Letter

# Universality in the tripartite information after global quenches

Vanja Marić and Maurizio Fagotti*
*Université Paris-Saclay, CNRS, LPTMS, 91405 Orsay, France*

(Received 4 October 2022; revised 9 February 2023; accepted 2 October 2023; published 31 October 2023)

We consider macroscopically large 3-partitions $(A, B, C)$ of connected subsystems $A \cup B \cup C$ in infinite quantum spin chains and study the Rényi-$\alpha$ tripartite information $I_3^{(\alpha)}(A, B, C)$. At equilibrium in clean 1D systems with local Hamiltonians it generally vanishes. A notable exception is the ground state of conformal critical systems, in which $I_3^{(\alpha)}(A, B, C)$ is known to be a universal function of the cross ratio $x = |A||C|/[(|A| + |B|)(|C| + |B|)]$, where $|A|$ denotes $A$'s length. We identify different classes of states that, under time evolution with translationally invariant Hamiltonians, locally relax to states with a nonzero (Rényi) tripartite information, which furthermore exhibits a universal dependency on $x$. We report a numerical study of $I_3^{(\alpha)}$ in systems that are dual to free fermions, propose a field-theory description, and work out their asymptotic behavior for $\alpha = 2$ in general and for generic $\alpha$ in a subclass of systems. This allows us to infer the value of $I_3^{(\alpha)}$ in the scaling limit $x \to 1^-$, which we call "residual tripartite information". If nonzero, our analysis points to a universal residual value $-\log 2$ independently of the Rényi index $\alpha$, and hence applies also to the genuine (von Neumann) tripartite information.

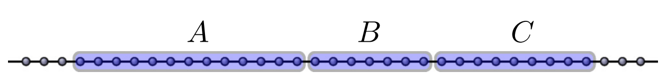
DOI: 10.1103/PhysRevB.108.L161116

*Introduction.* The concept of entanglement was introduced to distinguish quantum systems from classical ones [1,2]. With the development of quantum information theory [3,4], such a peculiarity of the quantum world was recognized as a resource, and the quantification of the entanglement became a key question [5]. Various measures of entanglement have then been put forward. In addition to their original purpose, the tools studied in quantum information attracted the attention of the community working on quantum many-body systems. It was indeed realized that the entanglement measures unveil universal properties. A famous example in one dimension (1D) is the von Neumann entropy of an interval in the ground state of a (conformal) critical system, which has a typical logarithmic growth with the length, proportional to the central charge of the underlying conformal field theory (CFT) [6–8]. Another quantity that attracted some attention (unfortunately under different names) is the *tripartite information*, which contains more information about the underlying CFT [9–11] but not only that. In two dimensions (2D), the tripartite information was shown to be sensitive to topological order and renamed for that reason *topological entanglement entropy* [12]. Concerning higher dimensions, we mention that, in Ref. [13], authors investigated the tripartite information in generic quantum field theories (QFTs; see also Ref. [14]), among which those with holographic duals hold a special place [15]. More recently, a type of tripartite information was proposed as a diagnostic of scrambling [16–18].

The tripartite information is defined as [4]

$$I_3(A, B, C) = I_2(A, B) + I_2(A, C) - I_2(A, B \cup C), \quad (1)$$

where $I_2(A, B) = S(A) + S(B) - S(A \cup B)$ denotes the mutual information and $S(A) \equiv S_1[\rho_A] = -\mathrm{tr}[\rho_A \ln \rho_A]$ is the von Neumann entropy of subsystem $A$ with density matrix $\rho_A$. It is defined to cancel the extensive and boundary contributions to the entropies. Moreover, just as the mutual information quantifies the extensiveness of the von Neumann entropy, so $I_3(A, B, C)$ quantifies the (bi)extensiveness of the mutual information after fixing one of the subsystems. We focus on the case in which $A$, $B$, and $C$ are adjacent intervals in an infinite spin chain and assume that their lengths are asymptotically large:




In noncritical 1D systems at equilibrium (and with clustering properties), $I_3(A, B, C)$ approaches 0 as the lengths approach infinity (we are not aware of exceptions). A more interesting behavior is observed in the ground state of a critical system with a low-energy CFT description, where conformal symmetry forces $I_3(A, B, C)$ to be a function of the cross ratio $x = \frac{|A||C|}{(|A|+|B|)(|B|+|C|)}$ [11]. That is to say, the limit

$$I_3(A, B, C) \xrightarrow[|A||C|/[(|A|+|B|)(|B|+|C|)]=x]{|A|,|B|,|C|\to\infty} G(x) \quad (2)$$

exists, is universal, and can in principle be computed within the underlying CFT. Some difficulties in calculation have not yet been overcome. The most important intermediate results concern the Rényi generalization of the tripartite information, which we will refer to as Rényi-$\alpha$ tripartite information and indicate by $I_3^{(\alpha)}(A, B, C)$. This has the same definition as $I_3(A, B, C)$ with the von Neumann entropy replaced by the Rényi-$\alpha$ entropy $S_\alpha[\rho] = \frac{1}{1-\alpha} \ln \mathrm{tr}[\rho^\alpha]$. Ideally, the tripartite information is recovered in the limit $\alpha \to 1^+$. Provided that

*maurizio.fagotti@universite-paris-saclay.fr

the intervals $A, B, C$ are adjacent [19], in a CFT, also the Rényi-$\alpha$ tripartite information depends only on the cross ratio $x$, i.e., $I_3^{(\alpha)}(A, B, C) = G_\alpha(x)$, and in some theories, it has been computed exactly for generic integer $\alpha > 1$ [10,22–24]. Just as $G(x)$ does for $I_3$, so $G_\alpha(x)$ describes the scaling limit of $I_3^{(\alpha)}$ in any spin chain with the same underlying CFT [25–29].

In this letter, we investigate the limit of infinite time of $I_3^{(\alpha)}$ after global quenches with local Hamiltonians. The entropies are known to become extensive [30–32], and often, the system exhibits typical features of thermal states [33–35]. We discuss when $I_3^{(\alpha)}$ should be expected not to vanish (in contrast to thermal states) and argue that it captures universal properties. We point out that the residual tripartite information with a discrete value $-\ln 2$ can emerge.

*The model.* We focus on two classes of noninteracting spin chains. The first is the generalized XY model [36], which is mapped into free fermions by a Jordan-Wigner transformation $\mathbf{a}_{2\ell-1} = \prod_{j<\ell} \boldsymbol{\sigma}_j^z\ \boldsymbol{\sigma}_\ell^x$, $\mathbf{a}_{2\ell} = \prod_{j<\ell} \boldsymbol{\sigma}_j^z\ \boldsymbol{\sigma}_\ell^y$, where $\mathbf{a}_\ell$ are Majorana fermions satisfying $\{\mathbf{a}_\ell, \mathbf{a}_n\} = 2\delta_{\ell n}\mathbf{I}$, and $\boldsymbol{\sigma}_\ell^\alpha$ are Pauli operators. The most studied models of this class are described by the quantum XY Hamiltonian [37] in a transverse field:

$$\mathbf{H} = \sum\nolimits_\ell J_x \boldsymbol{\sigma}_\ell^x \boldsymbol{\sigma}_{\ell+1}^x + J_y \boldsymbol{\sigma}_\ell^y \boldsymbol{\sigma}_{\ell+1}^y + h\boldsymbol{\sigma}_\ell^z. \quad (3)$$

This includes the XX model ($J_x = J_y$) and the transverse-field Ising model ($J_y = 0$).

The second class of systems is mapped into free fermions by the Kramers-Wannier transformation $\boldsymbol{\tau}_\ell^x = \prod_{j\leqslant\ell} \boldsymbol{\sigma}_j^x$, $\boldsymbol{\tau}_\ell^y = (\prod_{j<\ell} \boldsymbol{\sigma}_j^x)\boldsymbol{\sigma}_\ell^y \boldsymbol{\sigma}_{\ell+1}^z$ (for the sake of clarity, we have used a different notation $\boldsymbol{\tau}_\ell^\alpha$ for the Pauli operators), followed by the aforementioned Jordan-Wigner transformation. An example is the dual XY model [38,39]:

$$\mathbf{H} = \sum\nolimits_\ell \boldsymbol{\tau}_{\ell-1}^x \big(J_x \mathbf{I} - J_y \boldsymbol{\tau}_\ell^z\big) \boldsymbol{\tau}_{\ell+1}^x. \quad (4)$$

Hamiltonians like Eq. (4) possess semilocal conserved operators [39], which enable symmetry-protected topological order after global quenches [40]. Since the tripartite information was recognized as an indicator of topological order in 2D [12], how $I_3$ behaves after a quench in this second class of models is a compelling question.

*Homogeneous quench from a critical ground state.* The first example we consider is the paradigm of global quench: the ground state of a translationally invariant Hamiltonian is allowed to evolve under a different translationally invariant Hamiltonian [41]. We focus on generalized XY models. In generic situations, $I_3^{(\alpha)}$ vanish (both at the initial time and) at infinite time after the quench (cf. Refs [25,42]). We are about to uncover exceptions when the initial state is the ground state of a conformal critical system.

In noninteracting models, the corresponding central charge $c$ is a multiple of $\frac{1}{2}$. Our numerical analysis shows that generically $I_3^{(\alpha)}$ are nonzero also at late times, independently of whether the postquench Hamiltonian is critical or not, just provided that $c \geqslant 1$ in the initial state. This condition seems to be related to how slow the slowest spatial connected correlations decay. Specifically, after quenches from critical ground states there are two-point correlation functions that decay with the distance as a power law. In all the cases investigated with the initial state in the Ising universality class ($c = \frac{1}{2}$), we find that the spin-connected correlations in the stationary state do

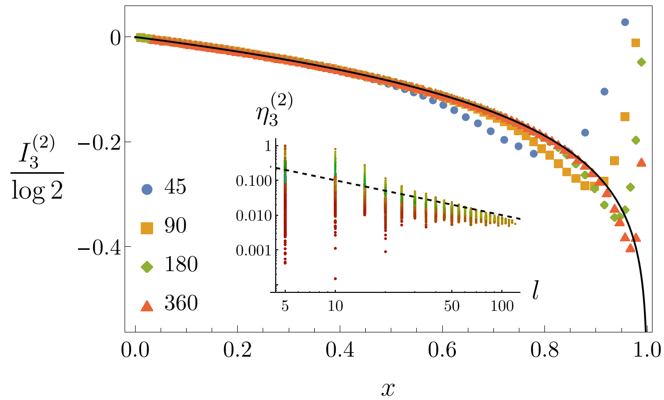



FIG. 1. Rényi-2 tripartite information for the quench in the XY model $(J_x, J_y, h) : (1, 1, 1) \to (1, 0.5, 0.5)$ for $|A| = |C| \in \{45, 90, 180, 360\}$ and variable $|B|$. The solid curve is a prediction from Eq. (9). Inset: Relative error $\eta_3^{(2)}$ as a function of $l = \min(|A|, |B|, |C|)$ for all configurations with length multiples of 5 in the range [5, 120]; the color and the size of the points varies linearly in $x$; the dashed line is a guide for the eye $\sim 1/l$.

not decay more slowly than $1/r^4$, where $r$ is the distance; this is not enough to generate nonzero tripartite information. With $c \geqslant 1$ in the initial state, we find instead that the slowest spin correlations generally decay as $1/r^2$, and $I_3^{(\alpha)}$ become nonzero (negative). Figure 1 shows $I_3^{(2)}$ at late times after a quench in the XY model. Remarkably, $I_3^{(2)}$ remains a function of the cross ratio. This is observed also for larger values of $\alpha$ and other choices of the Hamiltonian parameters; in addition, the data seem to approach curves that depend on few details of the system [43].

*Bipartitioning protocol.* Another type of global quench that has attracted a lot of attention is the time evolution after joining two globally different states [44]. We consider here the basic case in which the initial state consists of a domain of spins aligned along $z$ joined with a domain of spins aligned in the opposite direction. If we take the Hamiltonian of the XY model with $J_x \neq J_y$ the quench is global; indeed, the initial state is locally different from any excited state of the Hamiltonian. We stress that the initial state is not critical, and we are not aware of any CFT description of the infinite time limit. Figure 2 shows the Rényi-3 and Rényi-4 tripartite information in the nonequilibrium steady state emerging at infinite time. In agreement with the previous discussion, the slowest connected correlations decay as $1/r^2$, and we obtain nonzero $I_3^{(\alpha)}$. Again, the latter become functions of the cross ratio and seem to remain so even in more sophisticated bipartitioning protocols [43]. In the XY model considered here, the asymptotic curves do not even seem to depend on $J_x$ and $J_y$.

*Quench with symmetry-protected topological order.* We consider time evolution under Eq. (4) of two initial states: (a) the product state with all spins aligned in the $z$ direction (cf. Ref. [40]) and (b) the same as (a) with one spin flipped (cf. Ref. [39]). In case (a), $I_3^{(\alpha)}$ vanish. In case (b), $I_3^{(\alpha)}$ become nonzero and seem to approach the same curves as in the domain-wall quench above. Since the latter is dual to case (b) [39], our analysis suggests that $I_3^{(\alpha)}$ are not affected by the

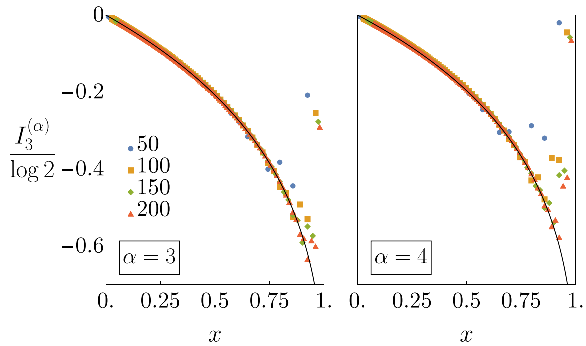



FIG. 2. Rényi-$\alpha$ tripartite information at infinite time after the quench from $|\dots\uparrow\uparrow\downarrow\downarrow\dots\rangle$ under the XY Hamiltonian in Eq. (3), with $(J_x, J_y, h) = (1, 0.5, 0)$ for $|A| = |C| \in \{50, 100, 150, 200\}$ and variable $|B|$. The solid curves are predictions from Eq. (6).

Kramers-Wannier duality. A more detailed analysis of global quenches into this kind of systems is reported in Ref. [45].

*Toward universality.* All systems investigated with nonzero $I_3^{(\alpha)}$ exhibit extensive entropies with logarithmic corrections, which can in turn be traced back to the presence of discontinuities in the filling function (or Fermi weight) $\vartheta(k)$ [46,47]. We remind the reader that, in an integrable model with a thermodynamic Bethe ansatz description [48,49], filling functions characterize excited states in the thermodynamic limit and represent the fraction of particle excitations per given momentum [50]; in noninteracting models, $\vartheta(k) \sim \langle \mathbf{b}_k^\dagger \mathbf{b}_k \rangle$ is the coarse-grained fermion occupation number. For example, in the Fermi sea equivalent to the ground state of the XX model in zero field, we have $\vartheta(k) = \frac{1+\mathrm{sgn}[\cos(k)]}{2}$. We are not aware of theorems connecting discontinuities in $\vartheta(k)$ with conformal invariance when $|2\vartheta(k) - 1| \neq 1$, but we stress that discontinuities produce nevertheless algebraically decaying correlations.

For the sake of simplicity, we restrict ourselves to filling functions with symmetric discontinuities $\lim_{k\to k_F^\pm}[2\vartheta(k) - 1] = \tanh(\eta_{k_F}^\pm)$, with $\eta_{k_F}^- = -\eta_{k_F}^+$. We find that the large-distance properties of the state can be described by a QFT of massless Dirac fermions, whose Euclidian action reads [43]

$$S = \int \mathrm{d}x \int \mathrm{d}\tau \sum_{k_F} \sum_{s=\pm} \psi_{s,k_F}^\dagger [\partial_\tau + isv(k_F)\partial_x]\psi_{s,k_F}, \quad (5)$$

where $k_F$ distinguishes theories emerging in the expansion around different discontinuities, $v(k_F)$ is the velocity, and the fields satisfy standard anticommutation relations $\{\psi_{s,k_F}(x), \psi_{s',k_F'}^\dagger(y)\} = \delta_{ss'}\delta_{k_F,k_F'}\delta(x-y)\mathrm{I}$, $\{\psi_{s,k_F}(x), \psi_{s',k_F'}(y)\} = 0$. The standard procedure to compute the Rényi entropies of subsystems starts with reinterpreting the moments of the reduced density matrix $\mathrm{tr}[\rho_X^\alpha]$ as partition functions (cf. fig. 1 of Ref. [11]) in models formed by $\alpha$ copies of the original one under the condition that each field is identified with the successive copy of itself when crossing the space-time lines corresponding to $X$ at fixed (imaginary) time $\tau = 0$. The moments are finally conveniently identified with the correlation functions of local branch twist fields (associated with the global symmetry of exchange of the copies) [51], which are localized at the boundaries of $X$ and implicitly defined through the partition functions.

In spin chains, if the subsystem consists of two disjoint blocks, there is an additional complication related to the nonlocality of the Jordan-Wigner transformation; indeed, there are spin operators in $X$ whose fermionic representation includes fermions lying in between the blocks. This problem can be overcome by writing the reduced density matrix as a linear combination of four density matrices [25]. One should then generalize the field theory description to capture such a richer structure—see, e.g., sec. 4 of Ref. [29]. For example, one also encounters terms of the form $\mathrm{tr}[\cdots P_A\rho_{A\cup C}^{n_j} P_A \rho_{A\cup C}^{n_{j+1}}\cdots]$, where $P_A$ counts the parity of the fermions in the first block $A$. In the QFT language, $P_A$ corresponds to the transformation that changes the sign of the field when crossing $A$. In general, there are also other contributions with the same root, but they are multiplied by expectation values of strings of Pauli matrices, which in our nonequilibrium setting decay exponentially with the separation. Accordingly, these terms can be dropped (see Ref. [43] for more details). This is why, despite the similarity with Fermi seas, condition $|\eta_{k_F}^\pm| \to \infty$ after global quenches generally leads to an unusual tripartite information: The value of the filling function at the discontinuities does not characterize the long-distance properties of all the relevant degrees of freedom, as a nontrivial behavior of the filling function between the discontinuities has the strong effect to turn some algebraically decaying correlations into exponentially decaying ones.

In our situation, in turn, a part of the structure behind the QFT procedure sketched above is lost. Specifically, the action of the $\alpha$-copy model is still simply the sum of $\alpha$ copies of $S$ in Eq. (5), but only a physical part of the fields, namely, $\Psi_{k_F}^{\mathrm{phys}}(x) = \sum_s [1 + \exp(2s\eta_{k_F}^+)]^{-1/2}\psi_{s,k_F}(x)$, satisfies the standard conditions relating different copies on $X$ at zero imaginary time (whereas the unphysical parts of the copies are independent). The same subtlety applies to $P_A$, which changes only the sign of the physical fields. A direct consequence of having physical and unphysical fields is that the energy density of the $\alpha$-copy model does change when crossing $X$ at $\tau = 0$, undermining, e.g., the interpretation of $\mathrm{tr}[\rho_X^\alpha]$ as the correlation function of local branch twist fields [51]. As detailed in Ref. [43], however, it is possible to work out the asymptotic behavior of the Rényi entropies of connected and disconnected spin blocks using the resolvent method, like what was done in Refs. [52–55] for fermions.

*Results.* We announce here the simplest results, as they are already sufficient to unveil the most striking feature of the tripartite information. To start with, we focus on the limit $|\eta_{k_F}^\pm| \to \infty$. Since the same condition is satisfied in a Fermi sea, we can use the correspondence with the ground state of a CFT and take advantage of the results of Ref. [29], which computed the terms $\mathrm{tr}[\cdots P_A\rho_{A\cup C}^{n_j} P_A \rho_{A\cup C}^{n_{j+1}}\cdots]$ contributing to the tripartite information in the CFT ground state one by one (together with terms that do not contribute in our case).

The resulting prediction reads

$$G_\alpha(x) \xrightarrow{|\eta_{k_F}^\pm|\to\infty} \frac{\ln\Big\{\sum_{\substack{\delta_j\in\{0,\frac{1}{2}\}\\ j=1,\dots,\alpha-1}} \Big[\frac{\Theta(\vec{\delta}|\hat{\tau}_x)}{\Theta(\vec{0}|\hat{\tau}_x)}\Big]^\nu\Big\}}{\alpha - 1} - \ln 2, \quad (6)$$

where $\hat{\tau}_x$ is the $(\alpha-1)\times(\alpha-1)$ period matrix of the Riemann surface $\mathcal{R}_\alpha$ with elements:

$$[\hat{\tau}_x]_{\ell n} = \frac{2i}{\alpha}\sum_{k=1}^{\alpha-1}\sin\left(\tfrac{\pi k}{\alpha}\right)\cos\left[\tfrac{2\pi k(\ell-n)}{\alpha}\right]\frac{P_{(k/\alpha)-1}(2x-1)}{P_{(k/\alpha)-1}(1-2x)}. \quad (7)$$

Here, $P_\mu(z)$ denotes the Legendre functions, $\Theta(\vec{z},M)=\sum_{\vec{m}\in\mathbb{Z}^{\alpha-1}}\exp(i\pi\vec{m}^tM\vec{m}+2\pi i\vec{m}\cdot\vec{\delta})$ is the Siegel theta function, and $\nu$ is the number of discontinuities of the filling function (assuming $|\eta^\pm_{k_F}|\to\infty$ for each one of them). The quench in Fig. 2 can be used to check this prediction; indeed, $|\eta^\pm_{k_F}|\to\infty$. The agreement between numerical data and prediction is excellent. We stress that, contrary to the CFT analog, our system does not exhibit crossing symmetry; indeed, $G_\alpha(x)\neq G_\alpha(1-x)$.

Remarkably, $\lim_{x\to 1^-} G_\alpha(x) = -\ln 2$ for every $\alpha$; therefore (by the replica trick), we conclude that the genuine tripartite information also approaches the same value:

$$\boxed{\mathfrak{I}_3 = \lim_{x\to 1^-} G(x) = -\ln 2}. \quad (8)$$

This limit corresponds to small separation between the blocks compared with their size, i.e., to the limit $1 \ll |B| \ll |A|, |C|$. We call it *residual tripartite information* because $x$ is exactly equal to 1 only when $|B|=0$, for which the tripartite information is zero by definition. Such unusual nonzero residual tripartite information should be contrasted to the ordinary zero value, which is found both in other nonequilibrium settings, such as after translationally invariant quenches from ground states of gapped Hamiltonians and critical Hamiltonians with $c=\frac{1}{2}$, and in equilibrium at any temperature, independently of criticality.

Equation (8) is the main result of these notes. In all investigated systems with nonzero tripartite information, our analysis points to a universal residual tripartite information equal to $-\ln 2$, irrespectively of the quench protocol. In support of it, we also announce $G_2(x)$ with generic $\eta^-_{k_F}$ and $\eta^+_{k_F}$:

$$G_2(x) = \ln\left[\tfrac{1+(1-x)^\gamma}{2}\right] \Rightarrow G_2(1^-) = -\ln 2, \quad (9)$$

where $\gamma=\sum_{k_F}\{\frac{1}{\pi}\arg[\sin(\frac{\pi}{4}+i\eta^+_{k_F})/\sin(\frac{\pi}{4}+i\eta^-_{k_F})]\}^2$. Figure 1 shows the excellent agreement of the prediction from Eq. (9) with numerical data.

*Numerical method.* The Rényi entropies have been numerically evaluated using their expressions in terms of the fermionic correlation matrix $\Gamma_{\ell n}=\delta_{\ell n}-\langle \mathbf{a}_\ell \mathbf{a}_n\rangle$. The latter has been computed directly in the generalized Gibbs ensemble emerging in the limit of infinite time [56,57]. For the entropy of connected subsystems we used $S_\alpha(X)=\mathrm{tr}\{\ln[(\frac{\mathrm{I}_X+\Gamma_X}{2})^\alpha+(\frac{\mathrm{I}_X-\Gamma_X}{2})^\alpha]\}/[2(1-\alpha)]$, where $\Gamma_X$ is the correlation matrix in $X$ [58,59]. For the entropy of disjoint blocks in the generalized XY model, we used the algorithm proposed in Ref. [25], which allows one to express $S_\alpha(A\cup C)$ in terms of four matrices: $\Gamma_1\equiv\Gamma_{A\cup C}$, $\Gamma_2\equiv P_A\Gamma_1P_A$, $\Gamma_3\equiv\Gamma_1-\Gamma_{A\cup C,B}\Gamma_B^{-1}\Gamma_{B,A\cup C}$, and $\Gamma_4=P_A\Gamma_3P_A$, where $\Gamma_{A,A'}$ is the correlation matrix in which the row and column indices run in $A$ and $A'$, respectively (we refer the reader to sec. 3 of Ref. [25] for the formula with $\alpha=2,3,4$ [60]). The terms claimed before to survive the infinite time limit are those constructed with $\Gamma_1$ and $\Gamma_2$ only. For disjoint blocks in the dual XY model, we have generalized the previous algorithm following Ref. [40], and it is detailed in Ref. [45].

*Discussion.* We have shown that the Rényi-$\alpha$ tripartite information captures universal properties in the limit of infinite time after a global quench whenever the correlations in the stationary state decay sufficiently slow with the distance. We have provided evidence that $I_3^{(\alpha)}$ can be obtained within a QFT, and in some cases, we have been able to predict their asymptotic behavior. We think that the proposed framework could be generalized at least to interacting integrable systems that can be described by thermodynamic Bethe ansatz, which have similar relaxation properties. If nonzero, we always found negative $I_3^{(\alpha)}$; we wonder how general this property is in our setting. Finally, we defined the residual tripartite information $\mathfrak{I}_3$, which we found either equal to 0 or to $-\ln 2$. We leave the question open of which values $\mathfrak{I}_3$ could attain with interactions.

*Acknowledgments.* We are grateful to Leonardo Mazza for collaboration at an early stage of this project. We thank Raoul Santachiara for valuable discussions. We also thank Pasquale Calabrese and Erik Tonni for useful correspondence. This letter was supported by the European Research Council under the Starting Grant No. 805252 LoCoMacro.

[1] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[3] R. S. Ingarden, Quantum information theory, Rep. Math. Phys. **10**, 43 (1976).
[4] N. J. Cerf and C. Adami, Information theory of quantum entanglement and measurement, Physica D **120**, 62 (1998).
[5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[6] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. B **424**, 443 (1994).
[7] P. Calabrese and J. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech.: Theory Exp. (2004) P06002.
[8] V. E. Korepin, Universality of entropy scaling in one dimensional gapless models, Phys. Rev. Lett. **92**, 096402 (2004).
[9] M. Caraglio and F. Gliozzi, Entanglement entropy and twist fields, J. High Energy Phys. 11 (2008) 076.
[10] S. Furukawa, V. Pasquier, and J. Shiraishi, Mutual information and Boson radius in a $c=1$ critical system in one dimension, Phys. Rev. Lett. **102**, 170602 (2009).
[11] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A: Math. Theor. **42**, 504005 (2009).
[12] A. Kitaev and J. Preskill, Topological entanglement entropy, Phys. Rev. Lett. **96**, 110404 (2006).

[13] H. Casini and M. Huerta, Remarks on the entanglement entropy for disconnected regions, J. High Energy Phys. 03 (2009) 048.

[14] C. A. Agón, P. Bueno, and H. Casini, Tripartite information at long distances, SciPost Phys. **12**, 153 (2022).

[15] P. Hayden, M. Headrick, and A. Maloney, Holographic mutual information is monogamous, Phys. Rev. D **87**, 046003 (2013).

[16] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, Chaos in quantum channels, J. High Energy Phys. 02 (2016) 004.

[17] O. Schnaack, N. Bölter, S. Paeckel, S. R. Manmana, S. Kehrein, and M. Schmitt, Tripartite information, scrambling, and the role of Hilbert space partitioning in quantum lattice models, Phys. Rev. B **100**, 224302 (2019).

[18] C. Sünderhauf, L. Piroli, X.-L. Qi, N. Schuch, and J. I. Cirac, Quantum chaos in the Brownian SYK model with large finite *N*: OTOCs and tripartite information, J. High Energy Phys. 18 (2019) 038.

[19] If the intervals are disjoint, the CFT prediction depends instead on three cross ratios—see, e.g., Refs. [20,21].

[20] A. Coser, L. Tagliacozzo, and E. Tonni, On Rényi entropies of disjoint intervals in conformal field theory, J. Stat. Mech.: Theory Exp. (2014) P01008.

[21] C. D. Nobili, A. Coser, and E. Tonni, Entanglement entropy and negativity of disjoint intervals in CFT: some numerical extrapolations, J. Stat. Mech.: Theory Exp. (2015) P06021.

[22] P. Calabrese, J. Cardy, and E. Tonni, Entanglement entropy of two disjoint intervals in conformal field theory, J. Stat. Mech.: Theory Exp. (2009) P11001.

[23] P. Calabrese, J. Cardy, and E. Tonni, Entanglement entropy of two disjoint intervals in conformal field theory: II, J. Stat. Mech.: Theory Exp. (2011) P01021.

[24] F. Ares, R. Santachiara, and J. Viti, Crossing-symmetric twist field correlators and entanglement negativity in minimal CFTs, J. High Energy Phys. 10 (2021) 175.

[25] M. Fagotti and P. Calabrese, Entanglement entropy of two disjoint blocks in XY chains, J. Stat. Mech.: Theory Exp. (2010) P04016.

[26] V. Alba, L. Tagliacozzo, and P. Calabrese, Entanglement entropy of two disjoint blocks in critical Ising models, Phys. Rev. B **81**, 060411(R) (2010).

[27] V. Alba, L. Tagliacozzo, and P. Calabrese, Entanglement entropy of two disjoint intervals in $c = 1$ theories, J. Stat. Mech.: Theory Exp. (2011) P06012.

[28] M. Fagotti, New insights into the entanglement of disjoint blocks, Europhys. Lett. **97**, 17007 (2012).

[29] A. Coser, E. Tonni, and P. Calabrese, Spin structures and entanglement of two disjoint intervals in conformal field theories, J. Stat. Mech.: Theory Exp. (2016) 053109.

[30] V. Alba and P. Calabrese, Entanglement and thermodynamics after a quantum quench in integrable systems, Proc. Natl. Acad. Sci. USA **114**, 7947 (2017).

[31] B. Bertini, K. Klobas, V. Alba, G. Lagnese, and P. Calabrese, Growth of Rényi entropies in interacting integrable models and the breakdown of the quasiparticle picture, Phys. Rev. X **12**, 031016 (2022).

[32] T. Zhou and A. Nahum, Entanglement membrane in chaotic many-body systems, Phys. Rev. X **10**, 031066 (2020).

[33] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).

[34] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. **83**, 863 (2011).

[35] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).

[36] M. Suzuki, The dimer problem and the generalized *X*-model, Phys. Lett. A **34**, 338 (1971).

[37] E. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. **16**, 407 (1961).

[38] L. Zadnik and M. Fagotti, The folded spin-$\frac{1}{2}$ XXZ model: I. Diagonalisation, jamming, and ground state properties, SciPost Phys. Core **4**, 10 (2021).

[39] M. Fagotti, Global quenches after localized perturbations, Phys. Rev. Lett. **128**, 110602 (2022).

[40] M. Fagotti, V. Marić, and L. Zadnik, Nonequilibrium symmetry-protected topological order: emergence of semilocal Gibbs ensembles, arXiv:2205.02221.

[41] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech.: Theory Exp. (2016) 064002.

[42] A. Coser, E. Tonni, and P. Calabrese, Entanglement negativity after a global quantum quench, J. Stat. Mech.: Theory Exp. (2014) P12017.

[43] V. Marić and M. Fagotti, Universality in the tripartite information after global quenches: (generalised) quantum XY models, J. High Energ. Phys. 06 (2023) 140.

[44] V. Alba, B. Bertini, M. Fagotti, L. Piroli, and P. Ruggiero, Generalized-hydrodynamic approach to inhomogeneous quenches: correlations, entanglement and quantum effects, J. Stat. Mech.: Theory Exp. (2021) 114004.

[45] V. Marić, Universality in the tripartite information after global quenches spin flip and semilocal charges, arXiv:2307.01842.

[46] V. Alba, M. Fagotti, and P. Calabrese, Entanglement entropy of excited states, J. Stat. Mech.: Theory Exp. (2009) P10020.

[47] F. Ares, J. G. Esteve, F. Falceto, and E. Sánchez-Burillo, Excited state entanglement in homogeneous fermionic chains, J. Phys. A: Math. Theor. **47**, 245301 (2014).

[48] C. N. Yang and C. P. Yang, Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction, J. Math. Phys. **10**, 1115 (1969).

[49] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1993).

[50] J. Mossel and J.-S. Caux, Generalized TBA and generalized Gibbs, J. Phys. A: Math. Theor. **45**, 255001 (2012).

[51] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, J. Stat. Phys. **130**, 129 (2007).

[52] H. Casini and M. Huerta, Entanglement entropy in free quantum field theory, J. Phys. A: Math. Theor. **42**, 504007 (2009).

[53] H. Casini and M. Huerta, Reduced density matrix and internal dynamics for multicomponent regions, Classical and Quantum Gravity **26**, 185005 (2009).

[54] P. Fries and I. A. Reyes, Entanglement and relative entropy of a chiral fermion on the torus, Phys. Rev. D **100**, 105015 (2019).

[55] D. Blanco, T. Ferreira Chase, J. Laurnagaray, and G. Pérez-Nadal, Rényi entropies of the massless Dirac field on the torus, Phys. Rev. D **105**, 045014 (2022).

[56] E. T. Jaynes, Information theory and statistical mechanics, Phys. Rev. **106**, 620 (1957).

[57] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a completely integrable many-body quantum system: An *ab initio* study of the dynamics of the highly excited states of 1D lattice hard-core bosons, Phys. Rev. Lett. **98**, 050405 (2007).

[58] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in quantum critical phenomena, Phys. Rev. Lett. **90**, 227902 (2003).

[59] B.-Q. Jin and V. E. Korepin, Quantum spin chain, Toeplitz determinants and the Fisher-Hartwig conjecture, J. Stat. Phys. **116**, 79 (2004).

[60] We warn the reader of a typo in Ref. [25]: $\det \Gamma_{B_1}$ in eqs. (57)–(59) should be replaced by $|\det \Gamma_{B_1}|$.