Letter

Global quench dynamics and the growth of entanglement entropy in disordered spin chains with tunable range interactions

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The nonequilibrium dynamics of disordered many-body quantum systems after a quantum quench unveils important insights about the competition between interactions and disorder, yielding, in particular, an interesting perspective toward the understanding of many-body localization. Still, the experimentally relevant effect of bond randomness in long-range interacting spin chains on their dynamical properties have so far not been investigated. In this Letter, we examine the entanglement entropy growth after a global quench in a quantum spin chain with randomly placed spins and long-range tunable interactions decaying with distance with power α. Using a dynamical version of the strong disorder renormalization group we find for $\alpha > \alpha_c$ that the entanglement entropy grows logarithmically with time and becomes smaller with larger α as $S(t) = S_p \ln(t)/(2\alpha)$. Here, $S_p = 2\ln 2 - 1$. We present results of numerical exact diagonalization calculations for system sizes up to $N \sim 16$ spins, in good agreement with the analytical results for sufficiently large $\alpha > \alpha_c \approx 1.8$. For $\alpha < \alpha_c$, we find that the entanglement entropy grows as a power law with time, $S(t) \sim t^{\gamma(\alpha)}$ with $0 < \gamma(\alpha) < 1$ a decaying function of the interaction exponent α.

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Introduction. Magnetic resonance experiments in doped semiconductors [1] motivated Anderson to address the issue of electron localization in disordered systems using a model of noninteracting electrons [2]. Later, Fleischman and Anderson [3] argued that short-range (SR) electron-electron interactions with localized single-particle states would preserve the existence of a localized phase for strong disorder. Localization in a many-body system was put on more rigorous footing in Refs. [4,5] using a perturbative approach, where the concept of many-body localization (MBL) was introduced. Since then, the study of MBL in disordered interacting systems has become a flourishing field. For recent reviews, see Refs. [6,7] and references therein.

In spin chains with random on-site potential and nonrandom long range (LR) power-law interactions strong evidence was found that a many-body localized phase persists, as long as the interactions fall-off faster than a certain critical power law$\alpha_c(E)$ [8–12]. In spin chains with SR bond-disorder and particle-hole symmetry at finite energy density a logarithmic divergence of the entanglement entropy (EE) with subsystem size n, $S \sim \ln n$ was found [13,14]. This was argued to characterize a so-called *quantum critical glass* (QCG). Recently, we found that this logarithmic divergence of the EE survives, both for the ground state and excited states, the introduction of random bond LR interactions, provided that their power-law decay has an exponent which exceeds a critical value $\alpha_c(E)$ [15,16]. The full characterization of the dynamical properties of such (marginally) localized phases in LR-coupled systems with bond disorder remains so far unexplored since previous studies focused rather on LR coupled systems with disordered random potential, even though bond-disordered LR interactions are ubiquitous in real quantum systems [2,17,18].

An insightful perspective of delocalization-localization transitions in random lattice spin models is provided by the entanglement entropy (EE) dynamics after a quantum quench. This has been widely used as a probe of many-body localization for both SR interacting archetypal random models [19–21], and LR interacting spin chains with random local magnetic fields [22–26]. For MBL systems with nearest neighbor interactions, it has been shown [19–21] that the EE grows logarithmically with time after a quantum quench, when starting in an unentangled high-energy state $S(t) \sim \ln(t)$. This EE growth continues until it reaches a saturation value, which is determined by the participation ratio of the initial state to the eigenstates of the subsystem [21]. In spin chains with LR deterministic interactions and random on-site potential numerical experiments found evidence for a power-law growth of EE with time, $S(t) \sim t^{1/\alpha}$ for some range of

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α [22,23]. On the other hand, a logarithmic growth of the EE was obtained for a model of Fermions with LR hoppings, LR interactions, and sufficiently strong random local fields for sufficiently large α [27]. More recently, it has been suggested that at the MBL transition in LR interacting spin models subject to random magnetic fields the EE grows also with a power law in time, albeit with a universal exponent $\delta \approx 0.33$ [28]. An analytical derivation of these results is still lacking, nor has the EE growth been studied in systems with sole bond randomness in the LR setup.

The strong disorder renormalization group procedure [7,29–33] has been extended to the RSRG-t scheme [32] to study the dynamics of EE in XX spin chains with random nearest-neighbor interactions, which are known to be in the quantum critical glass phase (QCG) [14], and more recently of Rényi entropies [34]. An ultra-slow dynamics was found thereby where the EE scales as $S(t) \sim \ln[\ln(t)]$. Whether random LR interactions change the EE dynamics in XX spin chains remains, however, unexplored.

To fill the gaps identified above, we will focus on the effect of LR bond randomness on the entanglement dynamics in XX spin chains, with the goal being to derive the growth of entanglement dynamics after a global quench, both analytically and numerically. To this end, we first introduce the model we will consider. Next, we will briefly review how the SDRG can be applied to these LR-coupled random spin chains [15,16]. We next present the RSRG-t scheme and show how to extend it to this LR-coupled model. We then present our main results, an analytical prediction for the EE growth, and numerical exact diagonalization results as function of time and exponent α. We then provide a detailed comparison of the results of both approaches and give our conclusions.

Model. In this Letter, we introduce an extension of the dynamical strong disorder renormalization group (SDRG), also known as RSRG-t [32,35,36] and apply it, together with numerical exact diagonalization (ED), to investigate the dynamics of the EE after a global quantum quench in a LR interacting XX spin chain with positional disorder, as defined by the Hamiltonian

$$H = \sum_{i<j} J_{ij}\left(S_i^x S_j^x + S_i^y S_j^y\right), \tag{1}$$

of N interacting $S = 1/2-$ spins, randomly placed at positions $\mathbf{r}_i$ on a lattice of length L and lattice spacing a with density $n = N/L$. The couplings between all pairs of sites i, j, are taken to be antiferromagnetic and long-ranged, decaying with a power law

$$J_{ij} = J_0|(\mathbf{r}_i - \mathbf{r}_j)/a|^{-\alpha}. \tag{2}$$

We consider open boundary conditions.

The entanglement properties of this model were previously investigated for both the ground state and generic excited eigenstates by means of SDRG and ED in Refs. [15,16]. It was found that its ground state is correctly captured by a random singlet phase, with a distribution of couplings which flows to a strong disorder fixed point (SDFP), as characterized by a finite dynamical exponent which was found to be related to the interaction power by $z = 2\alpha$ [15,37]. More recently, the eigenstates in the middle of the many-body spectrum of this model were studied [16]. A delocalized regime was found for $\alpha \leqslant \alpha_c$, characterized by an algebraic subvolume enhancement of the entanglement entropy with subsystem size n, $S \sim n^{\gamma(\alpha)}$, where $\gamma(\alpha) < 1$, which is equal to 1 only in the limit of infinite range coupling $\alpha = 0$. For $\alpha \geqslant \alpha_c \approx 1$ the eigenstates in the middle of the band were found to be marginally localized with a logarithmic scaling of the entanglement entropy with subsystem size $S_n \sim \ln(n)$, characteristic of a QCG phase.

SDRG. Let us recall how to apply the SDRG to the model Eq. (1) [15,37]. Since the pair (i, j) with largest coupling $J_{i,j}$ forms a singlet, we take the expectation value of the Hamiltonian in that particular singlet state. Within second-order perturbation theory in the couplings with all other spins this yields the renormalization rule for the couplings between all remaining spin pairs (l, m) in the LR antiferromagnetic (AFM) coupled XX model as [15,37]

$$\left(J_{lm}^x\right)' = J_{lm}^x - \frac{\left(J_{li}^x - J_{lj}^x\right)\left(J_{im}^x - J_{jm}^x\right)}{J_{ij}^x}. \tag{3}$$

In the SR spin chain the distribution of renormalized couplings gets wider at every RG step with a width W that diverges as the RG scale Ω is lowered, driven to the infinite randomness fixed point (IRFP) [7,29–33]. For LR couplings, however, the width W is found to saturate to the finite value $W = \Gamma = 2\alpha$ for the XX model. This characterizes the strong disorder fixed point (SDFP) [37]. For large number of spins $N \gg 1$, the resulting distribution function of renormalized couplings J converges at small RG scale Ω to [31]

$$P(J, \Omega) = \frac{1}{\Omega \Gamma_\Omega}\left(\frac{\Omega}{J}\right)^{1-1/\Gamma_\Omega}, \tag{4}$$

where $\Gamma_\Omega \rightarrow W = 2\alpha$ for $\Omega \rightarrow 0$.

RSRG-t. The time-dependent real-space renormalization group (RSRG-t) is an extension of the SDRG to nonequilibrium setups. RSRG-t is designed to capture the effective dynamics via iterative elimination of all degrees of freedom which oscillate at a particular RG step with highest frequency Ω. Thereby, the RG decimation does not project the spin pairs into singlet states, as in the SDRG case, but rather generates effective degrees of freedom which define the large-time dynamics of the system [32,35]. Via successive elimination of these fastest oscillating pair of spins on sites i and j, coupled by $J_{ij} = \Omega$, with $\Omega = \max\{J_{ij}\}$, which dominates the short time dynamics, RSRG-t yields an effective time-independent Hamiltonian which captures the longer time dynamics H_{eff}. In the presence of strong disorder, the frequency of the eigenmodes of the largest term in the Hamiltonian, is much larger than those of the undecimated spins. Hence, the sites i and j are seen by the remaining degrees of freedom as in a time-averaged state. The remaining degrees of freedom can then be treated perturbatively. In Ref. [36] the equivalence between this approach and the RSRG-X, the extension of the SDRG to excited eigenstates [38], was outlined and derived in the framework of Floquet high-frequency expansion.

Let us now apply this RSRG-t procedure to the Hamiltonian Eq. (1). In second-order perturbation theory we find that the couplings are renormalized as

$$(J_{lm})_r = J_{lm} - \frac{J_{il}J_{jm} + J_{im}J_{jl}}{J_{ij}}(P_1 + P_2) + \frac{(J_{li} + J_{lj})(J_{im} + J_{jm})}{J_{ij}}P_3 - \frac{(J_{li} - J_{lj})(J_{im} - J_{jm})}{J_{ij}}P_4, \qquad (5)$$

where we define the projectors associated to spins (i, j) as $P_\mu = |\mu\rangle\langle\mu|$, with $\mu = 1, 2, 3, 4$ and $|1\rangle = |\uparrow\uparrow\rangle$, $|2\rangle = |\downarrow\downarrow\rangle$, $|3\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|4\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ [36]. These RG rules correspond to the ones for the RSRG-X on the same model [16]. This is not surprising since the derivation of the effective Hamiltonian is equivalent [36].

Entanglement entropy growth after quantum quench. The entanglement dynamics is monitored by means of the EE $S(t) = -\mathrm{Tr}[\rho_A \ln(\rho_A)]$, where $\rho_A = \mathrm{Tr}_B(\rho)$ is the reduced density matrix of the the subsystem (A) when tracing over its complement (B). The fixed-point distribution of the couplings generated by the RSRG-t allows us to calculate the entanglement entropy growth with time t. Given an arbitrary bipartition of an infinite spin chain, the entanglement between the two parts of the bipartition is dominated by the oscillating pairs connecting the two parts of the system which formed at an RG scale $\Omega \sim \frac{1}{t}$. In Ref. [16] we showed that for $\alpha \geqslant \alpha_c$ the SDFP distribution Eq. (4) is obtained within the RSRG-X flow for excited eigenstates. To derive $S(t)$ it is thus sufficient to derive the number of singlets and entangled triplets n_Ω that formed at RG time $\Omega \sim 1/t$. This is found to be determined by the differential equation [39]

$$dn_\Omega = P(J = \Omega, \Omega)d\Omega. \qquad (6)$$

For $\Omega \to 0$ the SDFP distribution Eq. (4), yields $n_\Omega = 1/(2\alpha)\ln(\Omega)$. Since the only entanglement-generating mechanism is the decimation of pairs whose spins reside on opposite sides of the interface, one finally obtains with $\Omega \sim 1/t$,

$$S(t) = S_p \frac{1}{2\alpha} \ln(t). \qquad (7)$$

where S_p is the time-averaged entanglement contribution of a decimated pair of spins, which is found to be $S_p = 2\ln 2 - 1$ when the initial state is the Néel state [32], where only the singlet and the entangled triplet states are populated within the RSRG-t flow, which contribute to the entanglement entropy equally. For other initial states, the logarithmic growth of the EE Eq. (7) is expected to still hold for $\alpha > \alpha_c$, albeit with a different prefactor S_p.

Remarkably, we find that the obtained logarithmic EE growth, Eq. (7), is faster than the one obtained for the nearest-neighbor XX spin chain with random bonds after a global quench $S(t) \sim \ln[\ln(t)]$ [32]. A logarithmic increase with time was found previously in conventional MBL with SR interacting systems and random potential [19,20]. For Fermions with long-range hoppings and random local fields a logarithmic growth is obtained for $\alpha \geqslant \alpha_c \approx 2$ [27]. For spin chains with long-range (deterministic) interactions in the presence of random magnetic fields [22,23] for $\alpha \geqslant 1$ indications of a power-law increase with time $S(t) \sim t^{1/(\alpha)}$ were obtained in the MBL phase. Note, however, that for large $\alpha \gg 1$ and for the considered time range and system sizes in Refs. [22,23], this EE scaling $S(t) \sim \exp[1/(\alpha)\ln t]$ is hardly distinguishable from our result Eq. (7) $S(t) \sim (1/\alpha)\ln t$.

For $\alpha < \alpha_c$ the fixed-point of the coupling distribution is unknown. However, the excited eigenstates of this model were found to follow a subvolume law, with an algebraic growth of their EE with subsystem size n, $S \sim n^{\beta(\alpha)}$ with $0 < \beta(\alpha) < 1$ a decreasing function of α for $\alpha < 1$ with $\beta(\alpha = 0) = 1$ [16]. There, we traced the fact that it does not increase linearally with n, the volume law expected for extended states to the existence of localized regions, mostly dimers [16]. Repeating the argumentation above for this system with subvolume law, the half-chain EE at large time t is then expected to scale as $S(t) \sim t^{\gamma(\alpha)}$, where $\gamma(\alpha) \leqslant 1$ is a decreasing function of $\alpha < \alpha_c$. Moreover we can derive the scaling of the saturation value for the half-chain EE with the number of spins N and find

$$S(t \to \infty) = \ln 2(N/2)^{\gamma(\alpha)}. \qquad (8)$$

Exact diagonalization results. To check the validity of these analytical results, we use numerical exact diagonalization and examine the half-chain EE dynamics after a quench starting from a Néel state $|\psi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow \cdots\rangle$. Results are shown in Fig. 1 for different values of α. We consider two different system sizes $N = 12$ and $N = 16$ to trace finite-size effects. The density of spins is fixed to $N/L = 0.1$. Averaging was done over up to 1000 disorder realizations. We see that, for $\alpha > 1.8$, the EE for large times after a transient agrees with a logarithmic enhancement with time as it was obtained via RSRG-t. The prefactor is found to be a decaying function of α, consistent with our analytical prediction Eq. (7). For $\alpha = 1.9, 2, 3, 4$ Eq. (7) is in good agreement with ED. The analytical derivation is expected to become more precise with increasing $\alpha \gg 1$ where the corrections to the SDFP become smaller. Indeed, the standard deviation of the EE shows a strong decrease for larger times t, and decreases with larger number of spins N, which may be taken as confirmation of the convergence to a fixed point law at large times after the quench. We also observe oscillations, even after ensemble averaging, with an amplitude which increases with α and with a period only weakly depending on α. We have not found an explanation for such oscillations within our RSRG-t derivation.

For $\alpha \leqslant 1.8$ the EE is found to saturate more quickly, faster the smaller α. We observe a transient region where it grows faster than logarithmically, as a power-law in time $S(t) \sim t^{\gamma(\alpha)}$, where $\gamma(\alpha)$ is found to be a decreasing function of $\alpha \leqslant \alpha_c$. The dependence of the saturation value $S(t \to \infty)$ on the number of spins N is in quantitative agreement with the scaling obtained by applying the RSRG-X result of Ref. [16], Eq. (8). Remarkably, for $\alpha = \alpha_c \approx 1.8$, the exponent $\gamma(\alpha) = 0.34$ is similar to the observed universal exponent for MBL systems with random fields and long-range interactions at criticality [28].

Conclusion. Extending the strong disorder renormalization group (SDRG) to study quench dynamics, we find that the entanglement entropy (EE) grows with the time after a quench, starting from a nonentangled state for $\alpha > \alpha_c$ logarithmically with a prefactor, which is inversely proportional to α, Eq. (7), in good agreement with the results obtained by

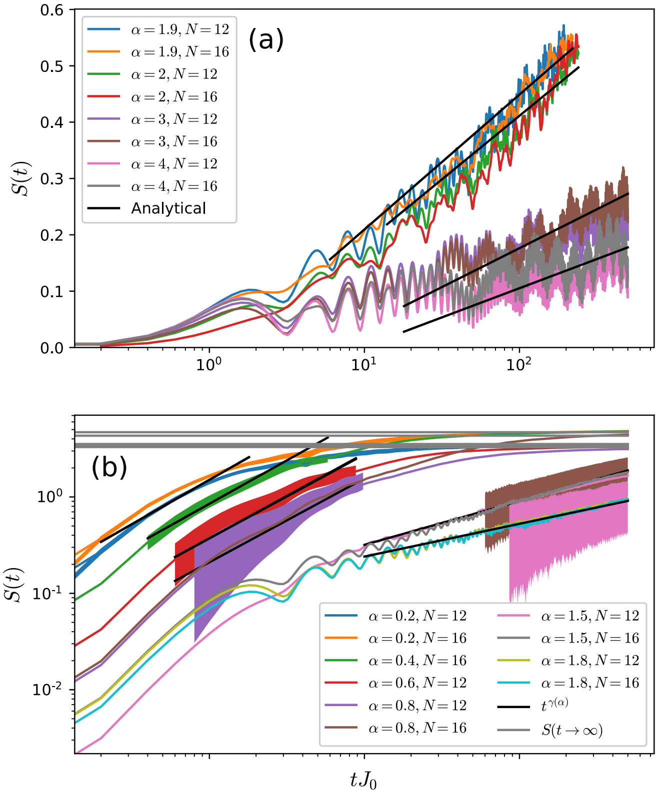

FIG. 1. (a) Half-chain EE as function of time for different values of $\alpha > \alpha_c \approx 1.8$ obtained via ED and compared with the analytical RSRG-t result Eq. (7) for $N = 12$, $N = 16$ spins, with density $N/L = 0.1$, coupling $J_0 = 1$ and up to 1000 disorder realizations. (b) Half chain EE as a function of time for $\alpha \leqslant \alpha_c$. After an initial transient time, the EE is found to grow as $S(t) \sim t^{\gamma(\alpha)}$ indicated by the solid black lines, reaching a limiting value $S(t \to \infty)$, Eq. (8). For $\alpha = 0.2, 0.4, 0.6, 0.8, 1.5, 1.8$ (grey lines), we find $\gamma = 0.92, 0.9, 0.88, 0.86, 0.45,$ and 0.34, respectively. The shaded regions show the standard deviation of the EE. We note the observation of oscillations, even after ensemble averaging, with an amplitude which increases with α and a period only weakly depending on α.

ED, Fig. 1. This logarithmic scaling with time differs from the one observed in SR XX spin chains with random bonds, where the EE grows logarithmically slower, as $S(t) \sim \ln\ln(t)$. The faster logarithmic growth of EE after a quench which we found here for exclusive bond randomness in LR couplings and in absence of magnetic fields could be a characteristic of quantum critical glasses with LR interactions. Indeed such logarithmic scaling of the EE with time was also observed in LR hopping fermions with random potential [27]. In spin chains with LR deterministic couplings and random on-site magnetic fields [22,23] indications of rather a power-law time dependence were found for $\alpha \gg 1$, but in the time and α range studied there by ED, there could be agreement with our result for the random bonded LR spin chains, Eq. (7), as well, but still needs to be checked. For $\alpha < \alpha_c$ delocalized eigenstates were found to exist previously [16], yielding to a power-law increase of the EE with subsystem size. Building on these results we derived the scaling for the large time saturation value of the half-chain EE, Eq. (8) with the number of spins N as $(N/2)^\gamma$ with $0 < \gamma(\alpha) < 1$ a decreasing function of α, in good agreement with our numerical exact diagonalization results, Fig. 1.

In conclusion, the entanglement dynamics is found to reveal different results for short-range [20] and long-range interacting random spin systems [22,23], while the EE scaling with subsystem size [15] and the energy-level spacing statistics [16] reveal similar results for random bond SR- and LR-coupled spin chains. Exploring the phase diagram of long-range interacting spin chains subject to both random on-site potentials and random bonds would help provide a more complete picture of this quantum entanglement dynamics. Recent advances in experimental setups allow to study XX spins with interactions that fall off as $1/r^3$, which was demonstrated by coupling Rydberg states with opposite parity [18,40,41]. Within this setup and exploiting the particle fluctuation and correlation technique [42,43] its nonequilibrium dynamics could thus be studied for $\alpha = 3.0$. Chains of trapped ions with power-law interactions, decaying as $1/r^\alpha$, with tunable $0 < \alpha < 1.5$ have also been realized recently [44–46], and may thereby open an experimental route to detect the quantum phase transition between phases with logarithmic and power-law growth of entanglement entropy.

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