Letter Editors' Suggestion

Spin-statistics relation for quantum Hall states

Alberto Nardin,[1,*] Eddy Ardonne,[2,†] and Leonardo Mazza[3,‡]
[1] *INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, 38123 Povo, Italy*
[2] *Department of Physics, Stockholm University, AlbaNova University Center, 106 91 Stockholm, Sweden*
[3] *Université Paris-Saclay, CNRS, LPTMS, 91405 Orsay, France*

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We prove a generic spin-statistics relation for the fractional quasiparticles that appear in Abelian quantum Hall states on the disk. The proof is based on an efficient way for computing the Berry phase acquired by a generic quasiparticle translated in the plane along a circular path, and on the crucial fact that once the gauge-invariant generator of rotations is projected onto a Landau level, it fractionalizes among the quasiparticles and the edge. Using these results we define a measurable quasiparticle fractional spin that satisfies the spin-statistics relation. As an application, we predict the value of the spin of the composite-fermion quasielectron proposed by Jain; our numerical simulations agree with that value. We also show that Laughlin's quasielectrons satisfy the spin-statistics relation, but carry the wrong spin to be the antianyons of Laughlin's quasiholes. We continue by highlighting the fact that the statistical angle between two quasiparticles can be obtained by measuring the angular momentum while merging the two quasiparticles. Finally, we show that our arguments carry over to the non-Abelian case by discussing explicitly the Moore-Read wave function.

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Introduction. The spin-statistics theorem is one of the pillars of our description of the world and classifies quantum particles into bosons and fermions according to their spin, integer, or half integer [1]. It was noted early on that in two spatial dimensions this relation is modified and intermediate statistics exist, called anyonic [2,3]. These objects too satisfy a generalized spin-statistics relation (SSR), and it is common nowadays to speak of fractional spin and statistics [4,5]. This type of SSR, which we also consider, arises in a nonrelativistic, non-field-theoretic context [6,7].

The quantum Hall effect (QHE) [8,9] is the prototypical setup where anyons have been studied, and several of their remarkable properties have also been experimentally observed [10,11]. Whereas the notion of fractional statistics has been earlier applied to the localized quasiparticles of the QHE [12–15], the notion of spin has been more controversial. The existence of a fractional spin satisfying a SSR has been established for setups defined on curved spaces owing to the coupling to the curvature of the surface [16–21]. The extension of this notion to planar surfaces has required more care and it is not completely settled yet [22–24].

In this Letter we prove an SSR for the Abelian quasiparticles of the QHE on a planar surface, for arbitrary filling fractions, directly from the microscopic Hamiltonian under the generic assumption that a QHE state satisfies the screening property. It does not require the notion of curvature and identifies an observable spin that is an emergent collective property unrelated to the physical SU(2) spin. Several applications are presented. First, we study the quasielectron (QE) wave functions proposed by Jain [25] and by Laughlin [26] for the filling factor $\nu = 1/M$. Second, we show how the fractional statistics affects the total angular momentum of the setup. Third, we discuss how our arguments carry over to the non-Abelian case. Finally, we remark on an intrinsic ambiguity in the definition of the spin.

The QHE model. We consider a two-dimensional (2D) system of N quantum particles with mass m and charge $q > 0$ traversed by a uniform and perpendicular magnetic field $\vec{B} = B\hat{e}_z$, $B > 0$. The cyclotron frequency and the magnetic length read $\omega = qB/m$ and $\ell_B = \sqrt{\hbar c/(qB)}$. We adopt the standard parametrization of the plane $z_j = x_j + iy_j = |z_j|e^{i\phi_j}$.

The Hamiltonian is

$$H_0 = \sum_{i=1}^{N}\left(\frac{\pi_{i,x}^2 + \pi_{i,y}^2}{2m} + v(|z_i|)\right) + \sum_{i<j} V_{\rm int}(|z_i - z_j|), \quad (1)$$

where $\pi_{i,a} = p_{i,a} - (q/c)A_a(z_i)$ and $v(|z|)$ is a central confining potential. We assume that the interaction potential $V_{\rm int}(|z|)$ is rotationally invariant, as it is the case for the Coulomb [27] and contact [28] interactions, relevant for electrons and for cold gases respectively. We also assume that the ground state of (1) is not degenerate and realizes an incompressible QHE state characterized by screening: In the presence of perturbations which do not close the energy gap the particles will arrange in such a way that the density of the system is the same everywhere except in an exponentially localized region close to the defects; gentle modifications of the confinement potentials fall into this class of perturbations, so that the specific form of $v(|z|)$ is not important if we are only interested in the bulk.

We assume the presence of $N_{\rm qp}$ pinning potentials located at positions $\mathbf{s}_\alpha$; using the complex-plane parametrization

*alberto.nardin@unitn.it
†ardonne@fysik.su.se
‡leonardo.mazza@universite-paris-saclay.fr

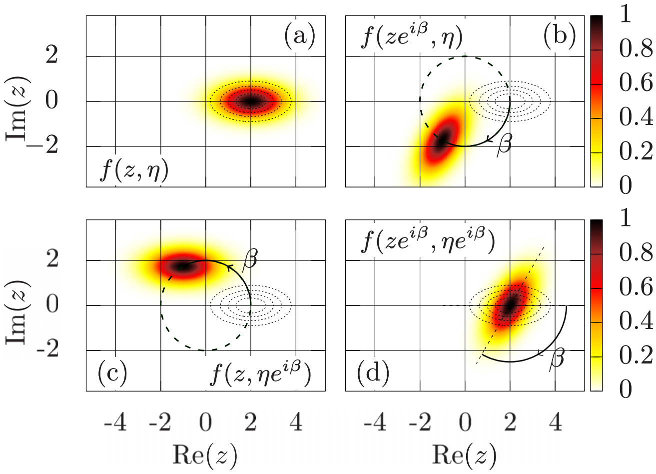

FIG. 1. Action of $\tilde{L}$. (a) Contour plot in z space of the function $f(z,\eta) = \exp[-\frac{1}{2}\mathrm{Re}(z-\eta)^2 - 2\,\mathrm{Im}(z-\eta)^2]$ for $\eta = 2$. (b) Contour plot of $f(ze^{i\beta},\eta)$ for $\beta = 2\pi/3$: With respect to (a), the plot is translated and rotated. (c) Contour plot of $f(z,\eta e^{i\beta})$: This time, the plot is only translated. (d) Contour plot of $f(ze^{i\beta},\eta e^{i\beta})$: The composition of the two is just a rotation and thus $\tilde{L}$ generates the self-rotations of the quasiparticles in the z plane.

$\eta_\alpha = s_{\alpha,x} + is_{\alpha,y} = |\eta_\alpha|e^{i\theta_\alpha}$ we write

$$H_1(\boldsymbol{\eta}) = \sum_{\alpha=1}^{N_{\mathrm{qp}}}\sum_{i=1}^{N} V_\alpha(|z_i - \eta_\alpha|), \tag{2}$$

where $\boldsymbol{\eta}$ is shorthand for $\eta_1,\dots,\eta_{N_{qp}}$. Since the pinning potentials might be different, we keep the subscript V_α; they are all assumed to be rotationally invariant.

The ground state of the model $H_{\boldsymbol{\eta}} = H_0 + H_1(\boldsymbol{\eta})$ is $|\Psi_{\boldsymbol{\eta}}\rangle$; we assume that it is unique and that it localizes N_{qp} quasiparticles at η_α. By virtue of screening, the density is the same everywhere as in the absence of pinning potentials, except close to the defects and at the boundary. Since the pinning potentials can be different, the quasiparticles need not be of the same kind. The set of η_α is completely arbitrary and rotational invariance is generically broken; in our discussion, we will assume that they are always kept far from the boundary. These assumptions imply that $|\Psi_{\boldsymbol{\eta}}\rangle$ can be a smooth function of $\boldsymbol{\eta}$.

Quasiparticle self-rotations. We introduce the operator that is the sum of the particle angular momentum and of its quasiparticle generalization measured in units of $\hbar$ (we use the symmetric gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\wedge\mathbf{r}$),

$$\tilde{L} = L_z + L'_z \quad \text{with} \quad L_z = -i\sum_{i=1}^{N}\frac{\partial}{\partial\phi_i}, \quad L'_z = -i\sum_{\alpha=1}^{N_{\mathrm{qp}}}\frac{\partial}{\partial\theta_\alpha}. \tag{3}$$

We also define the group operator $U_\beta = e^{i\beta\tilde{L}}$ that is generated by (3), with $\beta\in\mathbb{R}$. The physical meaning of U_β is best understood by considering its effect on a generic function $f(z,\eta)$ (see Fig. 1). Globally, U_β is the composition of the two transformations, and represents the quasiparticle self-rotations over an angle β.

Since the η_α are parameters, a gauge transformation $|\Psi_{\boldsymbol{\eta}}\rangle \to e^{ig(\boldsymbol{\eta})}|\Psi_{\boldsymbol{\eta}}\rangle$ using an arbitrary smooth function of the parameters $g(\boldsymbol{\eta})$ does not change the energy of the state. Our goal is to show that it is always possible to use a gauge such that the ground state is annihilated by $\tilde{L}$ and is thus invariant under the quasiparticle self-rotation operator U_β; this result is crucial for the proof of the SSR.

Let us first consider for simplicity the case of one quasiparticle, $N_{\mathrm{qp}} = 1$, so that $L'_z = -i\partial_\theta$ (we suppress the index $\alpha = 1$ for brevity). The Hamiltonian is explicitly invariant under the action of the group: $U_\beta H_\eta U_\beta^\dagger = H_\eta$. With a quasiparticle at η, the ground state satisfies the Schrödinger equation $H_\eta|\Psi_\eta\rangle = E_\eta|\Psi_\eta\rangle$. However, E_η can only depend on $|\eta|$, and not on θ; thus, $\partial_\theta E_\eta = 0$. We conclude that $H_\eta U_\beta|\psi_\eta\rangle = U_\beta E_\eta|\Psi_\eta\rangle = E_\eta U_\beta|\Psi_\eta\rangle$, namely that $U_\beta|\Psi_\eta\rangle$ is an eigenvector of H_η with energy E_η. If the ground state is unique, it must be an eigenvector of U_β and of its generator $\tilde{L}$; we dub the eigenvalue of the latter ℓ_η. For example, in the case of the normalized Laughlin state with a quasihole (QH), $\mathcal{N}(|\eta|)^{-1/2}\prod_i(z_i-\eta)\prod_{j<k}(z_j-z_k)^M e^{-\sum_i|z_i|^2/4\ell_B^2}$, the eigenvalue $\ell_\eta = \frac{M}{2}N(N-1)+N$ is the degree of the polynomial in z_i and η.

We now perform a gauge transformation that unwinds the generalized angular momentum ℓ_η moving along a trajectory at fixed $|\eta|$: $|\tilde{\Psi}_\eta\rangle = e^{i\tilde{g}(\eta)}|\Psi_\eta\rangle$ with $\tilde{g}(\eta) = -\int_0^\theta \ell_{|\eta|e^{i\theta'}}d\theta'$. In the aforementioned case, the Laughlin state gets multiplied by the phase $(\eta/\eta^*)^{-\frac{M}{4}N(N-1)-\frac{N}{2}}$. Let us show that

$$\tilde{L}|\tilde{\Psi}_\eta\rangle = 0. \tag{4}$$

By definition, $\tilde{L}|\tilde{\Psi}_\eta\rangle = e^{i\tilde{g}(\eta)}\tilde{L}|\Psi_\eta\rangle + (L'e^{i\tilde{g}(\eta)})|\Psi_\eta\rangle$. The first term of the sum is $\ell_\eta|\tilde{\Psi}_\eta\rangle$, the second term is obtained by differentiating the exponential, and equals $-\ell_\eta|\tilde{\Psi}_\eta\rangle$. This concludes the proof of the statement that one can find a gauge such that $\tilde{L}$ annihilates the ground state. Note that for a state satisfying Eq. (4), it is also true that $U_\beta|\tilde{\Psi}_\eta\rangle = |\tilde{\Psi}_\eta\rangle$ for any angle β. Choosing $\beta = 2\pi$ we obtain that this state is single valued in the η coordinate because $U_{2\pi}|\tilde{\Psi}_\eta\rangle$ is also equal to $|\tilde{\Psi}_{\eta e^{i2\pi}}\rangle$.

This reasoning can be easily extended to the case of several quasiparticles. We can define a reference angle θ_0 and express $\theta_\alpha = \theta_0 + \Delta\theta_\alpha$, treating the variables $\Delta\theta_\alpha = \theta_\alpha - \theta_0$ as independent from θ_0. The operator $-i\partial_{\theta_0}$ generates the group $e^{i\beta\times(-i\partial_{\theta_0})}$ that modifies the quasiparticle polar angles as follows, $\theta_\alpha \to \theta_\alpha + \beta$, leaving the radial distance unchanged; thus $\eta_\alpha \to \eta_\alpha e^{i\beta}$. This is exactly the action of L'_z, and thus we conclude that $L'_z = -i\partial_{\theta_0}$. With arguments paralleling those for one quasiparticle, one can (i) show that $\tilde{L}|\Psi_{\boldsymbol{\eta}}\rangle = \ell_{\boldsymbol{\eta}}|\Psi_{\boldsymbol{\eta}}\rangle$, (ii) make the dependence on θ_0, the $\Delta\theta_\alpha$ and the $|\eta_\alpha|$ explicit by writing $\ell_{\theta_0,\Delta\theta_\alpha,|\eta_\alpha|}$, and (iii) define $|\tilde{\Psi}_{\boldsymbol{\eta}}\rangle = e^{ig(\boldsymbol{\eta})}|\Psi_{\boldsymbol{\eta}}\rangle$ with $g(\boldsymbol{\eta}) = -\int_0^{\theta_0}\ell_{\theta'_0,\Delta\theta_\alpha,|\eta_\alpha|}d\theta'_0$, which is in the kernel of $\tilde{L}$.

Berry phase for the translation of the quasiparticles along a circle. We now compute the Berry phase corresponding to the translation along a closed circular path of the N_{qp} quasiparticle coordinates via $\theta_0 \to \theta_0 + 2\pi$ generated by L'_z, leaving all the $\Delta\theta_\alpha$ and $|\eta_\alpha|$ invariant. Using the fact that $|\tilde{\Psi}_{\boldsymbol{\eta}}\rangle$ is single valued in $\boldsymbol{\eta}$, this Berry phase is $\gamma_{\boldsymbol{\eta}} = \int_0^{2\pi}\langle\tilde{\Psi}_{\boldsymbol{\eta}}|i\partial_{\theta_0}|\tilde{\Psi}_{\boldsymbol{\eta}}\rangle d\theta_0$, where only the θ_0 coordinate is changed in the state inside the integral. Employing the definitions of $\tilde{L}$ and L'_z, and using (4), we get

$$\gamma_{\boldsymbol{\eta}} = \int_0^{2\pi}\langle\tilde{\Psi}_{\boldsymbol{\eta}}|L_z|\tilde{\Psi}_{\boldsymbol{\eta}}\rangle d\theta_0 = \int_0^{2\pi}\langle\Psi_{\boldsymbol{\eta}}|L_z|\Psi_{\boldsymbol{\eta}}\rangle d\theta_0. \tag{5}$$

The matrix element in the integral is manifestly gauge independent, as the L_z operator does not act on the $\boldsymbol{\eta}$; one can thus also use the original states. Finally, let us note that the integrand cannot be a function of θ_0, and thus we have an even simpler expression: $\gamma_{\boldsymbol{\eta}} = 2\pi \langle \Psi_{\boldsymbol{\eta}} | L_z | \Psi_{\boldsymbol{\eta}} \rangle$. This result was first established in Ref. [29] for the specific case of the Laughlin wave function and is here proved in full generality.

As any operator projected onto the lowest Landau level (LLL), the angular momentum L_z is a function of the guiding-center operators [8] $R_{j,x} = x_j + (\ell_B^2/\hbar)\pi_{j,y}$ and $R_{j,y} = y_j - (\ell_B^2/\hbar)\pi_{j,x}$, with $[R_{j,x}, R_{j',y}] = -i\ell_B^2 \delta_{j,j'}$, and it reads $L_z = \sum_j (R_j^2/\ell_B^2 - 1)/2$. Written in this projected form, L_z is the gauge-invariant generator of rotations, and it is just a function of the density of the gas $\rho_{\boldsymbol{\eta}}(z)$, which through the screening property can be split into a bulk contribution $\rho_b(z)$ (the state without quasiparticles), an edge contribution $\rho_e(z)$ (the difference at the edge with respect to the state without quasiparticles), and a quasiparticle contribution localized around the η_α, $\rho_{\mathrm{qp},\boldsymbol{\eta}}(z)$. We split the integrand into three parts: $\langle \Psi_{\boldsymbol{\eta}} | L_z | \Psi_{\boldsymbol{\eta}} \rangle = L_b + L_e(N_{\mathrm{qp}}) + L_{\mathrm{qp}}(\boldsymbol{\eta})$; as long as the quasiparticles are far from the edge, the screening property ensures that L_e can only depend on their number (more precisely, on how many quasiparticles of each kind), but not on their positions; in fact, it also does not change when two of them are put close by or stacked on top of each other.

Notice that L_b is an integer owing to rotational invariance; therefore we disregard this contribution to the Berry phase (5). The only relevant information is contained in the remaining pieces, which indeed depend, directly or indirectly, on the quasiparticles, and this constitutes the first main result of the Letter:

$$\gamma_{\boldsymbol{\eta}} = 2\pi \times [L_e(N_{\mathrm{qp}}) + L_{\mathrm{qp}}(\boldsymbol{\eta})]. \tag{6}$$

Compared to the direct computation of the integral, Eq. (6) is simpler to evaluate.

Let us consider now the case of a single quasiparticle at η; on the basis of very general arguments, γ_η should be the Aharonov-Bohm (AB) phase $qQ\pi|\eta|^2 B/(\hbar c)$, where Q is the charge of the quasiparticle in units of q. Let us compare Eq. (6) with this widely accepted result. In very general terms, the angular momentum of a rotationally invariant quasiparticle $L_{\mathrm{qp}}(\eta) = \int d^2r (r^2/2\ell_B^2 - 1)\rho_{\mathrm{qp},\eta}(r)$ can be split into an orbital part $\frac{Q|\eta|^2}{2\ell_B^2}$ and an intrinsic part,

$$J_{\mathrm{qp}} = \int d^2r \big(r^2/2\ell_B^2 - 1\big)\rho_{\mathrm{qp},\eta=0}(r) = L_{\mathrm{qp}}(0). \tag{7}$$

It follows that $\gamma_\eta = \pi Q|\eta^2|/\ell_B^2 + 2\pi[L_e(1) + J_{1\mathrm{qp}}]$. We recognize the AB phase, to which an apparently spurious contribution has been added; yet, we can show that it is an integer multiple of 2π, and thus inessential. To show that $L_e(1) + J_{1\mathrm{qp}}$ is an integer, we consider a system with a QP *at its center*, which is rotationally invariant, so its angular momentum $L_b + J_{1\mathrm{qp}} + L_e(1)$ is an integer; since $L_b \in \mathbb{Z}$, $J_{1\mathrm{qp}} + L_e(1)$ is also an integer. By the same logic $J_{n\mathrm{qp}} + L_e(n) \in \mathbb{Z}$ where $J_{n\mathrm{qp}}$ is the spin of the rotationally symmetric QP obtained by fusing n QPs together, stacking them on top of each other. Very generically, the gauge-invariant generator of rotations fractionalizes between the bulk quasiparticles and the edge, implying that the spin is robust to local circularly symmetric perturbations. Before continuing, we mention a set of earlier works that have studied the properties of the second moment of the depletion density of fractional quasiparticles, which is shown to be related to the conformal dimension [30–32].

Spin-statistics relation. We consider two identical quasiparticles placed at opposite positions η and $-\eta$ and far from each other and from the edge. In order to compute the statistical parameter κ, we consider a double exchange, that gives a gauge-invariant expression and avoids any discussion on the identity of the pinning potentials [12]. Accordingly, we study the difference between the Berry phase for exchanging two opposite particles and the single-particle AB phases [33]:

$$\kappa_{\mathrm{qp}} = \frac{1}{2\pi}(\gamma_{\eta,-\eta} - 2\gamma_\eta). \tag{8}$$

Using Eq. (6), we write $\kappa_{\mathrm{qp}} = L_e(2) + L_{\mathrm{qp}}(\eta, -\eta) - 2L_e(1) - 2L_{\mathrm{qp}}(\eta)$. As long as the QPs are well separated and since the liquid is screening, $\rho_{\mathrm{qp}}(\eta, -\eta) = \rho_{\mathrm{qp},\eta} + \rho_{\mathrm{qp},-\eta}$ and thus $L_{\mathrm{qp}}(\eta, -\eta) = 2L_{\mathrm{qp}}(\eta)$; since $L_e(n) + J_{n\mathrm{qp}} \in \mathbb{Z}$ we obtain the SSR:

$$\kappa_{\mathrm{qp}} = -J_{2\mathrm{qp}} + 2J_{1\mathrm{qp}} \pmod 1. \tag{9}$$

This result allows us to identify the intrinsic angular momentum with the fractional spin associated to the fractional statistics, and constitutes the second main result of the Letter. Interestingly, we have linked the statistics to a local property of the quasiparticles: If we assume screening, the fine details of the boundary do not matter, and one could probably prove (9) without requiring that $v(|z|)$ is a central potential.

With similar arguments, the SSR can be extended to the situation where the two quasiparticles are different: Calling J_a and J_b their spins, and J_{ab} the spin of the composite quasiparticle obtained by stacking them at the same place, we obtain the mutual statistics parameter: $\kappa_{ab} = -J_{ab} + J_a + J_b \pmod 1$. In the theory of modular tensor categories (see, for instance, Ref. [34]), a relation of this type is called a *ribbon identity*. Moreover, the fractionalization property allows us to read the phase κ directly at the edge; indeed, one easily obtains $\kappa_{\mathrm{qp}} = L_e(2) - 2L_e(1)$ and $\kappa_{ab} = L_e(a, b) - L_e(a) - L_e(b)$.

The spin of the QE. As a first application of our SSR (9), we consider the QE of the Laughlin state at filling $\nu = 1/M$. Numerical studies have highlighted that the composite-fermion wave function for the QE proposed by Jain [25,35] has the correct statistical properties when the QE is braided with another QE ($\kappa_{\mathrm{qe}} = 1/M$) or with a QH ($\kappa_{\text{qe-qh}} = -1/M$) [36–41]. Previous papers have already shown that LLL quasiparticles composed of p stacked QHs fractionalize the angular momentum $J_p = -p^2/(2M) + p/2$, and that these results are compatible via the SSR (9) with a correct QH statistics $\kappa_{\mathrm{qh}} = 1/M$ [21,24].

On the basis of these results and of the SSRs, it is easy to predict that Jain QE fractionalizes the same spin J_p, with $p < 0$ for QEs and $p > 0$ for QHs. We numerically verify this statement by performing a Monte Carlo analysis of Jain's wave function with one QE ($p = -1$) or two QEs ($p = -2$) [42]. Table I summarizes the expected values. The results of our simulations are in Figs. 2(a) and 2(b), and they agree perfectly with our theory. In the Supplemental Material [42] we show the same results obtained with the matrix-product-

TABLE I. The spin J_p of Jain's QE at filling factor ν.

	$\nu = \frac{1}{2}$	$\nu = \frac{1}{3}$	$\nu = \frac{1}{4}$
$p = -1$	$-\frac{3}{4}$	$-\frac{2}{3}$	$-\frac{5}{8}$
$p = -2$	-2	$-\frac{5}{3}$	$-\frac{3}{2}$

state formulation [41,43–46] using the Landau gauge. As we anticipated, our definition of quasiparticle spin is gauge invariant, and even if the two simulations are performed in different gauges (the symmetric and the Landau ones) the results coincide. Remarkably, this way of assessing the statistics of Jain's QE does not suffer from the undesired multiparticle position shift that needs to be taken into account in order to get the correct statistical phase [37,41].

Concerning the QE wave function proposed in the original paper by Laughlin [26], it was shown that it fractionalizes the correct charge, without making definitive statements about its braiding properties [33,36–38,40]. The results of our numerical simulations are in Figs. 2(c) and 2(d). The plateau values are described by the spin $J'_p = -p^2/(2M) + p(2-M)/(2M)$, that gives the correct braiding phase for the Laughlin QEs, but that also shows that it is not the antianyon of the Laughlin's QH.

Angular momentum of the gas. As a further application of the SSR, let us consider what happens when two QHs placed far apart are displaced radially in the sample. Let us call L_0 the

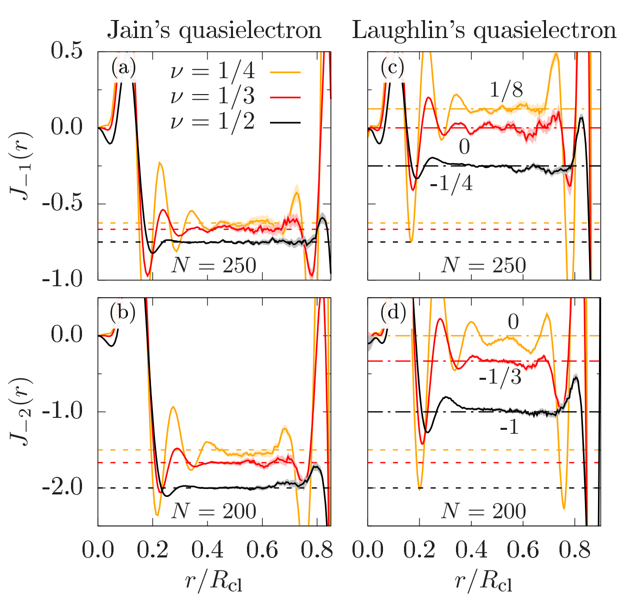

FIG. 2. Calculation of the QE spin via the integral $J(r) = \int_0^r (\frac{|r'|^2}{2\ell_B^2} - 1)\rho_{\rm qp}(r')2\pi r' dr'$; the spin of Eq. (7) coincides with the plateau appearing when r is far from the center and the boundary; $R_{\rm cl} = \sqrt{2N/\nu}$ is the classical radius of the droplet. (a) The spin of a single Jain's QE for $\nu = 1/2$, 1/3, and 1/4. (b) The case of two stacked Jain's QEs. (c) and (d) The same for one and two Laughlin's QEs, respectively. Theoretical predictions following from the SSR relation in Table I are marked with dashed lines and are only compatible with the spin of Jain's QE. Dashed-dotted lines in (c) and (d), together with their values, highlight the position of the spin plateau for Laughlin's QE.

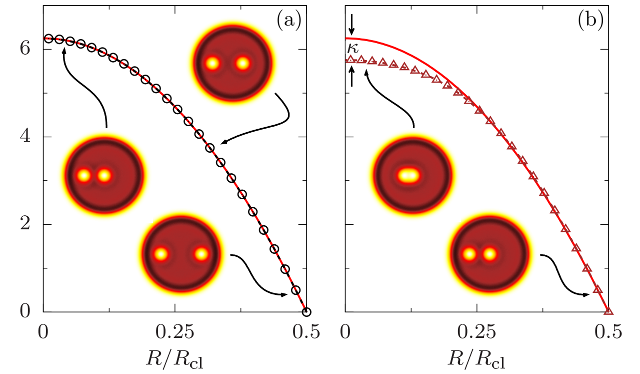

FIG. 3. Angular momentum $L(R_1, R_2)$ of a Laughlin state ($N = 25$, $\nu = 1/2$) with two QHs at distances R_1 and R_2 from the center, computed with Monte Carlo techniques [47,48]. (a) Displacement of the first QH; the angular momentum variation $L(R, R_0) - L(R_0, R_0)$ is plotted in black circles, and it is a quadratic function of R that agrees with the theory prediction $-\epsilon(R^2 - R_0^2)$ (red line). (b) Displacement of the second QH; the variation $L(0, R) - L(0, R_0)$ is plotted in brown triangles and it is a quadratic function of R only at large R; when the QHs fuse a deviation sets in that equals $-\kappa$, the statistical parameter.

angular momentum of the initial state with both quasiparticles at the same distance R_0 from the center. The first QH is then moved to the center: During this process the angular momentum increases and depends on the distance R as $L(R) = L_0 - \epsilon(R^2 - R_0^2)$ with $\epsilon = qQ\pi B/(hc)$ [42]. A gain in angular momentum of ϵR_0^2 is expected at the end of the process. The same is now done with the second QH. Whereas also in this case the angular momentum increases, it does not attain the value $L_0 + 2\epsilon R_0^2$ because when the two QHs fuse, their total spin changes. In fact, the final value is $L_0 + 2\epsilon R_0^2 - \kappa$. We verify this result with numerical simulations reported in Fig. 3. This provides an experimental procedure for measuring the mutual statistics of two generic quasiparticles in a controllable quantum simulator of the QHE.

The non-Abelian case. Our arguments carry over to non-Abelian QHE states (see Ref. [49] for some earlier ideas). When considering a state with two QPs in a definite fusion channel, the ground state is actually unique (we restrict ourselves to the case without fusion multiplicities); therefore, even the non-Abelian case is covered, because the hypotheses of derivation of the SSR are uniqueness of the ground state, screening, and rotational invariance. The non-Abelian nature shows up via the possibility that fusing two QPs can lead to different anyons, labeled by c. There is a different SSR for each possibility, $\kappa_{ab,c} = -J_c + J_a + J_b \pmod 1$.

As an example we discuss the SSR for the Moore-Read (MR) state [50]. We write the filling fraction of the state as $\nu = \frac{1}{q}$, where q is even in the fermionic case and odd in the bosonic one. The MR state is defined in terms of a chiral boson field φ and the fields of the Ising conformal field theory [51]. This means that we should label the quasiholes by their Ising sector (i.e., $\mathbf{1}$, σ, or ψ), and their charge. The smallest charge quasihole has the labels $(\sigma, \frac{1}{2q})$. Because the fusion of two σ fields has two possible outcomes, $\sigma \times \sigma = 1 + \psi$, the fusion of two quasiholes also leads to two possible results. In

TABLE II. The spin J_p of the MR quasiparticles.

$J_{(\mathbf{1},\frac{1}{q})}$	$J_{(\sigma,\frac{1}{2q})}$	$J_{(\psi,\frac{1}{q})}$
$\frac{1}{2}$	$\frac{1}{8q}+\frac{3}{16}$	0

particular, we have (the charge label is additive, as is the case for the Laughlin state)

$$\left(\sigma,\frac{1}{2q}\right)\times\left(\sigma,\frac{1}{2q}\right)=\left(\mathbf{1},\frac{1}{q}\right)+\left(\psi,\frac{1}{q}\right). \tag{10}$$

The first possible outcome $(\mathbf{1},\frac{1}{q})$ is the quasihole one obtains by piercing the sample with an additional flux, i.e., the "ordinary" Laughlin quasihole. The second possible outcome "contains" an additional neutral fermionic mode ψ. Table II summarizes the expected value of the spin J_a for each of the aforementioned quasiparticles [42,52]. We perform a Monte Carlo sampling of the MR wave function with the different quasiparticles localized in the center of the system. The numerical calculations reported in Fig. 4 agree with the expected results [42].

Alternative spins. Our definition of spin follows directly from the physical angular momentum $L=\mathcal{L}_R+\mathcal{L}_\pi$, where $\mathcal{L}_R=(R_x^2+R_y^2)/2l_B^2$ and $\mathcal{L}_\pi=-(\pi_x^2+\pi_y^2)l_B^2/2\hbar^2$; it is manifestly gauge invariant and is the generator of two-dimensional (2D) rotations, because it satisfies $[L,R_j]=i\epsilon_{jk}R_k$ and $[L,\pi_j]=i\epsilon_{jk}\pi_k$, with ϵ_{jk} the Levi-Civita tensor. The definition is ambiguous: Any operator $\mathcal{L}_c=L+c$, with c a c-number, has indeed the correct commutation properties; this is a peculiarity of U(1) rotations in 2D physics, as SU(2) ones do not leave room for such ambiguity. We conclude that any operator $\mathcal{L}_c$ defines a correct quasiparticle spin $J_p(c)$ [21]. In very general terms, J_p is composed of a part proportional to p^2 that determines the anyonic statistics [53], and of a part proportional to p that does not affect κ_{qp} [see (9)]. It is not difficult to prove that c can only appear in the prefactor multiplying p, as it is linear in the quasiparticle density. We consider this as an essential ambiguity that cannot be resolved, although different choices may have different physical meanings.

Conclusions. We have presented a SSR for the Abelian quasiparticles of the QHE on planar surfaces derived from very mild assumptions. We have shown that the quasiparticles fractionalize the gauge-invariant generator of rotations and that this quantity can be used to define a measurable spin. The fractional statistical properties of the quasiparticles follow from that. Our results carry over to non-Abelian quantum Hall states.

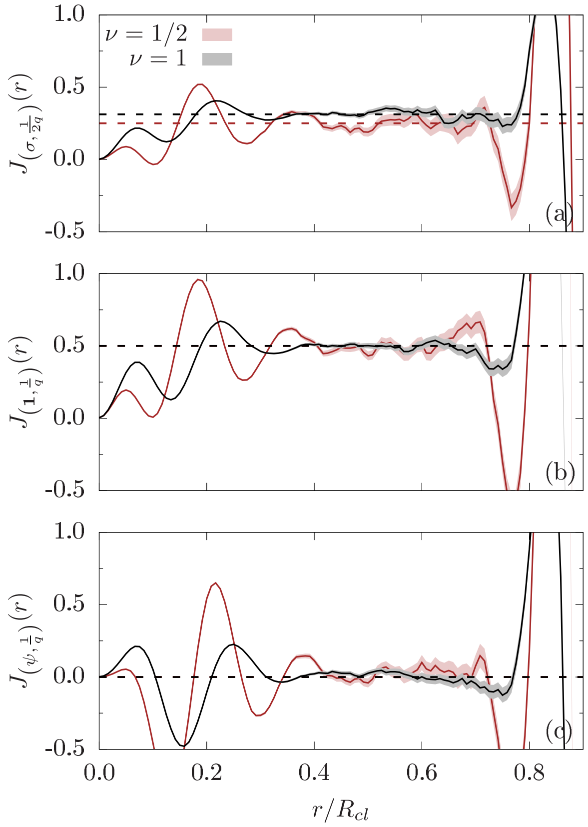

FIG. 4. Comparison of the quasihole spins $J(r)=\int_0^r(\frac{|r'|^2}{2\ell_B^2}-1)\rho_{\mathrm{qp}}(r')2\pi r'dr'$, for the different Moore-Read quasiholes: (a) the $(\sigma,\frac{1}{2q})$, (b) the $(\mathbf{1},\frac{1}{q})$, and (c) the $(\psi,\frac{1}{q})$, for the bosonic filling $\nu=1$ (corresponding to $q=1$) and the fermionic $\nu=\frac{1}{2}$ $(q=2)$. $R_{\mathrm{cl}}=\sqrt{2N/\nu}$ is the classical radius of the droplet and the number of particles is $N=200$.

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