Letter

$SO(8)$ unification and the large-N theory of superconductor-insulator transition of two-dimensional Dirac fermions

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Electrons on honeycomb or pi-flux lattices obey the effective massless Dirac equation at low energies and at the neutrality point, and should suffer quantum phase transitions into various Mott insulators and superconductors at strong two-body interactions. We show that 35 out of 36 such order parameters that provide Lorentz-invariant mass gaps to Dirac fermions belong to a single irreducible tensor representation of the $SO(8)$ symmetry of the two-dimensional Dirac Hamiltonian for the spin-$1/2$ lattice fermions. The minimal interacting Lagrangian away from the neutrality point has the $SO(8)$ symmetry reduced to $U(1) \times SU(4)$ by finite chemical potential, and it allows only two independent interaction terms. When the Lagrangian is nearly $SO(8)$ symmetric and the ground state insulating at the neutrality point, we show it turns superconducting at the critical value of the chemical potential through a "flop" between the tensor components, by exactly solving the $SU(4) \to SU(N)$ generalization in the large-N limit. A lattice Hamiltonian that may exhibit this transition, parallels with and the consequences for the Gross-Neveu model, and applicability to related electronic systems are briefly discussed.

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Introduction. Graphene at the neutrality point is a weakly interacting gapless Dirac semimetal. Multilayer graphene structures, in contrast, often appear to be Mott insulators. The insulating state in rhombohedral trilayer graphene, for example, with doping (and electric field) turns into a superconductor [1]. The latter phenomenon is reminiscent of the much studied but still incompletely understood behavior of cuprate superconductors. Cuprates are insulating antiferromagnets at half filling which become d-wave superconductors with high critical temperatures above critical doping with holes. Zhang [2,3] viewed this insulator-superconductor transition as a "flop" of a five-dimensional vector order parameter composed of the three Néel and two superconducting components, which is induced by the chemical potential which favors the superconducting directions. Unfortunately, the $SO(3) \times SO(2)$-symmetric three-dimensional Ginzburg-Landau theory is now understood [4,5] not to exhibit a particularly wide crossover regime near its unstable $SO(5)$-symmetric fixed point, as was originally hoped, which significantly reduces the range of relevance of such a unified theory. We argue below, however, that a different unification of physically disparate orders under the umbrella of an emergent internal symmetry becomes possible in two-dimensional (2d) Dirac systems.

As a paradigmatic example we take the electrons in graphene, which at low energies, at the neutrality point, and when assumed noninteracting, are described by the eight-dimensional Dirac Hamiltonian. The number eight comes from the honeycomb lattice being bipartite (two sublattices), there being two inequivalent Dirac cones, or valleys (fermion doubling), and finally the electrons having spin-$1/2$. By a judicious construction of the Dirac fermion the nontrivial matrix structure of the Dirac Hamiltonian can be completely stowed into the sublattice factor space, and then replicated four times for the two valley and the two spin components. When written like this the Dirac single-particle Lagrangian besides its hallmark space-time $SU(2)$ Lorentz symmetry clearly displays the internal $SU(4)$ symmetry, the latter acting on the spin-valley index. Of course, since the particle number is conserved, there is also the exact $U(1)$ gauge symmetry and the discrete time reversal symmetry. Both the Lorentz $SU(2)$ and the internal $SU(4)$ symmetries are emergent at low energies and broken by the lattice.

The long-range and Lorentz-violating nature of the electron-electron Coulomb interaction notwithstanding, it may still be profitable to ask, what is the minimal local *interacting* Lagrangian that would respect the larger $U(1) \times SU(4) \times SU(2)$ symmetry [6]? Such a Lagrangian would define the truly minimal (maximally symmetric) interacting field theory of 2d lattice spin-$1/2$ Dirac fermions. We show first that such a Lagrangian is remarkably simple, and contains only two independent quartic terms. Furthermore, the 36 Dirac fermion bilinears [7–9] that transform as Lorentz singlets, i.e., the average of which would be the order parameters that represent Dirac masses, can be understood as four distinct irreducible representations (irreps) 10, 10*, 15, and 1 under the $SU(4)$ [10]. The first two irreps correspond to x and y components of the 10 gapped superconducting order parameters [11–13] and transform as the symmetric second-rank tensor and its complex conjugate [14]. The irrep 15 is the adjoint representation comprising all insulating mass order parameters other than the quantum anomalous Hall (QAH) state [15]. The latter transforms as an $SU(4)$ singlet, i.e., as the irrep 1. The $U(1) \times SU(4) \times SU(2)$ interacting field theory already provides partial unification of dominant order parameters in the Dirac system.

Our main result is, however, that when a certain condition between the coupling constants of the two interaction terms is met, and when the chemical potential is at the Dirac point, the minimal interacting Lagrangian exhibits further enlargement from $U(1)\times SU(4)\times SU(2)$ symmetry to $SO(8)\times SU(2)$. The $SO(8)$ arises because the two-component 2d massive Dirac equation can be transformed into a "real" (Majorana) form, so that four copies of the usual two-component complex Dirac fermions are equivalent to eight copies of two-component Majorana fermions. Most importantly, the $SO(8)$ symmetry ultimately unifies *almost all* the 36 mass order parameters as

$$10 + 10^* + 15 + 1 \to 35 + 1, \tag{1}$$

where on the [left-hand] right-hand side we mean the irreps of [$SU(4)$] $SO(8)$. 35 stands for the irreducible, symmetric, second-rank $SO(8)$ tensor, and gathers together all the insulating and the superconducting mass gaps other than the QAH state. The $SO(8)$-symmetric field theory has a *single* interaction coupling constant, and we argue that, at strong coupling, and depending on its sign, the ground state (1) either through the canonical Gross-Neveu transition [6] becomes the QAH insulator, which preserves $SO(8)$ and spontaneously breaks the Z_2 (time reversal) symmetry, or (2) spontaneously breaks $SO(8)$ down to $SO(4)\times SO(4)$ through a new phase transition solved here in the large-N limit.

As the first application of the $SO(8)$ grand unified theory of graphene we demonstrate that the insulator-to-superconductor transition becomes induced by the chemical potential. Taking the sign of the interaction coupling in the $SO(8)$-symmetric theory so that $SO(8)\to SO(4)\times SO(4)$ transition is realized, we show that the theory becomes exactly solvable in the large-N limit, when the original $SU(4)$ is generalized to $SU(N)$. At the neutrality point and strong coupling the exact ground state may then be any of the gapped insulators (other than QAH) or the superconductors, including some of their linear combinations. To study the competition between the insulating and superconducting ground states we detune the above condition between the two interaction couplings in the $SU(N)$ theory to explicitly favor the insulating solution at the Dirac point. Finite chemical potential is then shown to benefit the superconducting solution, and to eventually cause a first-order transition between the two competing classes of states at its critical value. Our exact calculation provides a *proof of principle* that the same interaction may lead to insulating or superconducting state, depending on doping.

Dirac Lagrangian. The low-energy Dirac Lagrangian of the tight-binding nearest-neighbor-hopping Hamiltonian on the bipartite honeycomb lattice and at half filling can be written as [16,17]

$$L_0 = \psi^\dagger(1_4 \otimes (1_2\partial_\tau - i\sigma_1\partial_1 - i\sigma_2\partial_2))\psi, \tag{2}$$

where σ_i are the standard Pauli matrices, 1_N is a N-dimensional unit matrix, and the eight-component Grassmann field $\psi^T = (\psi_+^T(x), \psi_-^T(x))$, with $\psi_\sigma(x) = \int d^D q e^{iqx}\psi_\sigma(q)$ given by $\psi_\sigma^\dagger(q) = [u_\sigma^\dagger(K+q), v_\sigma^\dagger(K+q), iv_\sigma^\dagger(-K+q), -iu_\sigma^\dagger(-K+q)]$. u_σ and v_σ are electron variables on the triangular sublattices of the honeycomb lattice, and $\sigma=\pm$ is the third projection of spin-1/2. The $D=2+1$-dimensional energy-momentum vector $q=(\omega,\vec{q})$ collects together the Matsubara frequency ω and the wave vector $\vec{q}$, $K=(0,\vec{K})$, where $\pm\vec{K}$ are the inequivalent Dirac points, $|\vec{q}| < \Lambda \ll |\vec{K}|$, Λ is the momentum cutoff, and τ represents the imaginary time. The reference frame is chosen so that $q_x = \vec{q}\cdot\vec{K}/|\vec{K}|$, $q_y = (\vec{K}\times\vec{q})\times\vec{K}/|\vec{K}|^2$, and the Fermi velocity is set to unity.

Let us list some global symmetries of L_0:

(1) gauge $U(1)$,

$$\psi \to e^{i\phi}\psi, \quad \psi^\dagger \to e^{-i\phi}\psi^\dagger, \tag{3}$$

(2) antiunitary time reversal,

$$\psi \to (\sigma_2\otimes\sigma_2\otimes\sigma_2)\psi^*, \quad \psi^\dagger \to \psi^T(\sigma_2\otimes\sigma_2\otimes\sigma_2), \tag{4}$$

(3) internal $U\in SU(4)$,

$$\psi \to (U\otimes 1_2)\psi, \quad \psi^\dagger \to \psi^\dagger(U^\dagger\otimes 1_2), \tag{5}$$

(4) Lorentz $U\in SU(2)$,

$$\psi \to (1_4\otimes U)\psi, \quad \bar\psi \to \bar\psi(1_4\otimes U^\dagger), \tag{6}$$

accompanied by the corresponding rotation of the space-time vector $(\tau,\vec{x})$. Here, $\bar\psi = \psi^\dagger(1_4\otimes\sigma_3)$.

Interacting Lagrangian. Next, we exhibit the local interaction terms quartic in fermion fields that would respect the above symmetries, with the $SU(4)$ generalized to $SU(N)$. There are four such terms:

$$I_1 = [\bar\psi(1_N\otimes 1_2)\psi]^2, \quad I_2 = [\bar\psi(G_a\otimes 1_2)\psi]^2, \tag{7}$$

$$I_3 = [\bar\psi(1_N\otimes\sigma_i)\psi]^2, \quad I_4 = [\bar\psi(G_a\otimes\sigma_i)\psi]^2. \tag{8}$$

Here G_a, $a=1,\dots,N^2-1$, are the Hermitian generators of $SU(N)$, and $\mathrm{Tr}(G_aG_{a'}) = N\delta_{aa'}$. The summation convention is assumed.

For any $N\neq 2$, only two, and any two, of these four terms are linearly independent [18]. We may chose these two to be I_1 and I_2, so that the general $U(1)\times SU(N)\times SU(2)$-invariant local interaction term in the Lagrangian becomes

$$L_1 = g_1 I_1 + g_2 I_2. \tag{9}$$

When $g_2=0$, the Lagrangian $L=L_0+L_1$ is nothing but the canonical Gross-Neveu model in $D=2+1$ with N fermion flavors; for $N\to\infty$ and $g_1 < g_{1c} = -\pi/(4N\Lambda)$ one finds $\langle\bar\psi\psi\rangle\neq 0$, i.e., the QAH state. For $g_{1c} < g_1$, on the other hand, $\langle\bar\psi\psi\rangle = 0$ [19–22].

In [18] we derive the following Fierz identity:

$$-(N+1)I_1 = [\psi^\dagger(S_b\otimes\sigma_2)\psi^*][\psi^T(S_b\otimes\sigma_2)\psi] + I_2. \tag{10}$$

The index $b=1,\dots,N(N+1)/2$, and S_b are symmetric, N-dimensional real matrices, with $\mathrm{Tr}(S_bS_{b'}) = N\delta_{bb'}$. The interaction term L_1 can therefore also be written as

$$L_1 = -\frac{1}{N+1}\{g_1[\psi^\dagger(S_b\otimes\sigma_2)\psi^*][\psi^T(S_b\otimes\sigma_2)\psi] + \tilde g_1[\psi^\dagger(G_a\otimes\sigma_3)\psi]^2\}, \tag{11}$$

where $\tilde g_1 = g_1 - (N+1)g_2$. Let us now assume both $g_1, \tilde g_1 > 0$, so that the QAH state is suppressed ($\langle\bar\psi\psi\rangle = 0$). Using the Hubbard-Stratonovich (HS) transformation [23] the

Lagrangian $L = L_0 + L_1$ can be expressed as

$$L = L_0 + \frac{N+1}{4}\left(\frac{\Delta_b^*\Delta_b}{g_1} + \frac{m_a m_a}{\tilde{g}_1}\right) - m_a[\psi^\dagger(G_a\otimes\sigma_3)\psi] + \frac{\Delta_b}{2}[\psi^\dagger(S_b\otimes\sigma_2)\psi^*] + \frac{\Delta_b^*}{2}[\psi^T(S_b\otimes\sigma_2)\psi]. \quad (12)$$

The averages of the HS fields satisfy $(N+1)\langle\Delta_b\rangle = 2g_1\langle\psi^T(S_b\otimes\sigma_2)\psi\rangle$, and $(N+1)\langle m_a\rangle = 2\tilde{g}_1\langle\psi^\dagger(G_a\otimes\sigma_3)\psi\rangle$, and transform as the irreps $N(N+1)/2$ (symmetric tensor) and N^2-1 (adjoint) of the $SU(N)$, respectively. Both are singlets under Lorentz $SU(2)$. When $g_2 = 0$ Eq. (12) provides an alternative representation of the standard Gross-Neveu model.

Majorana representation. When $g_2 = 0$ the symmetry of the Lagrangian L is in fact $SO(2N)$. To see this, rotate $\psi \to \chi = 1_N \otimes e^{i(\pi/4)\sigma_1}\psi$, which makes the Dirac Hamiltonian for χ fully imaginary, and transforms the fermion bilinears as

$$\psi^\dagger(G_a\otimes\sigma_3)\psi \to \chi^\dagger(G_a\otimes\sigma_2)\chi, \quad (13)$$

$$\psi^T(S_b\otimes\sigma_2)\psi \to \chi^T(S_b\otimes\sigma_2)\chi. \quad (14)$$

The form of the second bilinear does not change, since the antisymmetric $SU(2)$ tensor σ_2 transforms as a singlet.

We decompose the new Dirac fermion as $\chi = \chi_1 - i\chi_2$, and $\chi^\dagger = \chi_1^T + i\chi_2^T$, where $\chi_{1,2}$ are "real", or Majorana fermions. In terms of the $4N$-component Majorana fermion $\phi^T = (\chi_1^T, \chi_2^T)$ the Lagrangian for $g_2 = 0$ now simplifies into

$$L = \phi^T[1_{2N}\otimes(1_2\partial_\tau - i\sigma_1\partial_1 - i\sigma_3\partial_2)]\phi + \frac{N+1}{8Ng_1}\mathrm{Tr}S^2 + \phi^T(S\otimes\sigma_2)\phi, \quad (15)$$

where the order parameter matrix S is

$$S = \Delta_b'\sigma_3\otimes S_b + \Delta_b''\sigma_1\otimes S_b + m_c 1_2\otimes G_c^S + m_d\sigma_2\otimes G_d^A, \quad (16)$$

and represents the general $2N$-dimensional, symmetric, real, traceless matrix, and $\Delta_b = \Delta_b' - i\Delta_b''$. G_c^S are the symmetric, and G_d^A are the antisymmetric generators of $SU(N)$, so the indices $c = 1,\ldots,(N-1)(N+2)/2$, and $d = 1,\ldots,N(N-1)/2$ [24]. The Lagrangian L in Eq. (15) is now manifestly invariant under the transformation $\phi \to (O\otimes 1_2)\phi$, $S \to OSO^T$, where $O \in SO(2N)$. The $(N+1)(2N-1)$ HS fields $(\Delta_b', \Delta_b'', m_a)$ transform as the *symmetric, traceless, second-rank tensor* under $SO(2N)$.

Large-N. In the limit $N\to\infty$ in Eq. (12) the theory becomes exactly solvable by the saddle-point method [23]. The details of relatively straightforward but somewhat long calculations are provided in [18]. Hereafter we take $N = 2^n$ and n integer. At the $SO(2N)$-symmetric point, for $g_2 = 0$ the minimum is the order parameter matrix S such that $S^2 = M^2 1_{2N}$, and

$$M = \Lambda - \frac{\pi}{g_1} \quad (17)$$

for $g_1\Lambda/\pi > 1$, and otherwise zero [18]. When $M \neq 0$ the $SO(2N)$ symmetry becomes broken to $SO(N)\times SO(N)$. If $[S, \sigma_2\times 1_N] = 0$ the ground state is an insulator; otherwise it is a superconductor. One can also show that S contains at most $N+1$ different and mutually anticommuting terms in the expansion in Eq. (16) [18,25,26]. The number of Goldstone excitations is N^2.

When the chemical potential $\mu > 0$, the Lagrangian becomes deformed as $L \to L + L_\mu$, and the term

$$L_\mu = \mu\psi^\dagger\psi = \mu\phi^T(\sigma_2\otimes 1_N\otimes 1_2)\phi \quad (18)$$

reduces $SO(2N)$ to $U(1)\times SU(N)$, and the Lorentz $SU(2)$ to $SO(2)$. We therefore restore $g_2 \neq 0$ to consider the case with the lower internal symmetry, while for simplicity still retaining the Lorentz $SU(2)$ of L_1. By selecting $g_2 < 0$ the symmetry at $\mu = 0$ is broken in favor of the insulating solution. The insulating order parameter matrix $G = m_a G_a$ at the minimum satisfies $G^2 = m^2 1_N$, with $m = \sqrt{m_a m_a}$ determined by [18]

$$\frac{\pi}{\tilde{g}_1} = \Lambda - \max(\mu, m). \quad (19)$$

The solution for m reduces to Eq. (17) if $\tilde{g}_1(\Lambda-\mu)/\pi > 1$, so that $\mu < m$, but is otherwise zero. The insulating solution, obtained at $\mu = 0$, would this way be suppressed completely beyond a certain value of the chemical potential [27]. The gauge $U(1)$ is preserved, but the internal $SU(N)$ is spontaneously broken as $SU(N)\to U(1)\times SU(N/2)\times SU(N/2)$, leading to $N^2/2$ Goldstone excitations. For $N = 4$, due to local isomorphisms $SU(4)\simeq SO(6)$ and $SU(2)\times SU(2)\simeq SO(4)$, this can also be understood as $SO(6)\to SO(2)\times SO(4)$.

At the superconducting minimum the order parameter matrix $S_\Delta = \Delta_b S_b$ is such that again $S_\Delta^2 = \Delta^2 1_N$, but with the amplitude Δ determined by [18]

$$\frac{\pi}{g_1} = \Lambda - \sqrt{\mu^2+\Delta^2} + \mu\ln\left[\frac{\mu}{\Delta} + \sqrt{1+\left(\frac{\mu}{\Delta}\right)^2}\right], \quad (20)$$

with the last term recognizable as the familiar Cooper log. The right-hand side is uniformly increased by a finite chemical potential, and the superconducting solution is consequently *enhanced* relative to its value at $\mu = 0$. One can show that the subgroup of the $SU(N)$ that leaves the superconducting ground state invariant is $SO(N)$. Since the particle-number $U(1)$ is also broken, the number of Goldstone excitations is now $N(N+1)/2$.

For the Lagrangian to be almost $SO(8)$-symmetric we now assume $0 < \tilde{g}_1 - g_1 \ll g_1$. The insulator-to-superconductor flop then occurs at $\mu = \mu_c$, where

$$\frac{\mu_c g_1}{\pi} = \sqrt{\frac{N|g_2|}{g_1}\left(\frac{g_1\Lambda}{\pi} - 1\right)}. \quad (21)$$

At $\mu = \mu_c$ the Dirac mass suffers a discontinuity from m in Eq. (19) to Δ in Eq. (20), with [18]

$$m - \Delta = \frac{\pi N|g_2|}{2g_1^2}. \quad (22)$$

Lattice model. A lattice model where the insulator-to-superconductor transition could be observable is

$$H = -t\sum_{\langle i,j\rangle} c_i^\dagger c_j + \lambda\sum_{\text{hex.}}\left\{\sum_{\langle\langle i,j\in\text{hex.}\rangle\rangle}\nu_{ij}c_i^\dagger c_j + \text{H.c.}\right\}^2, \quad (23)$$

where $c_i^\dagger = (c_{i,+}^\dagger, c_{i,-}^\dagger)$ creates an electron at the site i on the honeycomb lattice. Besides the first hopping term the second interaction term involves next-nearest-neighbor pairs of sites on each hexagon of the honeycomb lattice, with the phase factors $\nu_{ij} = -\nu_{ji} = \pm i$, as given by the Kane-Mele model [28]. At $\lambda = \lambda_{c1} < 0$ and at filling one half there should be the Dirac semimetal–QAH transition [29] described by the canonical Gross-Neveu field theory [16,20,30], studied by the large-N expansion [31–33] and the conformal bootstrap [34]. It represents the singlet version of the quantum anomalous spin Hall transition [30,35]. At $\lambda > 0$, on the other hand, the relevant field theory should be given by Eq. (15), so that for $\lambda > \lambda_{c2} > 0$ [29] the symmetry-breaking pattern is $SO(8) \to SO(4) \times SO(4)$, with the lattice terms deciding the precise ground state. It is conceivable that this phase transition becomes discontinuous for low N, as typical for matrix order parameters [36–38]. Our prediction is that the ground state flops into a gapped superconductor with doping.

Related systems. Any 2d $2N$-component Dirac Hamiltonian can be transformed into the Majorana form, and if the short-range interactions feature an emergent $SU(N)$ and Lorentz symmetries as well, be tuned into the $SO(2N)$-symmetric form. This includes spinless fermions hopping on honeycomb or pi-flux lattices ($N = 2$), and even the quasiparticles in the d-wave superconductor ($N = 4$) [8]. The interpretation of the ordered states that form the representations 1 and $(N + 1)(2N − 1)$ of the $SO(2N)$ depends on the physical context. Another related example is the rhombohedral trilayer graphene, which in the simplest approximation could be described by the Hamiltonian in Eq. (2) with the replacement (in the momentum space) $p_1 = p\cos(\theta) \to p^3\cos(3\theta)$ and $p_2 = p\sin(\theta) \to p^3\sin(3\theta)$. Oddness of the single-particle Hamiltonian in space and time derivatives allows it to adopt the Majorana form, but the lack of the Lorentz symmetry necessitates three independent contact interactions. The density of states is diverging at the neutrality point, however, and consequently the critical interaction vanishes. Details will be presented in a separate publication.

Conclusion. In reality the internal $SU(4)$ is broken by the lattice and the Coulomb interactions to $SU(2) \times SO(2)$, with $SU(2)$ as spin rotations, and the $SO(2)$ related to translations [6,8]. Likewise, the Lorentz symmetry is reduced to spatial rotations alone. Such a reduced symmetry allows nine independent local interaction terms [6,39,40]. On the other hand, the long-range Coulomb interaction is believed to be irrelevant near Gross-Neveu-like critical points [30,41,42]. It is conceivable that the ultraviolet complexity of real-world Dirac systems notwithstanding, the unified $SO(8)$ theory emerges as the effective low-energy description for a not-too-unrealistic choice of two-body interactions. This theory allows an exact solution in the well-defined large-N limit, which offers a proof of principle that the superconductor indeed may be an insulator flopped by the chemical potential [2].

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