Letter

Theory of fractionally magnetized quantum ferromagnet

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We present a theory to realize entangled quantum spin states with fractional magnetization. The origin of magnetization reduction is partly emergent antiferromagnetism, that is, the spin liquefaction of ferromagnetism. We study a ferromagnetic bilinear coupling region of the spin-S ($\geq$1) bilinear-biquadratic spin chain based on (i) a rigorous eigenstate correspondence between the spin-S model and spin-$\frac{1}{2}$ model and (ii) a numerical exact-diagonalization calculation up to $S = 3$. As a result, we obtain a fractional magnetized $M = 1 - 1/(2S)$ phase, where ground states have quantum entanglement-reflecting corresponding spin-$\frac{1}{2}$ antiferromagnetic ground states in a ferromagnetic background. This spin-liquefaction theory of ferromagnets can be generalized to any-dimensional lattices even under a magnetic field. This fractional ferromagnetism opens another research field of quantum ferromagnets.

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Entangled quantum states have been attracting not only researchers in physics but also developers in quantum computer science. In condensed-matter physics, antiferromagnets involve many interesting topics, including entangled gapped quantum spin-liquid states [1] in an integer spin-S chain with a Haldane gap [2], and fractionalized $S/2$ spins that form an entangled spin singlet on a bond in the valence-bond-solid picture of the Affleck-Kennedy-Lieb-Tasaki (AKLT) model [3]. On the other hand, ferromagnetically ordered states in quantum systems can be approximated as "classical" states in the sense that fully polarized local spins have no quantum entanglement. Is there any ferromagnet with an entangled quantum state?

A key to realizing an entangled ferromagnetic state is to partly create an antiferromagnetic quantum state in a ferromagnetic classical background, that is, "spin liquefaction" of a ferromagnet. When the total spin of a partly emergent "spin liquid" (a phase with nonmagnetic long-range Néel order) is zero, the coexistent states are fractionally magnetized. In this Letter, we propose a simple procedure to construct a quantum spin-S Hamiltonian that leads to the property of a phase transition from fully magnetized ground states to fractionally magnetized ground states under zero magnetic field. This transition is accomplished by flat-band one-magnon instability and magnetization changes from $M = 1$ to a fraction $M < 1$. Note that this is not a magnetization-plateau state under an external magnetic field but macroscopically degenerate ferromagnetic ground states with fractional magnetization under zero magnetic field, that is, a "fractional ferromagnet."

The realization of spin liquefaction is supported by a rigorous correspondence between a subset of eigenstates in the spin-S model and whole eigenstates in the spin-$\frac{1}{2}$ antiferromagnetic model. In other words, the rigorous correspondence is "eigensystem embedding." Thus, it might be interesting even in the context of quantum many-body scars [4–8]. As an example, we consider a spin-S ($S \geq 1$) bilinear-biquadratic (BLBQ) chain described by the Hamiltonian

$$\hat{H}_{\alpha}^{(S)} = \cos\alpha \sum_{i=1}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sin\alpha \sum_{i=1}^{N} (\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1})^2, \qquad (1)$$

with the periodic boundary condition $\hat{\boldsymbol{S}}_{N+1} = \hat{\boldsymbol{S}}_1$. The phase diagram for the $S = 1$ case, shown in Fig. 1, has been massively studied [9] and includes the AKLT point at $\alpha = \arctan\frac{1}{3}$ [3], the SU(3) point at $\alpha = \frac{\pi}{4}$ [10–12], and the other high-symmetry points at $\frac{5\pi}{4}$ [13], $\frac{3\pi}{2}$ [14–17], and $\frac{7\pi}{4}$ [18–21]. As explained later, for any S, the rigorous eigenstate correspondence between eigenstates consisting of S and $S-1$ spin states in the BLBQ chain and eigenstates in the spin-$\frac{1}{2}$ Heisenberg chain (i.e., spin-$\frac{1}{2}$ liquefaction) is realized at $\alpha = \alpha_r$ and $\alpha = \alpha_r + \pi$, where

$$\alpha_r = \begin{cases} -\arctan\left(\frac{1}{2S(S-2)+1}\right), & S \leq 3/2, \\ \pi - \arctan\left(\frac{1}{2S(S-2)+1}\right), & S \geq 2. \end{cases} \qquad (2)$$

For $S = 1$, $\alpha_r = \frac{\pi}{4}$ corresponds to the SU(3) point. In other words, α_r is a generalization of the $S = 1$ SU(3) point via the preservation of partial SU(2) symmetry for the spin-$\frac{1}{2}$ liquefaction. Note that, because this correspondence at α_r is for eigenstates, numerical evidence is required to obtain the ground-state properties. As a result of a numerical calculation of the BLBQ chain, we find that the ground state of the $S \geq 2$ BLBQ model at α_r is equivalent to that of the $S = \frac{1}{2}$ antiferromagnetic chain, and the fractionally magnetized state is stabilized in a finite parameter region for $S \geq 3/2$. The

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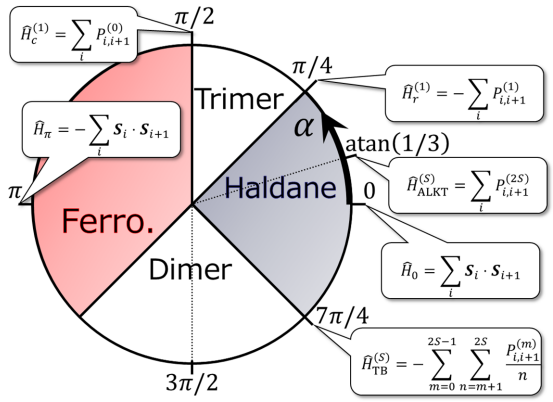


FIG. 1. Known phase diagram of $S = 1$ BLBQ chain. The high-symmetry points $\alpha_r = \pi/4$ and $\alpha_c = \pi/2$ are generalized to higher S in Eqs. (2) and (3). $\hat{P}_{ij}^{(s)}$ is a projection operator defined later in Eq. (4).

spin-liquefaction transition from the fully magnetized $M = 1$ phase around the ferromagnetic Heisenberg point $\alpha = \pi$ to the fractionally magnetized $M = 1 - \frac{1}{2S}$ phase occurs at

$$\alpha_c = \pi - \arctan\left(\frac{1}{2S(S-1)}\right) \tag{3}$$

for $S \geqq 3/2$, as shown schematically in Fig. 2(b). This spin-$\frac{1}{2}$ liquefaction of the spin-S system can be considered as a generalization of the "entire" spin liquefaction from the spin-$\frac{1}{2}$ ferromagnetic-ordered phase to the antiferromagnetic quantum-disordered phase of the $S = \frac{1}{2}$ Hamiltonian $J\sum_{i=1}^{N} \hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{s}}_{i+1}$ at $J = 0$, as shown in Fig. 2(a).

To explain the theoretical detail, let us start with a spin-projection Hamiltonian of a spin-S model on any lattice with general coefficients $J_{ij}^{(s)}$ defined as $\hat{H} = \sum_{ij}\sum_{s=0}^{2S} J_{ij}^{(s)} \hat{P}_{ij}^{(s)}$, where $\hat{P}_{ij}^{(s)}$ is a projection operator onto the subspace with total spin $s \in [0, 2S]$ for two spins at sites i and j. There is a general relation [22]

$$\hat{P}_{ij}^{(s)} = \prod_{\substack{n=0 \\ n\neq s}}^{2S} \frac{\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j - q_n}{q_s - q_n}, \quad (\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j)^n = \sum_{s=0}^{2S} {q_s}^n \hat{P}_{ij}^{(s)}, \tag{4}$$

with $q_s = s(s+1)/2 - S(S+1)$. Given $\sum_{s=0}^{2S} \hat{P}_{ij}^{(s)} = 1$, the $(2S+1)$-dimensional parameter space of $J_{ij}^{(0)}, J_{ij}^{(1)}, \ldots, J_{ij}^{(2S)}$ is reduced to $2S$ dimensions. By ignoring the positive energy scale factor, the intrinsic parameter space becomes a $2S$-dimensional sphere: For $S = 1$, a two-dimensional sphere is a circle parametrized by α, that is, the BLBQ Hamiltonian. Moreover, the spin-projection Hamiltonian can simply express the high-symmetry points of the $S = 1$ BLBQ chain, as summarized in Fig. 1, by ignoring the positive energy scale factor and energy shift. In previous studies for $S = 2$, $2S = 4$ independent parameters are assumed to be $J_{ij}^{(1)} = J_{ij}^{(3)} = 0$ [23–25] and $J_{ij}^{(0)} = J_{ij}^{(1)} = 0$ [26,27].

In this Letter, we consider the condition $J_{ij}^{(2S)} = J_{ij}^{(2S-2)}$ for spin liquefaction, which gives

$$\hat{H}_r^{(S)} = \sum_{ij}\sum_{s=0}^{2S} J_{ij}^{(s)} \hat{P}_{ij}^{(s)}\Big|_{J_{ij}^{(2S)} = J_{ij}^{(2S-2)}}, \tag{5}$$

where a subset of the eigensystem has a rigorous correspondence with the whole eigensystem in the spin-$\frac{1}{2}$ Heisenberg model $\hat{H}^{(1/2)} = \sum_{ij}(J_{ij}^{(2S)} - J_{ij}^{(2S-1)})\hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{s}}_j + \varepsilon_0$ with the $S = \frac{1}{2}$ operator $\hat{\boldsymbol{s}}_i$ and energy shift $\varepsilon_0 = \sum_{ij}(3J_{ij}^{(2S)} + J_{ij}^{(2S-1)})/4$. In short, for any eigenstate $|\psi\rangle$ of $\hat{H}^{(1/2)}$, the corresponding eigenstates of $\hat{H}_r^{(S)}$ are rigorously written as $|\Psi_0\rangle = \hat{C}|\psi\rangle$ with an intertwiner [28,29] $\hat{C} = \prod_{i=1}^{N}(|S\rangle_i\langle\uparrow| + |S-1\rangle_i\langle\downarrow|)$, which is a mapping operator from the spin-$\frac{1}{2}$ Hilbert space spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$, to the spin-S Hilbert space spanned by $|S\rangle, |S-1\rangle, \ldots, |-S\rangle$. The degeneracy in $\hat{H}_r^{(S)}$ is greater than that in $\hat{H}_r^{(1/2)}$ because of a ferromagnetic moment in $|\Psi_0\rangle$. The additional degenerate states are $|\Psi_s\rangle = (\hat{S}_{\rm tot}^{-})^s|\Psi_0\rangle$, where $\hat{S}_{\rm tot}^{\alpha} = \sum_i \hat{S}_i^{\alpha}$ is a total spin operator. This rigorous eigenstate correspondence is easily proved [30]. Note also that a numerical calculation is required to confirm that a ground state of $\hat{H}_r^{(S)}$ may also be written as $|\Psi_s\rangle$. For eigenstates, however, the correspondence is valid for a general lattice in any dimension, and even under a magnetic field.

The BLBQ chain, Eq. (1), is rewritten as $\hat{H}_\alpha^{(S)} = \sum_i \sum_{s=0}^{2S} J_{ii+1}^{(s)}(\alpha)\hat{P}_{ii+1}^{(s)}$, where $J_{ii+1}^{(s)}(\alpha) = q_s \cos\alpha + {q_s}^2 \sin\alpha$ based on Eq. (4) [32]. At the two points $\alpha = \alpha_r$ and $\alpha_r + \pi$, given by Eq. (2), the BLBQ chain satisfies the condition $J_{ii+1}^{(2S)}(\alpha) = J_{ii+1}^{(2S-2)}(\alpha)$. As a result, a subset of the eigensystem in $H_{\alpha_r}^{(S)}$ corresponds to the whole eigensystem in the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain. In addition, $\hat{H}_\alpha^{(S)}$ at α_r can be considered as a higher-S generalization of $\hat{H}_r^{(1)} = -\sum_i \hat{P}_{ii+1}^{(1)}$ at $\alpha_r = \pi/4$ in Fig. 1, which leads us to the spin-$\frac{1}{2}$ SU(2) model. This generalization is not the

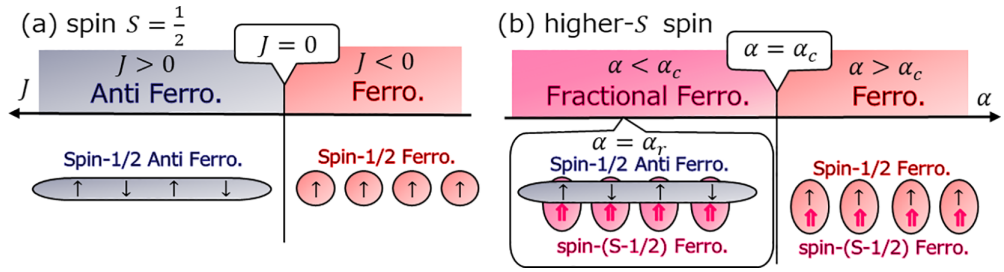
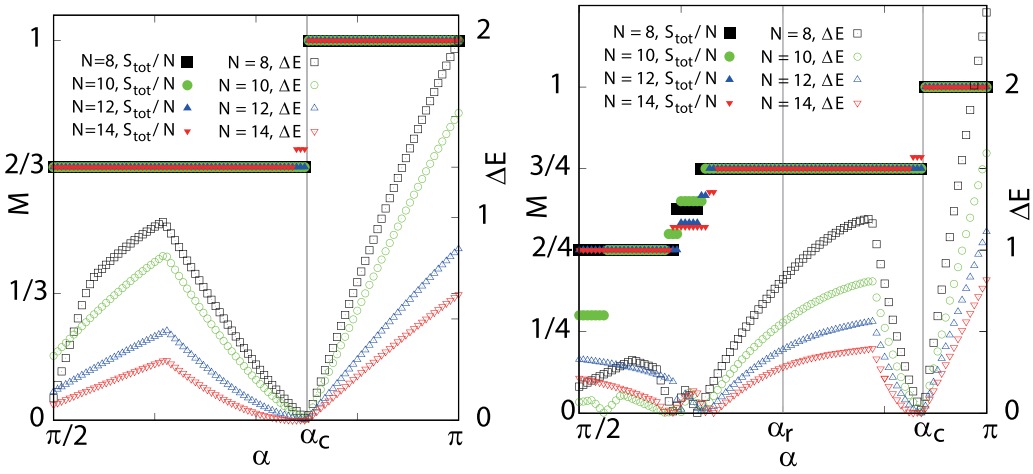


FIG. 2. (a) Phase transition from the ferromagnetic phase to antiferromagnetic phase in the $S = \frac{1}{2}$ Hamiltonian $J\sum_{i=1}^{N} \hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{s}}_{i+1}$. (b) Phase transition at α_c from the ferromagnetic $M = 1$ phase to the fractionally magnetized $M = 1 - \frac{1}{2S}$ phase in the higher-S BLBQ Hamiltonian $\hat{H}_\alpha^{(S)}$ described by Eq. (1). Rigorous ground-state correspondence with a spin-$\frac{1}{2}$ antiferromagnetic chain realized at α_r for $S \geqq 2$.



(a) $S = 3/2$ (b) $S = 2$

FIG. 3. Magnetization $M = \frac{S_{\rm tot}}{NS}$ and the energy gap ΔE for the spin-S BLBQ N-site chain Hamiltonian $\hat{H}_\alpha^{(S)}$ Eq. (1) in the $S^z_{\rm tot} = 0$ and $q = 0, \pi$ subspace. The phase transition from $M = 1$ to $M = 1 - 1/(2S)$ occurs at α_c corresponding to Eq. (3). (a) $S = 3/2$ and (b) $S = 2$.

usual SU$(2S+1)$ generalization with $J_{ij}^{(2S)} = J_{ij}^{(2S-2)} = \cdots = J_{ij}^{(0)}$, $J_{ij}^{(2S+1)} = J_{ij}^{(2S-1)} = \cdots = J_{ij}^{(1)}$ [10–12,24].

Similarly, as a higher-S generalization of $\hat{H}_c^{(1)} = \sum_i \hat{P}_{ij}^{(0)}$ at α_c in Fig. 1, let us introduce another limitation $J_{ij}^{(2S)} = J_{ij}^{(2S-1)}$ for the spin-projection Hamiltonian

$$\hat{H}_c^{(S)} = \sum_{ij}\sum_{s=0}^{2S} J_{ij}^{(s)} \hat{P}_{ij}^{(s)} \Bigg|_{J_{ij}^{(2S)} = J_{ij}^{(2S-1)} < J_{ij}^{(s)},\ (s \leqq 2S-2)}, \qquad (6)$$

which gives a phase boundary of the ferromagnetic phase ($J_{ij}^{(2S)} < J_{ij}^{(s)}$). For the BLBQ chain $\hat{H}_{\alpha_c}^{(S)}$, at the phase transition point α_c given by Eq. (3), the condition $J_{ii+1}^{(2S)}(\alpha) = J_{ii+1}^{(2S-1)}(\alpha) < J_{ii+1}^{(s)}(\alpha)$ is satisfied. For $S = \frac{1}{2}$, this is a quantum phase transition between the ferromagnetic and antiferromagnetic phases via a trivial Hamiltonian $\hat{H}_c^{(1/2)} = 0$, as shown in Fig. 2(a). In general, it is easy to check that the ferromagnetic state $|0\rangle = \prod_i |S\rangle_i$ and the one-magnon excited state $\hat{S}_i^- |0\rangle$ are ground states of $\hat{H}_c^{(S)}$ with eigenenergy $\sum_{ij} J_{ij}^{(2S)}$: that is, the one-magnon flat band is degenerate at the ground-state energy. In addition, other ground states are multisublattice Néel-like states, defined as $|m\rangle = (\prod_{i\in\mathcal{L}_m} \hat{S}_i^-)|0\rangle$ for any mth sublattice $\mathcal{L}_m$, where $|m\rangle$ has $S^z_{\rm tot} = NS - N_m$ and $N_m = |\mathcal{L}_m|$ is the number of mth sublattice sites. If a ground state for $J_{ij}^{(2S-1)} < J_{ij}^{(2S)}$ overlaps with $|m\rangle$, the ground state has $S^z_{\rm tot} = NS - N_m$ and $S_{\rm tot} \geqq S^z_{\rm tot}$, which becomes $S_{\rm tot} \geqq S^z_{\rm tot} = N(S - \frac{1}{2})$ for the BLBQ bipartite chain ($N_m = N/2$). It is naively expected that the magnetization jumps to $M = S_{\rm tot}/(NS) = (S - \frac{1}{2})/S$ from $M = 1$ at α_c, whereas numerical evidence is required because other states can be more stable.

To observe fractional magnetization for the spin-S BLBQ N-site chain Hamiltonian $\hat{H}_\alpha^{(S)}$ [Eq. (1)], we perform an exact diagonalization with the Lanczos method in the region $\pi/2 < \alpha < \pi$ up to $S = 3$ by using translational symmetry. Figure 3 shows the magnetization M of the ground state and energy gap ΔE of the first excited state in the subspace of $S^z_{\rm tot} = 0$ and wave number $q = 0$ or π for $S = 3/2$ and $S = 2$ and $N = 8$, 10, 12, and 14, which shows clear transitions at α_c [Eq. (3)]. The magnetization is fractionalized as $M = 1 - 1/(2S)$ in a certain region $\alpha < \alpha_c$. Here, $M = S_{\rm tot}/(NS)$ is calculated from $S_{\rm tot} = f(\langle \hat{\mathbf{S}}_{\rm tot} \cdot \hat{\mathbf{S}}_{\rm tot} \rangle)$ via $f(x) = (\sqrt{1+4x} - 1)/2$.

In most of the $M = (S - \frac{1}{2})/S$ phases in Fig. 3, wave vector q of the ground state depends on the system size N because $q = 0$ (π) for even (odd) $N/2$, while $q = 0$ for $\alpha > \alpha_c$. This even-odd effect of $N/2$ is consistent with that in the spin-$\frac{1}{2}$ Heisenberg chain [33]. In detail, in the vicinity of $\alpha \lesssim \alpha_c$ for large system size $N \geqq 14$, a state with $M = 1 - 1/(2S) + 1/(NS)$ has slightly lower energy than that with $M = 1 - 1/(2S)$, and the two states are almost degenerate, which reflects doubly degenerate $q = 0$ and $q = \pi$ modes in the thermodynamic limit ($N \to \infty$) [30].

At the rigorous point α_r, magnetization of the ground states becomes $M = 1 - 1/(2S)$ only for $S \geqq 2$ while $M \neq 1 - 1/(2S)$ for $S \leqq 3/2$. A main difference is whether the bilinear term in Eq. (1) is ferromagnetic ($S \geqq 2$) or antiferromagnetic ($S \leqq 3/2$). Since the eigenstate correspondence is rigorous for any S, the eigenstate of spin-$\frac{1}{2}$ liquefaction for $S \leqq 3/2$ can become stable under a magnetic field. For $S = 1$, the magnetization is $M = 0$ at $\alpha_r = \pi/4$, which is the critical point between the trimer and the Haldane phase [34], as shown in Fig. 1. However, a magnetic field induces a phase transition to the magnetized Haldane phase [35], which is known to have exact correspondence to the spin-$\frac{1}{2}$ model [36]. For general S, a rigorous correspondence between the ground state of the BLBQ model and that of the spin-$\frac{1}{2}$ antiferromagnetic model can be realized under an external magnetic field. For $S = 3/2$, magnetic-field-induced spin liquefaction occurs at $\alpha_r = \arctan(2) \simeq 0.35\pi$. However, this is left as a future problem.

The transition point α_c is at least the phase boundary of the fully magnetized ferromagnetic phase $M = 1$. The proof is simple because ground states and one-magnon excitation are written exactly [37]. As an exact result, the one-magnon band becomes flat at α_c, as is already known from spin-wave theory

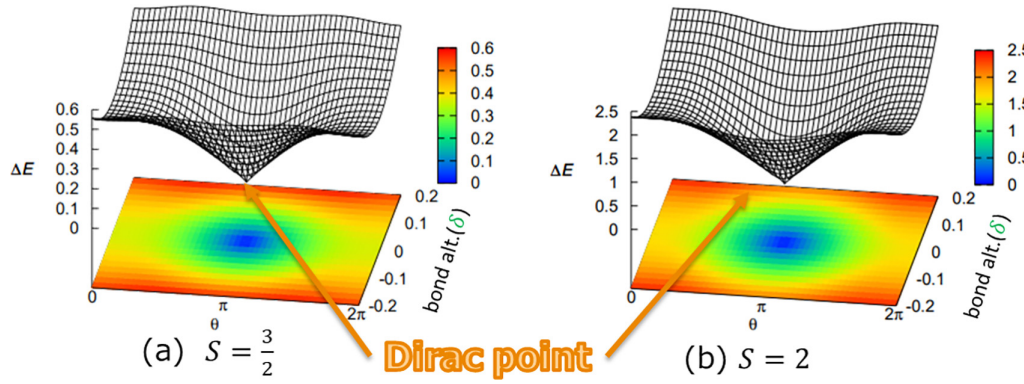


FIG. 4. Energy gap ΔE in the $S^z_{\rm tot} = N(S - \frac{1}{2})$ sector as a function of phase-twist angle θ and bond alternation δ at $\alpha_0 = \alpha_c - 2\pi \times 0.04$ ($\alpha_r < \alpha_0 \lesssim \alpha_c$) and system size $N = 8$ for (a) $S = 3/2$ and (b) $S = 2$.

[38]. Note that the continuous one-magnon excitation is not depicted in Fig. 3 because the energy gap ΔE is restricted in the sector $q = 0$ and π.

To confirm the thermodynamic limit under the existence of a finite-size gap, we adopt the twisted boundary condition [39] or quantized Berry phase [40], introducing $\hat{H}_{\alpha,\delta,\theta}$ with a bond alternation δ and boundary twist angle θ by using the δ-dependent coefficient $J^{(s)}_{i,i+1}(\alpha,\delta) = [1 + (-1)^i\delta]J^{(s)}_{i,i+1}(\alpha)$ and the θ-dependent boundary condition $\hat{S}^{\pm}_{N+1} = e^{\pm i\theta}\hat{S}^{\pm}_1$ and $\hat{S}^z_{N+1} = \hat{S}^z_1$. The energy gap ΔE in the sector for $S^z_{\rm tot} = N(S - \frac{1}{2})$ opens due to the finite system size $N = 8$ even in the uniform case ($\delta = 0$), while the finite gap closes under the twisted boundary condition ($\theta = \pi$) only at $\delta = 0$, as shown in Fig. 4 at $\alpha_0 = \alpha_c - 2\pi \times 0.04$ ($\alpha_r < \alpha_0 \lesssim \alpha_c$).

The result of Fig. 4 is identical to that of dimer singlets in a spin-$\frac{1}{2}$ dimerized Heisenberg chain. In the dimerized limit ($\delta = 1$), the unique ground state in the subspace for $S^z_{\rm tot} = N(S - \frac{1}{2})$ is given as a direct-product state of the two-site dimer $\prod_{i=1}^{N/2}(\hat{S}^-_{2i} - \hat{S}^-_{2i+1})|0\rangle$ for $\alpha < \alpha_c$ exactly [41]. Twist-angle θ dependence appears as $\hat{S}^-_N - \hat{S}^-_{N+1} = \hat{S}^-_N - e^{-i\theta}\hat{S}^-_1$ in the boundary dimer, while in the other dimerized limit ($\delta = -1$) the ground state $\prod_{i=1}^{N/2}(\hat{S}^-_{2i-1} - \hat{S}^-_{2i})|0\rangle$ does not depend on θ due to the absence of the boundary dimer. This difference of θ dependence results in the difference in Berry phase γ. The change in the quantized value $\gamma = 0, \pi$ is accompanied by the Dirac cone shown in Fig. 4. The twofold degenerate states at the Dirac point ($\theta = \pi$) adiabatically connect to two states separated by the finite-size gap at the periodic boundary condition ($\theta = 0$). These two states have $q = 0$ and π for the uniform case $\delta = 0$ depending on the even-odd parity of $N/2$. The scenario of the finite-size effect directly corresponds to the $S = \frac{1}{2}$ case, which is for the dimer-singlet state $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle = (\hat{S}^-_i - \hat{S}^-_{i+1})|\uparrow\uparrow\rangle = (\hat{S}^-_i - \hat{S}^-_{i+1})|0\rangle_{i,i+1}$ existing in the $S^z_{\rm tot} = N(S - \frac{1}{2}) = 0$ subspace; the finite-size gap disappears in the thermodynamic limit [42]. The Dirac point is observed in most of the $M = (S - \frac{1}{2})/S$ phases. However, an interesting discrepancy from the $S = \frac{1}{2}$ case occurs in the vicinity of $\alpha \lesssim \alpha_c$, where an additional Dirac cone appears at $\theta = 0$ and $\delta = \pm\delta_c$.

Apart from our numerical results on the chain, the general theory can be applied to previous studies on other lattices. On a square lattice [43], magnetic-field-induced spin-$\frac{1}{2}$ liquefaction of the $S = 1$ BLBQ model is realized. Moreover, on a $S = 1$ BLBQ triangular lattice [44], exact correspondence at $\alpha_r = \pi/4$ exists for $M \geqslant 2/3$; for example, the $M = 2/3$-plateau state must be regarded as the 1/3-plateau state of the spin-$\frac{1}{2}$ model and the $\uparrow\uparrow\downarrow$ state with spin-$\frac{1}{2}$ fully polarized in the $\uparrow\uparrow\uparrow$ background.

Generalizing $\hat{H}^{(S)}_c$ for the spin-$\frac{1}{2}$ liquefaction, it is naively expected that the spin-s liquefaction point is given by $J^{(2S)}_{ij} = J^{(2S-1)}_{ij} = \cdots = J^{(2S-2s)}_{ij} < J^{(m)}_{ij}$ ($m < 2S - 2s$) and perturbation from the point toward the other $2s + 1$ parameter space generates several phases, including the ferromagnetic phase ($M = 1$) and a fractionally magnetized phase ($M = 1 - \frac{s}{S}$).

In summary, we present herein the theory of entangled fractionally magnetized quantum states providing the viewpoint of spin liquefaction on a d-dimensional lattice. In the general discussion, the entangled states turn out to be antiferromagnetic entangled states in a ferromagnetic background. To address this fractional ferromagnet, the ferromagnetic region of the spin-S BLBQ chain was studied numerically. The fractional magnetization was revealed to have $M = 1 - 1/(2S)$ even under zero magnetic field; for example, $M = 2/3$ for $S = 3/2$, and $M = 3/4$ for $S = 2$. Numerous future problems remain. From a theoretical viewpoint, further calculations (using other numerical or analytical techniques) in the one-dimensional $S \geqq 3/2$ BLBQ model are required to clarify the magnetization curve as a function of external magnetic field, the boundary edge-spin problem (especially under open boundary conditions), the excitation spectrum as a function of q, and the entanglement entropy and spectrum. A more generic theoretical task is to establish the origin of the interaction in real materials or by optical-lattice experiments.

The spin-$\frac{1}{2}$ liquefaction at α_r opens up further discussion, for example, a comparison with ferrimagnetism [45]. In a ferrimagnet, spin-s and the S Hamiltonian break one-site translation symmetry because $s \neq S$, whereas a fractional ferromagnet holds that symmetry. The difference can induce anomalous low-energy excitations in the BLBQ model. In particular, the fractional ferromagnet at α_r exhibits linear magnon excitation, which reflects the twofold degeneracy of Néel-like states in the uniform Hamiltonian (i.e., the des Cloizeaux–Pearson mode [42]), and its existence is

guaranteed owing to the rigorous correspondence to the spin-$\frac{1}{2}$ antiferromagnetic chain.

As mentioned above, fractional ferromagnets are not conventional ferrimagnets. In addition, the fractional ferromagnetic state is not the classical ferromagnetic state near the quantum critical point [46]. Even after spontaneous magnetization, the ground state of a fractional ferromagnet has quantum entanglement corresponding to the spin-$\frac{1}{2}$ antiferromagnetic state. For the quantum entanglement in a fractional ferromagnet, the external magnetic field has the potential to be a tool to manipulate an entangled quantum state, which can be useful in the context of quantum computer science. From the viewpoint of condensed-matter physics, the key word "quantum magnet" has been used and accepted for antiferromagnets. Given that the present theory abolishes the prejudice that ferromagnetism is classical, quantum magnets will also be used for fractional ferromagnets.

To summarize, this Letter develops another frontier of quantum spin states (i.e., "quantum ferromagnet"), which opens the field not only in fundamental physics but also in quantum computer science.

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[1] F. D. M. Haldane, Nobel lecture: Topological quantum matter, Rev. Mod. Phys. **89**, 040502 (2017).
[2] F. D. M. Haldane, Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state, Phys. Rev. Lett. **50**, 1153 (1983).
[3] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Rigorous results on valence-bond ground states in antiferromagnets, Phys. Rev. Lett. **59**, 799 (1987).
[4] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. PapiAe, Weak ergodicity breaking from quantum many-body scars, Nat. Phys. **14**, 745 (2018).
[5] N. Shiraishi and T. Mori, Systematic construction of counterexamples to the eigenstate thermalization hypothesis, Phys. Rev. Lett. **119**, 030601 (2017).
[6] S. Moudgalya, S. Rachel, B. A. Bernevig, and N. Regnault, Exact excited states of nonintegrable models, Phys. Rev. B **98**, 235155 (2018).
[7] S. Moudgalya, N. Regnault, and B. A. Bernevig, Entanglement of exact excited states of Affleck-Kennedy-Lieb-Tasaki models: Exact results, many-body scars, and violation of the strong eigenstate thermalization hypothesis, Phys. Rev. B **98**, 235156 (2018).
[8] N. Shibata, N. Yoshioka, and H. Katsura, Onsager's scars in disordered spin chains, Phys. Rev. Lett. **124**, 180604 (2020).
[9] H. J. Mikeska and A. K. Kolezhuk, One-dimensional magnetism, in *Quantum Magnetism*, Lecture Notes in Physics Vol. 645 (Springer, Berlin, 2004).
[10] G. V. Uimin, One-dimensional problem for $S = 1$ with modified antiferromagnetic Hamiltonian, Sov. J. Exp. Theor. Phys. Lett. **12**, 225 (1970).
[11] C. K. Lai, Lattice gas with nearest-neighbor interaction in one dimension with arbitrary statistics, J. Math. Phys. **15**, 1675 (1974).
[12] B. Sutherland, Model for a multicomponent quantum system, Phys. Rev. B **12**, 3795 (1975).
[13] C. D. Batista, G. Ortiz, and J. E. Gubernatis, Unveiling order behind complexity: Coexistence of ferromagnetism and Bose-Einstein condensation, Phys. Rev. B **65**, 180402(R) (2002).
[14] I. Affleck, Exact critical exponents for quantum spin chains, non-linear σ-models at $\theta = \pi$ and the quantum Hall effect, Nucl. Phys. B **265**, 409 (1986).
[15] M. N. Barber and M. T. Batchelor, Spectrum of the biquadratic spin-1 antiferromagnetic chain, Phys. Rev. B **40**, 4621 (1989).
[16] M. T. Batchelor and M. N. Barber, Spin-S quantum chains and Temperley-Lieb algebras, J. Phys. A: Math. Gen. **23**, L15 (1990).
[17] A. Klümper, New results for q-state vertex models and the pure biquadratic spin-1 Hamiltonian, Europhys. Lett. **9**, 815 (1989).
[18] L. A. Takhtajan, The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins, Phys. Lett. A **87**, 479 (1982).
[19] H. M. Babujian, Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spins S, Phys. Lett. A **90**, 479 (1982).
[20] H. M. Babujian, Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model, Nucl. Phys. B **215**, 317 (1983).
[21] F. C. Alcaraz and M. J. Martins, Conformal invariance and critical exponents of the Takhtajan-Babujian models, J. Phys. A: Math. Gen. **21**, 4397 (1988).
[22] D. P. Arovas, A. Auerbach, and F. D. M. Haldane, Extended Heisenberg models of antiferromagnetism: Analogies to the fractional quantum hall effect, Phys. Rev. Lett. **60**, 531 (1988).
[23] P. Chen, Z. L. Xue, I. P. McCulloch, M. C. Chung, and S. K. Yip, Dimerized and trimerized phases for spin-2 bosons in a one-dimensional optical lattice, Phys. Rev. A **85**, 011601(R) (2012).
[24] P. Chen, Z. L. Xue, I. P. McCulloch, M. C. Chung, C. C. Huang, and S. K. Yip, Quantum critical spin-2 chain with emergent SU(3) symmetry, Phys. Rev. Lett. **114**, 145301 (2015).
[25] C. Li, V. L. Quito, E. Miranda, R. Pereira, I. Affleck, and P. L. S. Lopes, The case of SU(3) criticality in spin-2 chains, Phys. Rev. B **105**, 085140 (2022).
[26] D. Zheng, G. M. Zhang, T. Xiang, and D. H. Lee, Continuous quantum phase transition between two topologically distinct valence bond solid states associated with the same spin value, Phys. Rev. B **83**, 014409 (2011).
[27] J. Zang, H. C. Jiang, Z. Y. Weng, and S. C. Zhang, Topological quantum phase transition in an $S = 2$ spin chain, Phys. Rev. B **81**, 224430 (2010).
[28] P. D. Francesco and J.-B. Zuber, SU(N) lattice integrable models associated with graphs, Nucl. Phys. B **338**, 602 (1990).
[29] P. A. Pearce and Y.-K. Zhou, Intertwiners and A-D-E lattice models, Int. J. Mod. Phys. B **07**, 3649 (1993).

[30] The basic idea is that triplet (singlet) projection of spin-$\frac{1}{2}$ model corresponds to a spin-S operator $\hat{P}_{ij}^{(2S)} + \hat{P}_{ij}^{(2S-2)}$ ($\hat{P}_{ij}^{(2S-1)}$). The proof is based on the following formula, $(\hat{P}_{ij}^{(2S)} + \hat{P}_{ij}^{(2S-2)})|T_k\rangle_{ij} = |T_k\rangle_{ij}$, $\hat{P}_{ij}^{(2S-1)}|S_0\rangle_{ij} = |S_0\rangle_{ij}$, $(\hat{P}_{ij}^{(2S)} + \hat{P}_{ij}^{(2S-2)})|S_0\rangle_{ij} = 0$, $\hat{P}_{ij}^{(2S-1)}|T_k\rangle_{ij} = 0$, $\hat{P}_{ij}^{(s)}|T_k\rangle_{ij} = \hat{P}_{ij}^{(s)}|S_0\rangle_{ij} = 0$ $(s \leqslant 2S-3)$, for polarized triplet $|T_1\rangle_{ij} = |S\rangle_i|S\rangle_j = \hat{C}|\uparrow\rangle_i|\uparrow\rangle_j$, $|T_0\rangle_{ij} = \hat{C}\frac{|\uparrow\rangle_i|\downarrow\rangle_j + |\downarrow\rangle_i|\uparrow\rangle_j}{\sqrt{2}}$, $|T_{-1}\rangle_{ij} = \hat{C}|\downarrow\rangle_i|\downarrow\rangle_j$, and polarized singlet $|S_0\rangle_{ij} = \hat{C}\frac{|\uparrow\rangle_i|\downarrow\rangle_j - |\downarrow\rangle_i|\uparrow\rangle_j}{\sqrt{2}}$. The formula is valid not only for $S \geqslant 1$ but also for $S = \frac{1}{2}$ if we put $\hat{P}_{ij}^{(-1)} = 0$. Using the above formula, one can prove $\hat{H}_r^{(S)}\hat{C} = \hat{C}\hat{H}^{(1/2)}$. If $\hat{H}^{(1/2)}|\psi\rangle = \epsilon|\psi\rangle$, one finds $\hat{H}_r^{(S)}(\hat{C}|\psi\rangle) = \hat{C}\hat{H}^{(1/2)}|\psi\rangle = \epsilon(\hat{C}|\psi\rangle)$. More details of the proof and a demonstration are provided in Sec. 2 of the Supplemental Material [31].

[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.L140404 for demonstrations and details.

[32] The coefficients $J_{ij}^{(s)}$ as a function of one-parameter α are depicted in Sec. 1 of the Supplemental Material [31]. In addition, $J_{ij}^{(s)}(\alpha_r)$ and $J_{ij}^{(s)}(\alpha_c)$ are demonstrated for $S = 3/2, \ldots, 3$.

[33] Reference [42] showed that in the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain the ground state has $(q, S_{\rm tot}) = (0, 0)$ and the excited state has $(q, S_{\rm tot}) = (\pi, 1)$ for even $N/2$ and the ground state has $(q, S_{\rm tot}) = (\pi, 0)$ and the excited state has $(q, S_{\rm tot}) = (0, 1)$ for odd $N/2$. In the thermodynamic limit, these two states become degenerate, following the fact that the energy spectrum $\frac{\pi|J|}{2}|\sin q|$, that is, the des Cloizeaux–Pearson mode, becomes zero at $q = 0$ and π.

[34] A. Läuchli, G. Schmid, and S. Trebst, Spin nematics correlations in bilinear-biquadratic $S = 1$ spin chains, Phys. Rev. B **74**, 144426 (2006).

[35] S. R. Manmana, A. M. Läuchli, F. H. L. Essler, and F. Mila, Phase diagram and continuous pair-unbinding transition of the bilinear-biquadratic $S = 1$ Heisenberg chain in a magnetic field, Phys. Rev. B **83**, 184433 (2011).

[36] G. Fáth and P. B. Littlewood, Massless phases of Haldane-gap antiferromagnets in a magnetic field, Phys. Rev. B **58**, R14709 (1998).

[37] For $\alpha > \alpha_c$, the ground states are $(\hat{S}_{\rm tot}^-)^s|0\rangle$ $(s \in [0, 2NS])$, with eigenenergy $E_\alpha = NS^2(\cos\alpha + S^2\sin\alpha) = \langle 0|\hat{H}_\alpha^{(S)}|0\rangle$, which corresponds to the classical energy of the spin vector $\boldsymbol{S}_i \cdot \boldsymbol{S}_j = S^2\cos\theta$ for ferromagnetism $\theta = 0$. One-magnon exact excitation has eigenenergy $E_\alpha + W_\alpha(1 - \cos q)$ with q and the hopping element $W_\alpha = -2S[\cos\alpha + 2S(S-1)\sin\alpha]$. Excitation energy is positive due to $W_\alpha > 0$ for $\alpha > \alpha_c$. At α_c, $W_{\alpha_c} = 0$ means a one-magnon flat band. For $\alpha < \alpha_c$, a flipped band is realized due to $W_\alpha < 0$ and fully ferromagnetic states become excited states, that is, the end of a fully magnetized ferromagnetic phase $M = 1$.

[38] R. A. Muniz, Y. Kato, and C. D. Batista, Generalized spin-wave theory: Application to the bilinear-biquadratic model, Prog. Theor. Exp. Phys. **2014**, 083101 (2014).

[39] A. Kitazawa, Twisted boundary conditions of quantum spin chains near the Gaussian fixed points, J. Phys. A: Math. Gen. **30**, L285 (1997).

[40] Y. Hatsugai, Quantized Berry phases as local order parameters of quantum liquids, J. Phys. Soc. Jpn. **75**, 123601 (2006).

[41] Degenerate eigenstates of a two-site Hamiltonian at $\delta = 1$ written by the projections $\hat{P}_{2i,2i+1}^{(s)}$ are $|S_{2i,2i+1} = s, S_{2i,2i+1}^z = m\rangle_{2i,2i+1}$ with the eigenenergy $J_{2i,2i+1}^{(s)}$ and $2s+1$-fold degeneracy. For $\alpha < \alpha_c$, the minimum coefficient is $J_{2i,2i+1}^{(2S-1)}$ and the ground states are $|S_{2i,2i+1} = 2S-1, S_{2i,2i+1}^z = m\rangle_{2i,2i+1}$ $(m = -2S+1, m = -2S+2, \ldots, m = 2S-1)$ with the energy gap $J_{2i,2i+1}^{(2S)} - J_{2i,2i+1}^{(2S-1)} > 0$. The $S_{\rm tot}^z = N(S - \frac{1}{2})$ subspace considered in Fig. 4 has a unique ground state $\prod_{i=1}^{N/2}|S_{2i,2i+1} = 2S-1, S_{2i,2i+1}^z = 2S-1\rangle_{2i,2i+1} = C\prod_{i=1}^{N/2}(\hat{S}_{2i}^- - \hat{S}_{2i+1}^-)|0\rangle$, with a normalization constant C.

[42] J. des Cloizeaux and J. J. Pearson, Spin-wave spectrum of the antiferromagnetic linear chain, Phys. Rev. **128**, 2131 (1962).

[43] T. A. Tóth, A. M. Läuchli, F. Mila, and K. Penc, Competition between two- and three-sublattice ordering for $S = 1$ spins on the square lattice, Phys. Rev. B **85**, 140403(R) (2012).

[44] D. Yamamoto, C. Suzuki, G. Marmorini, S. Okazaki, and N. Furukawa, Quantum and thermal phase transitions of the triangular SU(3) Heisenberg model under magnetic fields, Phys. Rev. Lett. **125**, 057204 (2020).

[45] L. Néel, Propriétés magnétiques des ferrites; ferrimagnétisme et antiferromagnétisme, Ann. Phys. **12**, 137 (1948).

[46] M. Brando, D. Belitz, F. M. Grosche, and T. R. Kirkpatrick, Metallic quantum ferromagnets, Rev. Mod. Phys. **88**, 025006 (2016).