Letter

Thermodynamics and fractal Drude weights in the sine-Gordon model

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The sine-Gordon model is a paradigmatic quantum field theory that provides the low-energy effective description of many gapped one-dimensional systems. Despite this fact, its complete thermodynamic description in all its regimes has been lacking. Here, we fill this gap and derive the framework that captures its thermodynamics and serves as the basis of its hydrodynamic description. As a first application, we compute the Drude weight characterizing the ballistic transport of topological charge and demonstrate that its dependence on the value of the coupling shows a fractal structure, similar to the gapless phase of the XXZ spin chain. The thermodynamic framework can be applied to study other features of nonequilibrium dynamics in the sine-Gordon model using generalized hydrodynamics, opening the way to a wide array of theoretical studies and potential novel experimental predictions.

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I. INTRODUCTION AND SUMMARY

Transport properties such as electric conductivity provide fundamental insight into the dynamics of condensed matter [1]; however, their description is especially challenging in the case of strongly correlated systems [2]. In particular, integrable quantum many-body systems display many anomalous properties [3] due to ergodicity breaking. In the case of transport, the effects of nonergodicity are captured by the Mazur inequality [4,5], and are primarily characterized by ballistic transport and finite Drude weights [6,7]. It has been studied quite intensively for the XXZ spin chain [8–11], where a striking "fractal" structure of the spin Drude weight was observed in the gapless phase [12–16], a phenomenon also known as "popcorn" Drude weights [17]. Recently, the framework of generalized hydrodynamics (GHD) [18–22] has led to a different approach to transport properties and, in particular, the computation of Drude weights [23–27]. An experimental protocol to determine Drude weights was also proposed recently in Ref. [28].

In this work, we consider ballistic transport of topological charge in the sine-Gordon model, which is a paradigmatic integrable quantum field theory with numerous applications to condensed matter physics [29,30] with the Hamilton operator

$$H = \int \mathrm{d}x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 - \lambda \cos(\beta\phi) \right], \qquad (1)$$

where $\phi(x)$ is a real scalar field, β is the coupling, and λ sets the mass scale. In particular, it is a model for a one-dimensional (1D) Mott insulator with applications to quasi-1D antiferromagnets, carbon nanotubes, and organic conductors [31,32], where electric conduction corresponds to transport of the topological charge carried by kink excitations. It can also be realized in experiments with trapped ultracold atoms [33–36], which provide a well-controlled platform for studying nonequilibrium dynamics. In particular, the topological charge can be exploited in experiments to characterize the soliton dynamics [36]. Other proposed realizations are via quantum circuits [37] or coupled spin chains [38].

The main stumbling block preventing progress in modeling the nonequilibrium dynamics in this important model has been the absence of an explicit thermodynamic description for general couplings, despite its known exact scattering theory [39,40]. Thermodynamic Bethe ansatz (TBA) systems for the sine-Gordon model have so far only been formulated explicitly for special values of the coupling [41–43], although a corresponding set of functional relations (the so-called Y system) was conjectured for the general case in Ref. [44]. Our first main result consists of the TBA system for generic couplings with the corresponding dressing relations, which provides flexibility for applications to experiments by allowing generic values of the coupling parameter. We then exploit the TBA system to obtain the second main result: the Drude weight for charge transport in the sine-Gordon model. We demonstrate that the Drude weight considered as a function of the sine-Gordon coupling displays a fractal structure, as shown in Fig. 1.

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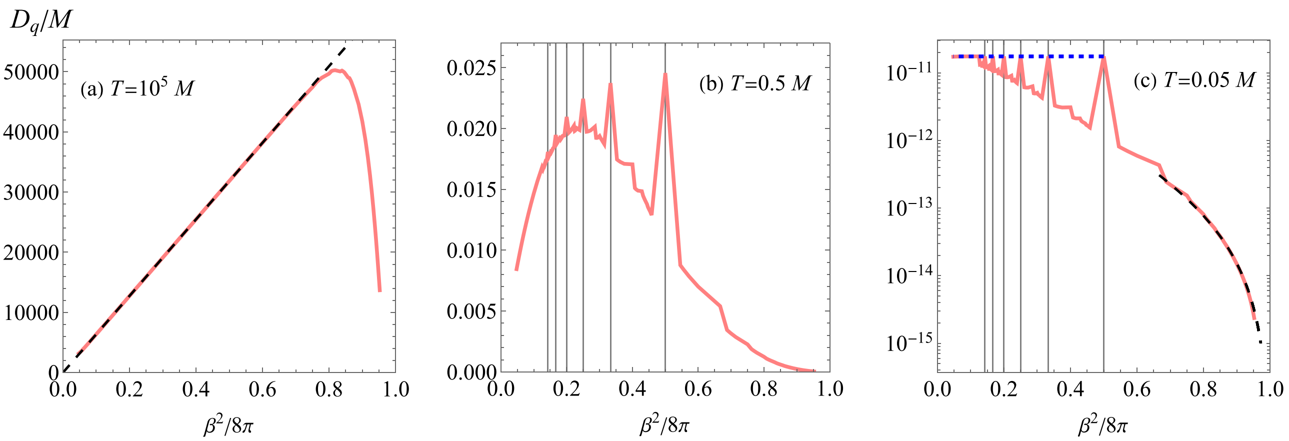

FIG. 1. Drude weight in the sine-Gordon model calculated from Eq. (7) as a function of the coupling strength $\beta^2/8\pi$ for (a) high, (b) moderate, and (c) low temperatures, at values of ξ with at most two magnonic levels in the TBA system (3). The Drude weight was computed at discrete points which are joined by a red line in the plot to emphasize their discontinuous nature. At high temperatures, the fractal structure is washed out, while it gets more pronounced as the temperature is lower. We verified that the results from the bipartition protocol Eq. (8) agree with the TBA results within 0.1% relative difference. The dashed line in (a) is the high-T limit Eq. (9), while in (c) the dotted blue line is the low-T prediction for the reflectionless points (11), and the black dashed line shows (10). The vertical grid lines in (b) and (c) indicate the reflectionless points. Note also the logarithmic scale in (c).

II. THERMODYNAMICS OF THE SINE-GORDON MODEL

The particle spectrum of the model is characterized in terms of the renormalized coupling constant $\xi = \beta^2/(8\pi - \beta^2)$. It consists of a doublet of topologically charged kinks of mass M, and, in the attractive regime $0 < \xi < 1$, also of their neutral bound states (breathers) of masses $m_{B_k} = 2M \sin\frac{k\pi\xi}{2}$ with $1 \leqslant k \leqslant n_B = \lfloor 1/\xi \rfloor$. The topological charge density and current are $\rho_q = \beta\partial_x\phi/2\pi$ and $j_q = -\beta\partial_t\phi/2\pi$.

Thermodynamics at a given temperature T and chemical potential μ conjugate to the topological charge can be computed using the TBA [45–47]. For the so-called reflectionless points $1/\xi = n_B - 1$ where all scattering is purely transmissive, it can be formulated in terms of the physical excitations (two kinks and n_B breathers) [47]. However, for generic couplings, it is given in terms of *three* kinds of excitations: a single solitonic excitation S accounting for the energy and momentum of the kinks, breathers B_i $(i = 1, \dots, n_B)$, and additional massless auxiliary excitations ("magnons") which account for the topological charge carried by the kinks. Magnons can be classified by writing ξ as a (unique) simple continued fraction,

$$\xi = \cfrac{1}{n_B + \cfrac{1}{\nu_1 + \cfrac{1}{\nu_2 + \cdots}}} \equiv \cfrac{1}{n_B + \cfrac{1}{\alpha}}, \tag{2}$$

with ν_k magnon species at level k. We restrict our attention to cases with at most two magnonic levels, but extending to the general case is straightforward [48,49].

Thermodynamic states corresponding to generalized Gibbs ensembles and containing an extensive number of particles can be characterized in terms of total and root densities $\rho_a^{\text{tot}}(\theta)$ and $\rho_a^{\text{r}}(\theta)$ describing the density of available and filled levels for excitation of type a per unit rapidity θ and per unit volume. It is convenient to introduce the so-called pseudoenergies $\epsilon_a(\theta)$ that parametrize the occupations as $[1 + e^{\epsilon_a(\theta)}]^{-1} = \rho_a^{\text{r}}/\rho_a^{\text{tot}}$ and satisfy equations following from the minimization of the free energy. We found that the TBA system of the sine-Gordon model can be written in the concise form

$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)}\epsilon_b - \sigma_b^{(2)}w_b + L_b\right), \tag{3}$$

where $L_a(\theta) = \log(1 + e^{-\epsilon_a(\theta)})$ and the star denotes convolution. The free energy f density is given by

$$f = -T \sum_a \int \frac{d\theta}{2\pi} M_a L_a(\theta) \cosh\theta\,. \tag{4}$$

The kernels $K_{ab}(\theta)$ originate from the scattering of the excitations and can be encoded in a diagram [41,50] as shown in Fig. 2 . The driving terms $w_a(\theta)$ carrying the dependence on the generalized chemical potentials (temperature T and the chemical potential μ in thermal equilibrium) and the factors $\sigma_a^{(1,2)}$ are given in Table I.

The above TBA system, our first main result, can be validated by cross-checking its predictions for the free energy in (neutral) thermal states against the so-called nonlinear integral equation [51,52] as reported in Ref. [48], and in the high-temperature limit [$\lambda = 0$ in (1)] we recover the free energy of the free massless boson.

Conserved charges with values $Q_a(\theta)$ assigned to excitation a with rapidity θ are also affected by the finite densities of excitations, and their dressed values Q_a^{dr} can be obtained by solving the *dressing equations* [48]

$$\eta_a Q_a^{\text{dr}} = Q_a + \sum_b K_{ab} * \left[\left(\sigma_b^{(1)} - \vartheta_b\right)\eta_b Q_b^{\text{dr}} - \sigma_b^{(2)} Q_b\right], \tag{5}$$

with $\vartheta_a(\theta) = \rho_a^{\text{r}}(\theta)/\rho_a^{\text{tot}}(\theta)$ denoting the *filling fractions*. The total densities themselves can be obtained as $2\pi\rho_a^{\text{tot}}(\theta) = (\partial_\theta p_a)^{\text{dr}}(\theta)$, where $p_a(\theta) = M_a \sinh\theta$ is the bare momentum of excitation type a with rapidity θ. The signs η_a ensure

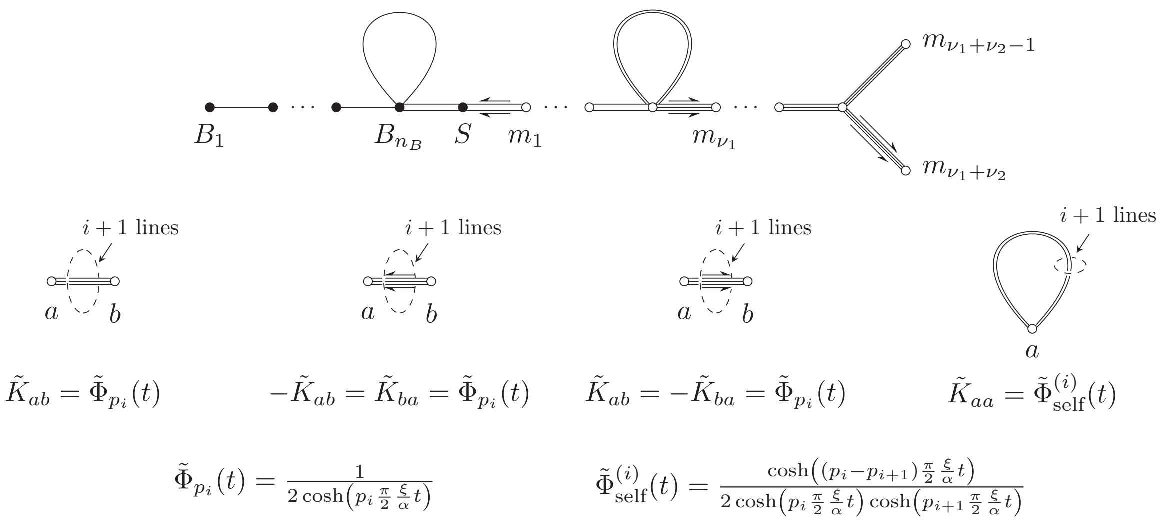

FIG. 2. Diagram encoding the TBA kernels $K_{ij}(\theta)$ in terms of their Fourier transforms $\tilde{K}_{ab}(t) = \int d\theta/2\pi\, K_{ab}(\theta) e^{-i\theta t}$ for the generic case with two magnonic levels (for the exceptions cf. Ref. [48]). The diagram is similar to the one presented in Ref. [50] for the boundary sine-Gordon model. The kernel parameters are $p_0 = \nu_1 + 1/\nu_2$, $p_1 = 1$, $p_2 = 1/\nu_2$.

the positivity of the densities and are given in Table I. The effective velocity of excitations as a function of their rapidity θ can be obtained as $v_a^{\text{eff}}(\theta) = (\partial_\theta e_a)^{\text{d}}(\theta)/(\partial_\theta p_a)^{\text{d}}(\theta)$, where $e_a(\theta) = M_a \cosh\theta$ is the bare energy [20,53,54].

III. DRUDE WEIGHT FOR THE TOPOLOGICAL CHARGE

Our TBA system can form the basis of the generalized hydrodynamics of the sine-Gordon model and can be used to compute various physical quantities in and out of equilibrium. We focus on the Drude weight of the topological charge defined from the connected current correlator as

$$D_q = \lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} dt \int dx\, \langle j_q(x,t) j_q(0,0)\rangle_c, \tag{6}$$

which can be expressed from the TBA as [23,24,27]

$$D_q = \sum_a \int d\theta\, \rho_a^{\text{tot}}(\theta)\vartheta_a(\theta)[1-\vartheta_a(\theta)]\left[v_a^{\text{eff}}(\theta) q_a^{\text{dr}}(\theta)\right]^2, \tag{7}$$

where $q_a^{\text{dr}}(\theta)$ is the dressed topological charge of excitation a with rapidity θ. Solving the TBA system (3) for different values of T and ξ (with $\mu = 0$), evaluating the dressed quantities using (5), and substituting them into (7) results in the data shown in Fig. 1.

Another way to obtain the Drude weights is to consider a bipartitioned initial state with a small chemical potential difference $\delta\mu$ between the half-systems $x > 0$ and $x < 0$. Initial densities on the two sides can be obtained from the dressing equation (5) using the appropriate solution of the TBA system (3) with finite temperature T and chemical potentials $\mu = \pm\delta\mu/2$, and the subsequent time evolution can be computed by a simple application of the GHD as described in Ref. [48]. The asymptotic current profile is a limit along "rays" $\mathrm{j}(\zeta) = \lim_{t\to\infty} j(x = \zeta t, t)$, and the Drude weight is [14,25]

$$D_q = \frac{\partial}{\partial\delta\mu} \int d\zeta\, \mathrm{j}(\zeta)\bigg|_{\delta\mu=0}. \tag{8}$$

We verified that this method gives the same result as Eq. (7) within the attainable numerical precision.

IV. SPECIAL CASES

Besides numerical results, we can also obtain analytic expressions in appropriate limits. For high temperatures $T/M \gg 1$, the fractal structure is suppressed and the Drude weight is

TABLE I. Driving terms, topological charges, and factors corresponding to the excitations in the TBA system (3) and dressing equation (5) for two magnonic levels in the generic case (see Ref. [48] for other cases).

Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	B_i, $i = 1, \ldots, n_B$	$M_{B_i}\cosh(\theta)/T$	0	+1	+1	+1
Soliton	S	$M\cosh(\theta)/T$	+1	+1	0	0
First-level intermediate magnons	m_j, $j = 1, \ldots, \nu_1 - 1$	0	$-2j$	−1	+1	0
First-level last magnon	m_{ν_1}	0	−2	+1	+1	0
Second-level intermediate magnons	m_{ν_1+k}, $k = 1, \ldots, \nu_2 - 2$	0	$-2(1 + k\,\nu_1)$	+1	+1	0
Second-level next-to-last magnon	$m_{\nu_1+\nu_2-1}$, $(k = \nu_2 - 1)$	$2(1 + \nu_1\,\nu_2)\mu/T$	$-2(1 + k\,\nu_1)$	+1	+1	0
Second-level last magnon	$m_{\nu_1+\nu_2}$	$2(1 + \nu_1\,\nu_2)\mu/T$	$-2\,\nu_1$	−1	0	0

a continuous function of the renormalized coupling ξ [48]:

$$D_q^{\text{high-}T} = \frac{2T}{\pi}\frac{\xi}{\xi+1} = \frac{T\beta^2}{4\pi^2}. \tag{9}$$

In this regime, the dynamics is independent of the coupling λ in (1) and is described by a free massless boson, which reproduces (9) by an easy derivation [48].

In the opposite limit of low temperature $T/M \ll 1$ and for a single magnonic level in the repulsive regime (i.e., $\xi = 2, 3, \ldots$), the analytic result is [48]

$$D_q^{\text{low-}T} = e^{-M/T}\frac{\sqrt{2}T^{5/2}}{\pi^{3/2}M^{3/2}}\frac{2\pi/\xi - \sin(2\pi/\xi)}{\xi\sin^2(\pi/\xi)}. \tag{10}$$

In the attractive regime, at reflectionless couplings $\xi = 1/(n_B - 1)$, we find in the low-temperature limit

$$D_q^{\text{low-}T} = 2\int\frac{\mathrm{d}\theta}{2\pi}\frac{M\cosh\theta\, e^{-M\cosh\theta/T}}{(1+e^{-M\cosh\theta/T})^2}\tanh^2\theta \tag{11}$$

independent of the coupling. This result can be obtained simply by considering the kinks and antikinks as noninteracting fermions with energy $M\cosh\theta$ and velocity $\tanh\theta$, since in the absence of the massless magnons all excitations are massive, and therefore the effects of the scattering embodied in the integral terms of Eqs. (3) and (5) are exponentially suppressed [48].

As shown in Fig. 1, these analytic limiting cases agree very well with the numerical values computed from (7).

V. DISCUSSION AND OUTLOOK

We have demonstrated the existence of a nonzero Drude weight of the topological charge in the sine-Gordon model, which under Mazur's inequality strongly suggests the existence of yet unknown conserved quantities that are odd under charge conjugation. Moreover, it has a fractal structure (as argued recently in Ref. [17]), which is the only appearance of such a commensurability effect so far besides the prototypical XXZ spin chain.

The Drude weight (cf. Fig. 1) has some interesting properties. First, we note that D_q approaches zero at the Kosterlitz-Thouless point $\beta^2/8\pi = 1$, as demonstrated by the numerical data and also by the explicit expression (10) for low temperatures. The numerical computations show that the Drude weight goes to zero at all finite T for $\xi \to \infty$, hence from Eq. (9) it follows that the limits $\xi \to \infty$ and $T \to \infty$ do not commute. Second, the values of D_q for a fixed number N of magnonic levels appear to form regular sequences when considered as a function of the deepest level integer ν_N in the continued fraction expansion (2) with all other (lower level) integers kept fixed, as demonstrated in Fig. 3. Third, increasing the depth of the continued fraction (2), i.e., the number of magnonic levels N, suppresses the fractal structure, suggesting that for irrational values of the coupling ξ corresponding to infinite continued fractions, the values of the Drude weight D_q lie on a limiting envelope curve. In fact, the Drude weight D_q for an irrational ξ is expected to be successively approximated by truncating the continued fraction expansion (2) progressively deeper, i.e., with an increasing number of magnonic levels [16].

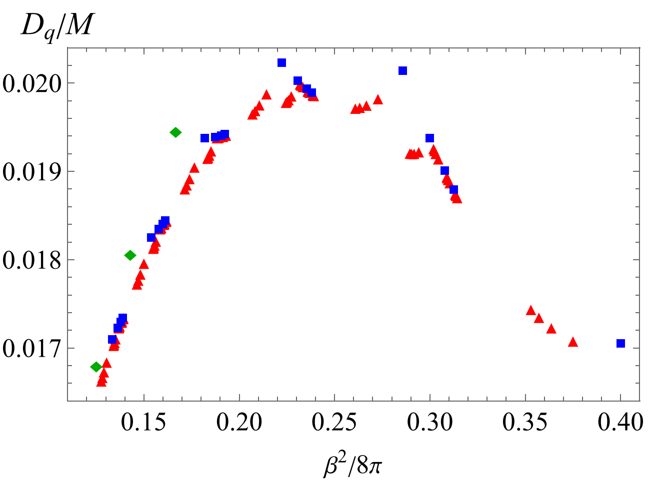

FIG. 3. A close-up view of the fractal structure of Drude weights from Fig. 1 (b) ($T = M/2$) with separate markings for different numbers of magnon levels. Green diamonds: reflectionless couplings ($N = 0$, no magnons), blue squares: $N = 1$ magnonic level, red triangles: $N = 2$ magnonic levels.

In addition to the results for the Drude weights, the TBA system (3) and the associated dressing equations (5) open the way to applications of generalized hydrodynamics to the sine-Gordon model at generic values of the coupling. We demonstrated the application to nonequilibrium dynamics by studying a partitioning protocol [48], but more general inhomogeneous setups can be studied providing predictions relevant to condensed matter and cold atom experiments. It is especially interesting in view of studies of generalized hydrodynamics in atom chip experiments [55], given the realization of the sine-Gordon model in this setting [35]. Phase imprinting and the ability of designing arbitrary space-dependent optical potentials in coupled atomic condensates allows for the study of controlled nonequilibrium situations [56,57]. Apart from the topological charge, transport of other quantities, such as the energy, can also be studied. Fluctuations, full counting statistics, and dynamical correlations of vertex operators can also be described via the ballistic fluctuation theory [58,59], extending the recent study [60], which we expect to provide another manifestation of the fractal structure. Finally, it would be very interesting to analyze the diffusive corrections to the ballistic behavior and the possibility of superdiffusion [61], which we plan to address in a forthcoming publication.

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[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.L241105 describing (1) details of the TBA system and the dressing equations, (2) the bipartitioning protocol, and (3) the analytic evaluation of limiting cases. The Supplemental Material also contains Refs. [62–67].
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