Letter

Eigenvector correlations across the localization transition in non-Hermitian power-law banded random matrices

Soumi Ghosh,* Manas Kulkarni,† and Sthitadhi Roy‡
International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India

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The dynamics of non-Hermitian quantum systems have taken on an increasing relevance in light of quantum devices which are not perfectly isolated from their environment. The interest in them also stems from their fundamental differences from their Hermitian counterparts, particularly with regard to their spectral and eigenvector correlations. These correlations form the fundamental building block for understanding the dynamics of quantum systems as all other correlations can be reconstructed from it. In this Letter, we study such correlations across a localization transition in non-Hermitian quantum systems. As a concrete setting, we consider non-Hermitian power-law banded random matrices which have emerged as a promising platform for studying localization in disordered, non-Hermitian systems. We show that eigenvector correlations show marked differences between the delocalized and localized phases. In the delocalized phase, the eigenvectors are strongly correlated as evinced by divergent correlations in the limit of vanishingly small complex eigenvalue spacings. On the contrary, in the localized phase, the correlations are independent of the eigenvalue spacings. We explain our results in the delocalized phase by appealing to the Ginibre random-matrix ensemble. On the other hand, in the localized phase, an analytical treatment sheds light on the suppressed correlations, relative to the delocalized phase. Given that eigenvector correlations are fundamental ingredients towards understanding real- and imaginary-time dynamics with non-Hermitian generators, our results open an avenue for characterizing dynamical phases in non-Hermitian quantum many-body systems.

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Ergodicity or the lack thereof, manifested in localization, in disordered, interacting quantum many-body systems is a question of immanent interest [1–6]. As many-body localized (MBL) systems fail to thermalize under their dynamics, they raise fundamental questions with regard to the statistical mechanical description as well as the precise nature of their dynamics when thrown out of equilibrium (see Refs. [3–6] for reviews on MBL and further references therein). Conventionally, these questions have been studied in the context of closed quantum systems where the dynamics is unitary.

More recently, however, understanding the dynamics of interacting quantum many-body systems described by non-Hermitian Hamiltonians has emerged as an extremely relevant question [7–16]. This is, in part, due to the advent of noisy intermediate-scale quantum (NISQ) devices [17–20], wherein the non-Hermiticity induced by external noise, or coupling to environments or measurement apparatuses, is inevitable and understanding its effect is of utmost importance. From a theoretical point of view, the interest lies in their fundamental differences from their Hermitian counterparts owing to the former's complex eigenvalue spectrum. This offers the possibility of realizing phase structures of quantum systems quite different than those in Hermitian systems [13,21–37].

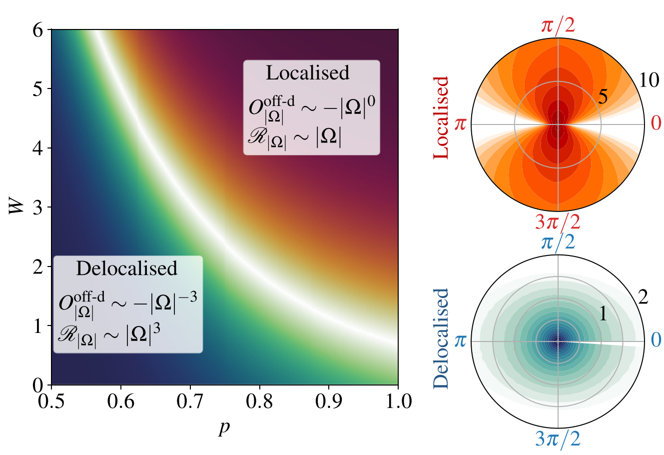

FIG. 1. Left: Schematic phase diagram of non-Hermitian power-law banded random-matrix (NH-PLBRM) ensemble in the p-W plane, where W denotes the disorder strength of the *complex* diagonal elements and p is the exponent of the power-law decay of off-diagonal, hopping matrix elements. The behaviors of the eigenvector and spectral correlations in the two phases are summarized. Right: The eigenvector correlations [Eq. (5)] as a heat map in the plane of complex eigenvalue spacings, in the localized (top) and delocalized (bottom) phases. Besides the difference in their scaling with $|\Omega|$, another stark difference is that in the delocalized phase the correlations are isotropic, whereas in the localized phase we find that they are strongly anisotropic.

*soumi.ghosh@icts.res.in
†manas.kulkarni@icts.res.in
‡sthitadhi.roy@icts.res.in

Under the umbrella of dynamics of non-Hermitian quantum systems, the physics of non-Hermitian many-body localization and the associated localization transition has been under intense investigation of late [8–11,38]. Spectral and eigenvector correlations constitute the basic building block for a theory of dynamics of any quantum system as all other dynamical correlations can be reconstructed from it. As far as spectral properties are considered, the ergodic phase of such systems displays universality, manifested in level repulsion [8,10] in the complex eigenvalue spectrum as well as a ramp in the dissipative spectral form factor [11,39,40], akin to Ginibre random-matrix ensembles [41–43]. On the other hand, the spectral properties in the localized phase show starkly different behavior and deviate significantly from random-matrix behavior. In fact, these as well as the participation ratios of eigenstates [44–46], which are a measure of how (de)localized are the eigenstates, have been extremely insightful diagnostics of the ergodicity or localization in disordered, non-Hermitian systems. However, one of the most fundamental ingredients to get a complete understanding of the dynamics of quantum systems are dynamical eigenvector correlations. While they have been studied extensively for Hermitian systems across localization transitions [47–49], they have been hitherto unexplored in non-Hermitian settings with results available only for random matrices [50,51]. This leads us to the central motivation of our work, namely, the behavior of eigenvector correlations across localization transitions in non-Hermitian systems.

As a concrete setting, we use power-law banded random matrices (PLBRMs), but in their non-Hermitian incarnation. For Hermitian systems, PLBRMs have long been used as an archetypal model for localization transitions in quantum systems [52–57]. In a very recent work, non-Hermitian power-law banded random matrices (NH-PLBRMs) were also shown to exhibit localization transitions [58]. In fact, NH-PLBRMs were shown to exhibit localization in parameter regimes where localization is forbidden in their Hermitian counterparts.

Our results show that eigenvector correlations (as well as spectral correlations) show stark differences in the delocalized and localized phases (see Fig. 1 for a summary of our main findings). In the delocalized phase, we find that the results for the correlations fall in the universality class of Ginibre random matrices [50,51,59]. An appropriately defined correlation between eigenvectors diverges as the complex eigenvalue spacing decreases, suggesting that the eigenvectors are very strongly correlated. By contrast, in the localized phase, we find that the correlations are independent of the spacing between the eigenvalues at small spacings which suggests that the correlations are strongly suppressed relative to those in the delocalized phase. We explain this behavior via an analytical calculation based on a simple perturbation theory where the bare resonances are renormalized appropriately. Within the limits of our numerical calculations we find an anomalous, intermediate behavior of the correlations in the critical regime.

The importance of our results lies in that the transient dynamics of non-Hermitian systems are controlled by the eigenvector correlations. Our results constitute a firm step towards understanding the spectral and dynamical properties of local observables across localization transitions in disordered, interacting, non-Hermitian quantum many-body systems.

To set the stage formally, consider a $N \times N$ non-Hermitian Hamiltonian matrix H with complex eigenvalues z_α. The corresponding left and right eigenvectors, $\langle L_\alpha|$ and $|R_\alpha\rangle$, which satisfy

$$\langle L_\alpha|H = \langle L_\alpha|z_\alpha, \quad H|R_\alpha\rangle = z_\alpha|R_\alpha\rangle, \tag{1}$$

form a complete, biorthonormal set with $\langle L_\alpha|R_\beta\rangle = \delta_{\alpha\beta}$. Requiring that eigenvector correlations are invariant under scale transformations, the simplest nontrivial measure of the Wcorrelations can be defined as [50]

$$O_{\alpha\beta} = \langle L_\alpha|L_\beta\rangle\langle R_\beta|R_\alpha\rangle. \tag{2}$$

The definition in Eq. (2) directly implies that $O_{\alpha\beta} = O^*_{\beta\alpha}$, and also from completeness, $\sum_\alpha O_{\alpha\beta} = 1$. It will be useful to resolve the correlations in Eq. (2) in terms of the eigenvalues, and define averaged diagonal and off-diagonal correlations, $O^{\rm d}$ and $O^{\text{off-d}}$, respectively, as

$$O^{\rm d}(z) = \left\langle N^{-1}\sum_\alpha O_{\alpha\alpha}\delta(z - z_\alpha)\right\rangle, \tag{3}$$

$$O^{\text{off-d}}(Z,\Omega) = \left\langle N^{-1}\sum_{\alpha\neq\beta} O_{\alpha\beta}\delta\left(Z - \frac{z_\alpha + z_\beta}{2}\right) \times \delta(\Omega - z_\alpha + z_\beta)\right\rangle. \tag{4}$$

The off-diagonal correlation, as defined above, depends on both the mean of the eigenvalues, Z, as well as their difference Ω. However, for simplicity, we will be interested in two specific versions of it. The first is where the mean is integrated over,

$$O^{\text{off-d}}(\Omega) \equiv \int dZ O^{\text{off-d}}(Z,\Omega) = \left\langle N^{-1}\sum_{\alpha\neq\beta} O_{\alpha\beta}\delta(\Omega - z_\alpha + z_\beta)\right\rangle, \tag{5}$$

and the second is where we restrict the sum over pairs of eigenvectors in Eq. (4) such that the mean of their eigenvalues is vanishing,

$$O^{\text{off-d}}_{Z=0}(\Omega) \equiv O^{\text{off-d}}(Z=0,\Omega). \tag{6}$$

Note that the averaged eigenvector correlations in Eqs. (5) and (6) are functions of Ω which is complex. In much of the following, we will find that it is sufficient to consider and focus on the respective correlations as a function of $|\Omega|$. With $\Omega = |\Omega|e^{i\theta}$, they are defined as

$$\tilde{O}^{\text{off-d}}_{|\Omega|}(|\Omega|) = |\Omega|\int_0^{2\pi} d\theta\, O^{\text{off-d}}(\Omega), \tag{7}$$

and similarly for $\tilde{O}^{\text{off-d}}_{Z=0,|\Omega|}$.

As we will show later, both $\tilde{O}^{\text{off-d}}_{|\Omega|}$ as well as $\tilde{O}^{\text{off-d}}_{Z=0,|\Omega|}$ exhibit the same universal behavior at small $|\Omega|$. However, this universal behavior is starkly different between delocalized and localized phases. In particular, in the thermodynamic limit $N \to \infty$ for $|\Omega| \ll 1$, in the delocalized phase both $\tilde{O}^{\text{offd}}_{|\Omega|}, \tilde{O}^{\text{offd}}_{Z=0,|\Omega|} \sim -|\Omega|^{-3}$ whereas in the localized phase, we find that both of them scale $\sim -|\Omega|^0$. This constitutes the central result of this Letter.

It is important to note here that if H were to be Hermitian, the eigenvector correlation defined in Eq. (2) would have been trivial with $O_{\alpha\beta} = \delta_{\alpha\beta}$. As such the diagonal correlation O^{d} in Eq. (3) would have simply been the density of states and the off-diagonal ones in Eq. (4) would have been identically zero. The nontriviality in the correlations arises purely from the non-Hermiticity. However, the crucial point is that the nature of the correlations depends on the phase in which the non-Hermitian system lies.

We now delve into the details of our results and start with describing the NH-PLBRM ensemble [58]. In an instance of the Hamiltonian from the ensemble, the element H_{mn} is given by

$$H_{mn} = \epsilon_n \delta_{mn} + j_{mn}, \tag{8}$$

where $j^*_{nm} = j_{mn}$ and ϵ_n, j_{mn} are complex random numbers. The real and imaginary parts of the diagonal elements are both chosen from uniform distributions, $\text{Re}[\epsilon_n], \text{Im}[\epsilon_n] \in [-W, W]$. The real (Re) and imaginary (Im) parts of the independent off-diagonal elements j_{mn} $(m > n)$ are chosen from uniform distributions, $\text{Re}[j_{mn}], \text{Im}[j_{mn}] \in [-\sigma_{|m-n|}, \sigma_{|m-n|}]$. The width σ_r decays with $r = |m - n|$ following a power law,

$$\sigma_r = [2(r^2 + b^2)]^{-p/2}, \tag{9}$$

where b is the bandwidth of the decay $(b \simeq 1)$ and p is the power of the off-diagonal power-law decay term. Here, the imaginary parts of the diagonal elements bring about the non-Hermiticity in the Hamiltonian. In the absence of those, the random matrices become Hermitian and the chaotic limit of the underlying model corresponds to the Gaussian unitary ensemble (strictly speaking, a complex Wigner matrix ensemble since they are sampled from uniform distributions). In an analogous way, the chaotic limit of the NH-PLBRM is shown to correspond to the Ginibre unitary ensemble (GinUE) [41]. The NH-PLBRM has a rich localization phase diagram in the p-W plane (see Fig. 1). While the model was introduced and studied in detail in Ref. [58], we summarize its salient features for completeness. Unlike its Hermitian counterpart where localization is forbidden for $p < 1$ [57], the NH-PLBRM hosts a localized phase and a disorder-driven localization transition for $1/2 \leqslant p \leqslant 1$. However, the localized phase is algebraic in nature, again in contrast to the Hermitian PLBRM. Finally, we note that the NH-PLBRM does not host a localized phase for $p < 1/2$ which can be understood via simple resonance counting argument [58].[1] Also, we find that the density of states of the NH-PLBRM in the complex eigenvalue plane is uniform (wherever finite) to a very good approximation [60]

[1]Since the complex energies live on a two-dimensional plane, the mean-level spacing of sites at distance r from any given site $\sim W/r^{1/2}$ which when compared to the power-law decaying matrix element $(\sim 1/r^p)$ implies that localization is forbidden for $p < 1/2$.

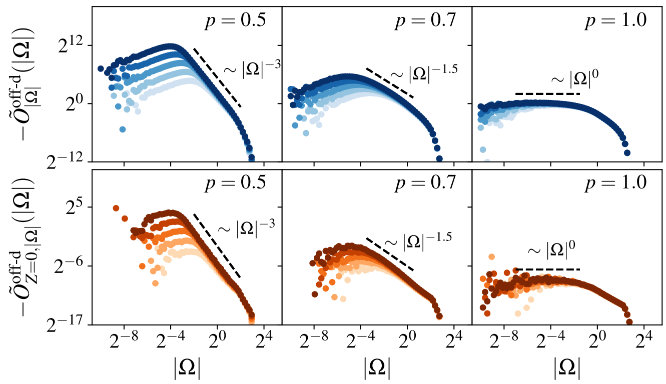

FIG. 2. Off-diagonal eigenvector correlations for three different values of p in the delocalized (left), critical (center), and localized (right) regime. The top and bottom rows correspond to $\tilde{O}^{\text{off-d}}_{|\Omega|}$ [Eq. (7)] and $\tilde{O}^{\text{off-d}}_{Z=0,|\Omega|}$, respectively. The dashed lines indicate the power laws, $|\Omega|^{-3}$ (delocalized), $|\Omega|^{-1.5}$ (critical), and $|\Omega|^0$ (localized). The different color intensities correspond to different system sizes, $N = 128, 256, 512, 1024, 2048$ (lighter to darker). Data are for $W = 3$.

which lets us conveniently avoid spectrum unfolding while defining the eigenvector correlations in Eq. (4).

The numerical results for the eigenvector correlations, obtained from exact diagonalization (ED) of the Hamiltonians in Eq. (8), are shown in Fig. 2. The top row corresponds to $\tilde{O}^{\text{off-d}}_{|\Omega|}$ defined in Eq. (7) and the bottom row to $\tilde{O}^{\text{off-d}}_{Z=0,|\Omega|}$. In the delocalized phase (left column), we find that both of them scale as $-|\Omega|^{-3}$. By contrast, in the localized phase (right column) they scale approximately as $-|\Omega|^0$. In the critical regime between the two phases ($p \approx 0.7$ for $W = 3$), our numerical results suggest an anomalous scaling $\sim -|\Omega|^{-1.5}$. In Fig. 3 (left), we show the off-diagonal correlation $\tilde{O}^{\text{off-d}}_{|\Omega|}$ for a fixed $N = 2048$ and $W = 3$ but for several values of $p \in [0.5, 1.0]$ straddling the critical point at $p_c \approx 0.7$. For a finite system, we observe that $\tilde{O}^{\text{off-d}}_{|\Omega|} \sim -|\Omega|^{-\nu}$ where the exponent ν sharply changes from $\nu = 3$ in the delocalized phase to vanishingly small $(\nu \to 0)$ in the localized phase. The right panel shows the variation of ν with p for several values of N with the data for different N showing a crossing at the putative critical point.

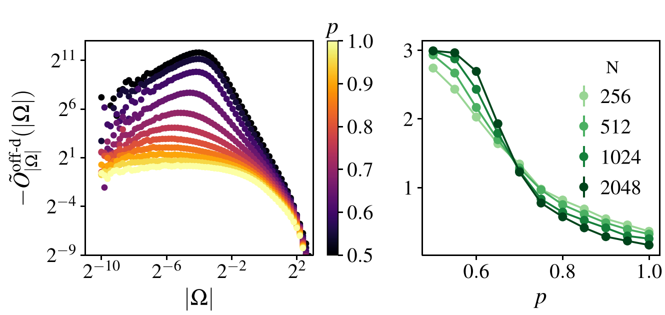

FIG. 3. Left: Off-diagonal correlations as a function of $|\Omega|$ for different values of p for $N = 2048$ and $W = 3$. As we move from the delocalized to the localized phase by increasing p from 0.5 to 1 (as indicated by the color bar), the correlation gets suppressed and the exponent ν (defined via $\tilde{O}^{\text{off-d}}_{|\Omega|} \sim -|\Omega|^{-\nu}$) changes from 3 to 0. Right: Variation of the exponent ν as a function of p for different N showing a crossing at the putative critical point $p_c \approx 0.7$ with $\nu_c \approx 1.5$.

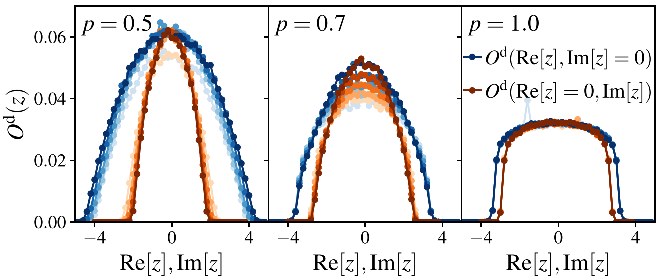

FIG. 4. Diagonal correlations $O^{\rm d}(z)$ defined in Eq. (3) in the delocalized (left), critical (center), and the localized (right) regime. The blue data points show the variation of $O^{\rm d}({\rm Re}[z], {\rm Im}[z] = 0)$ with Re[z], while the orange lines show the variation of $O^{\rm d}({\rm Re}[z] = 0, {\rm Im}[z])$ with Im[z]. As in Fig. 2, the different color intensities indicate different system sizes N.

While our main focus is on the off-diagonal eigenvector correlations, we also find sharp distinctions in the diagonal correlations, $O^{\rm d}(z)$ [defined in Eq. (3)] between the two phases, as shown in Fig. 4. In the delocalized phase (left column) we find an inverted parabolic profile of $O^{\rm d}$ symptomatic of GinUE universality [50]. By contrast, in the localized phase (right), $O^{\rm d}$ is significantly flatter and approximately mirrors the density of states profile. This can be understood as deep inside the localized phase the eigenvectors are sharply localized around $\mathcal{O}(1)$ nearby sites which gives rise $O_{\alpha\alpha} \sim \mathcal{O}(1)$ for all α irrespective of its eigenvalue. In the critical regime (center), we again observe an intermediate behavior in a similar spirit as the off-diagonal correlations.

Having established the numerical results for the eigenvector correlations, we next provide analytical insights into the results for the off-diagonal correlations in both the delocalized and localized phases. The delocalized phase of the NH-PLBRM can be understood by appealing to the GinUE universality class. The off-diagonal eigenvector correlations in GinUE matrices are given by [50,51]

$$O^{\text{off-d}}_{\rm GinUE}(Z, \Omega) = \frac{Z_+ Z_-^* - 1}{\pi^2 |\Omega|^4} \Theta(1 - |Z_+|)\Theta(1 - |Z_-|), \quad (10)$$

where $Z_\pm = Z \pm \Omega/2$. In the limit of $|\Omega| \ll 1$, Eq. (10) can be used to obtain

$$\tilde{O}^{\text{off-d}}_{|\Omega|}, \tilde{O}^{\text{off-d}}_{Z=0,|\Omega|} \sim -|\Omega|^{-3}, \quad (11)$$

which explains our results in the delocalized phase of the NH-PLBRM and demonstrates that it indeed lies in the GinUE universality class.

Deep in the localized phase, the eigenvectors can be well approximated by leading-order perturbative corrections to the site-localized states at infinite disorder. Denoting by $|\alpha\rangle$ a state localized on a single site,[2] the eigenvectors to leading order are given by

$$|R_\alpha\rangle = |\alpha\rangle + \sum_{\gamma\neq\alpha} \frac{H_{\alpha\gamma}}{\Delta_{\alpha\gamma}} |\gamma\rangle, \quad \langle L_\alpha| = \langle\alpha| + \sum_{\gamma\neq\alpha} \frac{H_{\gamma\alpha}}{\Delta_{\alpha\gamma}} \langle\gamma|, \quad (12)$$

where $\Delta_{\alpha\gamma} = \epsilon_\alpha - \epsilon_\gamma$. Also, at leading order, $z_\alpha = \epsilon_\alpha$. Using Eq. (12) and the definition in Eq. (2), we obtain

$$O^{\rm loc}_{\alpha\beta} = -4|H_{\alpha\beta}|^2 \left({\rm Im}\left[\Delta^{-1}_{\alpha\beta}\right]\right)^2, \quad (13)$$

for the eigenvector overlaps in the localized phase. Since the expression in Eq. (13) is obtained from an unrenormalized perturbative expansion [Eq. (12)], it allows for bare resonances due to ${\rm Im}[\Delta^{-1}_{\alpha\beta}] \to \infty$ which can result in a divergent overlap. While a mathematically rigorous renormalized perturbation theory, for example, *à la* Feenberg [61], is outside the scope of this Letter, we account for the bare resonances by imposing an empirical cutoff on the overlaps. Physically, this corresponds to setting the $O_{\alpha\beta}$ for the resonant pairs to an empirical $\mathcal{O}(1)$ threshold which is what a proper renormalization of the resonances would have done self-consistently, and leave the other $O_{\alpha\beta}$'s as they are. To this end, we define a renormalized overlap as

$$G_{\alpha\beta} = O^{\rm loc}_{\alpha\beta}\Theta\left(1 - \left|O^{\rm loc}_{\alpha\beta}\right|\right) - \Theta\left(\left|O^{\rm loc}_{\alpha\beta}\right| - 1\right), \quad (14)$$

and compute the off-diagonal correlations as $O^{\text{off-d}}(\Omega) = \langle N^{-1} \sum_{\alpha\neq\beta} G_{\alpha\beta}\delta(\Omega - \Delta_{\alpha\beta})\rangle$, and similarly for $O^{\text{off-d}}_{Z=0}$. Since the matrix elements of the Hamiltonian, $\{H_{\alpha\beta}\}$ and $\{\epsilon_\alpha\}$, are independent of each other, the eigenvector correlation can be expressed as $O^{\text{off-d}}(\Omega) = \sum_{r=1}^{N-1} Y_r(\Omega)$ where

$$Y_r(\Omega) = \int d\epsilon_\alpha P_\epsilon(\epsilon_\alpha) \int d\epsilon_\beta P_\epsilon(\epsilon_\beta) \times \left[\delta(\Omega - \Delta_{\alpha\beta}) \int dH_r P_{H_r}(H_r)\tilde{G}(H_r, \Delta_{\alpha\beta})\right]. \quad (15)$$

with $\tilde{G}(H_r, \Delta_{\alpha\beta}) \equiv G_{\alpha\beta}$ and $H_{\alpha\beta}$ set to H_r. The notation H_r refers to a hopping matrix element of the Hamiltonian between sites separated by distance r such that it is a random complex number with real and imaginary parts drawn from uniform distributions, ${\rm Re}[H_r], {\rm Im}[H_r] \in [-\sigma_r, \sigma_r]$ where σ_r is given by Eq. (9). Using the distributions for the ϵ_α's and H_r's, we obtain[3]

$$Y_r(\Omega) = \begin{cases} R(\Omega)\left(1 - \frac{1}{8(\sigma_r {\rm Im}[\Omega^{-1}])^2}\right), & \sigma_r \geqslant (2|{\rm Im}[\Omega^{-1}]|)^{-1}, \\ 2R(\Omega)\sigma_r^2({\rm Im}[\Omega^{-1}])^2, & \sigma_r < (2|{\rm Im}[\Omega^{-1}]|)^{-1}, \end{cases} \quad (16)$$

where $R(\Omega)$ is the probability that two uncorrelated random ϵ's are separated by Ω and it is given by

$$R(\Omega) = \frac{2}{\pi^2 W^2}\left[\cos^{-1}\left(\frac{|\Omega|}{2W}\right) - \frac{|\Omega|}{2W}\sqrt{1 - \left(\frac{|\Omega|}{2W}\right)^2}\right], \quad (17)$$

with $P_\epsilon(\epsilon) = (\pi W^2)^{-1}\Theta(W - |\epsilon|)$. Using Eq. (16), an analytical expression for $O^{\text{off-d}}(\Omega)$ in the localized phase can be obtained. It is, however, rather cumbersome and opaque and hence we omit it for brevity. Instead, we plot the result

[2]Deep in the localized phase, since every eigenvector is expected to be closely tied to a site, we use the same notation to index the sites and eigenvectors.

[3]For simplicity of expressions, we used a circular distribution of ϵ_α's and $H_{\alpha\beta}$'s. However, using numerical integration of Eq. (15), we checked that the results in the universal $|\Omega| \ll 1$ regime are identical for a box distribution.

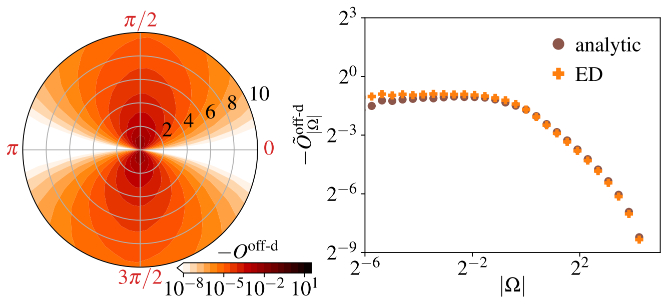

FIG. 5. Left: $O^{\text{off-d}}(\Omega)$ in the localized phase obtained from the analytic theory [from Eqs. (16) and (17)] as a color map in the complex Ω plane. Right: Comparison of $\tilde{O}^{\text{off-d}}_{|\Omega|}$ as a function of $|\Omega|$ from the analytic calculation with that from ED. Results are for deep in the localized phase with $W = 10$, $\alpha = 1$, and $N = 1024$.

for $O^{\text{off-d}}(\Omega)$ obtained analytically using Eqs. (16) and (17) as a color map in the complex Ω plane in Fig. 5 (left). The qualitative features of the exact ED result (see Fig. 1) are well captured. For a quantitative comparison, we derive the corresponding $\tilde{O}^{\text{off-d}}_{|\Omega|}(|\Omega|)$ and plot it in Fig. 5 (right); we find a remarkable agreement with the exact numerical results and the analytic calculation does indeed yield the approximately $|\Omega|$-independent behavior of $\tilde{O}^{\text{off-d}}_{|\Omega|}(|\Omega|)$ at small $|\Omega|$.

To conclude, we demonstrated that eigenvector correlations are starkly different between delocalized and localized phases in disordered, non-Hermitian systems. Using NH-PLBRM as a prototype, we showed, via extensive numerical calculations and analytical arguments, that eigenvectors are strongly correlated in the delocalized phase and the same are suppressed in the localized phase (see Fig. 1 for a summary). While eigenvector correlations were the focus of this Letter, for the sake of completeness, we also calculated spectral correlations, which were characterized by the presence and absence of complex level repulsion in the delocalized and localized phase, respectively [60].

Our findings will have a significant bearing on the characterization of dynamical phases of non-Hermitian, locally interacting quantum many-body Hamiltonians [62]. While eigenvector correlations, such as the ones discussed here, are definitely interesting in this context, it is equally interesting to understand the spectral properties of local observables in the same spirit. In particular, these quantities are expected to play a pivotal role in understanding fundamental issues such as (i) eigenstate thermalization (or lack thereof) in non-Hermitian systems [63,64] and (ii) non-Hermitian many-body localization. In fact, extending these ideas to open quantum systems in general, such as via the eigenvector correlations of the underlying Liouvillian operators [24–26,65–70], is topically interesting.

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