Letter

Spectral theorems for generalized Weyl nodes with impurities in a magnetic field

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We prove a few spectral theorems for the density of states of a Weyl node with arbitrary topology. We show that the density of extended states of a Weyl node with random impurity potentials remains gapless in the presence of a magnetic field. Therefore, a magnetic field precludes Anderson localization in Weyl semimetals, when internode transitions are suppressed for smooth enough potentials. We also provide a rigorous quantum mechanical proof of the chiral magnetic effect for arbitrary topology of a Weyl node.

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Introduction. Weyl semimetals, where certain band crossing points feature a topological character which is similar to relativistic massless fermions, provide us with an interesting bridge between condensed-matter and high-energy physics. While relativistic Weyl fermions [1] obey the constraints from Lorentz invariance, e.g., the alignment of spin and momentum, the quasiparticles in Weyl semimetals are not encumbered with such restrictions and can possess more diverse topological character [2,3]. Some of the most interesting physics stemming from nontrivial topology, such as chiral anomaly [4,5] and chiral transport phenomena [6–8], therefore find themselves naturally generalized in Weyl semimetals; a notable example is the chiral magnetic effect [9–12].

In condensed-matter systems, impurities due to imperfect lattices are common, and how they affect the spectral and transport properties of Weyl nodes has been a subject of great interest. It has been proposed that as the impurity strength increases, there is a quantum phase transition from the semimetal to the diffusive metal phases [13], where the density of states at the nodal point becomes nonzero [14–20] and a non-Fermi-liquid behavior emerges [21]. However, it was also pointed out that the rare region effects could make the transition a crossover instead [22–26] (see Refs. [27–30] for the current status of the issue). For a Dirac semimetal, i.e., two degenerate Weyl points with opposite topology, a further increase of impurity was shown to cause a three-dimensional (3D) Anderson metal-insulator transition [31]. Whether there is a similar transition, when each Weyl node is separated in momentum space and internode couplings are suppressed for smooth potentials, is an interesting question [32–34]. Obviously, the topology of the Weyl node is at the heart of these questions.

In this work, we hope to shed light on these questions by studying the spectral property of a single Weyl node with impurities in a nonzero external magnetic field. A nontrivial interplay of nodal topology and magnetic field is well known in the physics of chiral anomaly and the chiral magnetic effect [35,36], which offers a promising way to study the fate of topology in the presence of impurities.

The main spectral theorem and its consequences. We first derive the main spectral theorem that our subsequent discussions will rely upon. We consider a system described by a noninteracting quasiparticle, which carries a charge q for electromagnetic interactions with an external vector potential $\boldsymbol{A}$. We assume that the energy spectrum of the quasiparticle is described by a one-particle Hamiltonian $\hat{H}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \boldsymbol{p} - q\boldsymbol{A}$ is the gauge-invariant momentum. This includes the lattice models where $\boldsymbol{A}$ appears as Wilson lines in the hopping matrix and $\boldsymbol{p}$ is the lattice momentum. We also assume the thermodynamic limit of a large volume, $V \to \infty$. This means that we are focusing on the bulk spectral properties and ignoring boundary conditions. Therefore, our subsequent discussions and results apply only to the bulk properties of the system.

The primary object we consider is the spectral density in energy, weighted by the current operator, e.g., along the z direction,

$$\rho_J(\varepsilon) \equiv \lim_{V\to\infty} \frac{1}{V}\mathrm{Tr}[\hat{J}_z\delta(\hat{H} - \varepsilon)], \tag{1}$$

where the current operator is defined by $\hat{J}_z = -\frac{\partial \hat{H}(\boldsymbol{A})}{\partial A_z} = q\frac{\partial \hat{H}}{\partial \hat{p}_z}$, with a constant, auxiliary value of the z component of the vector potential $\boldsymbol{A}$, and $\delta(x)$ is the Dirac's delta function or any smooth function that is sufficiently narrow around $x = 0$ with unit area. Our subsequent results do not depend on the details of the shape of this function. Because of $\delta(\hat{H} - \varepsilon)$, only the energy eigenstates with the eigenvalues close to ε contribute to $\rho_J(\varepsilon)$. This ensures that $\rho_J(\varepsilon)$ is finite and well defined, even in the case where the spectrum of $\hat{H}$ has no lower or upper bounds in energy. Since it can be shown, in general, that $\langle \hat{J}_z \rangle = 0$ for a bound state, $\rho_J(\varepsilon)$ captures only the unbound extended states in the continuum.

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Our main theorem is that $\rho_J(\varepsilon)$ is a constant, independent of ε:

$$\text{Theorem 1}: \quad \rho_J(\varepsilon) = C_J \text{ (const)}. \tag{2}$$

We prove the theorem by showing that $\frac{\partial}{\partial\varepsilon}\rho_J(\varepsilon) = 0$, under reasonable assumption of finiteness of the density of states in an energy interval under consideration,

$$\rho(\varepsilon) \equiv \lim_{V\to\infty} \frac{1}{V}\text{Tr}[\delta(\hat{H} - \varepsilon)]. \tag{3}$$

We have $\frac{\partial}{\partial\varepsilon}\rho_J(\varepsilon) = -\lim_{V\to\infty}\frac{1}{V}\text{Tr}[\hat{J}_z\delta'(\hat{H}-\varepsilon)]$ and, using the definition of $\hat{J}_z = -\frac{\partial\hat{H}}{\partial A_z}$, we arrive at $\frac{\partial}{\partial\varepsilon}\rho_J(\varepsilon) = \frac{\partial}{\partial A_z}\rho(\varepsilon, \boldsymbol{A})$, where the dependence on $\boldsymbol{A}$ is implicit in the Hamiltonian $\hat{H}(\hat{\boldsymbol{\pi}})$. We now utilize the following observation: for any eigenstate $\psi(\boldsymbol{x})$ of energy ε of the Hamiltonian $\hat{H}(\hat{\boldsymbol{\pi}})$, the state $e^{i\boldsymbol{\alpha}\cdot\boldsymbol{x})/\hbar}\psi(\boldsymbol{x})$ with a constant $\boldsymbol{\alpha}$ is an eigenstate of the Hamiltonian $\hat{H}(\hat{\boldsymbol{\pi}} - \boldsymbol{\alpha})$ with the same energy. This corresponds to a shift of $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\alpha}/q$, which implies that the spectrum of $\hat{H}$, and hence the spectral density $\rho(\varepsilon)$, remains invariant under a constant shift of $\boldsymbol{A}$. This proves that $\frac{\partial}{\partial\varepsilon}\rho_J(\varepsilon) = \frac{\partial}{\partial A_z}\rho(\varepsilon, \boldsymbol{A}) = 0$.

Although the theorem and its proof are simple, the implications are rich. Suppose that the quasiparticles are fermions, the system has a chemical potential μ at zero temperature, and all eigenstates of energy $\varepsilon \leqslant \mu$ are occupied. The current density, i.e., the current per unit volume, of the system is then given by $j_z = \int_{-\infty}^{\mu}\rho_J(\varepsilon)d\varepsilon$, which can be shown by

$$j_z = \frac{1}{V}\sum_{\alpha}\langle\psi_\alpha|\hat{J}_z|\psi_\alpha\rangle\Theta(\mu - \varepsilon_\alpha), \tag{4}$$

where $|\psi_\alpha\rangle$ are the energy eigenstates with energy ε_α, labeled by α, and the fact that $\rho_J(\varepsilon)$ can be written as

$$\rho_J(\varepsilon) = \frac{1}{V}\sum_{\alpha}\langle\psi_\alpha|\hat{J}_z|\psi_\alpha\rangle\delta(\varepsilon_\alpha - \varepsilon). \tag{5}$$

The theorem then implies that $j_z = C_J\int_{-\infty}^{\mu}d\varepsilon$, which is infinite, unless $C_J = 0$.

Although the appearance of infinity for j_z looks troublesome, it is in fact expected for the following reason. Suppose the spectrum of $\hat{H}$ has an upper or lower bound. If ε is outside of these bounds, there is no state to contribute to $\rho_J(\varepsilon)$, and $\rho_J(\varepsilon) = 0$ for such ε. The theorem then implies that $\rho_J(\varepsilon) = 0$ for all ε, i.e., $C_J = 0$. Therefore, a nonvanishing C_J is possible only if the spectrum has no upper or lower bounds. An example that we will discuss in more detail later is provided by the spectrum of a relativistic Weyl fermion. We will also consider an exception to the statement, when the density of states $\rho(\varepsilon)$ is allowed to possess a nonintegrable singularity in ε. The necessity of an unbounded spectrum for nonvanishing $\rho_J(\varepsilon)$ can also be understood by considering $N(\varepsilon) \equiv \lim_{V\to\infty}\frac{1}{V}\text{Tr}[\Theta(\varepsilon - \hat{H})]$, where $\Theta(x)$ is the step function, which counts the number of states whose energy is less than ε. The similar steps as above show $\rho_J(\varepsilon) = \frac{\partial}{\partial A_z}N(\varepsilon) = 0$ since the energy spectrum, and hence $N(\varepsilon)$, do not depend on A_z. What saves a nonvanishing $\rho_J(\varepsilon)$ for the unbounded spectrum is that $N(\varepsilon)$ is infinite and ill defined.

The above discussion leads to the following corollary of the theorem, which is an alternative proof of the Bloch's theorem [37,38]: *The system with a lower bound in spectrum has no persistent current in any thermodynamic ensemble*. The proof relies on the fact that any thermodynamic ensemble is defined by the occupation number of one-particle states, which depends only on its energy eigenvalue ε. For example, a grand-canonical ensemble of temperature T and chemical potential μ gives the current density,

$$j_z = \frac{1}{V}\sum_{\alpha}\langle\psi_\alpha|\hat{J}_z|\psi_\alpha\rangle n_F(\varepsilon_\alpha) = \int_{-\infty}^{\infty}n_F(\varepsilon)\rho_J(\varepsilon)d\varepsilon, \tag{6}$$

where $n_F(x) = (1 + e^{(x-\mu)/(k_BT)})^{-1}$. A lower bound on the spectrum implies $C_J = 0$, i.e., $\rho_J(\varepsilon) = 0$ for all ε.

When the spectrum has no bounds, and $C_J \neq 0$, a proper way to proceed is to first define the ground state (or vacuum state) of the system, e.g., the state at $T = \mu = 0$ where all one-particle states of $\varepsilon \leqslant 0$ are occupied, and the current density j_z is measured with respect to the value of the ground state. This gives, for example, in the grand-canonical ensemble, $j_z = \int_{-\infty}^{\infty}[n_F(\varepsilon) - \Theta(-\varepsilon)]\rho_J(\varepsilon)d\varepsilon = C_J\mu$, which is linear in μ and, more interestingly, independent of T. This leads to the following corollary: *A persistent current in the grand-canonical ensemble, if it is nonvanishing, is strictly linear in μ and independent of T*. The theorem and its consequences do not assume any details of the Hamiltonian or external conditions.

The theorem implies a strong robustness of $\rho_J(\varepsilon)$ under any reasonable perturbation. Since $\rho_J(\varepsilon) = C_J$ is independent of ε, the constant C_J is a property of the one-particle states with arbitrarily large energy ε, and therefore its value is robust under any perturbation that could affect only the states with a finite ε. This is a nontrivial conclusion since a perturbation in general may affect the states of small ε in significant ways, e.g., the density of states and the current expectation values will be substantially modified for small ε. Yet, we find that $\rho_J(\varepsilon)$ should remain the same for *all* energy ε since it should be the same for $\varepsilon \to \infty$. We will establish this robustness more rigorously in the following discussions.

A toy example. To illustrate the theorem and its consequences in a simple example, we consider a one-dimensional system described by the Hamiltonian

$$\hat{H} = f(\hat{p}_z) + V_0(z) \equiv \hat{H}_0 + V_0(z), \tag{7}$$

where $f(\hat{p}_z)$ is an arbitrary function on the momentum operator $\hat{p}_z = -i\hbar\partial_z$, and $V_0(z)$ is a potential. Let us ignore the potential for a moment and consider the spectrum of $\hat{H}_0$, which is parametrized by the momentum eigenvalue p_z of the eigenstate $\psi_{p_z}(z) = e^{-ip_z z/\hbar}$, i.e., $\varepsilon(p_z) = f(p_z)$. The density of states in momentum space is $\frac{L}{2\pi\hbar}dp_z$, where L is the size of the system. Then, the density of states in energy is $\rho(\varepsilon) \equiv \frac{1}{L}\text{Tr}[\delta(\hat{H}_0 - \varepsilon)] = \frac{1}{2\pi\hbar}\int dp_z\delta[f(p_z) - \varepsilon] = \frac{1}{2\pi\hbar}\sum_\alpha\frac{1}{|f'(p_\alpha)|}$, where p_α are the solutions of $f(p_\alpha) = \varepsilon$. The current operator is $\hat{J}_z = qf'(\hat{p}_z)$, and our current weighted spectral density is $\rho_J(\varepsilon) = \frac{1}{L}\text{Tr}[\hat{J}_z\delta(\hat{H}_0 - \varepsilon)] = \frac{q}{2\pi\hbar}\sum_\alpha\frac{f'(p_\alpha)}{|f'(p_\alpha)|}$. Although $\rho(\varepsilon)$ depends on both ε and the shape of $f(p_z)$, it is easy to see that $\rho_J(\varepsilon)$, which counts the number of signed crossings of the curve $f(p_z)$ at energy ε, is constant in ε. It coincides with a well-known topological index of the one-dimensional Hamiltonian.

When the potential is present, p_z is no longer a good quantum number. The problem is still exactly solvable if $\hat{H}_0 = \hat{p}_z$. The energy eigenstates are $\psi_\varepsilon(z) = e^{\frac{i}{\hbar}[p_z z - \int_{z_0}^{z} V_0(z')dz']}$, with the spectrum $\varepsilon(p_z) = p_z$. The current operator is an identity $\hat{J}_z = q\frac{\partial \hat{H}}{\partial \hat{p}_z} = q$, and we find $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar}$, which is constant, independent of the potential $V_0(z)$. For a general $f(p_z)$ without bounds, i.e., $|f(p_z)| \to \infty$ as $p_z \to \pm\infty$, we can find the eigenstates of large energy in the eikonal approximation, $\psi_{p_z}(z) \sim e^{\frac{i}{\hbar}[p_z z + \phi(z)]}$, where $\phi(z) = -\frac{1}{f'(p_z)}\int_{z_0}^{z} V_0(z')dz'$, and the spectrum is $\varepsilon(p_z) = f(p_z)$ which defines p_z given ε. The small parameter for the approximation is $V_0(z)/f(p_z) \to 0$ as $p_z \to \infty$, and the approximation becomes exact in the $\varepsilon \to \infty$ limit. The current operator is $\hat{J} = qf'(\hat{p}_z)$, with the expectation value $\langle\psi_{p_z}|\hat{J}_z|\psi_{p_z}\rangle = qf'(p_z)$, up to the same corrections of $V_0(z)/f(p_z) \to 0$ in the eikonal limit. We then find that $\rho_J(\varepsilon)$ is given by the same expression, $\frac{q}{2\pi\hbar}\sum_\alpha \frac{f'(p_\alpha)}{|f'(p_\alpha)|}$, as $\varepsilon \to \infty$, and hence for all ε according to our theorem, irrespective of the potential $V_0(z)$.

As a corollary, our result proves *the absence of Anderson localization in one-dimensional systems if the kinetic operator dispersion relation behaves at large momentum as* $f(p_z) \sim p_z^{2n+1}$, where $n = 0, 1, 2, \ldots$, since we have $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar} \neq 0$, which implies the absence of a gap in the spectrum of extended states with random potentials. In particular, the $n = 0$ case corresponds to a relativistic 1D Weyl node.

Discussion on a counter example. In this short digression, we discuss the following counter example in one-dimensions: $\hat{H} = \varepsilon_0 \tanh(\hat{p}_z)$, where ε_0 is a constant. For the interval $|\varepsilon| < \varepsilon_0$, the spectrum is a monotonically increasing function of p_z and we have $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar}$, while outside of the interval, no states exist and $\rho_J(\varepsilon) = 0$. What causes the discontinuity in $\rho_J(\varepsilon)$ across $\varepsilon = \pm\varepsilon_0$ is the infinite number of states accumulated around the energy $\pm\varepsilon_0$. The density of states is $\rho(\varepsilon) = \frac{1}{2\pi\hbar\tanh'[\tanh^{-1}(\varepsilon/\varepsilon_0)]} = \frac{1}{2\pi\hbar[1-(\varepsilon/\varepsilon_0)^2]}$, which is nonintegrable at $\pm\varepsilon_0$. The theorem applies only when the density of states is a finite function in the energy interval of interest.

In the following, we discuss our main application of Theorem 1; we will compute the constant $C_J = \rho_J(\varepsilon)$ for a Weyl node of arbitrary topology in the presence of a magnetic field and impurities.

A clean Weyl node in a magnetic field. Let us first ignore impurities temporarily, and compute C_J for a clean Weyl node in the presence of a magnetic field. Our main technique is based on the following observation: since $\rho_J(\varepsilon) = C_J$ is independent of ε, we can express C_J by

$$C_J = \frac{1}{\sqrt{\pi}M}\int_{-\infty}^{\infty} d\varepsilon\, \rho_J(\varepsilon) e^{-\frac{\varepsilon^2}{M^2}} = \frac{1}{\sqrt{\pi}M}\frac{1}{V}\mathrm{Tr}\Big(\hat{J}_z e^{-\frac{\hat{H}^2}{M^2}}\Big), \tag{8}$$

for any $M > 0$. The $M \to 0$ limit reproduces the definition of $\rho_J(\varepsilon)$, while the $M \to \infty$ limit is similar to the common strategy in the proof of the index theorems.

Before considering the most general form of a Weyl node, let us consider a concrete example of the simplest (relativistic) Weyl Hamiltonian in the presence of a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is the unit vector in the z direction,

$$\hat{H} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}} = \sigma_x\hat{p}_x + \sigma_y(\hat{p}_y - qBx) + \sigma_z\hat{p}_z, \tag{9}$$

where σ_i are the Pauli matrices, and we work in the Landau gauge $\boldsymbol{A} = (0, Bx, 0)$. In this case, we are able to compute the right-hand side of Eq. (8) exactly for any M to find that it is, indeed, M independent, which demonstrates the validity of our Theorem 1. We have $\hat{H}^2 = \hat{H}_0 + \hat{H}_I$, where $\hat{H}_0 = (\hat{p}_x)^2 + (\hat{p}_y - qBx)^2 + (\hat{p}_z)^2$ and $\hat{H}_I = -\hbar qB\sigma_z$. Since $\hat{J}_z = q\sigma_z$, Eq. (8) becomes

$$C_J = \frac{q}{\sqrt{\pi}M}\frac{1}{V}\mathrm{Tr}\big(\sigma_z e^{-\frac{1}{M^2}(\hat{H}_0 - q\hbar B\sigma_z)}\big). \tag{10}$$

To evaluate this, first note that $\hbar qB\sigma_z$ commutes with $\hat{H}_0$ and that $e^{\frac{q\hbar B\sigma_z}{M^2}} = \cosh(\frac{q\hbar B}{M^2}) + \sigma_z\sinh(\frac{q\hbar B}{M^2})$. Taking the spin trace, we obtain

$$C_J = \frac{q}{\sqrt{\pi}M}\frac{2\sinh\left(\frac{q\hbar B}{M^2}\right)}{V}\mathrm{Tr}\big(e^{-\frac{\hat{H}_0}{M^2}}\big). \tag{11}$$

To evaluate the remaining trace in position-momentum space, we note that $\hat{H}_0$ is the classical Landau-level problem for a relativistic scalar particle. The eigenfunctions are given by $|\psi_{n,p_y,p_z}\rangle = e^{\frac{i}{\hbar}(p_y y + p_z z)}|\phi_{n,p_y}\rangle$, where $|\phi_{n,p_y}\rangle$ are the 2D Landau wave functions and the eigenvalues are $E_{n,p_z} = 2q\hbar B(n + 1/2\ ) + p_z^2$, with the degeneracy per unit transverse area $g_\perp = \frac{qB}{2\pi\hbar}$. From these, we obtain

$$\mathrm{Tr}\big(e^{-\frac{\hat{H}_0}{M^2}}\big) = \frac{qVB}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} dp_z e^{-\frac{p_z^2}{M^2}}\sum_{n=0}^{\infty} e^{-\frac{2q\hbar B}{M^2}(n+\frac{1}{2})}, \tag{12}$$

which is evaluated to be $\frac{qVB\sqrt{\pi}M}{(2\pi\hbar)^2}\frac{1}{2\sinh(q\hbar B/M^2)}$. Using this in Eq. (11), we indeed find the M-independent result,

$$C_J = q^2B/(2\pi\hbar)^2. \tag{13}$$

We now consider the most general form of a Weyl node in the presence of a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$,

$$\hat{H} = \boldsymbol{F}(\hat{\boldsymbol{\pi}})\cdot\boldsymbol{\sigma}, \tag{14}$$

where $\boldsymbol{F}(\hat{\boldsymbol{\pi}})$ is an arbitrary vector valued function in $\hat{\boldsymbol{\pi}}$. We will use Eq. (8) in the $M \to \infty$ limit to compute C_J since it is not possible to solve the eigenvalue problem exactly. Using $[\hat{\pi}_x, \hat{\pi}_y] = i\hbar qB$ and $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, we have $\hat{H}^2 = \boldsymbol{F}^2(\hat{\boldsymbol{\pi}}) - \hbar qB(\frac{\partial\boldsymbol{F}}{\partial\hat{\pi}_x}\times\frac{\partial\boldsymbol{F}}{\partial\hat{\pi}_y})\cdot\boldsymbol{\sigma} \equiv \hat{H}_0 + \hat{H}_I$. In obtaining this, we neglected the terms arising from position-momentum commutators since those terms possess less powers of momentum than the above leading term. As we will see, a finite contribution in the $M \to \infty$ limit comes only from the $\boldsymbol{p} \to \infty$ region, and the terms with less powers of momentum become irrelevant compared to the leading term. Following similar steps in the proofs of the index theorems, we invoke the heat kernel expansion,

$$e^{-\frac{\hat{H}^2}{M^2}} = e^{-\frac{\hat{H}_0}{M^2}}\left(1 - \frac{1}{M^2}\int_0^1 dt\, e^{t\frac{\hat{H}_0}{M^2}}\hat{H}_I e^{-t\frac{\hat{H}_0}{M^2}}\right) + O\left(\frac{1}{M^4}\right). \tag{15}$$

Since $\hat{J}_z = q(\frac{\partial\boldsymbol{F}}{\partial\hat{p}_z})\cdot\boldsymbol{\sigma}$, the first nonvanishing trace in spin space appears with the second term in the expansion, and we find

$$C_J = \frac{2\hbar q^2B}{\sqrt{\pi}M^3}\frac{1}{V}\mathrm{Tr}\left[e^{-\frac{\boldsymbol{F}^2(\hat{\boldsymbol{\pi}})}{M^2}}\left(\frac{\partial\boldsymbol{F}}{\partial\hat{\pi}_x}\times\frac{\partial\boldsymbol{F}}{\partial\hat{\pi}_y}\right)\cdot\frac{\partial\boldsymbol{F}}{\partial\hat{p}_z}\right], \tag{16}$$

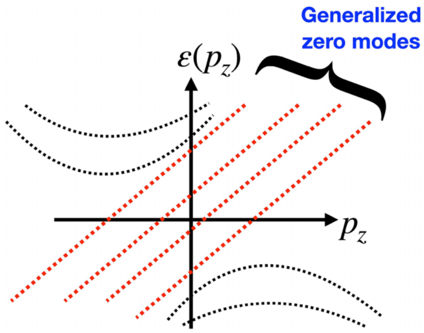

FIG. 1. A schematic example of spectral curves, $\varepsilon_{(n)}(p_z)$.

up to subleading corrections from position-momentum commutators. We then evaluate the remaining trace in momentum basis, and in the large-M limit, the trace is dominated by large momentum $\boldsymbol{p}$, for which $\boldsymbol{\pi}$ can be replaced by $\boldsymbol{p}$ up to subleading corrections, and we arrive at

$$C_J = \frac{2\hbar q^2 B}{\sqrt{\pi} M^3} \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\, e^{-\frac{F^2(p)}{M^2}} \left(\frac{\partial \boldsymbol{F}}{\partial p_x} \times \frac{\partial \boldsymbol{F}}{\partial p_y}\right) \cdot \frac{\partial \boldsymbol{F}}{\partial p_z}. \quad (17)$$

Noting that $(\frac{\partial \boldsymbol{F}}{\partial p_x} \times \frac{\partial \boldsymbol{F}}{\partial p_y}) \cdot \frac{\partial \boldsymbol{F}}{\partial p_z}$ is the signed Jacobian for the map $\boldsymbol{p} \to \boldsymbol{F}(\boldsymbol{p})$, which allows us to perform a change of variables $\boldsymbol{p} \to \boldsymbol{F}(\boldsymbol{p})$ in the above integration, we find

$$C_J = \frac{2\hbar q^2 B}{\sqrt{\pi} M^3} N_{\boldsymbol{F}} \int \frac{d^3\boldsymbol{F}}{(2\pi\hbar)^3}\, e^{-\boldsymbol{F}^2/M^2} = \frac{q^2 B}{(2\pi\hbar)^2} N_{\boldsymbol{F}}, \quad (18)$$

where $N_{\boldsymbol{F}} \in \pi_2(S^2) = \mathbf{Z}$ is the signed winding number of the map $\boldsymbol{p} \to \boldsymbol{F}(\boldsymbol{p})$ at asymptotic infinity, $S^2_\infty \to S^2_\infty$, which defines the topology of the Weyl node with the Hamiltonian $\hat{H} = \boldsymbol{F}(\hat{\boldsymbol{p}}) \cdot \boldsymbol{\sigma}$ [39]. The integer $N_{\boldsymbol{F}}$ counts the flux of the Berry's curvature on a Fermi surface around the Weyl node and also appears in the chiral anomaly relation near the Weyl node. $N_{\boldsymbol{F}}$ is essentially the number of times the $\boldsymbol{F}$ space at infinity is covered in the map $\boldsymbol{p} \to \boldsymbol{F}(\boldsymbol{p})$. As an example, $N_{\mathbf{F}} = N$ for $F^\pm = (p^\pm)^N$ and $F_z = p_z$, where $F^\pm = F_x \pm iF_y$ and $p^\pm = p_x \pm ip_y$. Only the asymptotic region of S^2_∞ at infinity in both $\boldsymbol{p}$ and $\boldsymbol{F}$ spaces matters since the surviving contribution in the $M \to \infty$ limit comes only from this region, as seen in the $\boldsymbol{F}$ integration in Eq. (18).

We can interpret the above result in the following interesting way. In the Landau gauge, the eigenstates of $\hat{H}$ can be written as $\psi(\boldsymbol{x}) = \psi^{(n)}_{p_z}[x - p_y/(qB)]e^{\frac{i}{\hbar}(p_y y + p_z z)}$, with good quantum numbers (p_y, p_z). The eigenvalue equation takes the form $\hat{H}_{p_z}\psi^{(n)}_{p_z}(x) = \varepsilon_{(n)}(p_z)\psi^{(n)}_{p_z}(x)$, where $\hat{H}_{p_z} = \boldsymbol{F}(\pi_x \to \hat{p}_x, \pi_y \to -qBx, \pi_z \to p_z) \cdot \boldsymbol{\sigma}$ is a Hamiltonian in the one-dimensional space of x, parametrized by p_z, and $\hat{p}_x = -i\hbar\partial_x$. The non-negative integer n is the discrete label for the spectral curves, $\varepsilon_{(n)}(p_z)$, as a function of the continuous parameter p_z, which may well be called the generalized Landau levels. The energy spectrum does not depend on p_y, and we have the usual density of states per unit transverse area, $g_\perp = \frac{qB}{2\pi\hbar}$. A typical shape of the spectral curves is depicted in Fig. 1. Using the Feynman-Hellmann theorem, the current expectation value is $\langle\psi^{(n)}_{p_z}|\hat{J}_z|\psi^{(n)}_{p_z}\rangle = q\langle\psi^{(n)}_{p_z}|\frac{\partial \hat{H}_{p_z}}{\partial p_z}|\psi^{(n)}_{p_z}\rangle = q\frac{\partial \varepsilon_{(n)}(p_z)}{\partial p_z}$, which implies that for each n, the contribution to $\rho_J(\varepsilon)$ is equal to that from a one-dimensional problem with a Hamiltonian $\hat{H}_n \equiv \varepsilon_{(n)}(\hat{p}_z)$, i.e., $\frac{q}{2\pi\hbar}\sum_\alpha \frac{\varepsilon'_{(n)}(p_\alpha)}{|\varepsilon'_{(n)}(p_\alpha)|}$, that we discussed as a toy example. A nonvanishing contribution comes only from the curves with no lower or upper bounds in energy, which we may call the generalized zero modes. We then conclude that $\rho_J(\varepsilon) = g_\perp \frac{q}{2\pi\hbar} N_0 = \frac{q^2 B}{(2\pi\hbar)^2} N_0$, where N_0 is the signed total number of zero modes of the Hamiltonian $\hat{H}_{p_z}$, which is a topological property of $\hat{H}_{p_z}$. Our result for C_J then proves the relation $N_0 = N_{\boldsymbol{F}}$.

For a special case of

$$\hat{H} = \boldsymbol{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) \cdot \boldsymbol{\sigma}_\perp + \hat{p}_z\sigma_z, \quad (19)$$

where $(\hat{\boldsymbol{\pi}}_\perp, \boldsymbol{\sigma}_\perp)$ have only the transverse components in (x, y) directions, we have $\hat{H}^2 = \hat{D}^2_\perp + \hat{p}^2_z$, where $\hat{D}_\perp \equiv \boldsymbol{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) \cdot \boldsymbol{\sigma}_\perp$ is a generalized Dirac operator in two dimensions. Our expression for C_J then factorizes as

$$C_J = \frac{1}{\sqrt{\pi}M}\frac{1}{V}\mathrm{Tr}(\hat{J}_z e^{-\hat{H}^2/M^2}) = \frac{q}{2\pi\hbar}\frac{1}{V_\perp}\mathrm{Tr}(\sigma_z e^{-\hat{D}^2_\perp/M^2}), \quad (20)$$

where we performed the trace over the z dimension, and the last trace is defined only in the transverse two dimensions. Noting that $\{\hat{D}_\perp, \sigma_z\}_+ = 0$, the trace coincides with the Atiyah-Singer index of the Dirac operator $\hat{D}_\perp$. Our result then gives a generalized version of the Atiyah-Singer index theorem,

$$\mathrm{Index}(\hat{D}_\perp) = \frac{qB}{2\pi\hbar} N_{\boldsymbol{F}} V_\perp = \frac{q}{2\pi\hbar} N_{\boldsymbol{F}} \int_{R^2_\perp} F_2, \quad (21)$$

where $N_{\boldsymbol{F}} \in \pi_1(S^1) = \mathbf{Z}$ is the winding number of the map $\boldsymbol{p}_\perp \to \boldsymbol{F}_\perp(\boldsymbol{p}_\perp)$ at asymptotic infinity, i.e., $S^1_\infty \to S^1_\infty$, which defines the topology of the Dirac operator $\hat{D}_\perp$.

Application to the chiral magnetic effect. Our result implies that in the system described by a Hamiltonian $\hat{H} = \boldsymbol{F}(\hat{\boldsymbol{\pi}}) \cdot \boldsymbol{\sigma}$, there is a nonvanishing current density along the direction of the magnetic field in the grand-canonical ensemble,

$$j_z = C_J\mu = \frac{q^2 B}{(2\pi\hbar)^2} N_{\boldsymbol{F}}\mu, \quad (22)$$

which is the chiral magnetic effect and is generalized to arbitrary topological number $N_{\boldsymbol{F}}$. As shown in Ref. [39], the same topological number also appears in the chiral anomaly relation, $\partial_\mu j^\mu = \frac{q^3}{(2\pi\hbar)^2} N_{\boldsymbol{F}}(\boldsymbol{E} \cdot \boldsymbol{B})$, where j^μ is the charge current density in relativistic notation. This affirms the connection between the chiral magnetic effect and chiral anomaly in the most general case of topology.

In real Weyl semimetals, as in all condensed-matter systems, the global spectrum has a lower bound, and our Theorem 1 dictates that there is no net persistent current in any thermodynamic ensembles. This is consistent with the Nielsen-Ninomiya theorem that the summation of $N_{\boldsymbol{F}}$ for all Weyl nodes vanishes [40]. Instead, one may consider a quasiequilibrium state, where each Weyl node, labeled by α, with a nonvanishing $N_{\boldsymbol{F}_\alpha}$, has its own effective chemical potential μ_α, and $j_z = \frac{q^2 B}{(2\pi\hbar)^2}\sum_\alpha N_{\boldsymbol{F}_\alpha}\mu_\alpha$ may not vanish. These quasiequilibrium states would make sense only if internode transitions are slow enough.

A Weyl node with impurities and a magnetic field. In this section, we explicitly examine whether C_J is robust under perturbations by considering several different perturbations and showing that C_J is indeed independent of these perturbations. Although these examples do not exhaust all possible perturbations, we expect that C_J is independent of many other reasonable perturbations. Here, we summarize the main elements and ideas of our proofs (see the Supplemental Material [41] for full details).

We examined the Hamiltonians with the general form, $\hat{H}_\lambda = \boldsymbol{\sigma} \cdot \boldsymbol{F}(\hat{\boldsymbol{\pi}}) + \lambda \hat{V}$, where $\hat{V}$ is the perturbation and λ is a parameter which controls the magnitude of the perturbation. The perturbations we consider have a general form $\hat{V}(\boldsymbol{x}) = V_0(\boldsymbol{x}) + \boldsymbol{V}(\boldsymbol{x}) \cdot \boldsymbol{\sigma}$, where $\boldsymbol{V}$ is a vector valued function representing a spin-dependent potential and V_0 is a spin-independent potential. Recall that in the Landau gauge, we have $\hat{\boldsymbol{\pi}} = (\hat{p}_x, \hat{p}_y - qBx, \hat{p}_z)$. The following cases have been examined:

(1) $\hat{H}_{1\lambda} = \boldsymbol{\sigma}_\perp \cdot \boldsymbol{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) + \sigma_z \hat{p}_z + \lambda \hat{V}(x, y)$,

(2) $\hat{H}_{2\lambda} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \lambda \hat{V}(x, y, z)$,

(3) $\hat{H}_{3\lambda} = \left[\sigma_x \hat{\pi}_x^{n_x} + \sigma_y \hat{\pi}_y^{n_y} + \sigma_z \hat{\pi}_z^{n_z}\right] + \lambda V_0(x, y, z)$,

where the $n_i \geqslant 1$ $(i = x, y, z)$ in case (3) are positive integers. Note that in case (1), $\hat{V}(x, y)$ is independent of z, and the momentum along the z direction, p_z, is a good quantum number, which makes the proof particularly simple. This proof is presented in Sec. 1 of the Supplemental Material [41].

Our main technique for the proofs is again the $M \to \infty$ limit and the heat kernel expansion. As an example, here we present an outline of the main analysis for case 2. We first write $\hat{H}_{2\lambda}^2 = \hat{\boldsymbol{p}}^2 + \hat{H}_c$ where $\hat{H}_c = \hat{H}_{2\lambda}^2 - \hat{\boldsymbol{p}}^2$, and do the heat kernel expansion for C_J treating $\hat{H}_c/M^2$ as a perturbation. Taking the trace in the momentum basis and using the fact that in the large-M limit we can ignore position-momentum commutators, the exponential term in the expression for C_J can be Taylor expanded as $e^{-\frac{p^2}{M^2}} e^{-\frac{\hat{H}_c}{M^2}} = e^{-\frac{p^2}{M^2}}(1 - \frac{\hat{H}_c}{M^2} + \frac{\hat{H}_c^2}{2!M^4} + \cdots)$. From this, we obtain $C_J = \sum_{m=0}^{\infty} C_{Jm}$, where C_{Jm} is given by

$$\frac{(-1)^m q}{M^{(2m+1)} m! \sqrt{\pi}} \frac{1}{V} \int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3} \int d^3\boldsymbol{x} e^{-\frac{p^2}{M^2}} \mathrm{Tr}_{\mathrm{spin}}\left(\sigma_z \hat{H}_c^m\right). \quad (23)$$

Since we are interested in the dependence of C_J on $\hat{V}$, we only consider the terms that include $\hat{V}$. The terms with odd powers of $\boldsymbol{p}$ drop out upon integration, and many terms vanish by spin trace. After this, it is found that in the limit $M \to \infty$, the only terms that remain are C_{J1} and C_{J2}. The $\hat{V}$-dependent parts of these terms are nonzero; however, they cancel in the sum $C_{J1} + C_{J2}$ quite nontrivially, proving that C_J is indeed independent of $\hat{V}$. Further details are given in Sec. II of the Supplemental Material [41]. Similar steps are used to prove that C_J is independent of V_0 for case 3 (see Sec. III of the Supplemental Material [41] for details).

The independence of C_J on the potential is a nontrivial fact, if viewed naively in quantum mechanics. The constant C_J as expressed in Eq. (8) is a complicated expression when expanded as a power series of $\hat{V}$. To obtain the claimed result requires precise cancellation of all terms containing $\hat{V}$. To explicitly demonstrate that these cancellations do indeed occur, we considered the simple case where $\hat{H}_{1a\lambda} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \lambda \hat{V}(x, y)$ and showed that the terms in C_J up to second order of $\hat{V}$ do cancel out exactly for any value of M. This nontrivial calculation is provided in Sec. IV of the Supplemental Material [41].

Implication for the spectral density. We discuss one immediate consequence of our results on the spectral density of a Weyl node in a magnetic field and impurities. We first state the following spectral theorem for the Hamiltonians with $|\frac{\partial \boldsymbol{F}(\hat{\boldsymbol{\pi}})}{\partial \hat{p}_z}| \leqslant C$ for all normalized states, where C is a nonzero positive constant,

$$\text{Theorem 2}: \quad \rho(\varepsilon) \geqslant |C_J|/(qC), \;\; \text{for all } \varepsilon. \quad (24)$$

An example is $\hat{H}_0 = \boldsymbol{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) \cdot \boldsymbol{\sigma}_\perp + \hat{p}_z \sigma_z$, where $|\frac{\partial \boldsymbol{F}(\hat{\boldsymbol{\pi}})}{\partial \hat{p}_z}| = 1$. The proof is based on $|\langle \hat{J}_z \rangle| = q|\langle \frac{\partial \boldsymbol{F}}{\partial \hat{p}_z} \cdot \boldsymbol{\sigma} \rangle| \leqslant q\langle |\frac{\partial \boldsymbol{F}(\hat{\boldsymbol{\pi}})}{\partial \hat{p}_z}| \rangle \leqslant qC$, and we have $|\rho_J(\varepsilon)| = \frac{1}{V}|\mathrm{Tr}[\hat{J}_z \delta(\hat{H} - \varepsilon)]| \leqslant qC \frac{1}{V} \mathrm{Tr}[\delta(\hat{H} - \varepsilon)] = qC\rho(\varepsilon)$. The theorem implies that the energy spectrum has no gap if $\rho_J(\varepsilon) = C_J$ is nonvanishing, which is the case for a Weyl node in a magnetic field and impurities.

Discussion. Our result implies the absence of a bulk spectral gap for a single Weyl node, as the impurity strength varies, when an external magnetic field is present. Whether this behavior is smooth in the $B \to 0$ limit or there is a discontinuity at $B = 0$ is an interesting question to study. In real Weyl semimetals, different Weyl nodes with opposite topological numbers are separated in momentum space, and the internode mixings can occur for sufficiently strong short-range potentials, which could invalidate our conclusion based on a single Weyl node. The robustness of C_J for a single Weyl node shows that the chiral magnetic effect is not affected by any random impurities, as long as internode transitions are suppressed. This should ultimately be related to the robustness of chiral anomaly [42,43] that is in effect near each Weyl node.

The same constant C_J is also responsible for the chiral energy transfer along a magnetic field, $T_{0z} = \int_{-\infty}^{\infty} d\varepsilon [f(\varepsilon) - \Theta(-\varepsilon)] \frac{1}{V} \mathrm{Tr}[\hat{v}_z \hat{H} \delta(\hat{H} - \varepsilon)] = \frac{1}{q} \int d\varepsilon [f(\varepsilon) - \Theta(-\varepsilon)] \varepsilon \frac{1}{V} \mathrm{Tr}[\hat{J}_z \delta(\hat{H} - \varepsilon)] = \frac{C_J}{q} \int d\varepsilon [f(\varepsilon) - \Theta(-\varepsilon)] \varepsilon = \frac{C_J}{q}(\frac{\mu^2}{2} + \frac{\pi^2 T^2}{6}) = \frac{qBN_{\mathrm{F}}}{(2\pi\hbar)^2}(\frac{\mu^2}{2} + \frac{\pi^2 T^2}{6})$, where we used $\hat{J}_z = q\hat{v}_z$ with the velocity operator $\hat{v}_z = \frac{\partial \hat{H}}{\partial p_z}$. The chiral energy transfer is equivalent to the chiral vortical effect in time-reversal invariant systems [44].

Our Theorem 1 is general enough to be applicable to interacting multiparticle systems and quantum field theories of charged particles, as it only relies upon gauge invariance. Moreover, the idea can easily be generalized to produce many similar versions. For example, for any gauge-invariant operator $\hat{\mathcal{O}}$ which is independent of $\hat{\pi}_z$ and commutes with $\hat{J}_z$, the weighted spectral density $\rho_{\mathcal{O}}(\varepsilon) = \lim_{V \to \infty} \frac{1}{V} \mathrm{Tr}[\hat{\mathcal{O}} \hat{J}_z \delta(\hat{H} - \varepsilon)]$ is independent of ε, and hence robust under perturbations. It is an interesting speculation whether this line of thinking might lead to a useful definition of topology for interacting multiparticle systems, *since these objects can be defined quite generally independent of the presence of interactions among multiparticles in the system.*

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