Letter

Disorder in the nonlinear anomalous Hall effect of $\mathcal{PT}$-symmetric Dirac fermions

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The study of the nonlinear anomalous Hall effect (NLAHE) in $\mathcal{PT}$-symmetric systems has focused on intrinsic mechanisms. Here, we show that disorder contributes substantially to NLAHE and often overwhelms intrinsic terms. We identify terms to zeroth order in the disorder strength involving the Berry curvature dipole, skew scattering, and side jump, all exhibiting a strong peak as a function of the Fermi energy, a signature of interband coherence. Our results suggest NLAHE at experimentally relevant transport densities in $\mathcal{PT}$-symmetric systems is likely to be extrinsic.

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Introduction. The past decade has witnessed the prediction and observation of the nonlinear anomalous Hall effect (NLAHE), an anomalous Hall response that is second order in the applied electric field. Initially motivated by the identification of a Hall response in the absence of time-reversal symmetry breaking [1], the bulk of research to date has focused on nonmagnetic materials [2–28]. However, in recent years there has been growing interest in the NLAHE in time-reversal breaking systems such as antiferromagnetic metals [29–34]. In particular, in systems with broken time-reversal ($\mathcal{T}$) and inversion ($\mathcal{P}$) symmetry, but with combined $\mathcal{PT}$ symmetry, the linear anomalous Hall effect vanishes and the NLAHE provides the leading contribution to the Hall response. Thus, while eliciting strong interest for applications in antiferromagnetic spintronics, the NLAHE also provides a tool for the investigation and classification of states with broken symmetries [29,33,35–39].

In a $\mathcal{PT}$-symmetric system, the leading-order contribution to the NLAHE is of order τ^0, where τ is an indicative momentum relaxation time used as a measure of the disorder strength. The $\mathcal{PT}$-symmetric case is in sharp contrast to the $\mathcal{T}$-symmetric systems, where the leading-order contribution begins at order τ. So far the intrinsic contribution has been considered as the only mechanism active at order τ^0 [32,33,38]. On the other hand, it is well known from the study of the linear anomalous Hall effect that extrinsic contributions such as skew scattering and side jump also manifest at order τ^0, which can compensate or even cancel the intrinsic contribution. Some of these have been addressed in $\mathcal{PT}$-broken systems [40–49]. Yet, to our knowledge, the role of disorder in the NLAHE in $\mathcal{PT}$-symmetric systems has thus far been neglected. This omission is difficult to justify: When seeking to extract intrinsic topological quantities from experimental data the effect of disorder must be incorporated.

In this Letter we determine the full expression for the NLAHE in the presence of disorder in systems with combined $\mathcal{PT}$ symmetry, taking two-dimensional (2D) tilted Dirac fermions as a prototype system. Defining the nonlinear current density $j_i = \chi_{ijk}E_jE_k$, with χ_{ijk} the nonlinear susceptibility, joint $\mathcal{PT}$ symmetry restricts the powers of τ that may appear in χ. In a $\mathcal{PT}$-symmetric system the allowed response scales with even powers of τ, and the susceptibility may be written as $[\chi_{ijk}] = [\chi^{(-2)}_{ijk}] + [\chi^{(0)}_{ijk}]$, where the superscripts indicate the second order and zeroth order in τ, respectively. Because current NLAHE experiments use moderately conducting channels, the τ^0 contribution is the most important, and our effort focuses primarily on $\chi^{(0)}_{ijk}$, where disorder competes directly with the intrinsic band structure contributions [33,38,50]. To second order in the electric field, disorder contributions involve a complex interplay between the band structure and disorder mechanisms. Our central result may be summarized in Fig. 1 as the quantitative comparison between intrinsic and disorder contributions. Our main findings are

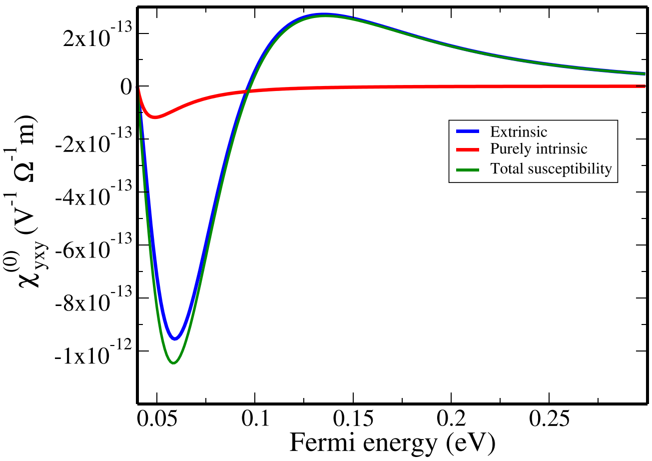

FIG. 1. Susceptibility $\chi^{(0)}_{yxy} \propto \tau^0$ with gap $\Delta = 40$ meV, tilt $t = 0.4$, and $v_F = 1.6 \times 10^6$ m/s. We approximate the Fermi velocity to be the same for all components, $v_{0x} = v_{0y}$.

as follows: (i) For realistic parameters [32,38] the disorder contributions generally overwhelm the intrinsic terms. This is evident from Fig. 1, where the total susceptibility essentially tracks the extrinsic contribution. (ii) The NLAHE exhibits a strong peak as a function of the Fermi energy ε_F, whose location is determined by the size of the gap. This peak is present in both $[\chi_{ijk}^{(0)}]$ and $[\chi_{ijk}^{(-2)}]$, and appears in all contributions to the NLAHE, whether intrinsic or induced by disorder. It is a signature of interband coherence [51], a factor that unifies all NLAHE mechanisms. (iii) We identify three main disorder contributions: skew scattering, side jump, and a contribution we term the *extrinsic Berry curvature dipole*. It consists of the Berry curvature dipole multiplied by a disorder term that is formally of zeroth order in τ. All the disorder terms are a consequence of an electric field correction to the collision integral. For our prototype model of 2D tilted Dirac fermions all nonlinear mechanisms are traced to the Fermi surface. Our quantum mechanical formalism also reveals the existence of additional intrinsic terms that a naive application of the semiclassical method misses, in analogy with Ref. [52].

Our findings suggest that the NLAHE signal at experimentally relevant transport densities is dominated by disorder, making an understanding of disorder indispensable in interpreting experimental data. They also provide a strong contrast with $\mathcal{PT}$-breaking systems studied so far. In that case, with $\mathcal{T}$ preserved, $\mathcal{PT}$ is necessarily broken in the second-order electrical response, and $\chi \propto \tau$. The NLAHE driven by the Berry curvature dipole (BCD) belongs to this category [1,37,53], and it was shown that disorder makes a contribution similar in magnitude to the intrinsic terms, without overwhelming them [53].

Quantum kinetic equation. The system is described by the density matrix $\rho(t)$, which obeys the quantum Liouville equation $\partial\rho/\partial t + (i/\hbar)[H, \rho] = 0$. The Hamiltonian has the form $H = H_0 + e\boldsymbol{E}\cdot\boldsymbol{r} + U(\boldsymbol{r})$, with H_0 the band Hamiltonian, $\boldsymbol{E}$ a constant, uniform electric field, and $U(\boldsymbol{r})$ the disorder scattering potential. We work in the crystal momentum representation spanned by Bloch states $|m, \boldsymbol{k}\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|u_{\boldsymbol{k}}^m\rangle$. The disorder model is defined through its correlation functions $\langle U(\boldsymbol{r})\rangle = 0$ and $\langle U(\boldsymbol{r})U(\boldsymbol{r}')\rangle = u_0^2\delta(\boldsymbol{r}-\boldsymbol{r}')$, where u_0^2 quantifies the strength of disorder. Alternatively, the disorder strength can be measured by the momentum relaxation time $1/\tau^m = \pi\rho(\epsilon_{\boldsymbol{k}}^m)u_0^2/\hbar$, with $\rho(\epsilon_{\boldsymbol{k}}^m)$ the density of states. In our calculation, we will assume that the Fermi energy is located in the conduction band, indexed here by a positive sign, but we will drop such a notation in our final results.

Following the methodology of Refs. [51,54,55], the density matrix is decomposed into a disorder-averaged part $f_{\boldsymbol{k}}$ and will be the focus of our attention, and a fluctuating part, which is integrated out to yield the scattering term in the Born approximation, assuming its time evolution to be Markovian. We do not consider here the nonlinear counterpart of the important issue about a possible reduction of the anomalous Hall response due to crossing diagrams in linear response [56–58]. Although it is expected to have a similar effect in a nonlinear regime and it is possible to include such diagrams within the density matrix formalism [51,59], this is beyond the scope of the present Letter. In this way we obtain the quantum kinetic equation

$$\frac{\partial f}{\partial t} + \frac{i}{\hbar}[H_0, f] + J_0(f) = \frac{e\boldsymbol{E}}{\hbar}\cdot\frac{Df}{D\boldsymbol{k}} - J_E(f) - J_{E2}(f). \tag{1}$$

The covariant derivative appearing above reads $\frac{Df_{\boldsymbol{k}}}{D\boldsymbol{k}} = \frac{\partial f_{\boldsymbol{k}}}{\partial\boldsymbol{k}} - i[\boldsymbol{\mathcal{R}}_{\boldsymbol{k}}, f_{\boldsymbol{k}}]$, with the Berry connection $\boldsymbol{\mathcal{R}}_{\boldsymbol{k}}^{mm'} = i\langle u_{\boldsymbol{k}}^m|\nabla_{\boldsymbol{k}}u_{\boldsymbol{k}}^{m'}\rangle$. The covariant derivative accounts for the momentum dependence of the basis functions. The density matrix $f_{\boldsymbol{k}}$ has both diagonal and off-diagonal elements in band index m. We represent the band-diagonal part by $n_{\boldsymbol{k}}$ and the off-diagonal part by $S_{\boldsymbol{k}}$, such that $f_{\boldsymbol{k}} = n_{\boldsymbol{k}} + S_{\boldsymbol{k}}$. The equilibrium density matrix is band-diagonal with matrix elements given by the Fermi-Dirac distribution $n_{\mathrm{FD}}(\epsilon_{\boldsymbol{k}}^m)$ for each band.

The bare collision integral is defined as $J_0(f) = (i/\hbar)\langle[U, g_0]\rangle$ with the function

$$g_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\epsilon\, G_0^R(\epsilon)[U, f]G_0^A(\epsilon) \tag{2}$$

and the electric field correction $J_E(f)$ defined as

$$J_E(f) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\epsilon \big\langle\big[U, G_0^R(\epsilon)[e\boldsymbol{E}\cdot\boldsymbol{r}, g_0]G_0^A(\epsilon)\big]\big\rangle. \tag{3}$$

The retarded Green's function is defined as $G_0^R(\epsilon) = -\frac{i}{\hbar}\int_0^\infty dt\, e^{-iH_0t/\hbar}e^{i\epsilon t/\hbar}e^{-\eta t}$, where the factor $e^{-\eta t}$ ensures convergence and the advanced Green's function $G_0^A(\epsilon)$ follows by Hermitian conjugation. The collision integral $J_{E2}(f) \to J_{E2}(n_{\mathrm{FD}})$ is by itself second order in an electric field and must be evaluated as a functional of the equilibrium Fermi-Dirac distribution. Its main role is to eliminate Fermi sea effects in the same way as in a linear response—this is explained in the Supplemental Material [60].

We solve Eq. (1) perturbatively in the electric field and disorder strength quantified by τ. The leading-order correction in the linear response comes from the first driving term on the right-hand side by taking $f \to n_{\mathrm{FD}}$. It will give the Boltzmann-like contribution $n_{E\boldsymbol{k}}^{(-1)}$, which in our notation refers to a band-diagonal term $\propto\tau$. The kinetic equation is solved iteratively in $\boldsymbol{E}$ up to second order, which is denoted by the subscript $E2$. As an example, the leading second-order response follows from the same driving term by using the linear Boltzmann equation. It will produce a contribution $n_{E2\boldsymbol{k}}^{(-2)}$, i.e., band-diagonal, second order in $\boldsymbol{E}$, and quadratic in τ. We will use a similar notation to represent the off-diagonal channel. Once the distribution is determined, the current follows from the trace $j = -e\sum_{\boldsymbol{k},mm'}[v_{\boldsymbol{k}}^{mm'}f_{E2\boldsymbol{k}}^{m'm}]$, namely, the velocity operator weighted by the density matrix.

Model Hamiltonian. We investigate a system that breaks both time-reversal $\mathcal{T}$ and parity $\mathcal{P}$ symmetry but preserves the joint $\mathcal{PT}$ symmetry. A generic paradigm is provided by a tilted Dirac cone. The band Hamiltonian for a single valley has the form

$$H_0 = \hbar v_t k_x\sigma_0 + \hbar v_{0x}k_x\sigma_x \pm \hbar v_{0y}k_y\sigma_y + \Delta\sigma_z, \tag{4}$$

where the first term is the tilt, σ_i are Pauli matrices, v_{0i} are Fermi velocities, and the term Δ is the energy gap. We replace $k_i \to v_{0i}k_i$ with the following rule for integrals $\sum_{\boldsymbol{k}}(\cdots) \to \frac{1}{(2\pi)^2}\frac{1}{v_{0x}v_{0y}}\int dk_xdk_y(\cdots)$ and derivatives $\frac{\partial}{\partial k_i} \to v_{0i}\frac{\partial}{\partial k_i}$. Since we assume the two $\mathcal{PT}$-symmetric states can be decoupled, we focus on the Hamiltonian with the positive sign. The eigenvalues read $\epsilon_{\boldsymbol{k}}^{\pm} = \hbar t k_x \pm \epsilon_{0k}$ with $\epsilon_{0k} = \sqrt{\hbar^2 k_x^2 + \hbar^2 k_y^2 + \Delta^2}$ and the dimensionless parameter $t = v_t/v_{0x}$. This dimensionless parameter controls the breaking of inversion symmetry necessary for the nonlinear response, hence the nonlinear susceptibility will be at least of first order in t.

Results. The components of the nonlinear susceptibility are related by $[\chi_{xxx}^{(0)}] = [\chi_{xyy}^{(0)}] + [\chi_{yxy}^{(0)}]$, apart from a proper velocity prefactor, meaning $[\chi_{xxx}^{(0)}] \propto v_{0x}^3$, while the right-hand side is $\propto v_{0x} v_{0y}^2$. The first index in χ is the direction of the current while the last two indices represent the two factors of the electric field. Below we show results for $[\chi_{yxy}^{(0)}] = [\chi_{xxx}^{(0)}] - [\chi_{xyy}^{(0)}]$.

Solving the kinetic equation to zeroth order in τ we identify a purely intrinsic contribution as well as three disorder corrections to the nonlinear anomalous Hall response of $\mathcal{PT}$-symmetric systems: a side-jump effect, a skew-scattering effect, and a Berry curvature dipole effect. The Berry curvature dipole and side-jump contributions emerge in the off-diagonal channel of the density matrix and follow from the equation

$$\frac{\partial S_{E2\boldsymbol{k}}^{(0)mm'}}{\partial t} + \frac{i}{\hbar}\left[H_{0\boldsymbol{k}}, S_{E2\boldsymbol{k}}^{(0)}\right]^{mm'} = \frac{e\boldsymbol{E}}{\hbar} \cdot \frac{DS_{E\boldsymbol{k},\mathrm{int}}^{(0)mm'}}{D\boldsymbol{k}} - i\frac{e\boldsymbol{E}}{\hbar} \cdot \left[\boldsymbol{\mathcal{R}}_{\boldsymbol{k}}, n_{E\boldsymbol{k},\mathrm{sj}}^{(0)}\right]^{mm'} - \left[J_0\left(n_{E2,\mathrm{sj}}^{(-1)}\right)\right]_{\boldsymbol{k}}^{mm'}. \tag{5}$$

The general solution reads $S_{E2\boldsymbol{k}}^{(0)mm'} = -i\hbar(\epsilon_{\boldsymbol{k}}^{m} - \epsilon_{\boldsymbol{k}}^{m'})^{-1} d_{E2\boldsymbol{k}}^{(0)mm'}$, where $d_{E2\boldsymbol{k}}^{(0)mm'}$ refers generically to the driving term on the right-hand side of the equation.

The first term in Eq. (5) produces the second-order intrinsic distribution. It is related to the covariant derivative of the intrinsic linear response that follows from the equation $S_{E\boldsymbol{k},\mathrm{int}}^{(0)mm'} = -(\epsilon_{\boldsymbol{k}}^{m} - \epsilon_{\boldsymbol{k}}^{m'})^{-1} e\boldsymbol{E} \cdot [\boldsymbol{\mathcal{R}}_{\boldsymbol{k}}, n_{\mathrm{FD}}]^{mm'}$. This is the only intrinsic contribution to the nonlinear response in the sense that it depends solely on the band structure. After tracing the off-diagonal velocity $v_{\boldsymbol{k},i}^{mm'} = i\hbar^{-1}(\epsilon_{\boldsymbol{k}}^{m} - \epsilon_{\boldsymbol{k}'}^{m'})\mathcal{R}_{\boldsymbol{k},i}^{mm'}$ with the intrinsic distribution we obtain the susceptibility

$$[\chi_{yxy}^{(0)}]_{\mathrm{int}} = -\frac{t}{8}\hbar^2 e^3 v_{0y}^2 v_{0x} \frac{\rho(\epsilon_F)}{\epsilon_F^3} \xi_F^2 (1 - \xi_F^2), \tag{6}$$

where $\rho(\epsilon_F)$ is the density of states and we defined the parameter $\xi_F = \Delta/\epsilon_F$. This contribution is an interband coherence effect where virtual transitions between the valence and conduction band are mediated by the product of off-diagonal terms in the Berry connection. It is also a Fermi surface response, vanishing at $\xi_F = 1$. This is in contrast to the intrinsic linear anomalous Hall effect, which is a Fermi sea response.[1]

[1]In some cases, when reducing a multiband Hamiltonian to an effective one, a correction to the position operator may be present, resulting in a correction to the velocity operator. This has been considered in the linear response in the spin Hall effect and in the anomalous Hall effect [61–63] and is likely to be present in the nonlinear regime. However, this correction is beyond the scope of our work.

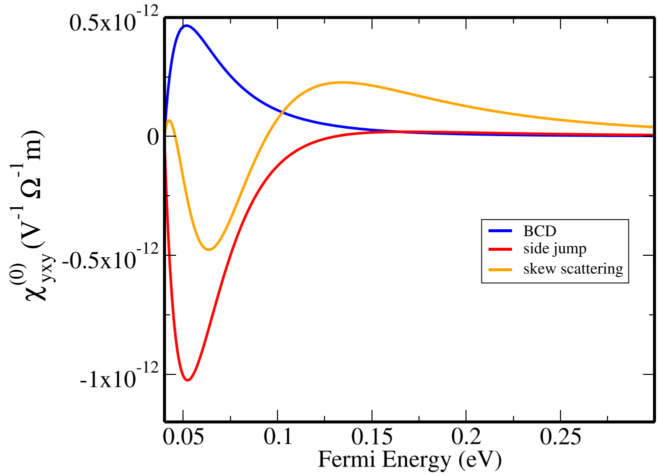

FIG. 2. Susceptibility $[\chi_{yxy}^{(0)}]$ with gap parameter $\Delta = 40$ meV and tilt $t = 0.4$. $v_F = 1.6 \times 10^6$ m/s. We show $[\chi_{yxy}^{(0)}]_{\mathrm{BCD}}$, $[\chi_{yxy}^{(0)}]_{\mathrm{sj}}$, and $[\chi_{yxy}^{(0)}]_{\mathrm{sk}}$

We turn our attention to the disorder corrections to the susceptibility. The band-diagonal term to zeroth order in τ reads $n_{Ey\boldsymbol{k},\mathrm{sj}}^{(0)++} = -eE_y v_{0y} A_0(\boldsymbol{k}) + \cdots$, where the coefficient is $A_0(\boldsymbol{k}) = t\frac{\tau_{sp}\hbar}{2\tau\epsilon_{0k}} \xi_k [(1 + \xi_k^2) + (1 - \xi_k^2)\epsilon_{0k}\frac{\partial}{\partial\epsilon_{0k}^{+}}]\delta(\epsilon_{0\boldsymbol{k}}^{+} - \epsilon_F)$. We have ignored higher harmonics irrelevant for transport. We have defined the transport time and the single-particle relaxation time as $1/\tau_{\mathrm{tr}} = (1 + 3\xi_k^2)/2\tau$ and $1/\tau_{\mathrm{sp}} = (1 + \xi_k^2)/\tau$, respectively. Tracing the off-diagonal velocity with this channel of the off-diagonal density matrix will produce a Berry curvature dipole (BCD)-like contribution given by the current $j_i = -(e^2/\hbar)\sum_{\boldsymbol{k},m}(\boldsymbol{E} \times \boldsymbol{\Omega}_{\boldsymbol{k}}^{mm})_i n_{E\boldsymbol{k}}^{(0)mm}$ where $\boldsymbol{\Omega}_{\boldsymbol{k}}^{mm}$ is the Berry curvature. Explicit evaluation yields

$$[\chi_{yxy}^{(0)}]_{\mathrm{BCD}} = \frac{te^3\hbar^2 v_{0x} v_{0y}^2}{2} \frac{\rho(\epsilon_F)\xi_F^2}{\epsilon_F^3} \frac{(1 - \xi_F^2)(2 + \xi_F^2)}{(1 + \xi_F^2)^2}. \tag{7}$$

This contribution is shown in Fig. 2 and given in full in the Supplemental Material [60]. The BCD susceptibility $[\chi_{yxy}^{(0)}]_{\mathrm{BCD}}$ is the analog of the BCD widely studied in $\mathcal{PT}$-broken systems. However, the BCD appears here as an interband coherence effect involving *disorder*, and we refer to this contribution as the *extrinsic Berry curvature dipole*. It is a consequence of the driving term arising from the electric field corrected collision integral. This response is also a Fermi surface effect. Previous studies on 2D [1,44] and 3D systems [37] focused on a Berry curvature dipole term, whose contribution to the NLAHE scales linearly with the momentum relaxation time. Such a contribution is, in principle, also extrinsic since it depends on disorder but is prohibited in $\mathcal{PT}$-symmetric materials. The important point for comparison is that the Berry curvature dipole in both cases is related to the anomalous velocity [64], which is due to an electric field correction to the ground states (it is intrinsic in this sense). It manifests as a nonlinear response when weighted by a linear response distribution, which will be manifestly due to a shift in the Fermi surface and then related to relaxation process (in this sense extrinsic).

Let us consider the last driving term in the kinetic equation. The first order in τ distribution in the collision integral in Eq. (6) reads

$$n^{(-1)++}_{xy\boldsymbol{k},\mathrm{sj}} = -\frac{e^2}{\hbar} v_{0y} E_y v_{0x} E_x \tau_{\mathrm{tr}} \cos(\theta_{\boldsymbol{k}}) \frac{\partial A_0(\boldsymbol{k})}{\partial k} + \cdots . \quad (8)$$

Tracing the off-diagonal velocity with $S^{(0)mm'}_{E2\boldsymbol{k}} = i\hbar(\epsilon^m_{\boldsymbol{k}} - \epsilon^{m'}_{\boldsymbol{k}})^{-1}[J_0(n^{(-1)}_{E2,\mathrm{sj}})]^{mm'}_{\boldsymbol{k}}$ yields

$$\begin{aligned}\left[\chi^{(0)}_{yxy}\right]^{(\mathrm{od})}_{\mathrm{sj}} = {} & \tfrac{1}{2} t e^3 v^2_{0y} v_{0x} \hbar^2 \frac{\rho(\epsilon_F)}{\epsilon^3_F} \xi^2_F\left(1-\xi^2_F\right)\Lambda_s\Lambda_t\left\{\left(1-\xi^2_F\right)^2\Lambda_s\right. \\ & - 8\xi^2_F\left(1-\xi^2_F\right)\Lambda_s\Lambda_t + 3\left(1-\xi^2_F\right) - 24\xi^2_F\Lambda_t \\ & \left. - 24\xi^2_F\left(1-\xi^2_F\right)\Lambda^2_t\right\}. \end{aligned} \quad (9)$$

The dimensionless parameters $\Lambda_s = (\tau_{\mathrm{sp}}/\tau)$ and $\Lambda_t = (\tau_{\mathrm{tr}}/2\tau)$. This coefficient represent the side jump in analogy to a linear response [55]. It is a Fermi surface response, as well as an interband coherence effect due to virtual transitions mediated by the Berry connection. Previous works on the nonlinear Hall effect [42,44–46] reported the counterpart of this side-jump-like contribution for $\mathcal{PT}$-broken systems. Here, it is reported for $\mathcal{PT}$-symmetric systems.

In the band-diagonal part of the density matrix to zeroth order τ we identify a skew-scattering contribution and a second side-jump contribution. They follow from $[J(n^{(0)}_{E2})]^{mm}_{\boldsymbol{k}} = -[J_0(S^{(0)}_{E2})]^{mm}_{\boldsymbol{k}} - [J_E(n^{(0)}_{E,\mathrm{sj}})]^{mm}_{\boldsymbol{k}}$, where the driving terms are to the right. Skew scattering follows from the first driving term while the second gives the side-jump contribution. This is identical to Eq. (9), hence the electric field corrected collision integral doubles the side jump, in analogy with the linear response [55,61]. We plot this side-jump term in Fig. 2.

Finally, the skew-scattering contribution is found as

$$\begin{aligned}\left[\chi^{(0)}_{yxy}\right]_{\mathrm{sk}} = {} & e^3 v^2_{0y} v_{0x} \tfrac{3}{2} t \hbar^2 \frac{\rho(\epsilon_F)}{\epsilon^3_F} \xi^2_F\left(1-\xi^2_F\right)\Lambda^2_t\left\{3\left(1-\xi^2_F\right)\right. \\ & - 16\xi^2_F\Lambda_t + 15\left(1-\xi^2_F\right)^2\Lambda_t - 112\xi^2_F\left(1-\xi^2_F\right)\Lambda^2_t \\ & - 12\xi^2_F\left(1-\xi^2_F\right)\Lambda_s\Lambda_t + 64\xi^4_F\left(1+\xi^2_F\right)\Lambda_s\Lambda^2_t \\ & \left. - 48\xi^4_F\left(1-\xi^2_F\right)^2\Lambda^3_t + 128\xi^6_F\Lambda^3_t\right\}. \end{aligned} \quad (10)$$

Although this quantity exhibits a similarly complex dependence on the Fermi energy, one can straightforwardly identify it as a Fermi surface effect. It represents interband coherence mediated by disorder through an extrinsic off-diagonal term in the density matrix. It is shown in Fig. 2. Again, we note that the counterpart of skew scattering for $\mathcal{PT}$-broken systems was reported in previous works [42,44–46] and reported here to zeroth order in relaxation processes in the first Born approximation. In a recent paper [65] a skew-scattering-like contribution was calculated in $\mathcal{PT}$-symmetric systems, with a different scaling with respect to disorder, but it requires going beyond the first Born approximation.

All contributions exhibit similar behavior, namely, $\propto 1/\epsilon^4_F$ that dominates for increasing Fermi energy and $\propto(1 - \Delta^2/\epsilon^2_F)$ that makes all the expressions zero when the Fermi energy approaches the gap (a signature of the Fermi surface effect in the prefactor of all the contributions we have calculated). This nonmonotonic behavior of the susceptibility makes the function bend in between these two limiting cases and to develop a peak. The sign of the peak is dictated by the dominant term in the terms inside the curly braces in each expression. For instances, for the side-jump susceptibility, the second and last terms scale similarly to the prefactor and make the function develop a negative peak. It is similar for the skew-scattering contribution. The peak develops in the vicinity of the gap, as the bands approach each other, revealing the effect of interband coherence. In fact, since this behavior is shared by all intrinsic and extrinsic contributions, we regard interband coherence as the unifying physical mechanism behind the NLAHE.

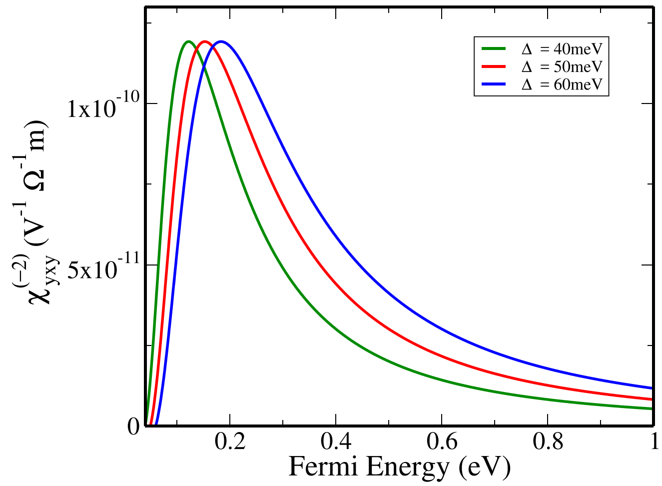

FIG. 3. Leading-order susceptibility $\chi^{(-2)}_{yxy} \propto \tau^2$ with $t = 0.4$, $v_F = 1.6 \times 10^6$ m/s, and $\tau = 1$ ps. We have approximated the Fermi velocity to be the same for all components.

To fully account for the transversal susceptibility of $\mathcal{PT}$-symmetric Dirac fermions, we also solved the Boltzmann-like equation $[J_0(n^{(-2)}_{E2})]^{mm}_{\boldsymbol{k}} = \frac{e\boldsymbol{E}}{\hbar} \cdot \nabla_{\boldsymbol{k}} n^{(-1)mm}_{E\boldsymbol{k}}$ for the leading-order susceptibility. Its behavior is shown in Fig. 3. It is also a Fermi surface effect that vanishes when we approach the gap and also vanishes in time-reversal symmetric systems. The peak shifts as a function of the gap, with a similar behavior noted in $[\chi^{(0)}_{ijk}]$.

Conclusions. We have calculated the electrical susceptibility to second order in the electric field in $\mathcal{PT}$-symmetric 2D tilted Dirac fermions. We have demonstrated the existence of intra- and interband disorder effects that are counterparts of the side-jump and skew-scattering terms in linear response, as well as another Berry curvature dipole correction, which is disorder dependent yet appears at zeroth order in the disorder strength. We showed that disorder corrections generally overwhelm the intrinsic contribution and are expected to play a vital role in realistic samples, where disorder is unavoidable.

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[1] I. Sodemann and L. Fu, Phys. Rev. Lett. **115**, 216806 (2015).
[2] H. Rostami and V. Juričić, Phys. Rev. Res. **2**, 013069 (2020).
[3] N. Nagaosa, T. Morimoto, and Y. Tokura, Nat. Rev. Mater. **5**, 621 (2020).
[4] B. T. Zhou, C.-P. Zhang, and K. T. Law, Phys. Rev. Appl. **13**, 024053 (2020).
[5] V. D. Esin, A. V. Timonina, N. N. Kolesnikov, and E. V. Deviatov, J. Exp. Theor. Phys. **133**, 792 (2021).
[6] Z.-S. Liao, H.-H. Zhang, and Z. Yan, Phys. Rev. B **103**, 235151 (2021).
[7] S. Li, C. M. Wang, Z. Z. Du, F. Qin, H.-Z. Lu, and X. C. Xie, npj Quantum Mater. **6**, 96 (2021).
[8] Y. D. Wang, Z.-G. Zhu, and G. Su, Phys. Rev. B **106**, 035148 (2022).
[9] D.-K. Zhou, Z.-F. Zhang, X.-Q. Yu, Z.-G. Zhu, and G. Su, Phys. Rev. B **105**, L201103 (2022).
[10] N. Kheirabadi and A. Langari, Phys. Rev. B **106**, 245143 (2022).
[11] Y. Wang, S. V. Mambakkam, Y.-X. Huang, Y. Wang, Y. Ji, C. Xiao, S. A. Yang, S. A. Law, and J. Q. Xiao, Phys. Rev. B **106**, 155408 (2022).
[12] J.-X. Hu, C.-P. Zhang, Y.-M. Xie, and K. T. Law, Commun. Phys. **5**, 255 (2022).
[13] C.-P. Zhang, J. Xiao, B. T. Zhou, J.-X. Hu, Y.-M. Xie, B. Yan, and K. T. Law, Phys. Rev. B **106**, L041111 (2022).
[14] R. Resta, Phys. Rev. Res. **4**, 033002 (2022).
[15] A. Chakraborty, K. Das, S. Sinha, P. C. Adak, M. M. Deshmukh, and A. Agarwal, 2D Mater. **9**, 045020 (2022).
[16] D. Kaplan, T. Holder, and B. Yan, Nat. Commun. **14**, 3053 (2023).
[17] S. Gholizadeh, J. H. Cullen, and D. Culcer, Phys. Rev. B **107**, L041301 (2023).
[18] Z.-Y. Zhuang and Z. Yan, Phys. Rev. B **107**, L161102 (2023).
[19] X.-G. Ye, H. Liu, P.-F. Zhu, W.-Z. Xu, S. A. Yang, N. Shang, K. Liu, and Z.-M. Liao, Phys. Rev. Lett. **130**, 016301 (2023).
[20] Y.-X. Huang, X. Feng, H. Wang, C. Xiao, and S. A. Yang, Phys. Rev. Lett. **130**, 126303 (2023).
[21] Q. Ma, S.-Y. Xu, H. Shen, D. MacNeill, V. Fatemi, T.-R. Chang, A. M. Mier Valdivia, S. Wu, Z. Du, C.-H. Hsu *et al.*, Nature (London) **565**, 337 (2019).
[22] K. Kang, T. Li, E. Sohn, J. Shan, and K. F. Mak, Nat. Mater. **18**, 324 (2019).
[23] Z. He and H. Weng, npj Quantum Mater. **6**, 101 (2021).
[24] C.-L. Zhang, T. Liang, Y. Kaneko, N. Nagaosa, and Y. Tokura, npj Quantum Mater. **7**, 103 (2022).
[25] P. He, G. K. W. Koon, H. Isobe, J. Y. Tan, J. Hu, A. H. C. Neto, L. Fu, and H. Yang, Nat. Nanotechnol. **17**, 378 (2022).
[26] S. Sinha, P. C. Adak, A. Chakraborty, K. Das, K. Debnath, L. D. V. Sangani, K. Watanabe, T. Taniguchi, U. V. Waghmare, A. Agarwal *et al.*, Nat. Phys. **18**, 765 (2022).
[27] J. Duan, Y. Jian, Y. Gao, H. Peng, J. Zhong, Q. Feng, J. Mao, and Y. Yao, Phys. Rev. Lett. **129**, 186801 (2022).
[28] L. Min, H. Tan, Z. Xie, L. Miao, R. Zhang, S. H. Lee, V. Gopalan, C.-X. Liu, N. Alem, B. Yan *et al.*, Nat. Commun. **14**, 364 (2023).
[29] T. Jungwirth, X. Marti, P. Wadley, and J. Wunderlich, Nat. Nanotechnol. **11**, 231 (2016).
[30] T. Jungwirth, J. Sinova, A. Manchon, X. Marti, J. Wunderlich, and C. Felser, Nat. Phys. **14**, 200 (2018).
[31] J. Godinho, H. Reichlová, D. Kriegner, V. Novák, K. Olejník, Z. Kašpar, Z. Šobáň, P. Wadley, R. P. Campion, R. M. Otxoa *et al.*, Nat. Commun. **9**, 4686 (2018).
[32] C. Xiao, Z. Z. Du, and Q. Niu, Phys. Rev. B **100**, 165422 (2019).
[33] C. Wang, Y. Gao, and D. Xiao, Phys. Rev. Lett. **127**, 277201 (2021).
[34] F. Mazzola, B. Ghosh, J. Fujii, G. Acharya, D. Mondal, G. Rossi, A. Bansil, D. Farias, J. Hu, A. Agarwal *et al.*, Nano Lett. **23**, 902 (2023).
[35] Y. Gao and D. Xiao, Phys. Rev. B **98**, 060402(R) (2018).
[36] H. Watanabe and Y. Yanase, Phys. Rev. Res. **2**, 043081 (2020).
[37] D.-F. Shao, S.-H. Zhang, G. Gurung, W. Yang, and E. Y. Tsymbal, Phys. Rev. Lett. **124**, 067203 (2020).
[38] H. Liu, J. Zhao, Y.-X. Huang, W. Wu, X.-L. Sheng, C. Xiao, and S. A. Yang, Phys. Rev. Lett. **127**, 277202 (2021).
[39] T. Nag and D. M. Kennes, Phys. Rev. B **105**, 214307 (2022).
[40] Z. Z. Du, C. M. Wang, H.-Z. Lu, and X. C. Xie, Phys. Rev. Lett. **121**, 266601 (2018).
[41] Y. Zhang, J. van den Brink, C. Felser, and B. Yan, 2D Mater. **5**, 044001 (2018).
[42] Z. Z. Du, C. M. Wang, S. Li, H.-Z. Lu, and X. C. Xie, Nat. Commun. **10**, 3047 (2019).
[43] O. Matsyshyn and I. Sodemann, Phys. Rev. Lett. **123**, 246602 (2019).
[44] S. Nandy and I. Sodemann, Phys. Rev. B **100**, 195117 (2019).
[45] Z. Z. Du, H.-Z. Lu, and X. C. Xie, Nat. Rev. Phys. **3**, 744 (2021).
[46] C. Ortix, Adv. Quantum Technol. **4**, 2100056 (2021).
[47] Z. Z. Du, C. M. Wang, H.-P. Sun, H.-Z. Lu, and X. C. Xie, Nat. Commun. **12**, 5038 (2021).
[48] S. S. Samal, S. Nandy, and K. Saha, Phys. Rev. B **103**, L201202 (2021).
[49] D. Kaplan, T. Holder, and B. Yan, SciPost Phys. **14**, 082 (2023).
[50] A. Gao, Y.-F. Liu, J.-X. Qiu, B. Ghosh, T. V. Trevisan, Y. Onishi, C. Hu, T. Qian, H.-J. Tien, S.-W. Chen *et al.*, Science **381**, 181 (2023).
[51] D. Culcer, A. Sekine, and A. H. MacDonald, Phys. Rev. B **96**, 035106 (2017).
[52] S. Lahiri, K. Das, R. B. Atencia, D. Culcer, and A. Agarwal, arXiv:2207.02178.
[53] S. Nandy, C. Zeng, and S. Tewari, Phys. Rev. B **104**, 205124 (2021).
[54] A. Sekine, D. Culcer, and A. H. MacDonald, Phys. Rev. B **96**, 235134 (2017).
[55] R. B. Atencia, Q. Niu, and D. Culcer, Phys. Rev. Res. **4**, 013001 (2022).
[56] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, Europhys. Lett. **111**, 37004 (2015).
[57] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, Phys. Rev. Lett. **117**, 046601 (2016).
[58] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, Phys. Rev. B **96**, 235148 (2017).
[59] H. Liu, R. B. Atencia, N. Medhekar, and D. Culcer, Mater. Quantum Technol. **3**, 025002 (2023).
[60] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.L201115 for detailed derivations of the main results.
[61] D. Culcer, E. M. Hankiewicz, G. Vignale, and R. Winkler, Phys. Rev. B **81**, 125332 (2010).

[62] X. Bi, P. He, E. M. Hankiewicz, R. Winkler, G. Vignale, and D. Culcer, Phys. Rev. B **88**, 035316 (2013).

[63] I. A. Ado, M. Titov, R. A. Duine, and A. Brataas, arXiv:2309.06416.

[64] D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).

[65] D. Ma, A. Arora, G. Vignale, and J. C. W. Song, Phys. Rev. Lett. **131**, 076601 (2023).