. In terms of qubit representation, these coherent states are given by $\otimes^N |0\rangle$ and $\otimes^N |+\rangle$, respectively [63]. These states have importance in the QKT. The first state is on the period-4 orbit, whereas the second one is on the fixed point in the classical phase space of the QKT [63].

Exact solution for five qubits. Using the basis in Eq. (5) for $N = 5$, the unitary operator $\mathcal{U}$ is given by

$$\mathcal{U} = \begin{pmatrix} \mathcal{U}_+ & 0 \\ 0 & \mathcal{U}_- \end{pmatrix}, \tag{7}$$

where $\mathcal{U}_\pm$ are 3×3 dimensional matrices and 0 is a null matrix of same dimension as $\mathcal{U}_\pm$. The $\mathcal{U}_+$ ($\mathcal{U}_-$) are written in the positive- (negative-)parity subspaces $\{\phi_0^+, \phi_1^+, \phi_2^+\}$ ($\{\phi_0^-, \phi_1^-, \phi_2^-\}$), respectively, and are obtained as follows:

$$\mathcal{U}_\pm = \frac{1}{4} e^{\pm \frac{i\pi}{4}} \begin{pmatrix} \mp 1 & i\sqrt{5} & \mp\sqrt{10} \\ -i\sqrt{5} & \pm 3 & -i\sqrt{2} \\ \pm\sqrt{10} & -i\sqrt{2} & \mp 2 \end{pmatrix}. \tag{8}$$

The eigenvalues of $\mathcal{U}_+$ and $\mathcal{U}_-$ are $e^{\frac{i\pi}{4}}\{1, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}\}$ and $e^{\frac{3i\pi}{4}}\{1, e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}\}$, respectively, whereas the eigenvectors of $\mathcal{U}_\pm$ are $[\pm i/\sqrt{5}, 1, 0]^T$, $[\pm i\sqrt{5/6}, -1/\sqrt{6}, 1]^T$, and $[\mp i\sqrt{5/6}, 1/\sqrt{6}, 1]^T$. For evolving an initial state, we need to get $\mathcal{U}^n$ and therefore $\mathcal{U}_\pm^n$, which is given as follows:

$$\mathcal{U}_\pm^n = (\pm 1)^n e^{\pm \frac{in\pi}{4}} \begin{bmatrix} [1 + 5\cos(2n\pi/3)]/6 & \pm i\sqrt{5}[\sin^2(n\pi/3)]/3 & -\sqrt{5/6}\sin(2n\pi/3) \\ \mp i\sqrt{5}[\sin^2(n\pi/3)]/3 & [5 + \cos(2n\pi/3)]/6 & \mp i\sin(2n\pi/3)/\sqrt{6} \\ \sqrt{5/6}\sin(2n\pi/3) & \mp i\sin(2n\pi/3)/\sqrt{6} & \cos(2n\pi/3) \end{bmatrix}. \tag{9}$$

It can be shown that the $\mathcal{U}$ is periodic with period 24, which is a signature of integrability (see Supplemental Material [106]). It is now straightforward to do the time evolution of any initial state. Let us first start with $|00000\rangle$. Its nth time evolution is given by

$$\begin{aligned} |\psi_n\rangle &= \mathcal{U}^n|00000\rangle = \mathcal{U}^n|w_0\rangle = \mathcal{U}^n(|\phi_0^+\rangle + |\phi_0^-\rangle)/\sqrt{2} = (\mathcal{U}_+^n|\phi_0^+\rangle + \mathcal{U}_-^n|\phi_0^-\rangle)/\sqrt{2} \\ &= (1/2)e^{\frac{in\pi}{4}}\{(1 + i^n)(\alpha_n |w_0\rangle - i\beta_n |\overline{w_1}\rangle + \gamma_n |w_2\rangle) + (1 - i^n)(i\alpha_n |\overline{w_0}\rangle + \beta_n |w_1\rangle + i\gamma_n |\overline{w_2}\rangle)\}, \end{aligned}$$

where $\alpha_n = [1 + 5\cos(2n\pi/3)]/6$, $\beta_n = i\sqrt{5}\sin^2(n\pi/3)/3$, and $\gamma_n = \sqrt{5/6}\sin(2n\pi/3)$. Using $|\psi_n\rangle$, one can obtain the reduced density matrix (RDM) of a single qubit $[\rho_1(n) = \mathrm{Tr}_{\neq 1}(|\psi_n\rangle\langle\psi_n|)]$ and the two qubits $[\rho_{12}(n) = \mathrm{Tr}_{\neq 1,2}(|\psi_n\rangle\langle\psi_n|)]$. For even time $n = 2m$, $\rho_1(2m)$ is diagonal and is given as follows:

$$\rho_1(2m) = \begin{bmatrix} \lambda_{2m} & 0 \\ 0 & 1 - \lambda_{2m} \end{bmatrix}, \tag{10}$$

where λ_{2m} and $1 - \lambda_{2m}$ are its eigenvalues with $\lambda_{2m} = [6 + 2\cos(4m\pi/3) + \cos(8m\pi/3)]/9$, whereas for odd time $n = 2m - 1$, we get

$$\rho_1(2m - 1) = \begin{bmatrix} 1/2 & h_{2m-1} \\ h_{2m-1} & 1/2 \end{bmatrix}, \tag{11}$$

where $h_{2m-1} = (2/9)\sin^2[(2m-1)\pi/3]\{2 + \cos[2(2m-1)\pi/3] - \sqrt{3}\sin[2(2m-1)\pi/3]\}$. Its eigenvalues are λ_{2m-1} and $1 - \lambda_{2m-1}$, with $\lambda_{2m-1} = (1/18)(9 - \{27 - 17\cos[2(2m-1)\pi/3] - 10\cos[4(2m-1)\pi/3] - 17\sqrt{3}\sin[2(2m-1)\pi/3] + 10\sqrt{3}\sin[4(2m-1)\pi/3]\}^{1/2})$. These eigenvalues give the linear entropy using $2\lambda_n(1 - \lambda_n)$ which is plotted in Fig. 1. It can be shown from the expressions and the figure that it has a periodic nature with period six. This periodic nature is also observed in previous works on integrable systems involving periodically kicked spin chains in Refs. [73,80,81]. In the context of QKT, time periodicity has been reported earlier for $N = 2$ [93], $N = 3$, and $N = 4$ for special values of k in Eq. (2) [63]. We analytically prove that the entanglement content is the same for consecutive odd and even values of n (see Supplemental Material [106]).

Similarly, we evolve the $|+++++\rangle$ state and obtain the linear entropy at the nth time. The eigenvalues of RDM $\rho_1(n)$ are λ_n and $1 - \lambda_n$, where $\lambda_n = [3 - 2\cos(2n\pi/3) - \cos(4n\pi/3)]/9$. The linear entropy using $2\lambda_n(1 - \lambda_n)$ is plotted in Fig. 1. Its again periodic in time with period three. Using ρ_{12}, it is shown that the pairwise concurrence is zero for all n for both of the initial states and is shown in the same figure (see Supplemental Material [106]) (see, also, Refs. [108,109] therein). This shows that the entanglement is of multipartite nature.

Exact solutions for six to 11 qubits. Following the similar procedure from the previous part, we solve for the cases from $N = 6$ to 11. We tabulate the explicit analytical expressions for the linear entropy for the initial states: $\otimes^N|0\rangle$ and $\otimes^N|+\rangle$ (refer to Table I). It must be noted that these expressions, including those for the $N = 5$ case, are true for any $n \in \mathcal{R}$. But the signatures of quantum integrability can be obtained

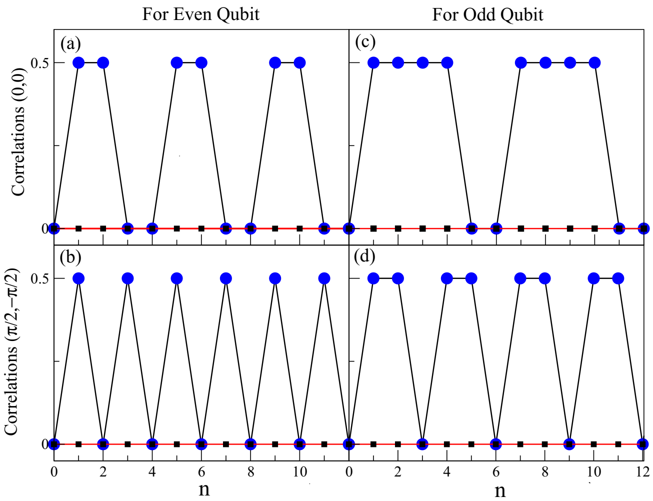

FIG. 1. Correlations using linear entropy (circles) and concurrence (squares) are plotted for (a),(b) even and (c),(d) odd number of qubits for various initial states.

by restricting n to positive integers only. The data points in all the figures of this paper are for $n \in \mathcal{N}^+$. For these cases also, entropy is the same for the consecutive odd and even values of n [63].

From these expressions, we find them to be periodic in time. We show that the time period T for an odd and even number of qubits for the initial states $\otimes^N |0\rangle$ ($\otimes^N |+\rangle$) is 6(3) and 4(2), respectively (see Fig. 1). We also show that the dynamics of the corresponding Floquet operator and its powers show a periodic nature in time, i.e., $\mathcal{U}^{n+T_1} = \mathcal{U}^n$, where $n \geqslant 1$ and T_1 is the period. For even N, $T_1 = 8$, while for $N = 5$, 7, 9 and 11, $T_1 = 24$, 12, 24 and 12, respectively. We have found the eigensystem analytically and observe that the spectrum is degenerate (see Supplemental Material [106]).

Results for general N. With our method, in principle, one can get the eigensystem and entanglement dynamics analytically for any finite N. But obtaining a general solution as a function of N is mathematically challenging, which can be observed from Ref. [80], where only the nearest-neighbor interaction is considered. Thus, we resort to numerical simulations and find various signatures of integrability for $N > 11$. For this purpose, we use the same signatures obtained for $N = 5$ to 11 and claim integrability for any $N > 11$ and $\tau = \pi/4$. To our surprise, we find the same signatures (depending only on the parity of N) as shown in Fig. 1. A similar N independence was observed in Ref. [80] for the case of the integrable kicked-Ising model in zero external magnetic field. When field is present in the NN interaction model, the time period shows dependence on N [73,81,82]. We also study the operator dynamics numerically. To quantify the periodic nature of time-evolved $\mathcal{U}$, we find its deviation from the original operator itself using $\delta(n) = \sum_{p,q} |\mathcal{U}^n_{p,q} - \mathcal{U}_{p,q}|/2N$. It is zero for any $n > 1$ if and only if $\mathcal{U}^n = \mathcal{U}$, thus confirming the time-periodic nature of $\mathcal{U}$. Numerically it is observed that division by $2N$ ensures the average of $\delta(n)$ is one. The results are plotted in Fig. 2 for even N. We find that for $\tau = \pi/4$ and N up to 400, the time evolution of $\mathcal{U}$ is periodic (checked for n as large as 5000), whereas for $\tau \neq \pi/4$, it is not. For odd N (results not plotted here), we find similar periodicity, but with different periods $T_1 = 12$ or 24. These periods are the same as that of our analytical ones for $N = 5$, 7, 9, and 11. Another signature of integrability is found from the eigenangle spectrum of $\mathcal{U}$ at $\tau = \pi/4$. We find it to be highly degenerate, taking values from the set $\{0, \pm\pi/4, \pm\pi/2, \pm 3\pi/4, \pm\pi\}$ for

TABLE I. The linear entropy $[S(n)]$ for different initial states and number of qubits, N.

N	The $S(n)$ for initial state $\otimes^N \lvert 0\rangle$	The $S(n)$ for initial state $\otimes^N \lvert +\rangle$
6	$S(n) = \frac{1}{512}\{[1-\cos(\frac{n\pi}{2})+\sin(\frac{n\pi}{2})]^2[284+229(\cos(\frac{n\pi}{2})$ $-\sin(\frac{n\pi}{2}))-96\sin(n\pi)-9(\cos(\frac{3n\pi}{2})+\sin(\frac{3n\pi}{2}))]\}$	$S(n) = \frac{1}{2}\{1-\frac{1}{4}[1+\cos(n\pi)]^2\}$
7	$S(2m-1) = \frac{1}{2}\{1-\frac{1}{81}\sin^2(\frac{\pi(2m-1)}{3})[-4\sqrt{3}\cos[\frac{\pi(2m-1)}{3}]$ $+\sqrt{3}\cos(\pi(2m-1))+6\sin(\frac{\pi(2m-1)}{3})+\sin((2m-1)\pi)]^2\}$ $S(2m) = \frac{1}{81}\{[7+2\cos(\frac{4m\pi}{3})][12+5\cos(\frac{4m\pi}{3})+\cos(\frac{2m\pi}{3})]\sin^2(\frac{2m\pi}{3})\}$	$S(n) = \frac{1}{2}\{1-\frac{1}{81}[3+5\cos(\frac{2n\pi}{3})+\cos(\frac{4n\pi}{3})]^2\}$
8	$S(n) = \frac{1}{2048}\{[1-\cos(\frac{n\pi}{2})+\sin(\frac{n\pi}{2})]^2[1212-448\sin(n\pi)$ $+1005(\cos(\frac{n\pi}{2})-\sin(\frac{n\pi}{2}))-49(\sin(\frac{3n\pi}{2})+\cos(\frac{3n\pi}{2}))]\}.$	$S(n) = \frac{1}{2}[1-\cos^2(\frac{n\pi}{2})]$
9	$S(2m-1) = \{\frac{1}{2}-\frac{8}{81}\sin^4(\frac{(2m-1)\pi}{3})[2+\cos(\frac{2(2m-1)\pi}{3})-\sqrt{3}\sin(\frac{2(2m-1)\pi}{3})]^2\}$ $S(2m) = \frac{8}{81}\{[2+\cos(\frac{4m\pi}{3})][6+2\cos(\frac{4m\pi}{3})+\cos(\frac{2m\pi}{3})]\sin^2(\frac{2m\pi}{3})\}$	$S(n) = \frac{1}{2}\{1-\frac{1}{81}[1+2\cos(\frac{2n\pi}{3})]^4\}$
10	$S(n) = 1-\frac{1}{8}\{4+[\frac{17}{32}\cos(\frac{n\pi}{4})+\cos(\frac{5n\pi}{4})+\frac{15}{32}\cos(\frac{9n\pi}{4})]^2\}$ $-\sin^2(\frac{n\pi}{2})[17(\cos(\frac{3n\pi}{4})+\sin(\frac{3n\pi}{4}))$ $-15(\cos(\frac{7n\pi}{4})+\sin(\frac{7n\pi}{4}))]^2/4096$	$S(n) = \frac{1}{2}\{1-\frac{1}{4}[1+\cos(n\pi)]^2\}$
11	$S(2m-1) = \frac{1}{2}\{1-\frac{1}{144}\sin^2(\frac{(2m-1)\pi}{3})[-2\sqrt{3}\cos(\frac{(2m-1)\pi}{3})+8\sin(\frac{(2m-1)\pi}{3})$ $+3(\sqrt{3}\cos((2m-1)\pi)+\sin((2m-1)\pi))]^2\}$ $S(2m) = \frac{1}{144}\{[11+6\cos(\frac{4m\pi}{3})][16+5\cos(\frac{4m\pi}{3})+3\cos(\frac{2m\pi}{3})]\sin^2(\frac{2m\pi}{3})\}$	$S(n) = \frac{1}{2}\{1-\frac{1}{144}[4+5\cos(\frac{2n\pi}{3})+3\cos(\frac{4n\pi}{3})]^2\}$

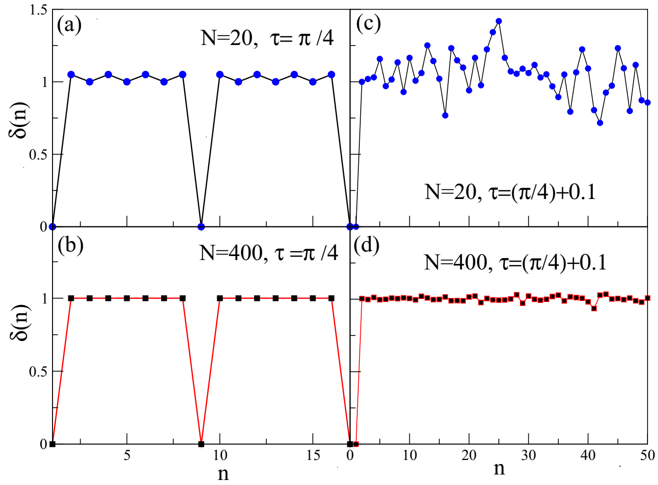

FIG. 2. Deviation $\delta(n)$ for various values of N and τ.

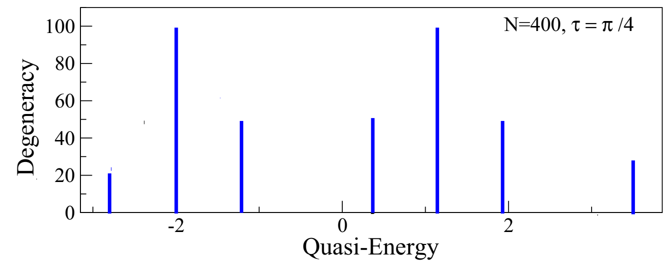

FIG. 3. Degeneracy of the quasienergies of $\mathcal{U}$.

even N (checked for N up to 400; refer to Fig. 3). Similarly, for odd N, we numerically find a degenerate spectrum. Thus, with these signatures, we can very well conjecture that the system is quantum integrable for any $N > 11$ number of qubits and the time period (τ) of the kick such that $\tau = \pi/4$.

Conclusions. Integrable models have played an important role in advancing our understanding of physical systems [110]. Our model of infinite range with Ising two-body interaction shows quantum integrability for any number of qubits. The cases involving $N = 5$ to 11 qubits are dealt with analytically, whereas a conjecture with sufficient evidence is presented for any $N \geqslant 11$. Previous work as far as integrability in our model (after mapping to QKT is concerned) was limited only up to four qubits. We have now generalized it to any N.

It must be noted that our conjecture is based on the circumstantial signatures of integrability. A more rigorous proof on quantum integrability in our case based on Bethe ansatz [111,112] and/or obtaining a transfer matrix by solving a Yang-Baxter relation in this case is highly warranted [69,113]. This transfer matrix can then be used to generate an infinite number of conserved quantities to prove integrability. Recent work involved the use of generalized Hubbard-Stratonovich transformation to get an exact solution for quantum strong long-range Ising chains [114]. Our model can be further investigated in this direction.

Our results (for the smaller number of qubits) can be experimentally verified in various setups from nuclear magnetic resonance (NMR) [78], superconducting qubits [45], and laser-cooled atoms [77], where the QKT is readily implemented, whereas for the larger number of qubits (of the order of 100s), one can use ion traps [8,10]. Our conjecture can be tested in this setup as well. With our findings, a search for similar quantum integrable spin systems can be initiated.

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