Abelian Higgs gauge theories with multicomponent scalar fields and multiparameter scalar potentials

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(Received 17 October 2023; revised 6 December 2023; accepted 6 December 2023; published 22 December 2023)

We consider multicomponent Abelian Higgs (AH) gauge theories with multiparameter scalar quartic potentials, which are extensions of the SU(N)-invariant AH theories, with a smaller global symmetry group. In particular, we consider an AH model with a two-parameter scalar potential and SO(N) global symmetry. We discuss the renormalization-group flow of the SO(N)-invariant AH field theory and the phase diagram and critical behavior of a corresponding three-dimensional (3D) noncompact lattice AH model. We argue that the phase diagram of 3D noncompact SO(N)- and SU(N)-symmetric lattice AH models are qualitatively similar. In both cases, there are three phases: the high-temperature Coulomb phase, and the low-temperature molecular and Higgs phases that differ for the topological properties of the gauge correlations. However, the main features of the low-temperature ordered phases and, in particular, of the Higgs phase, differ significantly in SO(N) and SU(N) models. In particular, in SO(N) models they depend on the sign of the self-interaction parameter v that controls the symmetry breaking from SU(N) to SO(N). As a consequence, the universal features of the transitions related with the spontaneous breaking of the global symmetry (those between the high-temperature Coulomb phase and the low-temperature molecular and Higgs phases) also depend on the sign of v.

DOI: 10.1103/PhysRevB.108.245154

I. INTRODUCTION

Many emergent collective phenomena in condensed-matter physics [1,2] admit an effective description in terms of Abelian Higgs (AH) gauge theories, in which charged scalar fields are minimally coupled with an Abelian gauge field. The phase structure and universal features of the transitions in this broad class of systems have been extensively studied [3–94], paying particular attention to the role of the topological features of the gauge correlations, to the interplay of gauge and scalar excitations, and to the role of the global symmetry whose spontaneous breaking is crucial for the emerging Higgs phases, see, e.g., Ref. [95]. Several lattice AH gauge models have been investigated, using both compact and noncompact gauge variables. They provide examples of topological transitions, which are driven by extended charged excitations without a local order parameter or by a nontrivial interplay between long-range scalar fluctuations and nonlocal topological gauge modes. Most studies on multicomponent systems have focused on AH theories with N-component fields and global SU(N) symmetry, see, e.g., Refs. [3,24,33,75,86,91,94]. In this paper, we consider field-theoretical and lattice AH models with more complex scalar self-interactions, which are invariant under a smaller group, focusing mainly on systems with a reduced SO(N) invariance. Some extensions of the SU(N)-symmetric AH field theories have already been discussed in Ref. [17], where a number of applications are also mentioned. We return to this issue and extend the analysis to the phase diagram and critical behaviors of their lattice counterparts and, in particular, to the SO(N)-symmetric lattice AH models with noncompact gauge variables.

In the SU(N)-symmetric AH field theory (UAHFT), an N-component complex scalar field $\boldsymbol{\phi}(\boldsymbol{x})$ is minimally coupled with a U(1) gauge field $A_\mu(\boldsymbol{x})$. The Lagrangian reads

$$\mathcal{L}_U = \frac{1}{4g^2} F_{\mu\nu}^2 + |D_\mu \boldsymbol{\phi}|^2 + r\, \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} + u\, (\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi})^2, \qquad (1)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \equiv \partial_\mu + iA_\mu$. Besides the Abelian U(1) gauge invariance, the UAHFT has a global SU(N) symmetry, $\boldsymbol{\phi} \to V\boldsymbol{\phi}$ with $V \in$ SU(N). The UAHFT is expected to describe the critical behavior of lattice models with analogous properties, i.e., a U(1) gauge invariance and an SU(N) global symmetry, but only when the U(1) gauge field develops critical correlations at the transition [91]. Transitions where the gauge fields play only the role of hindering the non-gauge-invariant modes from becoming critical are not described by a gauge field-theory model; they generically admit a description in terms of a gauge-invariant bilinear scalar order parameter only [81,91,96,97].

SU(N)-symmetric lattice AH models (ULAHM) have been extensively investigated, using compact and noncompact gauge fields, see, e.g., Refs. [7,41,81,85] and Refs. [4,6,12,14,15,31,32,86,91,94], respectively. In particular, a noncompact lattice formulation of the three-dimensional (3D) ULAHM is obtained by considering N-component unit-length complex vectors $z_{\boldsymbol{x}}$ (satisfying $\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}} = 1$) defined on the sites of a cubic lattice, noncompact gauge variables $A_{\boldsymbol{x},\mu} \in \mathbb{R}$ ($\mu = 1, 2, 3$) defined on the lattice links, and the nearest-neighbor Hamiltonian

$$H_U = \frac{\kappa}{2} \sum_{\boldsymbol{x},\mu>\nu} F_{\boldsymbol{x},\mu\nu}^2 - 2NJ \sum_{\boldsymbol{x},\mu} \mathrm{Re}\,(\lambda_{\boldsymbol{x},\mu} \bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}), \qquad (2)$$

where $\hat{\mu}$ are the lattice unit vectors, and

$$F_{\boldsymbol{x},\mu\nu} \equiv \Delta_\mu A_{\boldsymbol{x},\nu} - \Delta_\nu A_{\boldsymbol{x},\mu}, \quad \lambda_{\boldsymbol{x},\mu} \equiv e^{iA_{\boldsymbol{x},\mu}}, \tag{3}$$

where $\Delta_\mu A_{\boldsymbol{x},\nu} = A_{\boldsymbol{x}+\hat{\mu},\nu} - A_{\boldsymbol{x},\nu}$. The ULAHM has a local U(1) gauge invariance, $z_{\boldsymbol{x}} \to e^{i\Lambda_{\boldsymbol{x}}} z_{\boldsymbol{x}}$ and $A_{\boldsymbol{x},\mu} \to A_{\boldsymbol{x},\mu} + \Lambda_{\boldsymbol{x}} - \Lambda_{\boldsymbol{x}+\hat{\mu}}$ with $\Lambda_{\boldsymbol{x}} \in \mathbb{R}$, and a global SU($N$) symmetry $z_{\boldsymbol{x}} \to V z_{\boldsymbol{x}}$ with $V \in$ SU(N).

In this paper, we consider extensions of the SU(N)-symmetric AH field theories, such as those already introduced in Ref. [17], obtained by adding scalar gauge-invariant self-interactions that are not invariant under SU(N) transformations, but only under a smaller symmetry group. However, we require the global symmetry group to be such that $\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi}$ is the only quadratic combination of the scalar fields that is invariant under the global and local symmetry transformations. This requirement guarantees that the effective field theory describes a standard critical behavior [98,99], which can be observed by tuning a single Hamiltonian parameter. Indeed, in the presence of two or more quadratic terms that are invariant under the global symmetry group, the effective field theory describes a multicritical behavior [99–103]. In the context of scalar theories without gauge fields, this issue has been extensively studied, considering Landau-Ginzburg-Wilson (LGW) Φ^4 with complex symmetry-breaking patterns [98,99,104,105].

To be concrete, in most of the paper we will consider the simplest breaking of the SU(N) global symmetry, adding a term $|\boldsymbol{\phi} \cdot \boldsymbol{\phi}|^2$ to the Lagrangian Eq. (1). We therefore consider the SO(N)-symmetric AH field theory (OAHFT) with Lagrangian

$$\mathcal{L}_O = \frac{1}{4g^2} \sum_{\mu\nu} F_{\mu\nu}^2 + \sum_\mu |D_\mu \boldsymbol{\phi}|^2 + r\, \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} + V_O(\boldsymbol{\phi}), \tag{4}$$

$$V_O(\boldsymbol{\phi}) = u\, (\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi})^2 + v\, |\boldsymbol{\phi} \cdot \boldsymbol{\phi}|^2, \tag{5}$$

with $u \geqslant 0$ and $u + v \geqslant 0$, to guarantee the stability of the potential. The added term breaks the global SU(N) symmetry, making the theory invariant only under SO(N) transformations. However, the symmetry group is still large enough to guarantee that $\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi}$ is the only quadratic invariant allowed by the global and local symmetries. For $N = 1$, the two terms in V_O are equivalent, thus one recovers the standard one-component AH field theory.

An analogous extension can be considered for the ULAHM Eq. (2), considering the 3D SO(N)-symmetric lattice AH model (OLAHM) defined by the partition function

$$Z_O = \sum_{\{z,A\}} e^{-H_O(z,A)}, \tag{6}$$

$$H_O = H_U + v \sum_{\boldsymbol{x}} |z_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}}|^2. \tag{7}$$

One can easily check that the OAHFT Eq. (4) corresponds to the formal continuum limit of the OLAHM with $\kappa = g^{-2}$, after relaxing the unit-length constraint for the scalar field.

We show that some qualitative features of the phase diagram are the same in the ULAHM and OLAHM: In both models, three different phases occur that differ in the properties of the gauge correlations, in the confinement or deconfinement of the charged excitations, and in the behavior under the global symmetry transformations. However, the ordered phases, in particular, the Higgs phase, and the nature of the transition lines crucially depend on the global symmetry-breaking pattern that is determined by the specific form of the scalar self-interaction potential.

We remark that the study of the phase diagrams and critical behaviors of extensions of the AH gauge theories, allowing for more general scalar potentials and different global symmetry groups, may lead to a more thorough understanding of the possible critical behaviors that can be observed in the presence of an emergent Abelian gauge symmetry.

The paper is organized as follows. In Sec. II, we study the possible symmetry-breaking patterns and investigate the RG flow of the OAHFT Eq. (4) close to four dimensions. In Sec. III, we discuss the phase diagram and the nature of the phase transitions of the OLAHM Eq. (7). Finally, in Sec. IV we draw our conclusions and briefly discuss further extensions of the scalar potential.

II. SO(N)-SYMMETRIC ABELIAN HIGGS FIELD THEORIES

A. Global symmetry breaking patterns and order parameters

As in the ULAHM, also in SO(N) invariant models we expect transitions characterized by the breaking of the global symmetry. It is therefore important to determine the possible symmetry-breaking patterns. This analysis can be performed in the mean-field approximation, since space fluctuations are only relevant along the transition lines. To characterize the different phases within the mean-field framework, we need to determine the minima of the effective Hamiltonian:

$$H_{\rm mf} = r\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} + V_O(\boldsymbol{\phi}). \tag{8}$$

The analysis is straightforward. For $r > 0$, the minimum corresponds to $\boldsymbol{\phi} = 0$, so, for $r > 0$, the system is in the disordered phase in which the symmetry is unbroken. For $r < 0$, we find two nontrivial sets of minima. For $-u < v < 0$, the minimum of $H_{\rm mf}$ is obtained for

$$\boldsymbol{\phi} = e^{i\alpha} \boldsymbol{s}, \tag{9}$$

where $\boldsymbol{s}$ is a real N-component vector, and α an arbitrary phase. For $v > 0$, the fields corresponding to the minimum configurations can be parametrized as

$$\boldsymbol{\phi} = \frac{1}{\sqrt{2}}(\boldsymbol{s}_1 + i\boldsymbol{s}_2), \quad \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = 0, \tag{10}$$

where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are orthogonal real vectors satisfying $|\boldsymbol{s}_1| = |\boldsymbol{s}_2|$, so $\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} = s^2 \equiv s_1^2 = s_2^2$.

Using these results, we can determine the global symmetry of the broken phases. For $v < 0$, the broken phase is invariant under $O(N-1)$ transformations, while for $v > 0$ the residual symmetry is SO(2) $\otimes$ $O(N-2)$. Note that the SO(2) subgroup for $v > 0$ corresponds to the transformations that rotate the two vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ in the plane in which they lie together with a change of phase. To write these transformations explicitly, let us note that, by an appropriate SO(N) rotation, we can always take $\boldsymbol{\phi} = (A, \pm iA, 0, \ldots, 0)$. It is then immediate to verify that the vector $\boldsymbol{\phi}$ is left invariant by the transformation

$$\phi^a \to e^{\mp i\alpha} \sum_{ab} V^{ab} \phi^b, \tag{11}$$

where $V = V_2 \oplus I_{N-2}$, I_{N-2} is the $(N-2)$-dimensional identity matrix and

$$V_2 = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{12}$$

We should also consider discrete transformations, as they can also play a role at the transitions. The Lagrangian Eq. (4) and the mean-field Hamiltonian Eq. (8) are invariant under the $\mathbb{Z}_2$ transformation $\boldsymbol{\phi} \to \bar{\boldsymbol{\phi}}$.

Because of gauge invariance, minimum configurations that differ by gauge transformations are equivalent. For negative v, we can therefore set $\alpha = 0$ in Eq. (9) and consider only real vectors. The invariance group of the broken phase is $\mathbb{Z}_2 \otimes O(N-1)$, the $\mathbb{Z}_2$ transformations corresponding to $\boldsymbol{\phi} \to \bar{\boldsymbol{\phi}}$. For positive values of v, if we apply a gauge transformation to the minimum configurations, we obtain $\phi'^a = s'^a_1 + i s'^a_2$ with

$$s'^a_1 = \cos\alpha\ s^a_1 + \sin\alpha\ s^a_2,$$
$$s'^a_2 = -\sin\alpha\ s^a_1 + \cos\alpha\ s^a_2. \tag{13}$$

We can thus freely rotate the two vectors in the plane in which they lie. This implies that, modulo gauge transformations, minimum configurations are classified by the relative orientation of the two vectors (chirality) and by the plane in which $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ lie. The SO(2) symmetry Eq. (11) is irrelevant, as it also involves a gauge transformation, so the invariance group of the ordered phase is $O(N-2)$.

We wish now to identify appropriate order parameters for the two different symmetry-breaking patterns. In the UAHFT, an order-parameter field is provided by the complex gauge-invariant bilinear operator:

$$Q^{ab}(\boldsymbol{x}) = \bar{\phi}^a(\boldsymbol{x})\phi^b(\boldsymbol{x}) - \frac{1}{N}\bar{\boldsymbol{\phi}}(\boldsymbol{x})\cdot\boldsymbol{\phi}(\boldsymbol{x})\,\delta^{ab}. \tag{14}$$

We now show that

$$R^{ab}(\boldsymbol{x}) = \mathrm{Re}\,Q^{ab}(\boldsymbol{x}), \quad T^{ab}(\boldsymbol{x}) = \mathrm{Im}\,Q^{ab}(\boldsymbol{x}), \tag{15}$$

which transform under different representations of the SO(N) group, provide the order-parameter fields for the SO(N) theory. Indeed, if the minimum configurations are given by Eq. (9), thus for $v < 0$, we have $T^{ab} = 0$ and

$$R^{ab} = s^a s^b - \frac{s^2}{N}\delta^{ab}, \tag{16}$$

where $s = |\boldsymbol{s}|$. Instead, if the minimum configurations are given by Eqs. (10), thus for $v > 0$, we have

$$R^{ab} = \frac{1}{2}\left(s^a_1 s^b_1 + s^a_2 s^b_2\right) - \frac{s^2}{N}\delta^{ab},$$
$$T^{ab} = \frac{1}{2}\left(s^a_1 s^b_2 - s^b_1 s^a_2\right), \tag{17}$$

where $s^2 = |\boldsymbol{s}_1|^2 = |\boldsymbol{s}_2|^2$. Note that only the order parameter T^{ab} is sensitive to the breaking of the $\mathbb{Z}_2$ symmetry $\boldsymbol{\phi} \to \bar{\boldsymbol{\phi}}$. Moreover, Eq. (17) implies the relation

$$(T^2)^{ab} = -\frac{s^2}{2}R^{ab} - \frac{s^4}{2N}\delta^{ab}, \quad \mathrm{Tr}\,T^2 = -\frac{s^4}{2}. \tag{18}$$

In particular, for $N = 2$ we have that $R^{ab} = 0$ (taking into account that $\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = 0$), and we can write $T^{ab} = \frac{1}{2}\sigma s^2 \epsilon^{ab}$, where ϵ^{ab} is the two-index antisymmetric tensor ($\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$) and σ is a variable that takes only the values ± 1, which is related to the relative orientation (chirality) of the orthogonal pair $(\boldsymbol{s}_1, \boldsymbol{s}_2)$.

The analysis of the possible phases could have also been performed in the real formulation of the model. Indeed, the Lagrangian Eq. (4) can also be written in terms of a real matrix field φ_{ai} ($a = 1, ..., N$ and $i = 1, 2$) defined by $\phi_a = \varphi_{a1} + i\varphi_{a2}$. We obtain the equivalent Lagrangian

$$\mathcal{L} = \frac{1}{4g^2}\sum_{\mu\nu} F^2_{\mu\nu} + \sum_{\mu,ai}(D_\mu\varphi)^2_{ai} + r\sum_{ai}\varphi^2_{ai} + \tilde{u}\left(\sum_{ai}\varphi^2_{ai}\right)^2 + \tilde{v}\left[\sum_{ij}\left(\sum_a \varphi_{ai}\varphi_{aj}\right)^2 - \left(\sum_{ai}\varphi^2_{ai}\right)^2\right], \tag{19}$$

where $\tilde{v} = 2v$ and $\tilde{u} = u + v$, $(D_\mu\varphi)_{ai} = \partial_\mu\varphi_{ai} - iA_\mu\epsilon_{ab}\varphi_{bi}$. Of course, the results for the symmetry-breaking patterns reported in Ref. [106] are equivalent to those reported above. The real-field formulation with Lagrangian Eq, (19) shows that the OAHFT can be interpreted as the Abelian gauge theory obtained by gauging the SO(2) group of the global symmetry group $O(2) \otimes O(N)$.

B. RG flow and fixed points

Let us now discuss the RG flow of the OAHFT. Its main features can already be inferred from earlier field-theoretical perturbative analyses [5,17], in particular, from the one-loop perturbative computations within the minimal subtraction renormalization scheme of the dimensional regularizaton, reported in Ref. [5]. As we shall see, they indicate that, as it occurs for the UAHFT, the RG flow of the OAHFT has a stable fixed point for a sufficiently large number of scalar components, $N > N^*_O(d)$, where $N^*_O(d)$ depends on the dimension d of the system.

Close to four dimensions. the RG flow can be studied by using the β functions computed in dimensional regularization with the minimal subtraction renormalization scheme. Setting $\alpha \equiv g^2$ and $\varepsilon \equiv 4 - d$, the β functions associated with the couplings u, v, and α are given by [5][1]

$$\beta_u = -\varepsilon u + (N+4)u^2 + 4uv + 4v^2 - 18u\alpha + 54\alpha^2,$$
$$\beta_v = -\varepsilon v + Nv^2 + 6vu - 18v\alpha, \tag{20}$$
$$\beta_\alpha = -\varepsilon\alpha + N\alpha^2.$$

[1]The one-loop series reported in Eqs. (20) can be obtained by some straightforward manipulations of the one-loop series reported in Ref. [5] for scalar $O(N)$ gauge theories. Some checks are obtained by noting that (i) for $v = 0$, one should reobtain the UAHFT series [3]; (ii) for $\alpha = 0$, one should reproduce the series for the purely scalar model reported in Refs. [107,108], known up to five loops [107,108]; (iii) since the addition of the v-term changes the global symmetry of the model, the $v = 0$ plane must be a separatrix of the RG flow, which implies $\beta_v = -\varepsilon v + v f_v(u, v, \alpha)$; and (iv) since for $N = 1$ the two quartic terms are equivalent, for $N \to 1$ the relation $\beta_u(u, s-u, \alpha) + \beta_v(u, s-u, \alpha) = \beta_u(s, 0, \alpha)$ holds.

The normalizations of the renormalized couplings u, v, α have been chosen to simplify the formulas (they can be easily inferred from the above expressions).

Stable fixed points of the RG flow correspond to zeros of the β functions Eqs. (20), such that the eigenvalues of the stability matrix $\Omega_{ij} = \partial\beta_i/\partial g_j$, computed at the zero, are all positive. In the large-N limit, a stable fixed point occurs for

$$\alpha^* = u^* = v^* = \frac{\varepsilon}{N} + O(\varepsilon^2, N^{-2}). \qquad (21)$$

The analysis of the β functions shows that this stable fixed exists for $N > N^*_O(d)$, where $N^*_O(d) = N^*_{O,4} + O(\varepsilon)$ with $N^*_{O,4} \approx 210$, which is the only positive solution of the fourth-order equation [17]:

$$n^4 - 204n^3 - 1356n^2 - 864n - 15552 = 0. \qquad (22)$$

For comparison, we mention that the UAHFT has a stable fixed point for $N > N^*(d)$ with $N^*(d) = N^*_4 + O(\varepsilon)$ and $N^*_4 = 94 + 24\sqrt{15} \approx 183$. Therefore, for $\varepsilon \ll 1$, OAHFT and UAHFT have comparable boundary values for the existence of a stable fixed point. In the case of the UAHFT, $N^*(d)$ significantly decreases approaching $d = 3$: one finds $N^*(d = 3) \approx 7$ [78,86,91,93]. We also mention that $N^*(d)$ is expected to converge to a small value for $d \to 2$ [78], and that an analogous argument indicates that $|N^*_O(d \to 2)| = O(1)$ as well. This suggests that $N^*_O(d)$ has a d dependence similar to $N^*(d)$ for the UAHFT. Therefore, we may guess that $N^*_O(d = 3)$ is of order 10, as in the SU(N) case.

It is important to note that the stable fixed point for the OAHFT lies in the region $v > 0$. Therefore, this fixed point is only relevant for transitions between a disordered phase and a Higgs phase with a global $O(N-2)$ symmetry. On the other hand, on the basis of this RG analysis, no charged transition is expected if the Higgs phase is invariant under $\mathbb{Z}_2 \otimes O(N-1)$.

The analysis of the RG flow of the OAHFT provides information on the behavior of the transitions in 3D lattice AH systems, where gauge fields become critical. If the symmetry of the broken phase is the one observed in the OAHFT model for $v < 0$, only first-order transitions are possible. If, instead, the symmetry-breaking pattern is the one observed for $v > 0$, the behavior depends on the number of components of the scalar field. For $N > N^*_O(d = 3)$, continuous charged transitions are possible, provided the lattice system is effectively inside the attraction domain of the stable fixed point. Instead, for $N < N^*_O(d = 3)$, only first-order transitions are possible (unless there are additional 3D universality classes that are not analytically related with the 4D RG fixed points, as occurs for the one-component AH models [21,34,94]).

Beside the stable fixed point, the β functions admit other zeroes with unstable directions. The fixed point with $v^* = 0$ that is stable in the SU(N)-symmetric field theory (in the large-N limit, it corresponds to $\alpha^* = u^* \approx \varepsilon/N$) is unstable with respect to the perturbation proportional to v, with a negative eigenvalue $\lambda_v = -\varepsilon + O(\varepsilon^2, N^{-1})$. Therefore, the parameter v is a relevant perturbation of the SU(N)-symmetric fixed point, with positive RG dimension $y_v = -\lambda_v$. The addition of the term proportional to v drives the flow away from the stable SU(N) fixed point, either towards the O(N) stable fixed point (this is only possible for $v > 0$) or towards infinity (in this case first-order transitions are observed).

We also mention that the RG flow of the scalar model without gauge fields with Lagrangian Eq. (19) has been extensively studied, because it is relevant for the normal-to-planar superfluid transition in ^{3}He [107,109], and for transitions in some frustrated magnetic systems with noncollinear order [106,110,111]. The RG flow of the scalar theory close to four dimensions can be inferred from the analysis of the zeros of the β functions β_u and β_v, cf. Eqs. (20), setting $\alpha = 0$. Close to four dimensions, there is a stable fixed point for $N \gtrsim 22$ [111,112]. However, the analyses of high-order 3D perturbative expansions [106,107] show that, in 3D there are stable RG fixed points for $N = 2$ and $N = 3$ that are not connected with those existing close to four dimensions.

The stable fixed point of the scalar theory without gauge fields is unstable with respect to the gauge parameter $\alpha \sim \kappa^{-1}$. A simple analysis of the β functions Eqs. (20) shows that the RG dimension of the gauge perturbation is positive, i.e.,

$$y_\alpha = -\lambda_\alpha = -\left.\frac{\partial\beta_\alpha}{\partial\alpha}\right|_{\alpha=0,u=u^*,v=v^*} = \varepsilon = 4 - d, \qquad (23)$$

where λ_α is one of the eigenvalues of the stability matrix $\Omega_{ij} = \partial\beta_i/\partial g_j$ computed at the scalar fixed point with $\alpha = 0$. Note that $y_\alpha = 4 - d$ corresponds to the dimension of the gauge coupling $\alpha \sim \kappa^{-1}$ in d dimensions. This result holds to all order of the ε expansion, due to the fact that β_α has the general form $\beta_\alpha = -\varepsilon\alpha + \alpha^2 F(\alpha, u, v)$, where $F(\alpha, u, v)$ has a regular perturbative expansion [81]. Therefore, we find $y_\alpha = 1$ in three dimensions. Note that the relevance of the gauge fluctuations at the fixed points of the purely scalar theory, and therefore the crossover towards a different asymptotic behavior, is independent of the existence of the stable fixed point of the full theory, which is only relevant to predict the eventual asymptotic behavior.

The RG analysis reported above shows that SO(N)-symmetric Abelian gauge systems for large values of N may undergo continuous phase transitions. It is interesting to compute the corresponding critical exponents. The correlation-length exponent ν in the large-N limit can be computed by using the results of Ref. [5], which considered scalar-gauge theories obtained by gauging the $O(M)$ subgroup of the global symmetry $O(M) \otimes O(N)$. Assuming the existence of a critical transition, the critical exponents for fixed M and d were computed in the large-N limit. For $M = 2$ and $d = 3$, the correlation-length exponent ν is given by

$$\nu = 1 - \frac{176}{3\pi^2 N} + O(N^{-2}), \qquad (24)$$

which is numerically close to the large-N result for the UAHFT [3]:

$$\nu = 1 - \frac{48}{\pi^2 N} + O(N^{-2}). \qquad (25)$$

III. NONCOMPACT SO(N)-SYMMETRIC LATTICE ABELIAN HIGGS MODELS

We now discuss the phase diagram and the nature of the phase transitions in the OLAHM, whose Hamiltonian is given in Eq. (7). As we shall see, some qualitative features of the OLAHM phase diagram are analogous to those of the ULAHM. However, substantial changes are expected in the

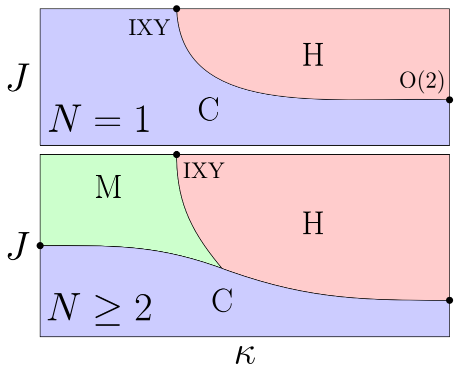

FIG. 1. The κ-J phase diagram of lattice AH models for $N = 1$ (top) and generic $N \geqslant 2$ (bottom). For $N \geqslant 2$, three phases are present: the small-J Coulomb (C) phase, in which the scalar field is disordered and gauge correlations are long ranged, and the large-J molecular (M) and Higgs (H) ordered phases, in which the global symmetry is spontaneously broken. For $N = 1$, there are only two phases: the Coulomb and the Higgs phase.

nature of the ordered low-temperature phases, in particular, of the Higgs phase and of the transition lines, which crucially depend on the global symmetry-breaking pattern that, in turn, depends on the specific form of the scalar self-interaction potential.

A. SU(N)-symmetric lattice Abelian-Higgs models

We begin by reviewing what is known for the SU(N)-symmetric ULAHM. A sketch of the κ-J phase diagram of the ULAHM is shown in Fig. 1. For $N \geqslant 2$, three phases occur. In the small-J Coulomb (C) phase, the scalar field is disordered and gauge correlations are long ranged. For large J two phases occur, the molecular (M) and the Higgs (H) ordered phase, in which the global symmetry is spontaneously broken from SU(N) to $U(N-1)$, with the emergence of $2N-2$ long ranged Goldstone modes [33]. An appropriate order parameter is the gauge-invariant bilinear operator

$$Q_{L,x}^{ab} = \bar{z}_x^a z_x^b - \frac{1}{N}\delta^{ab}, \tag{26}$$

which is the lattice analog of the field-theory bilinear operator (14). For $N = 1$, there is no global symmetry, therefore there are only two phases, see Fig. 1, the Coulomb phase and the Higgs phase, that are characterized by the behavior of nonlocal gauge-invariant charged operators [12,13,16,92], which are confined in the former and deconfined in the latter.

The two ordered phases of multicomponent systems are distinguished by the behavior of the gauge modes: the gauge field is long ranged in the M phase (small κ), while it is gapped in the H phase (large κ). Moreover, while the C and M phases are confined phases, the H phase charged excitations, represented by gauge-invariant nonlocal dressed scalar operators [12,13,92,113], are deconfined [12,13,16,92]. The transition lines may be of first order or continuous and, in the latter case, belong to universality classes that may depend on the number N of scalar components. The continuous transitions are related with the stable (charged or uncharged) fixed points of the RG flow, each one with its own attraction domain in the model parameter space.

The CH and CM transitions are both characterized by the spontaneous breaking of the global SU(N) symmetry, but differ in the role of the gauge fields. The continuous transitions along the CH line are charged transitions, where gauge fields become critical and are associated with the stable fixed point of the RG flow of the UAHFT (1). As already mentioned in Sec. II B, continuous transitions can only be observed for $N > N^\star$ with $N^\star = 7(2)$ [86]. At the CM transitions, gauge fields play no role (gauge correlations are long ranged on both sides of the transition) and thus an effective description can be obtained by considering an SU(N) symmetric LGW theory defined in terms of the complex Hermitian field Ψ^{ab} that corresponds to the bilinear operator Q^{ab}, without gauge fields [80,86]. This predicts that continuous transitions occur only in two-component systems, i.e., for $N = 2$. Their critical behavior belongs to the $O(3)$ vector (Heisenberg) universality class [98,99].

While transitions along the CM and CH lines are related with the spontaneous breaking of the global symmetry, the MH line separates two ordered phases that differ only in the behavior of the gauge correlations, without a local gauge-invariant order parameter. The continuous transitions along the MH line belong to the same universality class as those in the inverted XY (IXY) model [94], which is related with the standard XY model with Villain action by a duality transformation [34]. This is the same universality class controlling the topological critical behavior of the CH transitions in the one-component lattice AH model [94]. Note that this apparently simple behavior along the MH transition line is not obvious because of the simultaneous presence of the massless gauge modes that drive the IXY transitions, and of the long-ranged (zero-mass) Goldstone bosons, related with the breaking of the global SU(N) symmetry. The numerical analyses reported in Ref. [94] show that, along the MH line, the massless Goldstone modes effectively decouple from the massless gauge modes that drive the critical behavior, so the finite-J transitions belong to the IXY universality class.

B. SO(N)-symmetric lattice Abelian Higgs models

We now wish to discuss the general features of the phase diagram—the OLAHM. We will argue that the phase diagram is similar to that of the ULAHM, shown in Fig. 1. Indeed, the OLAHM presents three phases as well: one small-J disordered phase and two large-J phases in which the global SO(N) symmetry is spontaneously broken. As in the ULAHM, the ordered phases differ in the confinement properties of the nonlocal charged excitations [13,16,92,94] and in the nature of the gauge modes.

To characterize the spontaneous breaking of the SO(N) symmetry, we consider the lattice analog of the field-theory operators defined in Eqs. (15),

$$R_{L,x}^{ab} = \mathrm{Re}\, Q_{L,x}^{ab} = \frac{1}{2}\left(\bar{z}_x^a z_x^b + \bar{z}_x^b z_x^a\right) - \frac{1}{N}\delta^{ab}, \tag{27}$$

$$T_{L,x}^{ab} = \mathrm{Im}\, Q_{L,x}^{ab} = \frac{1}{2i}\left(\bar{z}_x^a z_x^b - \bar{z}_x^b z_x^a\right), \tag{28}$$

which transform under two different irreducible representations of the SO(N) group. As already discussed, in the disordered phase both order parameters vanish. Since the symmetry-breaking pattern depends on the sign of v, the behavior of $R^{ab}_{L,\boldsymbol{x}}$ and $T^{ab}_{L,\boldsymbol{x}}$ depends on the sign of v. For $v < 0$, $R^{ab}_{L,\boldsymbol{x}}$ condenses in the ordered phase, while $T^{ab}_{L,\boldsymbol{x}}$ still vanishes. For $v > 0$ and $N = 2$, $T^{ab}_{L,\boldsymbol{x}}$ condenses, while $R^{ab}_{L,\boldsymbol{x}}$ vanishes. Finally, for $v > 0$ and $N \geqslant 3$, both order parameters condense in the ordered phase.

1. The MH transition line at low temperature

The existence of two different large-J ordered phases which differ for the topological properties of the gauge field—charged excitations are confined/deconfined in the M and H phase, respectively—is suggested by the existence of a transition point for $J = \infty$. The argument is the same that holds for the ULAHM. For $J \to \infty$, the relevant configurations are those that maximize $\sum_{\boldsymbol{x},\mu} \mathrm{Re}\,(\bar{z}_{\boldsymbol{x}} \cdot \lambda_{\boldsymbol{x},\mu}\, z_{\boldsymbol{x}+\hat{\mu}})$, independently of the scalar potential. This implies $z_{\boldsymbol{x}} = \lambda_{\boldsymbol{x},\mu}\, z_{\boldsymbol{x}+\hat{\mu}}$, and therefore $\lambda_{\boldsymbol{x},\mu}\, \lambda_{\boldsymbol{x}+\hat{\mu},\nu}\, \bar{\lambda}_{\boldsymbol{x}+\hat{\nu},\mu}\, \bar{\lambda}_{\boldsymbol{x},\nu} = 1$ for each lattice plaquette. Then, by an appropriate gauge transformation, we can set $A_{\boldsymbol{x},\mu} = 2\pi n_{\boldsymbol{x},\mu}$, where $n_{\boldsymbol{x},\mu} \in \mathbb{Z}$, obtaining the IXY model, which has a transition in the XY universality class, at $\kappa_c = 0.076051(2)$ [34,86] (estimates of critical exponents can be found in Refs. [114–116]). Therefore, for $J \to \infty$ the OLAHM has a transition for any value of v.

A natural hypothesis is that the $J \to \infty$ IXY transition point is the starting point of a finite-J line (MH line) of transitions, whose nature, as in the ULAHM [94], is independent of J, at least for sufficiently large finite J (we cannot exclude that the transitions turn into first-order ones before the multicritical point, where the three transition lines meet). The transitions should belong to the IXY universality class, as the MH transitions in the ULAHM and the CH transitions in the one-component lattice AH model, see Fig. 1. This universal behavior of the MH transitions is possible if, as observed in the ULAHM, the gauge critical modes driving the IXY transitions decouple from the long-range Goldstone modes present in the ordered phases. If this occurs, the scalar degrees of freedom are irrelevant and so are the global symmetry and the symmetry-breaking pattern.

2. The CM transition line

As in the ULAHM, we expect transitions along the CM line, i.e., for small values of κ, to have the same universal features as those occurring for $\kappa = 0$. For $\kappa = 0$, the gauge variables can be integrated out in the partition function Eq. (6), obtaining the model with Hamiltonian

$$H_{O,\kappa=0} = -\sum_{\boldsymbol{x},\mu} \ln I_0(2NJ|\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|) + v\sum_{\boldsymbol{x}} |z_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}}|^2, \tag{29}$$

where $I_0(x)$ is the modified Bessel function. We recall that $I_0(x) = I_0(-x)$, $I_0(x) = 1 + x^2/4 + O(x^4)$ and $I_0(x) \approx e^x/\sqrt{2\pi x}$ for large x. In the absence of the v term, this Hamiltonian provides a lattice formulation of the CP^{N-1} model [8], which is equivalent, as far as the critical behavior is concerned, to the standard one with Hamiltonian

$$H_{CP} = -JN\sum_{\boldsymbol{x},\mu} |\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|^2. \tag{30}$$

For $\kappa = 0$, and for sufficiently small values of κ along the CM transition line, gauge fluctuations are not expected to play an active role at the transition. Indeed, the gauge properties of the C and M phases are the same: gauge modes are long ranged and charged excitations are confined in both of them. Therefore, the transition should be uniquely driven by the breaking of the global symmetry. Therefore, we expect that an effective description of the critical universal behavior can be obtained by considering a LGW theory for the gauge-invariant scalar order parameter that condenses at the transition, without considering the gauge fields [80,81,86].

For $v < 0$, the relevant order parameter is $R^{ab}_{L,\boldsymbol{x}}$, which is a real symmetric operator. Therefore, we expect the small-κ transitions along the CM line to be described by a LGW for a real symmetric traceless $N \times N$ matrix field $\Phi^{ab}(\boldsymbol{x})$ that represents a coarse-grained average of $R^{ab}_{L,\boldsymbol{x}}$ over a large, but finite, lattice domain. The corresponding LGW Lagrangian is obtained by considering all monomials in $\Phi^{ab}(\boldsymbol{x})$ that are allowed by the global SO(N) symmetry up to fourth order. We obtain

$$\begin{aligned}\mathcal{L} = &\,\mathrm{Tr}(\partial_\mu \Phi)^2 + r\,\mathrm{Tr}\,\Phi^2 + s\,\mathrm{tr}\,\Phi^3 \\ &+ u\,(\mathrm{Tr}\,\Phi^2)^2 + v\,\mathrm{Tr}\,\Phi^4.\end{aligned} \tag{31}$$

For $N = 2$, the cubic term vanishes and the two quartic terms are equivalent. The resulting LGW theory is equivalent to that of the $O(2)$-symmetric vector model, thus predicting that continuous transitions belong to the XY universality class. On the other hand, for $N \geqslant 3$ the cubic Φ^3 term is generally present. This is usually considered as the indication that phase transitions occurring in systems sharing the same global properties are of first order, as one can easily infer using mean-field arguments. We expect this behavior to hold for any $v < 0$, up to $v = 0$ where the SU(N) symmetry is restored, and we recover the CP^{N-1} model, whose transition is continuous for $N = 2$, in the $O(3)$ vector universality class, and of first order for any $N \geqslant 3$ [80,84].

We can explicitly verify the above predictions by considering the limit $v \to -\infty$. In this limit, the relevant configurations are those that minimize the potential V_O, as discussed in Sec. II A. Indeed, we should again minimize H_{mf}, where r now plays the role of the Lagrange multiplier that enforces the condition $\bar{z} \cdot z = 1$. For $v < 0$, we have

$$z_{\boldsymbol{x}} = e^{i\theta_{\boldsymbol{x}}} \boldsymbol{s}_{\boldsymbol{x}}, \tag{32}$$

where $\boldsymbol{s}_{\boldsymbol{x}}$ is a real unit-length N-component vector. This representation is redundant as the pair $\boldsymbol{s}_{\boldsymbol{x}}$, $\theta_{\boldsymbol{x}}$ and the pair $\boldsymbol{s}'_{\boldsymbol{x}} = -\boldsymbol{s}_{\boldsymbol{x}}$, $\theta'_{\boldsymbol{x}} = \theta_{\boldsymbol{x}} + \pi$ both correspond to $z_{\boldsymbol{x}}$. Thus, this parametrization of the scalar field introduces an additional kinematical $\mathbb{Z}_2$ gauge invariance. Using the parametrization Eq. (32), the scalar hopping term becomes

$$\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}} = e^{i(\theta_{x+\hat{\mu}} - \theta_x)}\, \boldsymbol{s}_{\boldsymbol{x}} \cdot \boldsymbol{s}_{\boldsymbol{x}+\hat{\mu}}. \tag{33}$$

If we choose the unitary gauge, we can set $\theta_{\boldsymbol{x}} = 0$, obtaining a theory in terms of $A_{\boldsymbol{x},\hat{\mu}}$ and $\boldsymbol{s}_{\boldsymbol{x}}$, which is invariant under $\mathbb{Z}_2$

gauge transformations that involve both $\boldsymbol{s}_{\boldsymbol{x}}$ and $A_{\boldsymbol{x},\mu}$. Substituting Eq. (33) into the $\kappa = 0$ Hamiltonian Eq. (29), we obtain

$$H = -\sum_{\boldsymbol{x},\mu} \ln I_0(2NJ|\boldsymbol{s}_{\boldsymbol{x}} \cdot \boldsymbol{s}_{\boldsymbol{x}+\hat{\mu}}|). \tag{34}$$

This Hamiltonian is invariant under global $O(N)$ and local $\mathbb{Z}_2$ transformations and is ferromagnetic (for large J the fields $\boldsymbol{s}_{\boldsymbol{x}}$ order). Thus, it represents a variant Hamiltonian of the so-called RP^{N-1} model, which is characterized by a global $O(N)$ invariance and a local $\mathbb{Z}_2$ gauge symmetry. This result is consistent with the LGW description given above. Indeed, in the RP^{N-1} models the relevant order parameter is

$$S^{ab}_{\boldsymbol{x}} = s^a_{\boldsymbol{x}} s^b_{\boldsymbol{x}} - \frac{1}{N}\delta^{ab}, \tag{35}$$

so the LGW fundamental field is a real symmetric traceless tensor Φ^{ab}. The corresponding Lagrangian is given in Eq. (31).

The behavior changes when we consider the opposite case $v > 0$. To understand some general features of the critical behavior, we begin by studying the lattice model in the limit $v \to \infty$. The relevant configurations are those that minimize the potential V_O for $v > 0$; see Sec. II A. Since $|z_{\boldsymbol{x}}| = 1$, they can be written as

$$z_{\boldsymbol{x}} = \frac{1}{\sqrt{2}}(s_{1,\boldsymbol{x}} + i s_{2,\boldsymbol{x}}), \quad s_{1,\boldsymbol{x}} \cdot s_{2,\boldsymbol{x}} = 0, \tag{36}$$

where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are two real orthogonal unit-length vectors ($s_1^2 = s_2^2 = 1$ so $\bar{z} \cdot z = 1$). One can easily check that the $\kappa = 0$ Hamiltonian Eq. (29) can be written in terms of the antisymmetric tensor field $T^{ab}_{L,\boldsymbol{x}}$ only. Indeed, since $Q_{L,\boldsymbol{x}} = R_{L,\boldsymbol{x}} + iT_{L,\boldsymbol{x}}$ and

$$\begin{aligned} |\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|^2 &= \mathrm{Tr}\, Q_{L,\boldsymbol{x}} Q_{L,\boldsymbol{x}+\hat{\mu}} + \frac{1}{N} \\ &= \mathrm{Tr}\, R_{L,\boldsymbol{x}} R_{L,\boldsymbol{x}+\hat{\mu}} - \mathrm{Tr}\, T_{L,\boldsymbol{x}} T_{L,\boldsymbol{x}+\hat{\mu}} + \frac{1}{N}, \end{aligned} \tag{37}$$

using Eqs. (18) with $s^2 = 1$, we obtain

$$|\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|^2 = -\mathrm{Tr}\, T_{L,\boldsymbol{x}} T_{L,\boldsymbol{x}+\hat{\mu}} + 4\mathrm{Tr}\, T^2_{L,\boldsymbol{x}} T^2_{L,\boldsymbol{x}+\hat{\mu}}. \tag{38}$$

This expression drastically simplifies for $N = 2$ and $N = 3$. For $N = 2$, the only relevant degree of freedom is the chirality of the configuration, which can be expressed in terms of the variable $\sigma_{\boldsymbol{x}} = \sum_{ab} \epsilon^{ab} s^a_{1,\boldsymbol{x}} s^b_{2,\boldsymbol{x}}$ that can only take the values ± 1. As expected, the gauge-invariant tensor $T^{ab}_{L,\boldsymbol{x}}$ depends only on $\sigma_{\boldsymbol{x}}$: $T^{ab}_{L,\boldsymbol{x}} = \frac{1}{2}\epsilon^{ab}\sigma_{\boldsymbol{x}}$. Substituting in Eq. (38), we obtain

$$|\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|^2 = \tfrac{1}{2}\sigma_{\boldsymbol{x}}\sigma_{\boldsymbol{x}+\hat{\mu}} + \tfrac{1}{2}. \tag{39}$$

We therefore obtain an Ising Hamiltonian.

For $N = 3$, configurations should be labeled by the unit-length vector $\boldsymbol{\tau} = \boldsymbol{s}_1 \times \boldsymbol{s}_2$ that encodes both the chirality of the two vectors and the plane in which they lie. The gauge-invariant tensor $T^{ab}_{L,\boldsymbol{x}}$ is related with $\boldsymbol{\tau}$ by $T^{ab}_{L,\boldsymbol{x}} = \frac{1}{2}\sum_c \epsilon^{abc}\tau^c_{\boldsymbol{x}}$. Substituting in Eq. (38), we obtain

$$|\bar{z}_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}+\hat{\mu}}|^2 = \tfrac{1}{2}\boldsymbol{\tau}_{\boldsymbol{x}} \cdot \boldsymbol{\tau}_{\boldsymbol{x}+\hat{\mu}} + \tfrac{1}{4}(\boldsymbol{\tau}_{\boldsymbol{x}} \cdot \boldsymbol{\tau}_{\boldsymbol{x}+\hat{\mu}})^2 + \tfrac{1}{4}. \tag{40}$$

We thus obtain a ferromagnetic Heisenberg model.

These results are confirmed by a standard LGW analysis. In the critical limit, the Hamiltonian with hopping term Eq. (38) becomes equivalent to the LGW model for an antisymmetric $N \times N$ field $\Psi^{ab}(\boldsymbol{x})$, which represents the coarse-grained average of $T^{ab}_{L,\boldsymbol{x}}$. The corresponding LGW Lagrangian reads

$$\begin{aligned} \mathcal{L} = &\ \mathrm{Tr}\, \partial_\mu \Psi^t \partial_\mu \Psi + r\, \mathrm{Tr}\, \Psi^t \Psi \\ &+ u\, (\mathrm{Tr}\, \Psi^t \Psi)^2 + v\, \mathrm{Tr}\, (\Psi^t \Psi)^2, \end{aligned} \tag{41}$$

where we have written the quadratic terms in terms of the transpose Ψ^t (since Ψ is antisymmetric $\Psi^t = -\Psi$) to show explicitly their positivity. Note that the cubic term is absent because $\mathrm{Tr}\Psi^n = 0$ for any odd n.

Using the above LGW theory, we can easily recover the results obtained in the large-v limit for $N = 2$ and 3, For $N = 2$, we can write Ψ^{ab} in terms of a single real scalar field ϕ, i.e., $\Psi^{ab} = \epsilon^{ab}\phi$. The two quartic terms are equivalent and we obtain the LGW model for a real scalar field that is associated with the Ising universality class. For $N = 3$, we can write $\Psi^{ab}(x)$ in terms of a single three-component vector as $\Psi^{ab} = \epsilon^{abc}\phi^c$, where ϵ^{abc} is the completely antisymmetric tensor. Again, the quartic terms are equivalent and we obtain the $O(3)$ vector LGW Hamiltonian. Thus, continuous transitions should belong to the $O(3)$ vector universality class.

No simplifications occur for $N \geqslant 4$. To determine the critical behavior one should therefore study the RG flow of the model Eq. (41) in the space of the quartic couplings u and v. The RG flow of the LGW theory Eq. (41) was studied in Ref. [117] within the perturbative ε expansion close to four dimensions, to one-loop order. This one-loop analysis shows that there are not stable fixed points for $N \geqslant 5$, while a stable fixed point exists for $N = 4$ in the region with $v > 0$. Actually, for $N = 4$ the LGW model Eq. (41) can be exactly mapped into one of the so-called mn models [104], more precisely the model with $m = 2$ and $n = 3$ [118]. The RG flow of mn models has been largely investigated by various methods, in particular, high-order pertubation theory within ϵ expansion and 3D schemes [99,119,120]. On the basis of these RG analyses, one can demonstrate the existence of a stable 3D fixed point for $N = 4$, describing two decoupled $O(3)$ critical behaviors [104,120]. Note that such a stable 3D fixed point is not related to the one found close to four dimensions reported in Ref. [117].

Assuming, as usual, that the LGW analysis is valid for any v, the previous results indicate that, for $N = 2$, an Ising behavior should also occur for finite positive values of v, down to the point $v = 0$, where the symmetry enlarges to SU(2) and the model undergoes a Heisenberg transition. The finite-v behavior for $N = 2$ can also be understood by considering a variant model in which the SO(N)-symmetric potential V_O is added to the standard CP^{N-1} Hamiltonian H_{CP} defined in Eq. (30). For this purpose, we parametrize the fields in terms of a real three-component unit vector $\boldsymbol{t}_{\boldsymbol{x}}$ defined by

$$t_{k,\boldsymbol{x}} = \bar{z}_{\boldsymbol{x}}\sigma_k z_{\boldsymbol{x}}, \tag{42}$$

where σ^k ($k = 1, 2, 3$) here represent the Pauli matrices. It is then easy to verify that

$$\begin{aligned} H_{CP} &+ v\sum_{\boldsymbol{x}} |z_{\boldsymbol{x}} \cdot z_{\boldsymbol{x}}|^2 \\ &= -\frac{1}{2}NJ\sum_{\boldsymbol{x}\mu}(\boldsymbol{t}_{\boldsymbol{x}} \cdot \boldsymbol{t}_{\boldsymbol{x}+\hat{\mu}} + 1) + v\sum_{\boldsymbol{x}}\left(1 - t^2_{2,\boldsymbol{x}}\right). \end{aligned} \tag{43}$$

Thus, the Hamiltonian is equivalent to a Heisenberg model with a symmetry-breaking term $-v t_{2,x}^2$. For positive v, only the second component of $\boldsymbol{t}$ becomes critical, so the transition is in the Ising universality class. On the other hand, for negative v the system magnetizes in the (1,3) plane, undergoing an XY transition, as already discussed. Transitions for $v = 0$ correspond to $O(3)$-symmetric multicritical points, belonging to the Heisenberg universality class (accurate estimates of the critical exponents and other universal features can be found in Refs. [121–124]), where the XY transition line for $v < 0$ and the Ising transition line for $v > 0$ meet.

To understand the finite-v behavior for $N = 3$, we recall that the lattice CP^2 model undergoes a first-order transition, see e.g., Ref. [80], so it is natural to expect first-order transitions also for small positive values of v. As a consequence, we predict the existence of a tricritical value v^*, such that the transition is in the Heisenberg universality class for $v > v^*$ and of first order for $v < v^*$.

We do not further discuss the more complicated cases with $N \geqslant 4$. Further work is clearly necessary to clarify their critical behavior for positive values of v.

3. *The CH transition line*

As CM transitions, CH transitions are also characterized by the spontaneous breaking of the SO(N) symmetry. As discussed in Sec. II A, the symmetry-breaking pattern depends on the sign of v and thus different Higgs phases are obtained for $v > 0$ and $v < 0$. However, at variance with the CM transitions, along the CH line the gauge-field modes are expected to play an active role, since the CH line separates a Coulomb phase with long-range gauge modes from a Higgs phase with massive gauge excitations. Therefore, as in the ULAHM [86,91], the critical behavior along the CH line is expected to be described by the OAHFT Eq. (4), which can be obtained by taking the formal continuum limit of the OLAHM.

The analysis of the RG flow of the OAHFT reported in Sec. II shows that a stable charged fixed point only exists for N larger than a critical value $N^*(d = 3)$, unless new universality classes emerge in three dimensions that are unrelated with the RG flow close to four dimensions, as occurs for the one-component AH models [21,34,94]. In analogy with what occurs in the ULAHM [86,91], we expect $N^*(d = 3)$ to be of order 10. This fixed point is located in the region $v > 0$ and is therefore different from the charged fixed point that controls the critical behavior of the ULAHM, which belongs to the line $v = 0$.

The field theory results allow us to predict the nature of the CH transitions. For $N < N^*(d = 3)$, all transitions along the CH line are expected to be of first order, as no stable charged fixed point exists. For $N > N^*(d = 3)$, the nature of the transition depends on the sign of v. Since the line $v = 0$ is a separatrix of the RG flow, and the charged fixed point lies in the region $v > 0$, the charged fixed point is unreachable for systems with negative v. Thus, CH transitions that separate a Coulomb disordered phase from a Higgs phase with residual $\mathbb{Z}_2 \otimes O(N-1)$ invariance—this is the symmetry-breaking pattern characterizing systems with $v < 0$—are expected to be of first order. Continuous transitions can only be observed in systems with $v > 0$, provided that the system is in the attraction domain of the stable fixed point.

For $\kappa = \infty$, the gauge variables freeze, thus we can fix all gauge variables to the trivial value $\lambda_{x,\mu} = 1$, obtaining an $O(2) \otimes O(N)$ model with Hamiltonian

$$H_{O,\kappa\to\infty} = -2NJ \sum_{x,\mu} \mathrm{Re}\,(\bar{z}_x \cdot z_{x+\hat{\mu}}) + v \sum_x |z_x \cdot z_x|^2. \quad (44)$$

As already mentioned in Sec. II, the corresponding LGW theory has 3D stable fixed points, and therefore continuous transitions are possible in the lattice model Eq. (44). However, as already discussed in Sec. II, any stable fixed point of the purely scalar theory is unstable with respect to gauge fluctuations. Therefore, for finite κ, one can never observe the same asymptotic critical behavior as for $\kappa = \infty$. However, the $\kappa = \infty$ criticality may give rise to substantial preasymptotic crossover effects for finite large values of κ.

IV. CONCLUSIONS

We have investigated some generalizations of the standard multicomponent AH models with SU(N) symmetry, such as the AH field theory defined in Eq. (1) and the lattice AH model defined in Eq. (2). By adding an appropriate scalar potential, we obtain models with a reduced symmetry, which, therefore, may undergo transitions belonging to different universality classes and have Higgs phases with different symmetries. In this paper, we focus on AH models with SO(N) invariance, the field theory with Lagrangian Eq. (4) and the lattice AH model with Hamiltonian Eq. (7). We determine the possible symmetry-breaking patterns and the symmetry of the Higgs phases, finding that they depend on the sign of the Hamiltonian parameter v. Thus, for $v > 0$ and $v < 0$, we observe different ordered phases characterized by the condensation of different gauge-invariant order parameters. In particular, there are different Higgs phases.

The analysis of the RG flow of the OAHFT, using the perturbative and large-N computations of Refs. [5,17], indicates that the quartic scalar term that breaks the SU(N) symmetry is a relevant perturbation of the SU(N)-symmetric fixed point. Therefore, in the absence of exact SU(N) symmetry, under RG transformations the system flows away from the SU(N) fixed point, possibly toward a stable fixed point—in this case one may observe an SO(N)-symmetric critical behavior—or toward infinity—in this case, a first-order transition would occur. The analysis of the RG flow close to four dimensions and in the large-N limit [5] shows that an SO(N)-symmetric stable fixed point exists for $N > N_O^*(d)$ and lies in the region $v > 0$. We expect this fixed point to also exist in three dimensions for N sufficiently large, i.e., for $N > N_O^*(d = 3)$ [by analogy with the SU(N)-symmetric case, we guess that $N_O^*(d = 3)$ is of order 10]. It would control the behavior of lattice systems at transitions where the gauge degrees of freedom are critical. We should note, however, that we cannot exclude the existence of 3D stable fixed points that are not related with the fixed points identified by the ε expansion close to four dimensions, as happens in the one-component 3D ULAHM [21,34,94] and in LGW theories that are effective models of the $O(2) \otimes O(N)$ scalar models that are obtained by taking the $\kappa \to \infty$ limit of the multicomponent OLAHM [106,107].

Concerning the 3D lattice models, we argue that some features of the phase diagram of SO(N)-symmetric lattice AH models are independent of the Hamiltonian parameter v and similar to those of the SU(N)-symmetric models. In all cases, the qualitative phase diagram is the one reported in Fig. 1, with three different phases—the Coulomb, the molecular, and the Higgs phase—which differ in the properties of the gauge correlations, in the confinement or deconfinement of the charged excitations, and in the residual symmetry of the ordered (molecular and Higgs) phases.

As far as the nature of the phase transitions, we argue that MH transitions always belong to the IXY universality class, as in SU(N) invariant models. This is due to the fact that MH transitions are topological transitions only driven be gauge modes. Scalar fields play no role at the transition. On the other hand, the nature of the CM and CH transitions depends on v and different behaviors are observed for positive and negative values of v. This is due to the fact that the breaking of the SO(N) symmetry differs for $v > 0$ and $v < 0$ [the residual symmetry is $O(N-2)$ and $\mathbb{Z}_2 \otimes O(N-1)$ in the two cases, respectively], with the condensation of different order parameters, such as $R^{ab}_{L,\boldsymbol{x}}$ and $T^{ab}_{L,\boldsymbol{x}}$ defined in Eqs. (27) and (28).

The behavior along the CH line is essentially determined by the presence or absence of the field-theoretical fixed point. For $N < N^*(d=3)$, we expect only first-order transitions along the CH line. For $N > N^*(d=3)$, if the Higgs phase is symmetric under $\mathbb{Z}_2 \otimes O(N-1)$ transformations, which is the residual symmetry for $v < 0$, the CH line is again a line of first-order transitions. Indeed, if the RG flow starts in a point $v < 0$, it necessarily runs toward infinity. Instead, if the Higgs phase is $O(N-2)$-symmetric, i.e., v is positive, CH transitions may be continuous in the universality class associated with the field-theory fixed point.

Also, the behavior along the CM line depends on v. For negative v, transitions are driven by the condensation of $R^{ab}_{\boldsymbol{x}}$, defined in Eq. (27). Gauge modes play no role, so one can perform a simple LGW analysis to determine the nature of the phase transitions. It predicts first-order transitions for any $N \geqslant 3$. For $N = 2$, continuous transitions in the XY universality class are possible. For positive v, we argue that the critical behavior is effectively described by the $O(N)$-symmetric LGW theory for an antisymmetric rank-two tensor, which is the coarse-grained analog of $T^{ab}_{\boldsymbol{x}}$, see Eq. (41). This allows us to predict that, for $N = 2$, CM transitions should belong to the Ising universality class for all positive values of v. For $N = 3$, instead, we expect the existence of a tricritical value $v^* > 0$, such that the transition is of first order for $v < v^*$ and in the Heisenberg universality class for $v > v^*$. The existence of a tricritical value is due to the first-order nature of the CM transitions in SU(3) invariant AH models, i.e., for $v = 0$. The behavior for larger N is less clear and currently under investigation.

Of course, numerical checks of the above predictions would be useful and welcome. However, we believe that the theoretical arguments reported in this paper, that also rely on the known behavior of SU(N) invariant AH models (which have been carefully studied numerically, see, e.g., Refs. [86,91,94]), are sound and can be easily extended to AH models with more general scalar interactions. We also stress that the predictions reported above should not only apply to the OLAHM but also to more general models, in which the unit-length constraint for the lattice variable $\boldsymbol{z_x}$ is relaxed.

We remark that one may also consider a more general quartic scalar potential. For instance, one may consider the AH field theory with quartic potential [17]

$$V_{\mathbb{P}}(\boldsymbol{\phi}) = u\,(\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi})^2 + v\,|\boldsymbol{\phi} \cdot \boldsymbol{\phi}|^2 + w \sum_{a=1}^{N} (\bar{\phi}^a \phi^a)^2, \quad (45)$$

which is only invariant under the permutation group $\mathbb{P}_N$. A one-loop analysis of the RG flow close to four dimensions was reported in Ref. [17], showing that a stable fixed point appears only for very large values of N, more precisely for $N \geqslant 5494$. Note that for $N = 2$ the quartic potential $V_{\mathbb{P}}(\boldsymbol{\phi})$ is the most general one, preserving the U(1) gauge invariance and the uniqueness of the quadratic $\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi}$ term. For $N > 2$ there are other quartic terms satisfying these conditions, such as $\sum_{a=1}^{N} \bar{\phi}^a \phi^a \bar{\phi}^{a+1} \phi^{a+1}$ (we identify $\phi^{N+1} = \phi^1$), which leaves a residual $\mathbb{Z}_N$ symmetry only. Analogous terms can be added to the lattice AH models.

Lattice AH counterparts with residual $\mathbb{P}_N$ global symmetry can be simply obtained by adding a term $w \sum_{\boldsymbol{x}} \sum_{a=1}^{N} (\bar{z}^a z^a)^2$ to the SO(N)-symmetric Hamiltonian Eq. (7). We expect the phase diagram of these lattice AH models (keeping the quartic parameters fixed) to be qualitatively the same as that of SU(N)- and SO(N)-symmetric theories, with three phases and three transition lines, as sketched in Fig. 1. MH transitions are always expected to belong to the IXY universality class. Instead, the nature of the transitions, and the universality classes of the continuous ones along the CM and CH lines, are expected to change. In particular, in the $N = 2$ lattice AH model with quartic potentials analogous to $V_{\mathbb{P}}(\boldsymbol{\phi})$, the residual symmetry is $\mathbb{P}_2 = \mathbb{Z}_2$. Thus, we expect CM transitions to be Ising transitions. This behavior can be easily confirmed by rewriting the $\mathbb{Z}_2$-symmetric scalar potential in terms of the variable Eq. (42), obtaining $V_P(z) = v(1 - t_2^2) + \frac{1}{2} w (1 + t_3^2)$. Thus, for generic values of v and w the system undergoes Ising transitions. On the planes $v = 0$, $w = 0$, and $2v + w = 0$ one can observe both Ising and XY transitions, depending on the symmetry of the low-temperature phase. The nature of the CH transitions is less clear, but we believe these transitions to be of first order, since no stable fixed points are found in the corresponding AH field theory, at least close to four dimensions [17]. Although we believe that the phase diagram and critical behaviors of these extended lattice AH theories are worth being investigated, we have not pursued this study further.

We finally stress that the understanding of the possible extensions of the AH gauge theories, allowing for more general scalar potentials, may be useful to get a more thorough understanding of the possible phases and critical behaviors that can be observed in critical phenomena characterized by an emerging Abelian gauge field.

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