Letter

Hydrodynamics in long-range interacting systems with center-of-mass conservation

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In systems with a conserved density, the additional conservation of the center of mass (dipole moment) has been shown to slow down the associated hydrodynamics. At the same time, long-range interactions generally lead to faster transport and information propagation. Here, we explore the competition of these two effects and develop a hydrodynamic theory for long-range center-of-mass-conserving systems. We demonstrate that these systems can exhibit a rich dynamical phase diagram containing subdiffusive, diffusive, and superdiffusive behaviors, with continuously varying dynamical exponents. We corroborate our theory by studying quantum lattice models whose emergent hydrodynamics exhibit these phenomena.

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Hydrodynamic theories are coarse-grained descriptions of the flow of conserved densities. They help us understand large-scale behavior and universality without knowing how these emerge from microscopic details [1,2]. The price for this is the introduction of unknown macroscopic parameters such as diffusivity or viscosity, which are difficult to compute from first principles [3–8]. Understanding how hydrodynamics emerges in quantum systems is of great interest, and recent progress has been made in this direction [9–13]. It is also of interest to understand what kinds of hydrodynamic behavior can possibly arise in quantum many-body systems, including both chaotic and integrable models [14–30], as well as models with symmetries or kinetic constraints [31–37].

For highly excited quantum chaotic lattice models, the default expectation is often that standard diffusion will emerge. However, the dynamics can be systematically slowed down by imposing further constraints on the allowed microscopic transitions such that higher moments of the density distribution are conserved. A common example of this is the subdiffusion that results from conserving the center of mass (c.m.), also known as the dipole moment [38–50]. This can occur in "tilted" systems with a strong linear potential [51–54], as realized in cold-atom setups [55–57], and is relevant also for quantum Hall systems [58–61]. The additional conservation law can not only modify the hydrodynamics but can result in a frozen phase where the state space is fragmented into dynamically disconnected sectors [62–71]. While we focus on the highly excited dynamics of such systems, there has also been interest in their low-temperature equilibrium properties [72–84].

In contrast to constrained dynamics, faster dynamics can be achieved in systems with long-range interactions [85,86]. Interactions that decay as a power law of the distance between particles—including Coulomb, dipolar, and van der Waals—are ubiquitous, and can nowadays be explored using, e.g., ultracold atoms [87–90], polar molecules [91–93], or trapped ions [94]. In such systems, Lieb-Robinson bounds can become superballistic, and hydrodynamics superdiffusive, depending on the power-law exponent and the dimensionality of the underlying lattice [95–105].

Given their competing effects, it is natural to examine the interplay between higher-order conservation laws and long-range interactions, however, this problem has only been briefly touched on in the literature [39]. Here, we bridge the gap and consider a class of models where "hops" are long range but always occur in pairs such that the c.m. is conserved [Fig. 1(a)]. We develop a hydrodynamic theory of such models and map out its phase diagram [Figs. 1(b) and 1(c)]. More specifically, our model is characterized by two power-law exponents, α and β, that control the suppression of pair-hopping amplitudes. We calculate the dynamical exponent z, governing long-wavelength relaxation, as a function of these two exponents, as well as the scaling functions for density-density correlations. When α and β are large enough, we recover the known $z = 4$ subdiffusion of short-range c.m.-conserving systems in one dimension. When α or β is small enough, z continuously varies. The scaling function of the correlations also continuously varies with the exponents. We find consistent results when examining a less-tractable quantum spin-1 model (a long-range version of the systems studied in Refs. [64,65]). We also discuss the relevance of our findings to long-range many-body quantum systems in a tilted potential.

Setup. As a starting point, we consider a one-dimensional (1D) quantum spin model

$$H = \sum_x \sum_{s,r} J_{sr}(S_x^+ S_{x+r}^- S_{x+r+s}^- S_{x+s+2r}^+ + \text{H.c.}), \qquad (1)$$

where $S_x^\pm$ are raising and lowering operators at site x, and J_{sr} depend on the distances $s \geqslant 0$ and $r > 0$. The Hamiltonian conserves the magnetization $\sum_x S_x^z$ and its dipole moment $\sum_x x S_x^z$, i.e., the "center of mass" of the S^z density. Equation (1) can be understood as a generalization of the short-range "pair-hopping" models of Refs. [64,65,67]. When those systems thermalize, they exhibit subdiffusion with dynamical exponent $z = 4$ and density correlations that are nonmonotonic in space [38–40]. Here, we are interested in

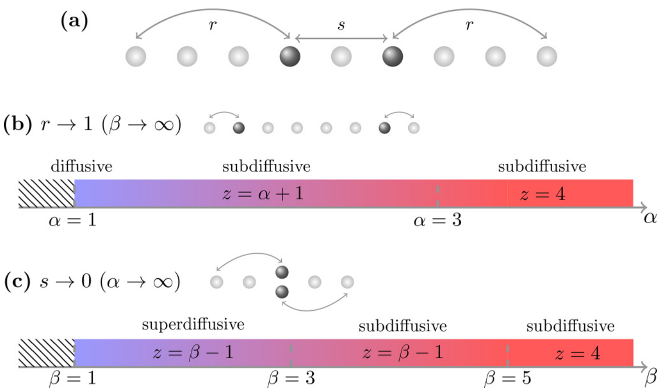

FIG. 1. (a) We consider models involving long-range interactions and c.m. conservation. The dynamics are governed by pairs of equal and opposite currents (or the underlying "pair-hopping" processes depicted) acting on lattice sites separated by the distances $s \geqslant 0$ and $r \geqslant 1$ (the "interaction" and "hopping" distances). The strengths of the currents are suppressed by power laws $(s+1)^{-\alpha} r^{-\beta}$, so α controls the range of interactions and β the range of hopping. (b), (c) Schematic cuts through the dynamical phase diagram for the limiting cases $r \to 1$ $(\beta \to \infty)$ and $s \to 0$ $(\alpha \to \infty)$.

how the hydrodynamics is affected by long-range interactions, particularly for J_{sr} with power-law decay.

Since directly computing the emergent hydrodynamics of quantum models is generally intractable, here we also introduce a solvable hydrodynamic model that is inspired by Eq. (1). We consider a density $n_x \in \mathbb{R}$ which is slightly perturbed from its uniform equilibrium value. The dynamics of the model are governed by pairs of equal but opposite currents that displace density from positions $x+r$ and $x+r+s$ to positions x and $x+s+2r$ (and vice versa), where $s \geqslant 0$ and $r \geqslant 1$ [Fig. 1(a)]. Such "pair currents" occur in superposition for all s and r, but the magnitudes of each are driven by the density at the four positions involved. The analog to Fick's law is that each pair current is driven with strength proportional to the curvature of n_x on those four sites [38]. The proportionality for a given s and r is denoted by C_{sr} and it encodes the suppression of such pair currents as a function of s and r. The equation of motion for the density profile is

$$\begin{aligned}\dot{n}_x = \sum_{r,s} C_{sr}[&-(n_{x-2r-s} - n_{x-r-s} - n_{x-r} + n_x)\\ &+ (n_{x-r-s} - n_{x-s} - n_x + n_{x+r})\\ &+ (n_{x-r} - n_x - n_{x+s} + n_{x+s+r})\\ &- (n_x - n_{x+r} - n_{x+s+r} + n_{x+s+2r})]. \qquad (2)\end{aligned}$$

The four lines correspond to the ways in which four sites with spacings s and r can intersect site x. As a simple tractable choice, we will focus on the separable form

$$C_{sr} = (s+1)^{-\alpha} r^{-\beta}, \qquad (3)$$

where α and β are tuning parameters. Later, we will also discuss the nonseparable form of C_{sr} that arises in strongly tilted systems.

While this classical hydrodynamic model is our main focus, we expect that our results qualitatively carry over to the transport behavior of appropriate quantum systems. Indeed,

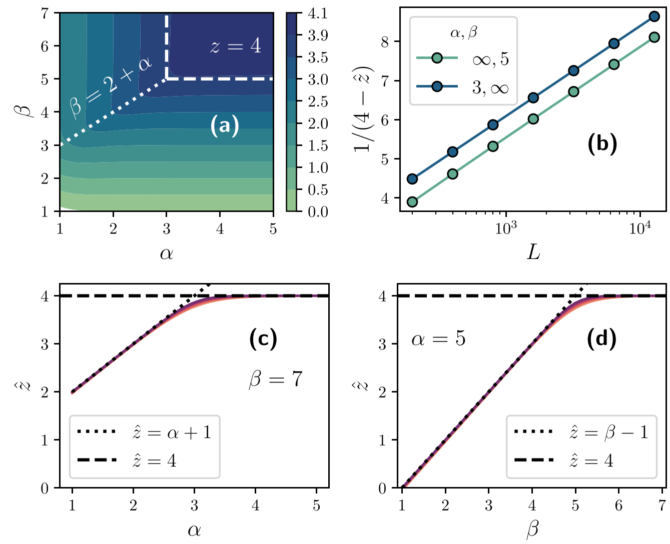

FIG. 2. The dynamical exponent z. (a) Contour plot of $\log_2(\gamma_{2k}/\gamma_k) \cong \hat{z}$ on the (α, β) plane, numerically evaluated using $k = 2\pi/3200$. The dashed box marks $z = 4$ subdiffusion. The dotted line along $\beta = 2 + \alpha$ marks where the behavior of z changes from $z = \beta - 1$ to $z = \alpha + 1$. (b) Demonstration of the boundary behavior $\hat{z}(k) = 4 - \ln(1/k)^{-1}$ where the horizontal axis is $L = 2\pi/k$, the lattice sizes used in the estimates. The vertical shift between the two curves is from different subleading additive contributions. (c), (d) Cuts along the top and right edges of the contour plot. Colors (light to dark) represent estimates using $k \geqslant 2\pi/3200$, i.e., lattices of size $L \leqslant 3200$.

after solving our hydrodynamic model, below we numerically study the dynamics generated by Eq. (1) for small spin-1 chains and obtain behavior consistent with our hydrodynamic theory if the pair-hopping amplitudes in the Hamiltonian are chosen as $J_{sr} \sim \sqrt{C_{sr}}$, in accordance with Fermi's golden rule [101] (cf. Fig. 4).

Hydrodynamic theory. Equation (2) is a linear equation with a basis of decaying plane-wave solutions $n_x = e^{ikx - \gamma_k t}$, where the decay rates are $\gamma_k = \sum_{s,r} 16 C_{sr} \sin^2(\frac{kr}{2}) \sin^2(\frac{kr+ks}{2})$. We are interested in the small-k behavior of γ_k as a function of α and β. In that limit, the lattice approximates a continuum such that

$$\gamma(k) = \int_1^\infty \int_0^\infty \frac{16 \sin^2\left(\frac{kr}{2}\right) \sin^2\left(\frac{kr+ks}{2}\right)}{(s+1)^\alpha r^\beta} ds dr. \qquad (4)$$

At small enough k we expect $\gamma(k) \propto k^z$ where z is a dynamical exponent that could dependent on α and β. It is then useful to define an effective exponent

$$\hat{z}(k) = \frac{d \ln \gamma(k)}{d \ln k}. \qquad (5)$$

To get an immediate picture of the model's behavior, in Fig. 2 we numerically evaluate $\log_2(\gamma_{2k}/\gamma_k) \cong \hat{z}$ on the (α, β) plane [Fig. 2(a)], and along two cuts through it [Figs. 2(c) and 2(d)]. We identify three phases: (A) $\alpha > 3$ and $\beta > 5$. The power laws are sufficiently short ranged that $z = 4$, consistent with the known behavior in finite-range c.m.-conserving systems in one dimension [38–40,55]. (B) $\alpha > \beta - 2$ and $\beta < 5$. The long-range "hopping" dominates. The result is a dynamical exponent $z \in [0, 4]$ that continuously varies with β but not α. This is similar to Refs. [100,101] where the hydrodynamics of models with long-range hopping (but no higher-moment conservation laws) was studied. In particular, once $\beta \leqslant 3$ such that $z \leqslant 2$, the hydrodynamics of our center-of-mass conserving model is equivalent to the model in Refs. [100,101]. (C) $\beta > \alpha + 2$ and $\alpha < 3$. The long-range "pairing" dominates, and hopping is short ranged. This again leads to a continuously varying dynamical exponent, but this time it varies with α and not β. Notably, the fastest relaxation possible in this phase is diffusive ($z = 2$). This results from the effective loss (locally) of the c.m. conservation that follows from allowing the paired short-ranged currents to be far away from each other.

While we focus on $\alpha, \beta > 1$ to maintain locality of both interaction and hopping distances, here we briefly comment on the cases where $\alpha \leqslant 1$ or $\beta \leqslant 1$ (cf. hatched areas in Fig. 1). For $\alpha \leqslant 1$, the two left sites and two right sites in the pair-hopping process can be arbitrarily far away from each other. Thereby, center-of-mass conservation is effectively lost and the system behaves as a system with only charge conservation and hopping governed by β. In contrast, for $\beta \leqslant 1$, the hops can be arbitrarily long ranged such that the system effectively becomes an all-to-all model where hydrodynamics breaks down.

These phases and boundaries can be studied in more detail by analyzing Eq. (4): At large α and β, the integral is dominated by $s, r \ll k^{-1}$ and we can expand the sin functions to leading order which yields $\gamma(k) \propto k^4$ as expected, i.e., $z = 4$. The resulting integral is only convergent when $\alpha > 3$ and $\beta > 5$, consistent with Fig. 2.

For $\alpha = \infty$ ($s = 0$), the asymptotic small-k behavior of Eq. (4) is

$$\gamma(k) = \begin{cases} (\beta - 5)^{-1} k^4, & \beta > 5, \\ k^4 \ln(1/k), & \beta = 5, \\ (2^\beta - 8)\Gamma(1-\beta)\sin\left(\frac{\pi\beta}{2}\right) k^{\beta-1}, & \beta < 5, \end{cases} \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function. Thus, for $\beta > 5$ the long-wavelength behavior is characterized by dynamical exponent $z = 4$, while for $\beta < 5$ it is $z = \beta - 1$, in accordance with Fig. 2. At the phase boundary $(\alpha, \beta) = (\infty, 5)$, the dynamics incurs a logarithmic correction that corresponds to $\hat{z}(k) = 4 - \ln(1/k)^{-1}$. This scaling of $\hat{z}$ at the phase boundary is demonstrated in Fig. 2(b).

For $\beta = \infty$ ($r = 1$), we similarly get

$$\gamma(k) = \begin{cases} (\alpha - 3)^{-1} k^4, & \alpha > 3, \\ k^4 \ln(1/k), & \alpha = 3, \\ 2[-\Gamma(1-\alpha)]\sin\left(\frac{\pi\alpha}{2}\right) k^{\alpha+1}, & \alpha < 3, \end{cases} \tag{7}$$

that is, the behavior changes from $z = 4$ to $z = \alpha + 1$ at $\alpha = 3$, and at the boundary we obtain the same logarithmic correction.

In contrast, across $\beta = 2 + \alpha$ [dotted line in Fig. 2(a)] there is a qualitative change of behavior, from $z = \beta - 1$ to $z = \alpha + 1$, but the effective z converges to a constant without a log correction. The dashed and dotted boundaries in Fig. 2(a) are therefore distinct in that respect. It should be noted that the boundaries between the "phases" are not continuous phase transitions with a diverging length scale, so there is no criticality of that type.

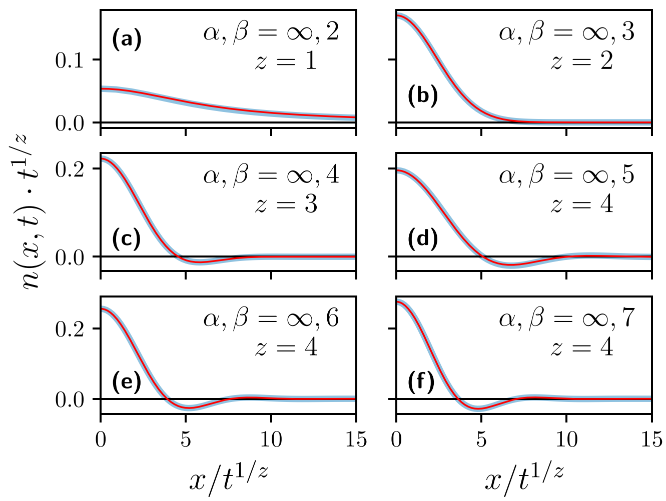

FIG. 3. Scaled density profiles obtained from time evolving $n(x, 0) = \delta_{x,0}$ under Eq. (2) (red), and comparison with $F_z(\xi)$. The system size is $L = 3200$ and the time is $t \sim \gamma_{2\pi/L}^{-1}$ for each panel. Position x extends from $-\frac{L}{2}$ to $\frac{L}{2}$ but we show $x \geqslant 0$. We set $\alpha = \infty$, and β varies over (a)–(f). The annotated values of z are used for the scaling of the axes. The slowly vanishing logarithmic correction to $z = 4$ [$\hat{z}(k) = 4 - \ln(1/k)^{-1}$] that is applicable to (d) is small enough to ignore. Data at other late times collapse well onto the same curves, but we show curves from one time for clarity. Data are in red, and $D^{-1/z} F_z$ is plotted in faint blue for comparison, where D is extracted from $n(0, t) = Dt^{1/z}$, but these are directly behind the data curves due to excellent agreement.

Distinct spatial correlations develop as a result of the various different z values. In Fig. 3 we show the late-time density profile resulting from evolving $n(x, 0) = \delta_{x,0}$, and compare with our theoretical understanding. At late enough times, the relaxation is governed by $\gamma(k) \cong Dk^z$, so $n(x, t) \cong (Dt)^{-1/z} F_z(x/(Dt)^{1/z})$, where D is the generalized diffusion constant (depends on α and β) and the only parameter controlling the scaling function $F_z(\cdot)$ is the value of z governing the long-wavelength relaxation. More specifically, we have $F_z(\xi) = \int_{-\infty}^{\infty} e^{i\kappa\xi - |\kappa|^z} \frac{d\kappa}{2\pi}$. In Fig. 3 we test this scaling, with z set to its predicted asymptotic value in each panel, and find excellent agreement. When $z = 4$ the sign of correlations is positive at smaller distances, negative at intermediate distances, and positive again at larger distances [Figs. 3(d)–3(f)]. At $z = 3$, the correlations no longer become positive again at larger distances; they approach zero from below [Fig. 3(c)]. At $z = 2$, we regain standard Gaussian diffusion at long times [Fig. 3(b)], which does not have any negative lobe in the spatial profile. Finally, at $z = 1$ we have a heavy-tailed Lorentzian profile [Fig. 3(a)].

Dynamics of a quantum spin-1 chain. Having established the properties of our classical hydrodynamic model, we now turn to less-tractable microscopic many-body quantum systems. Given a quantum spin Hamiltonian H as in Eq. (1) with couplings J_{sr}, transition rates $|\langle a|H|b\rangle|^2$ between two spin configurations $|a\rangle$ and $|b\rangle$ will scale as $\propto J_{sr}^2$ due to Fermi's golden rule [101]. Thus, in order to compare the transport properties of H to our hydrodynamic theory, it is natural to

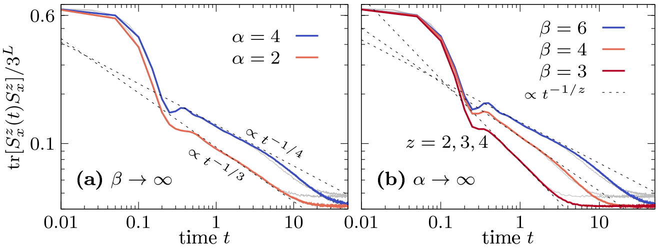

FIG. 4. Infinite-temperature autocorrelation function for a long-range c.m.-conserving spin-1 chain as in Eq. (1) with $J_{sr} \propto s^{-\alpha/2} r^{-\beta/2}$ and system size $L = 16$. (a) Varying α for $\beta \to \infty$. (b) Varying β for $\alpha \to \infty$. The dashed curves indicate power laws $\propto t^{-1/z}$ with z set according to the prediction of our hydrodynamic theory. The thin gray curves are data for smaller $L = 14$.

choose $J_{sr} \propto \sqrt{C_{sr}}$ [103,105,106]. More specifically, we study Eq. (1) for spin-1 and set $J_{sr} = 1$ if $s + 2r \leqslant 3$, while $J_{sr} \propto s^{-\alpha/2} r^{-\beta/2}$ for terms of longer range. This choice of J_{sr} avoids signatures of strong Hilbert-space fragmentation and, under $\alpha, \beta \to \infty$, recovers the pair-hopping model of Ref. [64].

In Fig. 4, we show the infinite-temperature autocorrelation function $\mathrm{tr}[S_x^z(t)S_x^z]/3^L$, obtained using quantum typicality [107,108], for chains with $L = 16$. The specific site x is irrelevant due to periodic boundaries. We consider the limiting cases of $\beta \to \infty$, i.e., short hopping distance r [Fig. 4(a)], and $\alpha \to \infty$, i.e., short pairing distance s [Fig. 4(b)]. In all cases, we find a hydrodynamic tail $\propto t^{-1/z}$, where the value of the dynamical exponent z is consistent with the prediction of our hydrodynamic theory (Fig. 1). Given the small systems in the quantum case, a more detailed comparison is challenging and the regime of smaller α, β is not accessible. Nevertheless, the data in Fig. 4 substantiate that our hydrodynamic model indeed correctly captures the dynamics of generic quantum systems with long-range pair-hopping processes.

Quantum systems in a tilted lattice. Our study of the hydrodynamics in long-range models with c.m. conservation is also motivated by potential experimental realizations. Consider, e.g., a long-range spin-1/2 XY model in a "tilted" potential,

$$H_{\mathrm{XY}} = \sum_{i<j} \frac{J}{|i-j|^{\nu}} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+) + F \sum_i i n_i, \qquad (8)$$

where F is the tilt strength, $\nu > 1/2$ the power-law exponent, and $n_i = (\sigma_i^z + \mathbb{1})/2$. Such models with power-law interactions can nowadays be achieved in experiments with Rydberg atoms, polar molecules, or trapped ions, at least for certain values of ν, e.g., Refs. [100,109,110].

When F/J is large, the model has a long-lived conservation of a dressed version of $F \sum_i i n_i$. As we show in the Supplemental Material (SM) [111] using a Schrieffer-Wolff (SW) transformation, the dynamics in the strong-tilt regime are governed by pair-hopping processes $\sigma_i^+ \sigma_{i+r}^- \sigma_{i+r+s}^- \sigma_{i+2r+s}^+$ (and H.c.). These terms occur with amplitude

$$J_{\mathrm{eff}} \propto \frac{J^3}{F^2} \frac{1}{r^{\nu+1}(r+s)^{\nu+1}} \left(\frac{1}{s^{\nu}} - \frac{1}{(2r+s)^{\nu}} \right), \qquad (9)$$

and can be understood as third-order processes via, e.g., combining $\sigma_{i+2r+s}^+ \sigma_{i+r+s}^-$, $\sigma_{i+r+s}^+ \sigma_{i+r}^-$, and $\sigma_i^+ \sigma_{i+r+s}^-$, with the intermediate states being off-shell by energies Fr and $F(r+s)$ [111]. Therefore, strongly tilted long-range models are indeed governed by pair hopping analogous to our hydrodynamic model. However, we note that Eq. (9) is not separable and depends only on one tuning exponent ν.

Similar to our results in the context of Fig. 2, we can make predictions about the hydrodynamics of the strongly tilted XY model by studying the small-k behavior of the decay rates $\gamma(k)$, but now for a hydrodynamic model with the more complicated pair-current strengths $C_{sr} \propto J_{\mathrm{eff}}^2$. Doing so, we find that $z = 4$ subdiffusion in fact remains stable even for originally long-range systems with small $\nu = 1/2$. This stability occurs because the exponents in the denominator of J_{eff}^2 are larger due to the pair-hopping processes being generated at third order. Realizing the dynamical regimes with continuously varying z found for our simplified model in Fig. 1 thus seems challenging in strongly tilted systems; see also the SM [111].

Conclusion. Using a hydrodynamic model inspired by pair hopping, we have investigated the interplay of long-range interactions and c.m. conservation on the dissipative hydrodynamics of 1D systems. We have also corroborated the predictions of our theory using simulations of small spin-1 chains. When the interactions decay sufficiently quickly, the dynamics are subdiffusive with dynamical exponent $z = 4$, consistent with earlier work. As the power-law interactions are made longer ranged, our model enters a phase where z, and the associated spatial profile of density correlations, vary continuously. One possible extreme for this is when the "pairing distance" of the pair currents is long range but the "hopping distance" is short range, where transport can become as fast as diffusive ($z = 2$). Another extreme is where the pairing distance is short and the hopping distance is long, then z can be made arbitrarily small.

While we have discussed the connections and differences of our pair-hopping model to potential experimental realizations in strongly tilted systems, we note that in the case of short-range systems, Ref. [55] observed and explained subdiffusive relaxation at long wavelengths even with weak lattice tilts. Indeed, as we numerically illustrate in the SM [111], taking F to be large appears not to be necessary to study the interplay of long-range interactions and c.m. conservation. Nonetheless, it appears that the regimes of $z < 4$ found in this work will be difficult to realize using tilted systems. How to realize these regimes in experimental systems is therefore an interesting question for future research.

Note added. Recently, we became aware of related independent works by Gliozzi *et al.* [112] and Ogunnaike *et al.* [113]. These contain generalizations of our results for higher-moment conservation laws and higher dimensions.

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[1] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, UK, 1995).

[2] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Theoretical and Mathematical Physics (Springer, Berlin, 2012).

[3] E. Leviatan, F. Pollmann, J. H. Bardarson, D. A. Huse, and E. Altman, Quantum thermalization dynamics with matrix-product states, arXiv:1702.08894.

[4] B. Kloss, Y. B. Lev, and D. Reichman, Time-dependent variational principle in matrix-product state manifolds: Pitfalls and potential, Phys. Rev. B **97**, 024307 (2018).

[5] C. D. White, M. Zaletel, R. S. K. Mong, and G. Refael, Quantum dynamics of thermalizing systems, Phys. Rev. B **97**, 035127 (2018).

[6] T. Prosen and M. Znidaric, Matrix product simulations of non-equilibrium steady states of quantum spin chains, J. Stat. Mech.: Theory Exp. (2009) P02035.

[7] T. Rakovszky, C. W. von Keyserlingk, and F. Pollmann, Dissipation-assisted operator evolution method for capturing hydrodynamic transport, Phys. Rev. B **105**, 075131 (2022).

[8] C. von Keyserlingk, F. Pollmann, and T. Rakovszky, Operator backflow and the classical simulation of quantum transport, Phys. Rev. B **105**, 245101 (2022).

[9] S. Mukerjee, V. Oganesyan, and D. Huse, Statistical theory of transport by strongly interacting lattice fermions, Phys. Rev. B **73**, 035113 (2006).

[10] V. Khemani, A. Vishwanath, and D. A. Huse, Operator Spreading and the Emergence of Dissipative Hydrodynamics under Unitary Evolution with Conservation Laws, Phys. Rev. X **8**, 031057 (2018).

[11] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive Hydrodynamics of Out-of-Time-Ordered Correlators with Charge Conservation, Phys. Rev. X **8**, 031058 (2018).

[12] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems, Phys. Rev. B **98**, 220303(R) (2018).

[13] T. Banks and A. Lucas, Emergent entropy production and hydrodynamics in quantum many-body systems, Phys. Rev. E **99**, 022105 (2019).

[14] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium, Phys. Rev. X **6**, 041065 (2016).

[15] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. **117**, 207201 (2016).

[16] B. Doyon, Lecture notes on generalised hydrodynamics, SciPost Phys. Lect. Notes **18** (2020).

[17] M. Ljubotina, M. Žnidarič, and T. Prosen, Spin diffusion from an inhomogeneous quench in an integrable system, Nat. Commun. **8**, 16117 (2017).

[18] M. Ljubotina, M. Žnidarič, and T. Prosen, Kardar-Parisi-Zhang Physics in the Quantum Heisenberg Magnet, Phys. Rev. Lett. **122**, 210602 (2019).

[19] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Bethe-Boltzmann hydrodynamics and spin transport in the XXZ chain, Phys. Rev. B **97**, 045407 (2018).

[20] V. B. Bulchandani, Kardar-Parisi-Zhang universality from soft gauge modes, Phys. Rev. B **101**, 041411(R) (2020).

[21] S. Gopalakrishnan and R. Vasseur, Kinetic Theory of Spin Diffusion and Superdiffusion in XXZ Spin Chains, Phys. Rev. Lett. **122**, 127202 (2019).

[22] E. Ilievski, J. De Nardis, S. Gopalakrishnan, R. Vasseur, and B. Ware, Superuniversality of Superdiffusion, Phys. Rev. X **11**, 031023 (2021).

[23] J. D. Nardis, S. Gopalakrishnan, and R. Vasseur, Non-linear fluctuating hydrodynamics for KPZ scaling in isotropic spin chains, arXiv:2212.03696.

[24] S. Hild, T. Fukuhara, P. Schauß, J. Zeiher, M. Knap, E. Demler, I. Bloch, and C. Gross, Far-from-Equilibrium Spin Transport in Heisenberg Quantum Magnets, Phys. Rev. Lett. **113**, 147205 (2014).

[25] P. N. Jepsen, J. Amato-Grill, I. Dimitrova, W. W. Ho, E. Demler, and W. Ketterle, Spin transport in a tunable Heisenberg model realized with ultracold atoms, Nature (London) **588**, 403 (2020).

[26] A. Scheie, N. E. Sherman, M. Dupont, S. E. Nagler, M. B. Stone, G. E. Granroth, J. E. Moore, and D. A. Tennant, Detection of Kardar–Parisi–Zhang hydrodynamics in a quantum Heisenberg spin-1/2 chain, Nat. Phys. **17**, 726 (2021).

[27] D. Wei, A. Rubio-Abadal, B. Ye, F. Machado, J. Kemp, K. Srakaew, S. Hollerith, J. Rui, S. Gopalakrishnan, N. Y. Yao, I. Bloch, and J. Zeiher, Quantum gas microscopy of Kardar-Parisi-Zhang superdiffusion, Science **376**, 716 (2022).

[28] B. Bertini, F. Heidrich-Meisner, C. Karrasch, T. Prosen, R. Steinigeweg, and M. Žnidarič, Finite-temperature transport in one-dimensional quantum lattice models, Rev. Mod. Phys. **93**, 025003 (2021).

[29] V. B. Bulchandani and D. A. Huse, Hot band sound, arXiv:2208.13767.

[30] C. M. Langlett and S. Xu, Noise induced universal diffusive transport in fermionic chains, arXiv:2304.02671.

[31] P. Glorioso, L. V. Delacrétaz, X. Chen, R. M. Nandkishore, and A. Lucas, Hydrodynamics in lattice models with continuous non-Abelian symmetries, SciPost Phys. **10**, 015 (2021).

[32] H. Singh, B. A. Ware, R. Vasseur, and A. J. Friedman, Subdiffusion and Many-Body Quantum Chaos with Kinetic Constraints, Phys. Rev. Lett. **127**, 230602 (2021).

[33] M. Dupont and J. E. Moore, Universal spin dynamics in infinite-temperature one-dimensional quantum magnets, Phys. Rev. B **101**, 121106(R) (2020).

[34] M. Ljubotina, J.-Y. Desaules, M. Serbyn, and Z. Papić, Superdiffusive Energy Transport in Kinetically Constrained Models, Phys. Rev. X **13**, 011033 (2023).

[35] P. Brighi, M. Ljubotina, and M. Serbyn, Hilbert space fragmentation and slow dynamics in particle-conserving quantum east models, arXiv:2210.15607.

[36] J. Iaconis, S. Vijay, and R. Nandkishore, Anomalous subdiffusion from subsystem symmetries, Phys. Rev. B **100**, 214301 (2019).

[37] J. Richter and A. Pal, Anomalous hydrodynamics in a class of scarred frustration-free Hamiltonians, Phys. Rev. Res. **4**, L012003 (2022).

[38] A. Morningstar, V. Khemani, and D. A. Huse, Kinetically constrained freezing transition in a dipole-conserving system, Phys. Rev. B **101**, 214205 (2020).

[39] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Res. **2**, 033124 (2020).

[40] J. Feldmeier, P. Sala, G. De Tomasi, F. Pollmann, and M. Knap, Anomalous Diffusion in Dipole- and Higher-Moment-Conserving Systems, Phys. Rev. Lett. **125**, 245303 (2020).

[41] J. Iaconis, A. Lucas, and R. Nandkishore, Multipole conservation laws and subdiffusion in any dimension, Phys. Rev. E **103**, 022142 (2021).

[42] A. G. Burchards, J. Feldmeier, A. Schuckert, and M. Knap, Coupled hydrodynamics in dipole-conserving quantum systems, Phys. Rev. B **105**, 205127 (2022).

[43] P. Glorioso, J. Guo, J. F. Rodriguez-Nieva, and A. Lucas, Breakdown of hydrodynamics below four dimensions in a fracton fluid, Nat. Phys. **18**, 912 (2022).

[44] J. Guo, P. Glorioso, and A. Lucas, Fracton Hydrodynamics without Time-Reversal Symmetry, Phys. Rev. Lett. **129**, 150603 (2022).

[45] P. Glorioso, X. Huang, J. Guo, J. F. Rodriguez-Nieva, and A. Lucas, Goldstone bosons and fluctuating hydrodynamics with dipole and momentum conservation, J. High Energy Phys. 05 (2023) 022.

[46] C. Stahl, M. Qi, P. Glorioso, A. Lucas, and R. Nandkishore, Fracton superfluid hydrodynamics, arXiv:2303.09573.

[47] J. H. Han, E. Lake, and S. Ro, Scaling and localization in kinetically constrained diffusion, arXiv:2304.03276.

[48] A. Głódkowski, F. Peña-Benítez, and P. Surówka, Hydrodynamics of dipole-conserving fluids, Phys. Rev. E **107**, 034142 (2023).

[49] J. Armas and E. Have, Ideal fracton superfluids, arXiv:2304.09596.

[50] A. Jain, K. Jensen, R. Liu, and E. Mefford, Dipole superfluid hydrodynamics, arXiv:2304.09852.

[51] E. van Nieuwenburg, Y. Baum, and G. Refael, From Bloch oscillations to many-body localization in clean interacting systems, Proc. Natl. Acad. Sci. USA **116**, 9269 (2019).

[52] M. Schulz, C. A. Hooley, R. Moessner, and F. Pollmann, Stark Many-Body Localization, Phys. Rev. Lett. **122**, 040606 (2019).

[53] S. Mandt, A. Rapp, and A. Rosch, Interacting Fermionic Atoms in Optical Lattices Diffuse Symmetrically Upwards and Downwards in a Gravitational Potential, Phys. Rev. Lett. **106**, 250602 (2011).

[54] P. Zhang, Subdiffusion in strongly tilted lattice systems, Phys. Rev. Res. **2**, 033129 (2020).

[55] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and Heat Transport in a Tilted Two-Dimensional Fermi-Hubbard System, Phys. Rev. X **10**, 011042 (2020).

[56] S. Scherg, T. Kohlert, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Observing non-ergodicity due to kinetic constraints in tilted Fermi-Hubbard chains, Nat. Commun. **12**, 4490 (2021).

[57] T. Kohlert, S. Scherg, P. Sala, F. Pollmann, B. Hebbe Madhusudhana, I. Bloch, and M. Aidelsburger, Exploring the Regime of Fragmentation in Strongly Tilted Fermi-Hubbard Chains, Phys. Rev. Lett. **130**, 010201 (2023).

[58] E. J. Bergholtz and A. Karlhede, Half-Filled Lowest Landau Level on a Thin Torus, Phys. Rev. Lett. **94**, 026802 (2005).

[59] E. J. Bergholtz and A. Karlhede, Quantum Hall system in Tao-Thouless limit, Phys. Rev. B **77**, 155308 (2008).

[60] E. J. Bergholtz and A. Karlhede, 'One-dimensional' theory of the quantum Hall system, J. Stat. Mech.: Theory Exp. (2006) L04001.

[61] Z.-Y. Wang, S. Takayoshi, and M. Nakamura, Spin-chain description of fractional quantum Hall states in the Jain series, Phys. Rev. B **86**, 155104 (2012).

[62] S. Pai, M. Pretko, and R. M. Nandkishore, Localization in Fractonic Random Circuits, Phys. Rev. X **9**, 021003 (2019).

[63] S. Pai and M. Pretko, Dynamical Scar States in Driven Fracton Systems, Phys. Rev. Lett. **123**, 136401 (2019).

[64] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity Breaking Arising from Hilbert Space Fragmentation in Dipole-Conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).

[65] V. Khemani, M. Hermele, and R. Nandkishore, Localization from Hilbert space shattering: From theory to physical realizations, Phys. Rev. B **101**, 174204 (2020).

[66] T. Rakovszky, P. Sala, R. Verresen, M. Knap, and F. Pollmann, Statistical localization: From strong fragmentation to strong edge modes, Phys. Rev. B **101**, 125126 (2020).

[67] S. Moudgalya, A. Prem, R. Nandkishore, N. Regnault, and B. A. Bernevig, Thermalization and its absence within Krylov subspaces of a constrained Hamiltonian, in *Memorial Volume for Shoucheng Zhang* (World Scientific, Singapore, 2021), pp. 147–209.

[68] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and Hilbert space fragmentation: A review of exact results, Rep. Prog. Phys. **85**, 086501 (2022).

[69] S. Moudgalya and O. I. Motrunich, Hilbert Space Fragmentation and Commutant Algebras, Phys. Rev. X **12**, 011050 (2022).

[70] X. Feng and B. Skinner, Hilbert space fragmentation produces an effective attraction between fractons, Phys. Rev. Res. **4**, 013053 (2022).

[71] C. Pozderac, S. Speck, X. Feng, D. A. Huse, and B. Skinner, Exact solution for the filling-induced thermalization transition in a one-dimensional fracton system, Phys. Rev. B **107**, 045137 (2023).

[72] C. Stahl, E. Lake, and R. Nandkishore, Spontaneous breaking of multipole symmetries, Phys. Rev. B **105**, 155107 (2022).

[73] E. Lake, M. Hermele, and T. Senthil, Dipolar Bose-Hubbard model, Phys. Rev. B **106**, 064511 (2022).

[74] E. Lake, H.-Y. Lee, J. H. Han, and T. Senthil, Dipole condensates in tilted Bose-Hubbard chains, Phys. Rev. B **107**, 195132 (2023).

[75] E. Lake and T. Senthil, Non-Fermi liquids from kinetic constraints in tilted optical lattices, arXiv:2302.08499.

[76] P. Zechmann, E. Altman, M. Knap, and J. Feldmeier, Fractonic Luttinger liquids and supersolids in a constrained Bose-Hubbard model, Phys. Rev. B **107**, 195131 (2023).

[77] A. Anakru and Z. Bi, Non-Fermi liquids from dipolar symmetry breaking, arXiv:2304.01181.

[78] J.-K. Yuan, S. A. Chen, and P. Ye, Fractonic superfluids, Phys. Rev. Res. **2**, 023267 (2020).

[79] S. A. Chen, J.-K. Yuan, and P. Ye, Fractonic superfluids. II. Condensing subdimensional particles, Phys. Rev. Res. **3**, 013226 (2021).

[80] S. Sachdev, K. Sengupta, and S. M. Girvin, Mott insulators in strong electric fields, Phys. Rev. B **66**, 075128 (2002).

[81] S. Pielawa, T. Kitagawa, E. Berg, and S. Sachdev, Correlated phases of bosons in tilted frustrated lattices, Phys. Rev. B **83**, 205135 (2011).

[82] O. Dubinkin, J. May-Mann, and T. L. Hughes, Theory of dipole insulators, Phys. Rev. B **103**, 125129 (2021).

[83] J. May-Mann and T. L. Hughes, Topological dipole conserving insulators and multipolar responses, Phys. Rev. B **104**, 085136 (2021).

[84] O. Dubinkin, J. May-Mann, and T. L. Hughes, Lieb-Schultz-Mattis-type theorems and other nonperturbative results for strongly correlated systems with conserved dipole moments, Phys. Rev. B **103**, 125133 (2021).

[85] M. A. Baranov, M. Dalmonte, G. Pupillo, and P. Zoller, Condensed matter theory of dipolar quantum gases, Chem. Rev. **112**, 5012 (2012).

[86] N. Defenu, T. Donner, T. Macri, G. Pagano, S. Ruffo, and A. Trombettoni, Long-range interacting quantum systems, arXiv:2109.01063.

[87] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, The physics of dipolar bosonic quantum gases, Rep. Prog. Phys. **72**, 126401 (2009).

[88] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, Dipolar physics: a review of experiments with magnetic quantum gases, Rep. Prog. Phys. **86**, 026401 (2023).

[89] M. Saffman, T. G. Walker, and K. Mølmer, Quantum information with Rydberg atoms, Rev. Mod. Phys. **82**, 2313 (2010).

[90] A. Browaeys and T. Lahaye, Many-body physics with individually controlled Rydberg atoms, Nat. Phys. **16**, 132 (2020).

[91] L. D. Carr, D. DeMille, R. V. Krems, and J. Ye, Cold and ultracold molecules: Science, technology and applications, New J. Phys. **11**, 055049 (2009).

[92] B. Gadway and B. Yan, Strongly interacting ultracold polar molecules, J. Phys. B: At., Mol. Opt. Phys. **49**, 152002 (2016).

[93] S. A. Moses, J. P. Covey, M. T. Miecnikowski, D. S. Jin, and J. Ye, New frontiers for quantum gases of polar molecules, Nat. Phys. **13**, 13 (2017).

[94] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano, P. Richerme, C. Senko, and N. Y. Yao, Programmable quantum simulations of spin systems with trapped ions, Rev. Mod. Phys. **93**, 025001 (2021).

[95] T. Matsuta, T. Koma, and S. Nakamura, Improving the Lieb–Robinson bound for long-range interactions, Ann. Henri Poincaré **18**, 519 (2017).

[96] L. Colmenarez and David J. Luitz, Lieb-Robinson bounds and out-of-time order correlators in a long-range spin chain, Phys. Rev. Res. **2**, 043047 (2020).

[97] Dominic V. Else, F. Machado, C. Nayak, and Norman Y. Yao, Improved Lieb-Robinson bound for many-body Hamiltonians with power-law interactions, Phys. Rev. A **101**, 022333 (2020).

[98] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Non-local propagation of correlations in quantum systems with long-range interactions, Nature (London) **511**, 198 (2014).

[99] C. Zu, F. Machado, B. Ye, S. Choi, B. Kobrin, T. Mittiga, S. Hsieh, P. Bhattacharyya, M. Markham, D. Twitchen, A. Jarmola, D. Budker, C. R. Laumann, J. E. Moore, and N. Y. Yao, Emergent hydrodynamics in a strongly interacting dipolar spin ensemble, Nature (London) **597**, 45 (2021).

[100] M. K. Joshi, F. Kranzl, A. Schuckert, I. Lovas, C. Maier, R. Blatt, M. Knap, and C. F. Roos, Observing emergent hydrodynamics in a long-range quantum magnet, Science **376**, 720 (2022).

[101] A. Schuckert, I. Lovas, and M. Knap, Nonlocal emergent hydrodynamics in a long-range quantum spin system, Phys. Rev. B **101**, 020416(R) (2020).

[102] B. Kloss and Y. Bar Lev, Spin transport in a long-range-interacting spin chain, Phys. Rev. A **99**, 032114 (2019).

[103] T. Zhou, S. Xu, X. Chen, A. Guo, and B. Swingle, Operator Lévy Flight: Light Cones in Chaotic Long-Range Interacting Systems, Phys. Rev. Lett. **124**, 180601 (2020).

[104] T. Kuwahara and K. Saito, Strictly Linear Light Cones in Long-Range Interacting Systems of Arbitrary Dimensions, Phys. Rev. X **10**, 031010 (2020).

[105] J. Richter, O. Lunt, and A. Pal, Transport and entanglement growth in long-range random clifford circuits, Phys. Rev. Res. **5**, L012031 (2023).

[106] M. Block, Y. Bao, S. Choi, E. Altman, and N. Y. Yao, Measurement-Induced Transition in Long-Range Interacting Quantum Circuits, Phys. Rev. Lett. **128**, 010604 (2022).

[107] T. Heitmann, J. Richter, D. Schubert, and R. Steinigeweg, Selected applications of typicality to real-time dynamics of quantum many-body systems, Z. Naturforsch. A **75**, 421 (2020).

[108] F. Jin, D. Willsch, M. Willsch, H. Lagemann, K. Michielsen, and H. De Raedt, Random state technology, J. Phys. Soc. Jpn. **90**, 012001 (2021).

[109] P. Scholl, H. J. Williams, G. Bornet, F. Wallner, D. Barredo, L. Henriet, A. Signoles, C. Hainaut, T. Franz, S. Geier, A. Tebben, A. Salzinger, G. Zürn, T. Lahaye, M. Weidemüller, and A. Browaeys, Microwave engineering of programmable *XXZ* Hamiltonians in arrays of Rydberg atoms, PRX Quantum **3**, 020303 (2022).

[110] L. Christakis, J. S. Rosenberg, R. Raj, S. Chi, A. Morningstar, D. A. Huse, Z. Z. Yan, and W. S. Bakr, Probing site-resolved correlations in a spin system of ultracold molecules, Nature (London) **614**, 64 (2023).

[111] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.108.L020304 for details on the Schrieffer-Wolff transformation, as well as numerical results on dynamics in moderately tilted lattice models of classical spins, which includes Refs. [114,115].

[112] J. Gliozzi, J. May-Mann, T. L. Hughes, and G. De Tomasi, Hierarchical hydrodynamics in long-range multipole-conserving systems, arXiv:2304.12342.

[113] O. Ogunnaike, J. Feldmeier, and J. Y. Lee, Unifying emergent hydrodynamics and Lindbladian low energy spectra

across symmetries, constraints, and long-range interactions, arXiv:2304.13028.

[114] V. Oganesyan, A. Pal, and D. A. Huse, Energy transport in disordered classical spin chains, Phys. Rev. B **80**, 115104 (2009).

[115] D. Schubert, J. Richter, F. Jin, K. Michielsen, H. D. Raedt, and R. Steinigeweg, Quantum versus classical dynamics in spin models: Chains, ladders, and square lattices, Phys. Rev. B **104**, 054415 (2021).