Universal conductivity at a two-dimensional superconductor-insulator transition: The effects of quenched disorder and Coulomb interaction

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We calculate the zero-temperature dc electrical conductivity in the collisionless $\hbar\omega/k_BT \to \infty$ limit at superconductor-insulator transitions in the $(2+1)$d XY model universality class. We use a dual model consisting of a single Dirac fermion at zero density, coupled to a Chern-Simons gauge field and in the presence of a quenched random mass, with or without an unscreened Coulomb interaction. Our calculation is performed in a $1/N_f$ expansion, where N_f is the number of Dirac fermions. At the fixed point without Coulomb interaction, we obtain the universal conductivities $(\sigma_{xx}, \sigma_{xy}) = (0.97 - 0.52/N_f, -0.24 + 1.64/N_f) \cdot (2e)^2/h$. At the fixed point with Coulomb interaction, we find $(\sigma_{xx}, \sigma_{xy}) = (0.97 + 1.09/N_f, -0.24 + 0.93N_f) \cdot (2e)^2/h$. At zeroth order, the model exhibits particle-vortex self-dual electrical transport with $\sigma_{xx} \lesssim (2e)^2/h$ and small, but finite σ_{xy}. Corrections of $\mathcal{O}(1/N_f)$ due to fluctuations in the Chern-Simons gauge field and disorder produce violations of self-duality. These fluctuations reduce/enhance the longitudinal conductivity at the fixed point without/with the Coulomb interaction.

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I. INTRODUCTION

Continuous quantum phase transitions [1,2] are some of the most intriguing phenomena in condensed matter physics, due to the possibility of emergent behaviors that are different from the proximate phases. Superconductor to insulator transitions (SITs) in disordered two-dimensional thin films, the subject of this paper, provide some of the best examples (see [3,4], for reviews). Here, as a tuning parameter, such as the charge density, disorder, or external magnetic field, is varied, the electrons in the disordered film transition between a superconducting and insulating ground state. At a critical value of the tuning parameter, the electrons form a metal with finite, nonzero dc resistance (measured as the temperature $T \to 0$) on the order of the quantum of Cooper-pair resistance $R_Q \equiv h/4e^2 \approx 6.45\ k\Omega/\Box$.

How should we understand this? One hypothesis [5,6] is that such transitions are due to the phase disordering of the superconducting order parameter, rather than the loss of superconducting pairing amplitude [7]. The critical properties should then be described by a model of interacting charge-$2e$ Cooper-pair bosons moving in a random potential in two spatial dimensions [5,6]. In this theory—generally known as the dirty boson model—the critical resistivity is identified as a universal critical amplitude [5,6,8], which takes a value $\sim 1/R_Q$ reflective of the delocalized charge carriers. The relevance of the dirty boson model to field-tuned SITs is directly implied by ac conductance measurements that find a nonzero superfluid stiffness across the transition and into insulating phase [9,10]. By contrast, the relevance of the boson model to SITs in zero magnetic field has been questioned (see, e.g., [4]). (Measurements such as those in [9,10] have not been carried out for SITs with time-reversal symmetry.) In this paper, we restrict our attention to the simplified setting of boson-only models, which ignore possible fermionic quasiparticle excitations.

One of the more intriguing possibilities suggested by the dirty boson model is that the transition might be self-dual, i.e., the critical Hamiltonian for Cooper-pair bosons on the brink of localization is the same as the dual critical Hamiltonian for vortices [11–13] on the brink of condensation. A consequence of self-duality is the so-called semicircle law for the dc electrical conductivity,

$$\sigma_{xx}^2 + \sigma_{xy}^2 = \left(\frac{4e^2}{h}\right)^2. \tag{1.1}$$

This relation is particularly interesting because it relates the dissipative (σ_{xx}) and nondissipative (σ_{xy}) parts of the conductivity to a universal constant, R_Q. There is strong experimental evidence [14] that field-tuned transitions are self-dual with $\sigma_{xx} \approx 1/R_Q$ and $\sigma_{xy} \approx 0$. This may be surprising, given the presence of the nonzero magnetic field, and suggests an emergent particle-hole symmetry [15]. Measurements [3,4] of the longitudinal resistance on the order of R_Q at charge density or disorder-tuned SITs are suggestive of an underlying self-duality, so long as the impurities are nonmagnetic.

The argument for self-duality within the boson model is indirect [5,6]. Here, we will consider an alternative description, given in terms of $(2+1)$d quantum electrodynamics with a single Dirac fermion coupled to a Chern-Simons gauge field [16–19]. In close analogy to theories [20–22] for the half-filled Landau level, we will refer to the Dirac fermions in this description as composite fermions. In this composite fermion model, self-duality is a consequence of an unbroken particle-vortex symmetry that (loosely speaking) acts on the Hamiltonian of the model as a time-reversal symmetry and therefore fixes the composite fermion Hall conductivity to be zero [19]. (The composite fermion conductivity σ^{ψ}_{ij} is distinct from the electrical conductivity σ_{ij}. We will recall the correspondence between the two in Sec. IV B.) This means that, so long as particle-vortex symmetry is preserved, the composite fermion theory will yield self-dual response. Note, however, that the precise manner in which self-duality (1.1) is realized depends on the specific value of the composite fermion longitudinal conductivity. For instance, a composite fermion longitudinal conductivity of $\frac{1}{2}R_Q^{-1}$ (with $\sigma^{\psi}_{xy} = 0$) reproduces (1.1) with $\sigma_{xy} = 0$, while any other value of the composite fermion longitudinal conductivity gives self-dual transport with nonzero σ_{xy}.

We will study a limit of the composite fermion theory that corresponds to lattice bosons at integer filling with charge-conserving disorder and vanishing external magnetic field [8]. (Simplifying the model in this way unfortunately takes us further away from the experiments reviewed in [3,4].) In this limit, the critical properties are those of the dirty $(2+1)$d XY model [8], recently clarified in [23]. (In the classical 3d XY model, the disorder we consider reflects a position-dependent critical temperature that is nonuniform with respect to two of the three spatial dimensions.) In the fermion dual, the Dirac composite fermions lie at zero density in the presence of a quenched random mass.

An unscreened Coulomb interaction is believed to be important for the experimentally realized SITs [5,6]; in the $(2+1)$d XY model, the Coulomb interaction corresponds to a relevant perturbation. General constraints on the transport properties of a self-dual SIT, with Coulomb interactions, were derived in [24,25]. The stability of these results to fluctuations in the Coulomb interaction, in the presence of disorder, is not addressed in these papers.

In Ref. [26], we showed that the composite fermion theory we consider here admits renormalization group (RG) fixed points at finite disorder, with or without the Coulomb interaction, when the mean of the random mass is tuned to zero. Other symmetry classes of disorder did not yield accessible fixed points. The random mass fixed points are controlled within a $1/N_f$ expansion, where N_f is the number of fermion flavors. Related works studying the effects of quenched randomness on theories of Dirac fermions coupled to a fluctuating boson include [27–32]. In this paper, we determine the electrical conductivity at the random mass fixed points of [26], with or without the unscreened Coulomb interaction, to $\mathcal{O}(1/N_f)$. Our results are summarized in Eqs. (4.18) and (4.19) and Figs. 5(a) and 5(b) (see below). (Such calculations of the quantum critical conductivity are notoriously challenging, e.g., [33–35].) At $\mathcal{O}(1/N_f^0)$, the theory exhibits self-dual electrical transport with $\sigma_{xx} \lesssim 1/R_Q$ and finite, nonzero σ_{xy}. The leading $\mathcal{O}(1/N_f)$ corrections, due to fluctuations in the Chern-Simons gauge field and disorder, violate self-duality. The nature of this violation depends on whether or not the Coulomb interaction is present. At the unstable fixed point without Coulomb interaction and only short-ranged interactions, fluctuations reduce the longitudinal conductivity, while at the stable fixed point with Coulomb interaction, the longitudinal conductivity is increased (above $1/R_Q$).

There are two important things to note about our results. First, the random mass does not preserve particle-vortex symmetry. This is not the underlying reason why we find nonzero composite fermion Hall conductivity: The $\mathcal{O}(1/N_f)$ violation of self-duality is due to the fluctuations of the Chern-Simons gauge field. There is a nonzero composite fermion Hall conductivity in the pure limit without disorder. We are unaware of a calculable model of a disordered fixed point that is self-dual. Second, we calculate the conductivity in the phase-coherent regime $\hbar\omega/k_BT \to \infty$ (where ω is the measuring frequency). This regime gives a universal conductivity tensor that, in general, differs from conductivity in the incoherent regime $\hbar\omega/k_BT \to 0$ [35]. Extending our results to the incoherent regime using quantum Boltzmann methods, such as those recently employed in [36], is of great interest.

The structure of this paper is as follows. In Sec. II, we define the model that we study. In Sec. III, we sketch the calculation of the composite fermion conductivity for general values of the coupling constants. In Sec. IV, we evaluate these composite fermion conductivities at two RG fixed points, corresponding to disordered quantum critical points with or without the unscreened Coulomb interaction, and use the duality dictionary to translate these quantum critical composite fermion conductivities to electrical conductivities. In Sec. V, we discuss our results. Four appendices contain the technical details of the calculations summarized in the main parts of the paper: Appendix A contains the derivation of the effective gauge field propagator; Appendices B, C, and D contain details for the evaluation of loop diagrams.

II. THE MODEL

In this section, we define the Dirac composite theory of the SIT. See [26] for additional details.

A. The effective action

The total Euclidean effective action for the Dirac composite fermion theory has three parts,

$$S_{\text{tot}} = S_0 + S_c + S_{\text{dis}}. \tag{2.1}$$

We will define each part in the following three paragraphs.

To begin, S_0 is a Euclidean action of quantum electrodynamics in 3d. It consists of a two-component Dirac fermion ψ that is coupled to a dynamical Chern-Simons gauge field a_μ ($\hbar = 2e = 1$),

$$\begin{aligned} S_0 = \sum_{I=1}^{N_f} \int d^2x d\tau\, \bar{\psi}_I \gamma^0 \left(\partial_\tau - i\frac{g}{\sqrt{N_f}} a_\tau\right)\psi_I \\ + \bar{\psi}_I \gamma^j \left(\partial_j - i\frac{g}{\sqrt{N_f}} a_j\right)\psi_I \\ + \frac{i\kappa}{2} ada + \frac{i}{4\pi}(-2Ada + AdA). \end{aligned} \tag{2.2}$$

In Eq. (2.2), $\bar{\psi} \equiv \psi^\dagger \gamma^0$ and the 2×2 gamma matrices γ^μ are chosen such that the anticommutator $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbf{1}_{2\times2}$, where $\mu, \nu \in \{0, 1, 2\} = \{\tau, x, y\}$ and $\mathbf{1}_{2\times2}$ is a 2×2

identity matrix in the 2d spinor space. We follow the Einstein summation convention, with, for instance, the spatial indices $j \in \{x, y\}$ summed over above. (Being in Euclidean signature, upper and lower indices are equivalent and will be used interchangeably.) The Chern-Simons term $ada \equiv \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$ (and similarly for the other Chern-Simons terms), where the Levi-Civita symbol $\epsilon^{012} \equiv +1$. Here, N_f is the number of Dirac fermion flavors. S_0 is dual to the critical 3d XY model when $N_f = 1$ and the Chern-Simons coefficient $\kappa = \frac{1}{4\pi}$ [16,17,19], with the Dirac mass playing the role as the critical tuning parameter. We take $N_f > 1$ in order to study the theory within a controlled $1/N_f$ expansion. A_μ is a probe, nondynamical electromagnetic gauge potential that serves to define the electrical current and corresponding electrical conductivity of the model. The gauge coupling g is set to unity in the infrared; we keep g general for the moment.

Next, we introduce the Coulomb interaction. Following Refs. [26,37], the Coulomb interaction dualizes into the composite fermion theory as

$$S_c = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} a_T(k)(|\boldsymbol{k}|\, w_x) a_T(-k), \tag{2.3}$$

where we introduced the effective Coulomb coupling $w_x \equiv \frac{e_*^2}{8\pi^2}$. Here, e_* is the charge of the Cooper-pair bosons (nominally, $e_* = 2e$). The Fourier space integration measure is $d^3k \equiv dk_x dk_y dk_0$, where k_0 is the zero-temperature Matsubara frequency. We adopt the following notation throughout this paper:

$$\boldsymbol{k} = (k_x, k_y), \quad k_\mu = (k_0, k_x, k_y), \quad |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2},$$
$$k = \sqrt{k_0^2 + k_x^2 + k_y^2}. \tag{2.4}$$

The T subscript indicates that a_T is the transverse component of the Chern-Simons gauge field: a_T is related to the Cartesian components (a_x, a_y) via $a_x(q) = i\frac{q_y}{|\boldsymbol{q}|} a_T(q)$ and $a_y(q) = -i\frac{q_x}{|\boldsymbol{q}|} a_T(q)$, provided we choose Coulomb gauge, wherein the longitudinal component of a_j is set to zero.

Finally, we turn to the introduction of the quenched randomness. Previous analyses [26,31] show that if all types of disorder (quenched random couplings that couple to composite fermion bilinears), regardless of the symmetry they individually preserve, are added to the theory, the system flows to strong coupling, out of reach of any analytic control. However, if charge-conjugation symmetry is imposed, only a random mass term $m(\boldsymbol{x})\bar{\psi}\psi$ is allowed and the theory flows to an accessible disordered fixed point [26]. Here, following [26], we make this symmetry assignment based on the fact that $\bar{\psi}\psi$ is even under charge-conjugation and odd under time-reversal. Because the fermion theory with random mass is believed to be dual to the bosonic description of the XY model with random mass, which has both charge-conjugation and time-reversal symmetries, the fermion model should possess an emergent time-reversal symmetry, in addition to the explicit charge-conjugation symmetry. Applying the standard replica trick to disorder average the theory, the action picks up the term,

$$S_{\text{dis}} = \frac{-1}{2}\int d^2x d\tau d\tau' \sum_{r,r'=1}^{n_r} g_m\, (\bar{\psi}_r\psi_r)_{x,\tau}\, (\bar{\psi}_{r'}\psi_{r'})_{x,\tau'}, \tag{2.5}$$

where r, r' are the replica indices and the number of replicas n_r is to be set to zero in the last step of any calculation. The replica indices are not important in the later calculations, so we will not write them out explicitly anymore (this is also why we did not include them in the earlier parts of the total action). The parameter g_m is the disorder strength, which is always non-negative. The random mass does not preserve particle-vortex symmetry. This will be apparent later when we discuss the quantum critical conductivity of the model.

B. The effective gauge field propagator

In the Cartesian basis, the one-loop gauge field self-energy induced by the Dirac fermions is

$$\Pi^{1-\text{loop}}_{\mu\nu} = \frac{-g^2}{16}\frac{\delta_{\mu\nu}k^2 - k_\mu k_\nu}{k}. \tag{2.6}$$

This self-energy is $\mathcal{O}(N_f^0)$, i.e., the same order as the bare terms in the total action that are quadratic in the gauge field. Adding together this self-energy S_c, and the Chern-Simons term in S_0 involving a_μ only, we obtain the one-loop gauge field action,

$$S_{\text{gauge}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} (a_0 \quad a_T)_k \begin{pmatrix} \frac{g_X^2 \boldsymbol{k}^2}{k} & i\kappa\,|\boldsymbol{k}| \\ i\kappa\,|\boldsymbol{k}| & g_X^2\, k + |\boldsymbol{k}|\, w_x \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_T \end{pmatrix}_{-k}, \tag{2.7}$$

where $g_X^2 \equiv \frac{g^2}{16}$. This action defines the one-loop corrected gauge field propagator.

The computation we summarize in the next section is performed with the gauge field, expressed in the Cartesian basis. We therefore translate the gauge field propagator, defined by Eq. (2.7), to the Cartesian basis,

$$D_{00}(k_0, \boldsymbol{k}) = \frac{1}{\boldsymbol{k}^2}\frac{1}{\kappa^2 + g_X^4 + \frac{w_x g_X^2 |\boldsymbol{k}|}{k}}\left(g_X^2\, k + |\boldsymbol{k}|\, w_x\right), \tag{2.8}$$

$$D_{0i}(k_0, \boldsymbol{k}) = -D_{i0} = \frac{1}{\boldsymbol{k}^2}\frac{1}{\kappa^2 + g_X^4 + \frac{w_x g_X^2 |\boldsymbol{k}|}{k}}\left(\frac{\kappa\,\epsilon_{ij}k_j}{1}\right), \tag{2.9}$$

$$D_{ij}(k_0, \boldsymbol{k}) = \frac{1}{\kappa^2 + g_X^4 + \frac{w_x g_X^2 |\boldsymbol{k}|}{k}}\left(\frac{g_X^2}{k}\right)\left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right). \tag{2.10}$$

To obtain $D_{\mu\nu}$ above, we have used the gauge fixing term $\frac{(\nabla a)^2}{2\lambda}$. See Appendix A for details.

III. COMPOSITE FERMION CONDUCTIVITY

In this section, we sketch the computation of the composite fermion conductivity, to $\mathcal{O}(1/N_f)$. The details of this calculation are in Appendices B, C, and D. This composite fermion conductivity is related to the electrical conductivity considered in the next section.

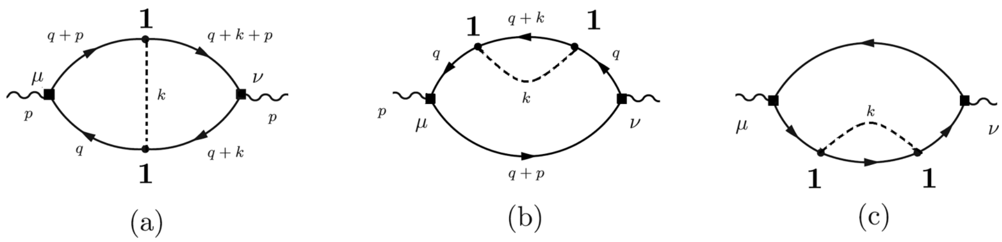

FIG. 1. Two-loop diagrams with random mass corrections, denoted as $\hat{\Pi}_{ij}$.

A. Definition of the composite fermion conductivity

The composite fermion conductivity $\sigma^{\psi}_{ij}(\omega, \mathbf{p})$ is determined by the real-frequency retarded gauge field self-energy Π^R_{ij}, which is related to the imaginary-time self-energy Π^{tot}_{ij} as follows:

$$\sigma^{\psi}_{ij}(\omega, \boldsymbol{p}) = \frac{i}{\omega}\Pi^R_{ij}(\omega + i0^+, \boldsymbol{p}) = \frac{i}{\omega}\Pi^{\text{tot}}_{ij}(ip_0 \to \omega + i0^+, \boldsymbol{p}). \tag{3.1}$$

Our definition of σ^{ψ}_{ij} includes the contributions from the composite fermions; it does not include the tree-level Chern-Simons term. We decompose Π^{tot}_{ij} into symmetric $\Pi^{\text{tot}}_{(\text{S})}$ and antisymmetric $\Pi^{\text{tot}}_{(\text{A})}$ components as

$$\Pi^{\text{tot}}_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi^{\text{tot}}_{(\text{S})}(p) + \frac{p_\kappa}{|p|}\epsilon_{\kappa\mu\nu}\Pi^{\text{tot}}_{(\text{A})}(p). \tag{3.2}$$

To extract the symmetric component of Π^{tot}_{ij}, we will set $(\mu, \nu) = (x, x)$ and evaluate

$$\Pi^{\text{tot}}_{(\text{S})} = \frac{1}{1 - \frac{p_x^2}{p^2}}\Pi^{\text{tot}}_{xx}. \tag{3.3}$$

The antisymmetric component will be obtained by contracting Π^{tot}_{ij} with $\epsilon_{\mu\nu\lambda}p^\lambda$,

$$\Pi^{\text{tot}}_{(\text{A})} = \frac{p_\lambda \epsilon_{\mu\nu\lambda}}{2|p|}\Pi^{\text{tot}}_{\mu\nu}. \tag{3.4}$$

B. Two-loop corrections to the conductivity

The $\mathcal{O}(1/N_f)$ corrections to the composite fermion conductivity σ^{ψ}_{ij}, obtained through $\mathcal{O}(1/N_f)$ corrections to Π^{tot}_{ij} through Eq. (3.1), involves two-loop and three-loop diagrams. The one-loop corrections have already been included in the one-loop corrected gauge boson self-energy.

We begin here with the two-loop diagrams. The two-loop corrections result from self-energy corrections due to the random mass $\hat{\Pi}_{ij}$ to $\mathcal{O}(g_m)$ and the fluctuations of the gauge boson Π_{ij} to $\mathcal{O}(1/N_f)$. The diagrams contributing to these corrections are shown in Figs. 1 and 2. In each figure, diagrams (b) and (c) are equal.

We start with the corrections due to the random mass. The diagrams in Fig. 1 are

$$\hat{\Pi}^{(a)}_{\mu\nu}(p) = -\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f \int_{k,q} \text{Tr}[\gamma_\mu G(q)G(q+k)\gamma_\nu G(q+k+p)G(p+q)] \times 2\pi g_m \delta(k_0), \tag{3.5}$$

$$\hat{\Pi}^{(b+c)}_{\mu\nu}(p) = -2\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f \int_{k,q} \text{Tr}[\gamma_\mu G(q)G(k+q)G(q)\gamma_\nu G(p+q)] \times 2\pi g_m \delta(k_0), \tag{3.6}$$

where we have used the shorthand $\int_{k,q} \equiv \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}$ and the trace is performed over the Dirac indices. Here, $G(k)$ is the composite fermion propagator,

$$-iG(k) = \frac{k_0\gamma^0 + k_i\gamma^i}{k_0^2 + k_ik^i}. \tag{3.7}$$

To evaluate the gauge field diagrams in Fig. 2, we replace the disorder-induced four-point interaction by the gauge field propagator and disorder strength coupling by the fermion-gauge field coupling,

$$\Pi^{(a)}_{\mu\nu}(p) = -\frac{g^4}{N_f}\int_{k,q} \text{Tr}[\gamma_\mu G(q)\gamma_\alpha G(q+k)\gamma_\nu G(q+k+p)\gamma_\beta G(p+q)]\, D_{\alpha\beta}(k), \tag{3.8}$$

$$\Pi^{(b+c)}_{\mu\nu}(p) = -\frac{2g^4}{N_f}\int_{k,q} \text{Tr}[\gamma_\mu G(q)\gamma_\alpha G(k+q)\gamma_\beta G(q)\gamma_\nu G(p+q)]\, D_{\alpha\beta}(k), \tag{3.9}$$

where $D_{\alpha\beta}$ is the gauge field propagator.

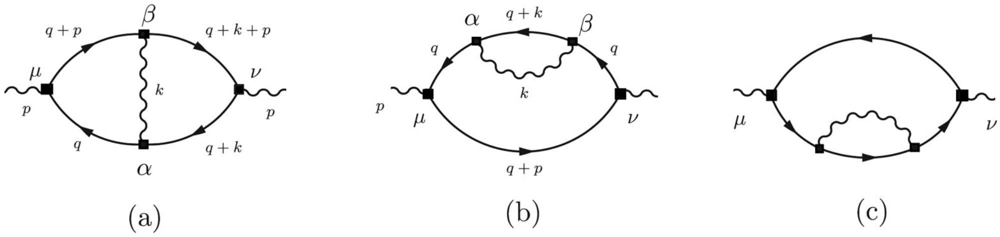

FIG. 2. Two-loop diagrams with gauge field corrections, denoted as Π_{ij}.

If the Coulomb interaction is present, the 3d Euclidean symmetry that rotates the temporal and spatial directions into one another is lost in the gauge field propagators; this symmetry is already broken by the random mass. (This is due to the fact that the Coulomb interaction, considered as an instantaneous interaction, is fundamentally due to an electron density-density interaction that is mediated by the temporal component of the electromagnetic gauge field. We have translated this interaction into the composite fermion theory [26].) This makes the evaluation of these integrals by a naive Feynman parametrization difficult. Instead, we reexpress all terms in the integrands involving q using partial fractions in a way that allows us to carry out the UV-finite integrals over q, after which we perform the k integrals. The lengthy calculations that do this are relegated to Appendices B and C. We now summarize the results.

The only nonzero contribution of the random mass diagrams is to $\hat{\Pi}_{xx}$,

$$\hat{\Pi}^{(a+b+c)}_{xx}(p_0, \boldsymbol{p}=0) = 3\times(-1)\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \times \frac{1}{2\pi}\times\frac{|p_0|}{96}. \tag{3.10}$$

This result was first obtained by Thomson and Sachdev in [31]. Note that the disorder strength $g_m \sim 1/N_f$ when evaluated at the fixed points, described in the next section.

The gauge field diagrams result in nonzero contributions to both the symmetric and antisymmetric components of Π_{ij}. The antisymmetric component is

$$\Pi^{(a+b+c)}_{(\mathrm{A})}(p) = -\frac{1}{2|p|}\frac{g^4}{N_f}\int_{-1}^{1} dz\, p^2\, \frac{-z+(-1+z^2)\mathrm{ArcTanh}[z]}{8\pi^2 z}\times\frac{1}{A_X + B_X\sqrt{1-z^2}}, \tag{3.11}$$

$$A_X \equiv \frac{\kappa^2+g_X^4}{\kappa},\quad B_X \equiv \frac{w_x g_X^2}{\kappa}. \tag{3.12}$$

The symmetric component is

$$\Pi^{(a+b+c)}_{(\mathrm{S})}(p_0, \boldsymbol{p}=0) = -\frac{g^4}{N_f}\,|p|\int_{-1}^{1} dz \frac{f_0^{w_x}(z)+f_0^{w_x}(-z)}{2}, \tag{3.13}$$

$$\frac{f_0^{w_x}(z)+f_0^{w_x}(-z)}{2} \equiv \frac{-5\,C_Y\, z(-1+z^2)+z\sqrt{1-z^2}(9+z^2)}{192\pi^2\, z\,[\,A_Y\sqrt{1-z^2}+B_Y(1-z^2)]} + \frac{6(-1+z^2)(C_Y - C_Y\, z^2 + 2\sqrt{1-z^2})\,\mathrm{arctanh}[z]}{192\pi^2\, z\,[\,A_Y\sqrt{1-z^2}+B_Y(1-z^2)]}, \tag{3.14}$$

where

$$A_Y \equiv \frac{\kappa^2+g_X^4}{g_X^2},\quad B_Y \equiv w_x,\quad C_Y \equiv \frac{w_x}{g_X^2}. \tag{3.15}$$

As a consistency check, we note the precise agreement of our results in Eqs. (3.11) and (3.13) for the gauge field diagrams, evaluated at $w_x = 0$, with the earlier computations of these same diagrams, computed in the absence of the Coulomb interaction, in [38–40],

$$\Pi^{(a+b+c)}_{(\mathrm{A})}(p; w_x=0) = \frac{1}{2|p|}\frac{g^4}{N_f}\frac{\kappa}{\kappa^2+g_X^4}\left(p^2\,\frac{4+\pi^2}{32\pi^2}\right), \tag{3.16}$$

$$\Pi^{(a+b+c)}_{(\mathrm{S})}(p; w_x=0, \kappa=0) = \frac{-g^4}{N_f}\times 0.00893191. \tag{3.17}$$

C. Three-loop corrections to the conductivity

The three-loop diagrams that contribute to the composite fermion conductivity at $\mathcal{O}(1/N_f)$ are given in Fig. 3. As the figure indicates, these diagrams involve a combination of disorder and gauge field fluctuations. Diagrams of the same form in which the two horizontal internal lines are of the same type, i.e., both disorder or both gauge field lines, vanish by charge-conjugation symmetry (Furry's theorem).

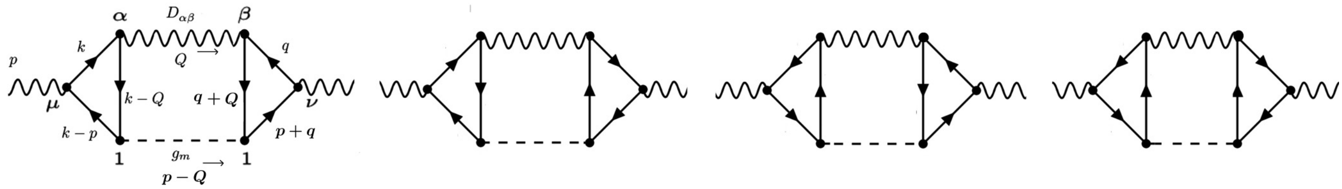

FIG. 3. Three-loop diagrams with both random mass and gauge field corrections, denoted as $\tilde{\Pi}_{\mu\nu}$. We use a convention in which different diagrams are differentiated by the direction of momentum flow in fermion lines in the left and right triangles, rather than using diagrams with fixed triangle orientations and twisted or line-crossed internal horizontal lines.

Each of the four diagrams in Fig. 3 are equal. Summing them gives

$$\tilde{\Pi}_{\mu\nu}(p) = 4 \times (-1)^2 \left(\frac{ig}{\sqrt{N_f}}\right)^4 \times N_f^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3Q}{(2\pi)^3} \mathrm{Tr}[\mathbf{1}G(k-Q)\gamma_\alpha G(k)\gamma_\mu G(k-p)] \\ \times D_{\alpha\beta}(Q) \times 2\pi\, g_m \delta(-Q_0+p_0)\, \mathrm{Tr}[\mathbf{1}G(q+Q)\gamma_\beta G(q)\gamma_\nu G(q+p)]. \tag{3.18}$$

We define the first trace integral appearing in (3.18) as

$$C_{\alpha\mu}(Q,p) \equiv \int \frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\mathbf{1}G(k-Q)\gamma_\alpha G(k)\gamma_\mu G(k-p)] = \int \frac{d^3k}{(2\pi)^3}(i^3)\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\tau\gamma_\mu\gamma_\rho]\frac{(k-Q)_\sigma k_\tau (k-p)_\rho}{(k-Q)^2\, k^2\, (k-p)^2}. \tag{3.19}$$

The second trace integral has the same structure, but different momentum dependency,

$$\int \frac{d^3q}{(2\pi)^3}\mathrm{Tr}[\mathbf{1}G(q+Q)\gamma_\beta G(q)\gamma_\nu G(q+p)] = C_{\beta\nu}(-Q,-p) = -C_{\beta\nu}(Q,p). \tag{3.20}$$

The last equality uses the fact that the fermion Green's function $G(k)$ is odd with respect to k, $G(-k) = -G(k)$.

Computing the exact form of the triangle subdiagram $C_{\alpha\mu}$ is the primary step, which can be done by appropriate integral reductions. Once obtained, the evaluation of the Q integral in Eq. (3.18) is straightforward. The calculation details are relegated to Appendix D.

Here we summarize the results of these calculations. The symmetric component $\tilde{\Pi}_{(\mathrm{S})}$ of $\tilde{\Pi}_{ij}$ is found from

$$\tilde{\Pi}_{xx}(p_0, \boldsymbol{p}=\boldsymbol{0}) = \tilde{\Pi}_{yy} = 4\, g_m\, |p_0| \int_0^\infty dz \frac{1}{16\pi\sqrt{1+z^2}[F^2 + (F^2 - w_x^2)\,z^2]} \\ \times (-1)\left(z^2[-w_x\, z + F\sqrt{1+z^2}\,]\left[w_x + 2w_x\, z^2 - 4g_X^2(1+z^2)\right] \right. \\ \left. + 2z[(1+z^2)\,F - w_x\, z\sqrt{1+z^2}]\left[-w_x\, z^2 + g_X^2(1+2z^2)\right]\right), \tag{3.21}$$

where $F \equiv \frac{\kappa^2 + g_X^4}{g_X^2}$, $g_X^2 = \frac{1}{16}$. Taking the zero external momentum limit of Eq. (3.3), we find $\tilde{\Pi}_{(\mathrm{S})} = \tilde{\Pi}_{xx}$. The antisymmetric component $\tilde{\Pi}_{(\mathrm{A})}$ of $\tilde{\Pi}_{ij}$ is found from

$$\tilde{\Pi}_{xy}(p_0, \boldsymbol{p}=\boldsymbol{0}) = -4 g_m p_0 \int_0^\infty dz \frac{\kappa}{\kappa^2 + g_X^4 + w_x g_X^2 \frac{z}{\sqrt{1+z^2}}} \frac{z(1 + 2z^2 - 2z\sqrt{1+z^2})}{128\pi\sqrt{1+z^2}}. \tag{3.22}$$

We use Eq. (3.4) to extract the antisymmetric component, $\tilde{\Pi}_{(\mathrm{A})} = \frac{2p_0\tilde{\Pi}_{xy}}{2|p_0|}$.

D. Summary of the composite fermion conductivity to $\mathcal{O}(1/N_f)$

We now collect the results of the previous two subsections. Using the definition of the composite fermion conductivity in Eq. (3.1) and $\Pi^{\mathrm{tot}}_{ij} = \hat{\Pi}_{ij} + \Pi_{ij} + \tilde{\Pi}_{ij}$, we find that to $\mathcal{O}(\frac{1}{N_f}, g_m)$, the composite fermion conductivity equals

$$\sigma^{\psi}_{xx}(\omega) = \frac{1}{16} - \frac{3g_m}{96\pi} + \frac{i}{\omega}\left(\Pi^{(a+b+c)}_{(\mathrm{S})} + \tilde{\Pi}_{(\mathrm{S})}\right) \\ \times (|p_0| \to i\omega, \boldsymbol{p}=0), \tag{3.23}$$

$$\sigma^{\psi}_{xy}(\omega) = \frac{i}{\omega}\left(\Pi^{(a+b+c)}_{(\mathrm{A})} + \tilde{\Pi}_{(\mathrm{A})}\right)(|p_0| \to i\omega, \boldsymbol{p}=0). \tag{3.24}$$

We note that $g_m \sim 1/N_f$ at the finite disorder fixed points.

IV. QUANTUM CRITICAL ELECTRICAL TRANSPORT

A. RG flow and critical composite fermion conductivity

The perturbative RG flows for this system S_{tot} (2.1) were found in [26]. In that study, we allowed for the possibility of a dissipative Coulomb interaction. We will not consider this possibility here. With the Fermi velocity set equal to unity, the beta functions ($\beta_X \equiv -\mu\frac{\partial X}{\partial\mu}$) are

$$\beta_{w_x} = w_x(z-1) = w_x\left[\frac{g_m}{2\pi} - F_w(w_x,\kappa)\right], \tag{4.1}$$

$$\beta_{g_m} = 2g_m(z - 1 - \gamma_{\bar{\psi}\psi}) = -2g_m\left[\frac{g_m}{2\pi} + F_m(w_x,\kappa)\right]. \tag{4.2}$$

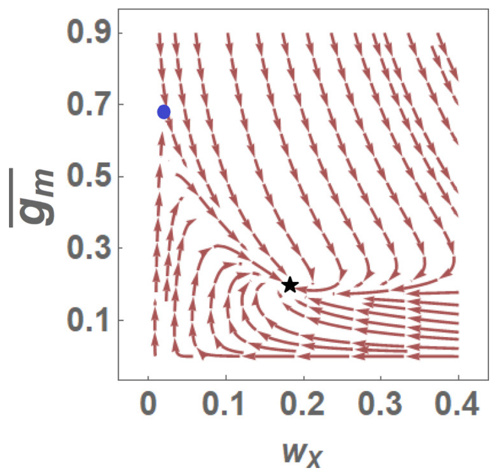

FIG. 4. RG flow of the strength of the random mass ($g_m = 2\pi\overline{g_m}$) and Coulomb coupling (w_x), evaluated at $N_f = 2$. The circle denotes the fixed point without the Coulomb interaction. The star is the fixed point with finite Coulomb interaction.

The dynamical exponent z, fermion anomalous dimension $\gamma_{\bar{\psi}\psi}$, and loop-functions F_w, F_m are defined as follows:

$$z = 1 + \frac{g_m}{2\pi} - F_w(w_x, \kappa), \tag{4.3}$$

$$\gamma_{\bar{\psi}\psi} = 2\frac{g_m}{2\pi} + F_m - F_w, \tag{4.4}$$

$$F_w(w_x, \kappa) = \frac{1}{4\pi^2 N_f}\int_{-\infty}^{\infty} dy \frac{g_1(2y^2-1) + w_x\sqrt{1+y^2}}{(1+y^2)^2\left[\sqrt{1+y^2}(g_1^2+\kappa^2) + g_1 w_x\right]}, \tag{4.5}$$

$$F_m(w_x, \kappa) = \frac{1}{4\pi^2 N_f}\int_{-\infty}^{\infty} dy \left[\frac{g_1(-2y^2-3) - w_x\sqrt{1+y^2}}{(1+y^2)^2\left[\sqrt{1+y^2}\left(g_1^2+\kappa^2\right) + g_1 w_x\right]} + \frac{\sqrt{1+y^2}\left(g_1^2-\kappa^2\right) + g_1 w_x}{2(1+y^2)\left[\sqrt{1+y^2}\left(g_1^2+\kappa^2\right) + g_1 w_x\right]^2}\right], \tag{4.6}$$

where $g_1 \equiv g_X^2 = \frac{g^2}{16}$. The dynamical critical exponent and fermion anomalous dimension determine the inverse correction length exponent: $\nu^{-1} = z - \gamma_{\bar{\psi}\psi}$. Figure 4 depicts the RG flow diagram for these beta functions (4.1) and (4.2).

As Fig. 4 indicates, a finite-disorder fixed point exists whether or not the Coulomb interaction is present. The disordered fixed point without Coulomb interaction is unstable to the inclusion of the Coulomb interaction.

In the absence of the Coulomb interaction ($w_x = 0$), β_{w_x} in Eq. (4.1) vanishes. At $w_x = 0$, the y integral in Eq. (4.2) can be evaluated analytically. The solution g_m^* is given by

$$\frac{g_m^*}{2\pi} \equiv -F_m(0,\kappa) = \frac{g_1^2(-3+16g_1) + (3+16g_1)\kappa^2}{12\pi^2 N_f (g_1^2+\kappa^2)^2} \approx \frac{1.41091}{N_f}, \tag{4.7}$$

where we plugged in $\kappa = \frac{1}{4\pi}$ in the rightmost equality. Substituting these fixed point values of w_x and g_m into the expressions for the conductivity, Eqs. (3.23) and (3.24), we obtain at $\omega = 0$,

$$\sigma_{xx}^{\psi} = \frac{1}{16} + \frac{0.0946518}{N_f} \quad (w_x = 0), \tag{4.8}$$

$$\sigma_{xy}^{\psi} = \frac{0.0577949}{N_f} \quad (w_x = 0). \tag{4.9}$$

The correlation length and dynamical critical exponents at this fixed point are [26] $(\nu, z) = (1, \frac{1.4}{N_f})$.

In the presence of the Coulomb interaction ($w_x \neq 0$), we have to solve the flow equations in (4.1) and (4.2) together. The fixed point solution is found to lie at

$$(w_x^*, g_m^*) \approx \left(0.184193, 2\pi\frac{0.393787}{N_f}\right). \tag{4.10}$$

Using Eqs. (3.23) and (3.24), this gives the dc composite fermion conductivity,

$$\sigma_{xx}^{\psi} = \frac{1}{16} + \frac{0.0744856}{N_f} \quad (w_x \neq 0), \tag{4.11}$$

$$\sigma_{xy}^{\psi} = \frac{-0.0538345}{N_f} \quad (w_x \neq 0), \tag{4.12}$$

The critical exponents at this fixed point are [26] $(\nu, z) = (1, 1)$.

We observe that the combination of disorder and Chern-Simons gauge field fluctuations enhance the composite fermion longitudinal conductivity, whether or not the Coulomb interaction is present. Further, these fluctuations produce a nonzero composite fermion Hall conductivity, Eqs. (4.9) and (4.11). We note that the composite fermion Hall conductivity—a measure of broken self-duality—remains nonzero in the pure limit ($g_m = 0$) at leading order in $1/N_f$. We believe this to be a deficiency of the perturbative calculation presented in this paper. We will comment further on this below.

B. Electrical response

To determine the quantum critical electrical transport at the fixed points described in the previous subsection, we must first recall the dictionary that relates the composite fermion and electrical conductivities [19]. The two- and three-loop calculations described previously can be understood to give rise to the following quadratic effective action, in which the composite fermion is integrated out:

$$S_{\rm DCF}^{\rm eff} = \int \frac{d^2k\, d\omega}{(2\pi)^3}\frac{-i\omega}{2} \times \left(a_i\,\sigma_{ij}^{\psi}\,a_j + \frac{1}{2\pi}\epsilon_{0ij}\left(\frac{1}{2}a_i a_j - 2A_i a_j + A_i A_j\right)\right). \tag{4.13}$$

Here, we have used our gauge freedom to set $a_0 = A_0 = 0$. Terms that are higher order in the gauge field a_i are ignored. We integrate over a_i to obtain the effective electrical response

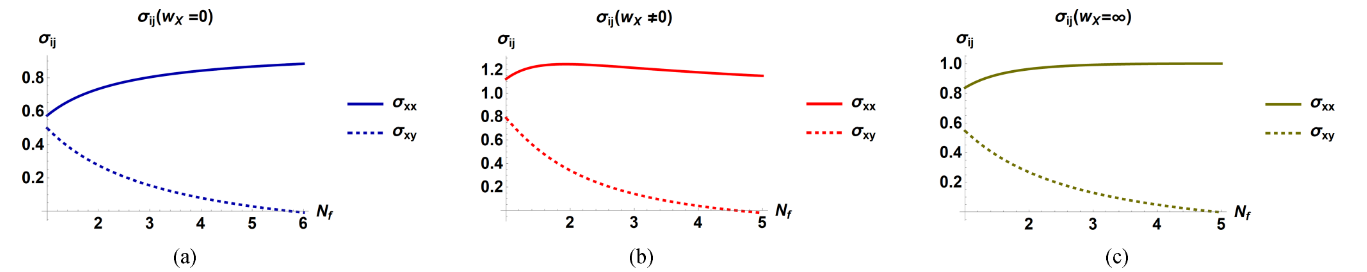

FIG. 5. Electrical response components for two fixed points with varying number of fermion flavor N_f. The values in the figures are all in the unit of $\frac{1}{2\pi}$. The curves are depicted based on Eqs. (4.15) and (4.16) without any expansion of $\frac{1}{N_f}$. (a) Zero Coulomb interaction ($w_x = 0$) and finite disorder, (b) Finite Coulomb interaction ($w_x \neq 0$) and finite disorder, and (c) Infinite Coulomb interaction ($w_x = \infty$) without disorder.

action,

$$S^{\rm eff}_{\rm DCF} = \int \frac{d^2k d\omega}{(2\pi)^3} \frac{-i\omega}{2} \epsilon_{0ij} A_i\, \sigma_{ij}\, A_j, \tag{4.14}$$

where the electrical conductivity is

$$\sigma_{xx} = \frac{\sigma^{\psi}_{xx}}{\left(\sigma^{\psi}_{xx}\right)^2 + \left(\sigma^{\psi}_{xy} + \frac{1}{4\pi}\right)^2} \cdot \left(\frac{1}{2\pi}\right)^2, \tag{4.15}$$

$$\sigma_{xy} = \frac{1}{2\pi} - \frac{\sigma^{\psi}_{xy} + \frac{1}{4\pi}}{\left(\sigma^{\psi}_{xx}\right)^2 + \left(\sigma^{\psi}_{xy} + \frac{1}{4\pi}\right)^2} \cdot \left(\frac{1}{2\pi}\right)^2. \tag{4.16}$$

An immediate consequence of this dictionary is that

$$\sigma^2_{xx} + \sigma^2_{xy} = \frac{\left(\sigma^{\psi}_{xx}\right)^2 + \left(\sigma^{\psi}_{xy} - \frac{1}{4\pi}\right)^2}{\left(\sigma^{\psi}_{xx}\right)^2 + \left(\sigma^{\psi}_{xy} + \frac{1}{4\pi}\right)^2} \cdot \left(\frac{1}{2\pi}\right)^2. \tag{4.17}$$

This shows that self-dual electrical transport (1.1) occurs when the composite fermion Hall conductivity $\sigma^{\psi}_{xy} = 0$. Our system, however, is not self-dual, since $\sigma^{\psi}_{xy} \neq 0$ [see Eqs. (4.9) and (4.12)].

We now use (4.15) and (4.16) and the composite fermion conductivities computed in the previous subsection to obtain the electrical conductivities. We begin with the fixed point without the Coulomb interaction. Plugging in the composite fermion conductivities (4.8) and (4.9) into Eqs. (4.15) and (4.16), we find the dc electrical conductivities to $\mathcal{O}(1/N_f)$,

$$(\sigma_{xx}, \sigma_{xy}) = \left(0.97 - \frac{0.524133}{N_f}, -0.24 + \frac{1.64228}{N_f}\right) \cdot \frac{1}{2\pi},$$
$$(w_x = 0). \tag{4.18}$$

Similarly, for the fixed point with Coulomb interaction, we use the composite fermion conductivities (4.11) and (4.12) to obtain the dc electrical conductivities,

$$(\sigma_{xx}, \sigma_{xy}) = \left(0.97 + \frac{1.08735}{N_f}, -0.24 + \frac{0.926541}{N_f}\right) \cdot \frac{1}{2\pi},$$
$$(w_x \neq 0). \tag{4.19}$$

At $\mathcal{O}(1/N_f^0)$, we see that self-duality (1.1) occurs with finite, but nonzero σ_{xy} and a longitudinal conductivity $\sigma_{xx} \lesssim R_Q^{-1}$. The $\mathcal{O}(1/N_f)$ corrections, due to disorder and the fluctuating Chern-Simons gauge field, lead to violations of self-duality. These violations are due to the nonzero composite fermion Hall conductivity σ^{ψ}_{xy}, either without (4.9) and with (4.12) the Coulomb interaction. We notice that the disorder and gauge field fluctuations suppress/enhance the longitudinal conductivity when the Coulomb interaction is absent/present. This behavior arises from the difference in sign of the composite fermion Hall conductivities. The expressions for the quantum critical electrical conductivities, (4.18) and (4.19), are reliable at large N_f. To interpolate these results to small, but finite N_f, we use the exact expressions that obtain from substituting the composite fermion conductivities, either without (4.8) and (4.9) or with (4.11) and (4.12) the Coulomb interaction, into Eqs. (4.15) and (4.16). The results are plotted in Figs. 5(a) and 5(b).

We end this section with two speculative comments.

The first comment concerns another way to obtain self-duality. Examining the composite fermion Hall conductivity (3.24), we observe that $\sigma^{\psi}_{xy} \sim \frac{1}{\text{finite}+w_x}$, where "finite" refers to an $\mathcal{O}(1)$ constant independent of w_x. Taking $w_x \to \infty$, with all other parameters held fixed, results in $\sigma^{\psi}_{xy} = 0$. We plot the corresponding electrical conductivities in Fig. 5(c). We do not have an argument for why the Coulomb coupling would flow to strong coupling; $w_x = \infty$ is not a fixed point of the beta functions (4.1) and (4.2). We note that in [26] we studied, in addition, the effect of dissipation, which allowed for fixed points with infinite strength dissipative Coulomb interaction. The gauge field propagator at such fixed points is different from that considered in this paper and so a determination of the electrical conductivity at such fixed points is left for future work.

The second speculative comment concerns the lack of manifest time-reversal invariance of the composite fermion action (2.2). As we mentioned, because this theory is believed to be dual to the (2 + 1)d XY model, the composite fermion theory should have both charge-conjugation and time-reversal symmetries. The lack of manifest time-reversal invariance implies that time reversal must be emergent. The leading-order perturbative calculations presented here (using the $1/N_f$ expansion) do not realize emergent time-reversal invariance since $\sigma^{\psi}_{xy} \neq 0$. We speculate that the following "averaging" procedure produces a conductivity for the theory that is "more faithful" to both the manifest and emergent symmetries. We consider the conductivities of the model defined by (2.1), as already presented, and its time-reversal conjugate. [The disorder and Coulomb parts of the total action are invariant under time-reversal, assuming zero-mean white noise disorder, and so this is tantamount to considering (2.2) and its time-reversal conjugate, in which the signs of the first and third Chern-Simons levels are flipped. The beta functions of the time-reversed theory are identical to (4.1) and (4.2), since they are even functions of the ada Chern-Simons coefficient.] We can easily adapt the transport calculations presented earlier to the time-reversed theory. As might be expected, but we have also checked, the result is the same longitudinal conductivity and opposite sign Hall conductivity. (This statement holds for both the electrical and composite fermion conductivities.) We then declare the "more faithful" conductivity to be the average of the conductivities obtained from (2.1) and its time-reversal conjugate. The results of this "averaging" are as follows. Either with or without the Coulomb interaction, the electrical Hall conductivity is zero, reflective of unbroken time-reversal symmetry. The longitudinal conductivities without or with the Coulomb interaction are simply given by the longitudinal conductivities in (4.18) or (4.19). Notice that in both cases, self-duality is violated, because disorder and gauge field fluctuations reduce/enhance the longitudinal conductivity to a value different from $1/R_Q$, at vanishing ("averaged") Hall conductivity. This self-duality violation might be expected since the random mass does not preserve particle-vortex symmetry. Whether this "averaging" procedure can be justified in a principled way remains to be seen.

V. DISCUSSION

In this paper, we calculated the quantum critical electrical conductivity at a superconductor-insulator transition in two spatial dimensions, in the universality class of the dirty $(2+1)$d XY model with vanishing external magnetic field. For our study, we used a composite fermion theory, consisting of a single Dirac fermion, coupled to a Chern-Simons gauge field and in the presence of a quenched random mass. This theory is believed to be a "fermionic dual" to the usual "bosonic" description of the XY model. There are at least two reasons to prefer the composite fermion description. The first is that it can exhibit a particle-vortex symmetry at the level of its Lagrangian. A consequence of this symmetry is the semicircle law (1.1), which appears to be realized in some experiments [14]. A second reason to prefer the composite fermion theory is that an unscreened Coulomb interaction is straightforwardly included in the model. This allows for a comparison of the properties of the theory with or without the Coulomb interaction.

Our calculation of the critical conductivity was performed in an expansion in $1/N_f$, where N_f is the number of fermion flavors. Our results are summarized by Eqs. (4.18) and (4.19) and in Figs. 5(a) and 5(b). At order $1/N_f^0$, the theory is self-dual, realizing the semicircle law (1.1) with $\sigma_{xx} \lesssim (2e)^2/h$ and small, finite σ_{xy}. Fluctuation effects due to the gauge field and the random mass contribute corrections of order $1/N_f$ to the conductivity that violate self-duality. The nature of these corrections depends on whether or not the Coulomb interaction is present. We find that fluctuations reduce/increase the longitudinal conductivity at the fixed point without/with the Coulomb interaction. There does not appear to be a qualitative difference between the Hall conductivities at the two fixed points.

In the previous section, based on the leading corrections to the composite fermion Hall conductivity, we speculated that infinite strength Coulomb interactions of the sort we studied in [26] might produce self-dual transport. Such fixed points required a dissipative Coulomb interaction, a feature not considered in this paper. We leave an explicit calculation of the critical conductivity at such infinite strength Coulomb interaction fixed points, with dissipation, for future work. We note that such fixed points have dynamical critical exponent $z > 1$.

It would be interesting to carry out these electrical transport calculations at the dirty XY model fixed point found in [23], i.e., within the usual bosonic description, and to compare the results to those in this paper. The fixed point in [23] does not include the unscreened Coulomb interaction. The bosonic description of the dirty XY model fixed point gives critical exponents that are in closer agreement with numerical experiment [41] when $N_f \to 1$, than those of the composite fermion theory [at least to $\mathcal{O}(1/N_f)$].

There are two aspects as to why we focused on a theory corresponding to a model of lattice bosons with charge-conserving disorder. First, regarding the type of disorder within this lattice boson model, we found in [26] that other types of disorder do not yield accessible disordered fixed points with or without a finite Coulomb interaction. These calculations were performed to leading order in a $1/N_f$ expansion. It is possible that higher-order calculations or calculations performed using a different choice of artificial expansion parameter (perhaps one that preserves the same symmetries as the $N_f = 1$ theory) may find nontrivial fixed points. Second, regarding the commensurability constraint, the dual composite fermions acquire a finite chemical potential if this is relaxed. It would be extremely interesting to find disordered fixed points in this more general situation, as we believe it more closely resembles the experimentally-realized superconductor-insulator transitions, in particular, magnetic field-tuned transitions. Recent work has shown [42,43] how the optical conductivity is constrained by the anomaly structure of emergent symmetries in a class of non-Fermi liquids (so-called "Hertz-Millis" theories) consisting of a Fermi surface coupled to a gapless bosonic order parameter. Such models are closely related to the theory we studied (see [44] for a related theory studied recently), when the composite fermion density is nonzero. It would be worthwhile to understand the interplay of this emergent anomaly structure with particle-vortex symmetry.

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APPENDIX A: GAUGE BOSON PROPAGATOR

Start from Eq. (2.7), with $A \equiv \frac{+g_X^2 \boldsymbol{k}^2}{k}$, $B \equiv i\kappa\, |\boldsymbol{k}|$, and $C \equiv g_X^2\, k + |\boldsymbol{k}|\, w_x$. This one-loop effective action includes the leading-order polarization,

$$\Pi_{00} = -g_X^2 \frac{\boldsymbol{k}^2}{k}, \tag{A1}$$

$$\Pi_{TT} = \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right)\Pi_{ij} = -g_X^2\, k. \tag{A2}$$

We work in Coulomb gauge (vanishing longitudinal component of the gauge field). Including the gauge-fixing term, $\int_x \frac{(\nabla_j a_j)^2}{2\lambda}$, which preserves $SO(2)$ rotation invariance, the one-loop effective action is

$$S_{\text{gauge}}^{(\text{gauge-fixed})} \equiv \frac{1}{2}\int_k (a_0 \quad a_T)_k \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} a_0 \\ a_T \end{pmatrix}_{-k} \tag{A3}$$

$$= \lim_{\lambda\to 0} \frac{1}{2}\int_k (a_0 \quad a_x \quad a_y)_k \left(\begin{pmatrix} A & B\frac{ik_y}{|\boldsymbol{k}|} & B\frac{-ik_x}{|\boldsymbol{k}|} \\ B\frac{-ik_y}{|\boldsymbol{k}|} & \frac{k_y^2}{|\boldsymbol{k}|^2}C & \frac{-k_x k_y}{|\boldsymbol{k}|^2}C \\ B\frac{+ik_x}{|\boldsymbol{k}|} & \frac{-k_x k_y}{|\boldsymbol{k}|^2}C & \frac{k_x^2}{|\boldsymbol{k}|^2}C \end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix} 0 & 0 & 0 \\ 0 & k_x k_x & k_x k_y \\ 0 & k_x k_y & k_y k_y \end{pmatrix} \right) \begin{pmatrix} a_0 \\ a_x \\ a_y \end{pmatrix}_{-k}, \tag{A4}$$

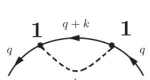
where $\int_k \equiv \int \frac{d^3k}{(2\pi)^3}$. We obtain the gauge field propagators (2.8)–(2.10), upon first inverting the 3×3 matrix kernel and then taking the limit $\lambda \to 0$.

APPENDIX B: TWO-LOOP RANDOM MASS CORRECTIONS TO THE COMPOSITE FERMION CONDUCTIVITY

In this Appendix, we detail the evaluation of Eqs. (3.5) and (3.6), following [31].

1. $\hat{\Pi}^{(b+c)}$

We first consider the subdiagram of Eq. (3.6),



$$= \int \frac{d^3k}{(2\pi)^3} \mathbf{1}\gamma_\sigma\, \mathbf{1}(+i)\frac{(q+k)_\sigma}{(q_0+k_0)^2 + (\boldsymbol{k}+\boldsymbol{q})^2} 2\pi\delta(k_0) \tag{B1}$$

$$= \int \frac{d^2k}{(2\pi)^2}\gamma_\sigma(+i)\frac{(q_0)\delta_{\sigma 0} + (q_a + k_a)\delta_{\sigma a}}{(q_0+0)^2 + (\boldsymbol{k}+\boldsymbol{q})^2} \tag{B2}$$

$$= \int \frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\gamma_\sigma(+i)\frac{(q_0)\delta_{\sigma 0} + (\ell_a)\delta_{\sigma a}}{(q_0+0)^2 + (\boldsymbol{\ell})^2} \equiv i\,\gamma_0\, I_X(q_0)\,\delta_{\sigma 0} + 0, \tag{B3}$$

Then, the full diagram of Eq. (3.6) reads

$$\Pi_{xx}^{(b+c)}(p_0, \boldsymbol{p} = 0) \tag{B4}$$

$$= -2\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f \int_q \text{Tr}\left[\gamma_x G(q)\left(\int \frac{d^3k}{(2\pi)^3}\mathbf{1}G(k+q)\mathbf{1} \times 2\pi\delta(k_0) g_m\right) G(q)\gamma_x G(p+q)\right] \tag{B5}$$

$$= -2\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f \int_q \text{Tr}[\gamma_x G(q)(i\,I_X\gamma_0 + 0)G(q)\gamma_x G(p+q)] \times g_m \tag{B6}$$

$$= -2\left(\frac{ig}{\sqrt{N_f}}\right)^2 N_f \int_q \text{Tr}\left[\gamma_x \frac{(+i)q_\rho\gamma_\rho}{q_0^2 + \boldsymbol{q}^2}(i\,I_X\gamma_0 + 0)\frac{(i)q_\tau\gamma_\tau}{q_0^2+\boldsymbol{q}^2}\gamma_x(i)\frac{(q+p)_\kappa\gamma_\kappa}{(q_0+p_0)^2 + (\boldsymbol{q}+\boldsymbol{0})^2}\right] \times g_m. \tag{B7}$$

Using

$$\text{Tr}[\gamma_x\gamma_\rho\gamma_0\gamma_\tau\gamma_x\gamma_\kappa]\, q_\rho\, q_\tau\, (q+p)_\kappa = 2(p_0+q_0)\left(\boldsymbol{q}^2 - q_0^2\right) \tag{B8}$$

we obtain

$$\begin{aligned}
\Pi_{xx}^{(b+c)}(p_0,\boldsymbol{p}=0) &= 2(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \int \frac{d^2\boldsymbol{q}dq_0}{(2\pi)^3}\frac{1}{\left[q_0^2+\boldsymbol{q}\right]^2}\frac{2(p_0+q_0)\left(\boldsymbol{q^2}-q_0^2\right)}{(q_0+p_0)^2+\boldsymbol{q}^2} I_X(q_0)\\
&= 2(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \int \frac{d^2\boldsymbol{q}dq_0}{(2\pi)^3}\frac{2(p_0+q_0)\left(\boldsymbol{q^2}-q_0^2\right)}{\left[q_0^2+\boldsymbol{q}\right]^2[(q_0+p_0)^2+\boldsymbol{q}^2]}\int^{\Lambda}\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{q_0}{q_0^2+\boldsymbol{\ell}^2}\\
&= 2(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \int \frac{dq_0}{2\pi}\Bigg(2(p_0+q_0)q_0\\
&\quad\times\frac{1}{4\pi}\frac{2q_0^2-2(p_0+q_0)^2+\left[(p_0+q_0)^2+q_0^2\right]\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]}{\left[(p_0+q_0)^2-q_0^2\right]^2}\Bigg)\frac{1}{4\pi}\log\left[\frac{\Lambda^2}{q_0^2}\right].
\end{aligned} \tag{B9}$$

Here, we used

$$\int_0^{\Lambda}\frac{d^2\ell}{(2\pi)^2}\frac{1}{q_0^2+\ell^2}=\frac{1}{4\pi}\log\left[\frac{\Lambda^2+q_0^2}{q_0^2}\right]\to\frac{1}{4\pi}\log\left[\frac{\Lambda^2}{q_0^2}\right]. \tag{B10}$$

We simplify the integrand by defining

$$\begin{aligned}
K(q_0) &\equiv \left(2(p_0+q_0)q_0\times\frac{1}{4\pi}\frac{2q_0^2-2(p_0+q_0)^2+\left[(p_0+q_0)^2+q_0^2\right]\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]}{\left[(p_0+q_0)^2-q_0^2\right]^2}\right)\frac{1}{4\pi}\left(-\log\left[q_0^2\right]\right)\\
&= \left(\frac{-\log\left[q_0^2\right]}{4\pi}\right)\frac{(p_0+q_0)q_0}{\pi(p_0+2q_0)p_0}\left(-1+\frac{1}{2}\frac{\left[(p_0+q_0)^2+q_0^2\right]\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]}{(p_0+2q_0)^2\,p_0^2}\right).
\end{aligned} \tag{B11}$$

A change of variables gives

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}K(q_0=xp_0) &= |p_0|\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{-1}{4\pi}\left(\log[x^2]+\log\left[p_0^2\right]\right)\frac{x(1+x)}{1(1+2x)}\frac{-1}{\pi}\left[1+\frac{x^2+(x+1)^2}{2(1+2x)}\log\left[\frac{x^2}{(x+1)^2}\right]\right]\\
&= |p_0|\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{-1}{4\pi}(\log[x^2])\frac{x(1+x)}{1(1+2x)}\frac{-1}{\pi}\left[1+\frac{x^2+(x+1)^2}{2(1+2x)}\log\left[\frac{x^2}{(x+1)^2}\right]\right]\\
&\quad+|p_0|\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{-1}{4\pi}\left(\log\left[p_0^2\right]\right)\frac{x(1+x)}{1(1+2x)}\frac{-1}{\pi}\left[1+\frac{x^2+(x+1)^2}{2(1+2x)}\log\left[\frac{x^2}{(x+1)^2}\right]\right].
\end{aligned} \tag{B12}$$

The principal value of the second term, $\int dx\log[p_0^2]\times[...]$ is zero. If we make the plot, we find it is an odd function with respect to $x=\frac{1}{2}$, so after we shift $x\to x-\frac{1}{2}$, the integral vanishes. For the same reason, the $\log\Lambda^2$ term in Eq. (B9) vanishes.

Hence, we symmetrize the first line of Eq. (B12), the $\log q_0^2$ term of Eq. (B9),

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\frac{K(q_0)+K(-q_0)}{2} &= |p_0|\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{\log[x^2]}{4\pi^2}\left[\frac{x(1+x)}{1(1+2x)}\left(1+\frac{x^2+(x+1)^2}{2(1+2x)}\log\left[\frac{x^2}{(x+1)^2}\right]\right)\right.\\
&\quad\left.+\frac{-x(1-x)}{1(1-2x)}\left(1+\frac{x^2+(-x+1)^2}{2(1-2x)}\log\left[\frac{x^2}{(-x+1)^2}\right]\right)\right]
\end{aligned} \tag{B13}$$

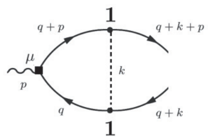
$$= \frac{1}{2\pi}\times\frac{|p_0|}{96}. \tag{B14}$$

Thus, we conclude that

$$\Pi_{xx}^{(b+c)}(p_0,\boldsymbol{p}=0)=2(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m\times\frac{1}{2\pi}\times\frac{|p_0|}{96}. \tag{B15}$$

2. $\hat{\Pi}^{(a)}$

Next we turn to Eq. (3.5). Again, consider the subdiagram,



$$= \int \frac{d^3k}{(2\pi)^3} i\frac{(q_0+k_0+p_0)\gamma_0 + (\boldsymbol{q}+\boldsymbol{k}+\boldsymbol{p})\cdot\vec{\gamma}}{(q_0+k_0+p_0)^2+(\boldsymbol{q}+\boldsymbol{k}+\boldsymbol{p})^2}\mathbf{1}\, i\frac{(q_0+p_0)\gamma_0+(\boldsymbol{q}+\boldsymbol{p})\cdot\vec{\gamma}}{(q_0+p_0)^2+(\boldsymbol{q}+\boldsymbol{p})^2}$$

$$\times\gamma_\mu\, i\frac{(q_0)\gamma_0+(\boldsymbol{q})\cdot\vec{\gamma}}{(q_0)^2+(\boldsymbol{q})^2}\mathbf{1}\, i\frac{(q_0+k_0)\gamma_0+(\boldsymbol{q}+\boldsymbol{k})\cdot\vec{\gamma}}{(q_0+k_0)^2+(\boldsymbol{q}+\boldsymbol{k})^2}\times 2\pi\delta(k_0)\, g_m \tag{B16}$$

$$= g_m\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\gamma_\tau\mathbf{1}\gamma_\kappa\gamma_\mu\gamma_\rho\mathbf{1}\gamma_\sigma\,\frac{(q+k+p)_\tau}{(q+k+p)^2}\frac{(q+p)_\kappa}{(q+p)^2}\frac{q_\rho}{q^2}\frac{(q+k)_\sigma}{(q+k)^2}\bigg|_{k_0=0}. \tag{B17}$$

Consider the terms involving k. Setting $\boldsymbol{p}=0$, we find

$$I_{\tau\sigma}(q_0,p_0)\equiv\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{(q+k+p)_\tau}{(q+k+p)^2}\frac{(q+k)_\sigma}{(q+k)^2}\bigg|_{k_0=0,\boldsymbol{p}=0} \tag{B18}$$

$$=\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{(q_0+k_0+p_0)\delta_{\tau 0}+(\boldsymbol{\ell}+\boldsymbol{p})_\tau\delta_{\tau a}}{(q_0+k_0+p_0)^2+\boldsymbol{\ell}^2}\;\frac{(q_0+k_0)\delta_{\sigma 0}+(\boldsymbol{\ell})_\sigma\delta_{\sigma b}}{(q_0+k_0)^2+\boldsymbol{\ell}^2}\bigg|_{k_0=0,\boldsymbol{p}=0} \tag{B19}$$

$$=\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{(q_0+p_0)\delta_{\tau 0}+(\boldsymbol{\ell})_\tau\delta_{\tau a}}{(q_0+p_0)^2+\boldsymbol{\ell}^2}\;\frac{(q_0+0)\delta_{\sigma 0}+(\boldsymbol{\ell})_\sigma\delta_{\sigma b}}{(q_0+0)^2+\boldsymbol{\ell}^2} \tag{B20}$$

$$=\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{(q_0+p_0)q_0\,\delta_{\tau 0}\delta_{\sigma 0}}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\,[q_0^2+\boldsymbol{\ell}^2]}+\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{\boldsymbol{\ell}^2\delta_{ab}\delta_{\tau a}\delta_{\sigma b}}{d}\frac{1}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\,[q_0^2+\boldsymbol{\ell}^2]}. \tag{B21}$$

Now we set $\mu=\nu=x$. First consider the temporal component of the gamma index trace in Eq. (3.5),

$$\mathrm{Tr}[\gamma_\tau\gamma_\kappa\gamma_{\mu=x}\gamma_\rho\gamma_\sigma\gamma_{\nu=x}]\,\delta_{\tau 0}\delta_{\sigma 0}=\mathrm{Tr}[\gamma_0\gamma_\kappa\gamma_x\gamma_\rho\gamma_0\gamma_x]=(-1)\mathrm{Tr}[\gamma_\kappa\gamma_x\gamma_\rho\gamma_x]. \tag{B22}$$

Contracting this with the momentum gives

$$(-1)\mathrm{Tr}[\gamma_\kappa\gamma_x\gamma_\rho\gamma_x]\,(q+p)_\kappa q_\rho=2q_0(p_0+q_0). \tag{B23}$$

Next consider the spatial components of the gamma index trace. We perform the trace in $2-\epsilon$ spatial dimensions, rather than two spatial dimensions,

$$\mathrm{Tr}[\gamma_\tau\gamma_\kappa\gamma_{\mu=x}\gamma_\rho\gamma_\sigma\gamma_{\nu=x}]\,\delta_{ab}\delta_{\tau a}\delta_{\sigma b}=\sum_{a=2-\epsilon\text{ number of indices}}\mathrm{Tr}[\gamma_a\gamma_\kappa\gamma_x\gamma_\rho\gamma_a\gamma_x]=\mathrm{Tr}\left[\gamma_\kappa\gamma_x\gamma_\rho\sum_a(\gamma_a\gamma_x\gamma_a)\right] \tag{B24}$$

$$=\mathrm{Tr}[\gamma_\kappa\gamma_x\gamma_\rho(2\gamma_x-(2-\epsilon)\gamma_x)]=\epsilon\;\mathrm{Tr}[\gamma_\kappa\gamma_x\gamma_\rho\gamma_x]. \tag{B25}$$

Contracting this with the momentum yields

$$\epsilon\;\mathrm{Tr}[\gamma_\kappa\gamma_x\gamma_\rho\gamma_x]\,(q+p)_\kappa q_\rho=\epsilon\times(-2)q_0(p_0+q_0). \tag{B26}$$

Putting this all together, we find

$$\Pi^{(a)}_{\mu\nu}(p_0,\boldsymbol{p}=0)=(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\,g_m\int\frac{d^3q}{(2\pi)^3}\frac{2q_0(p_0+q_0)}{[(q_0+p_0)^2+(\boldsymbol{q}+\boldsymbol{0})^2]\,\left[q_0^2+\boldsymbol{q}^2\right]}$$

$$\times\left(\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{(q_0+p_0)q_0}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\left[q_0^2+\boldsymbol{\ell}^2\right]}-\epsilon\int\frac{d^{2-\epsilon}\boldsymbol{\ell}}{(2\pi)^2}\frac{\boldsymbol{\ell}^2}{2}\frac{1}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\left[q_0^2+\boldsymbol{\ell}^2\right]}\right). \tag{B27}$$

We wish to extract finite parts of this expression. The finite part of the first $d^2\boldsymbol{\ell}$ integral is

$$\int\frac{d^2\boldsymbol{\ell}}{(2\pi)^2}\frac{(q_0+p_0)q_0}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\left[q_0^2+\boldsymbol{\ell}^2\right]}=\frac{1}{4\pi}\frac{(q_0+p_0)q_0}{p_0(p_0+2q_0)}\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]. \tag{B28}$$

For the second integral, we need to extract the $\frac{1}{\epsilon}$ part. Using the usual Feynman parametrization,

$$\int \frac{d^{2-\epsilon}\boldsymbol{\ell}}{(2\pi)^2}\frac{\boldsymbol{\ell}^2}{2}\frac{1}{[(q_0+p_0)^2+\boldsymbol{\ell}^2]\left[q_0^2+\boldsymbol{\ell}^2\right]} = \int \frac{d^{2-\epsilon}\boldsymbol{\ell}}{(2\pi)^2}\int_0^1 dx \frac{1}{2}\frac{\boldsymbol{\ell}^2}{[\boldsymbol{\ell}^2+\Delta(x,q_0,p_0)]^2} \tag{B29}$$

$$= \int_0^1 dx\frac{1}{2}\frac{1}{2\pi\epsilon}+\mathcal{O}(\epsilon^0) = \frac{1}{4\pi\epsilon}+\mathcal{O}(\epsilon^0). \tag{B30}$$

It remains to perform the $d^2\boldsymbol{q}$ integral, which has the same structure as the one above,

$$\int \frac{d^3q}{(2\pi)^3}\frac{2q_0(p_0+q_0)}{[(q_0+p_0)^2+\boldsymbol{q}^2]\,[q_0^2+\boldsymbol{q}^2]} = \int \frac{dq_0}{2\pi}\frac{1}{2\pi}\frac{(q_0+p_0)q_0}{p_0(p_0+2q_0)}\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]. \tag{B31}$$

Putting this together, we have

$$\Pi^{(a)}_{\mu\nu}(p_0,\boldsymbol{p}=0) = (-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\frac{1}{2\pi}\frac{(q_0+p_0)q_0}{p_0(p_0+2q_0)} \times \log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]\left(\frac{1}{4\pi}\frac{(q_0+p_0)q_0}{p_0(p_0+2q_0)}\log\left[\frac{(q_0+p_0)^2}{q_0^2}\right]-\frac{1}{4\pi}\right) \tag{B32}$$

$$= (-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \times \frac{1}{2\pi}\times\frac{|p_0|}{96}. \tag{B33}$$

Summarizing the results of this section, we find the random mass diagrams equal

$$\Pi^{(a+b+c)}_{\mu\nu}(p_0,\boldsymbol{p}=0) = 3\times(-1)\left(\frac{+ig}{\sqrt{N_f}}\right)^2 N_f\, g_m \times \frac{1}{2\pi}\times\frac{|p_0|}{96}. \tag{B34}$$

Notice there is no off-diagonal component. By Eq. (3.1), this self-energy contributes the following to the composite fermion conductivity:

$$\sigma^{g_m}_{xx} = \lim_{p_0\to 0}\frac{\Pi^{(S)}(\boldsymbol{p}=0)}{p_0} = \frac{i}{\omega}\Pi^{xx}(\boldsymbol{p}=0,|p_0|\to i|\omega|) = \frac{-3g_m}{96\times 2\pi}, \tag{B35}$$

where we set the $g^2=1$ and $\omega>0$.

APPENDIX C: TWO-LOOP GAUGE FIELD CORRECTIONS TO THE COMPOSITE FERMION CONDUCTIVITY

In this Appendix, we detail the evaluation of Eqs. (3.8) and (3.9).

1. Useful integrals

To evaluate gauge field corrections to the composite fermion conductivity, we need the formulas given in the next two subsections. In the expressions below, Λ is a momentum integral UV cutoff.

a. Basic integral building blocks

The simplest integrals we will need are the following:

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} = \int_0^\Lambda dk\frac{4\pi k^2 dk}{(2\pi)^3 k^2} = \frac{\Lambda}{2\pi^2}, \tag{C1}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2(k+p)^2} = \frac{1}{8|p|}, \tag{C2}$$

$$J_2(p,q) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{(k+p)^2(k+q)^2} = \int \frac{d^3K}{(2\pi)^3}\frac{1}{(K)^2(K-p+q)^2} = \frac{1}{8|p-q|}, \tag{C3}$$

$$J_3(p,q) \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2(k+p)^2(k+q)^2} = \frac{1}{8|p|\,|q|\,|p-q|}, \tag{C4}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{(k+p)^2(k+q)^2} = \frac{\Lambda}{2\pi^2}+\frac{p\cdot q}{8|p-q|}. \tag{C5}$$

Next we consider the integral

$$I(p,q) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{(k+p)^2(k+q)^2(k+p+q)^2} = \int \frac{d^3k}{(2\pi)^3} \frac{(k+q)^2+(k+p)^2-(k+p+q)^2+2p\cdot q}{(k+p)^2(k+q)^2(k+p+q)^2}$$
$$= \frac{1}{8|q|} + \frac{1}{8|p|} - \frac{1}{8|p-q|} + \frac{2p\cdot q}{8|p||q||p-q|}. \tag{C6}$$

It is symmetric, $I(p,q) = I(q,p)$.

Next we consider the more complicated integral,

$$J_4(p,q) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k)^2(k+p)^2(k+q)^2(k+p+q)^2} = \frac{1}{8|p||q|(p\cdot q)}\left(\frac{1}{|p-q|} - \frac{1}{|p+q|}\right). \tag{C7}$$

To prove this result, we reexpress the integrand using partial fractions,

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{(k)^2(k+p)^2(k+q)^2(k+p+q)^2} \tag{C8}$$
$$= \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{(k)^2(k+p)^2(k+q)^2} - \frac{1}{(k)^2(k+p)^2(k+p+q)^2}\right.$$
$$\left. + \frac{1}{(k+q)^2(k+p+q)^2(k+p)^2} - \frac{1}{(k+q)^2(k+p+q)^2\,k^2}\right)\frac{1}{2p\cdot q} \tag{C9}$$
$$= \frac{1}{8(p\cdot q)\,|p|\,|q|}\left(\frac{1}{|p-q|} - \frac{1}{|p+q|}\right). \tag{C10}$$

Now consider the three-propagator integral

$$L(p_5,p,k) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{q\cdot p_5}{(q)^2(q+p)^2(q+k)^2} = (-1)\frac{(|k|+|p|-|k-p|)\,(|p|\,(p_5\cdot k)+|k|\,(p_5\cdot p))}{16\,|k|\,|p|\,(k\cdot p+|k|\,|p|)\,|k-p|}. \tag{C11}$$

Consider the four-propagator integral

$$M(k;p) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2(q+k)^2(q+k+p)^2(q+p)^2}. \tag{C12}$$

$M(k;p)$ is not symmetric with respect to k, p. It is difficult to evaluate this four-propagator integral using the usual Feynman parametrization. [The three-propagator integral (C11) can be evaluated in this way.] Instead, we use the partial fraction trick in Eq. (C9) to rewrite it as multiple three-propagator integrals, and then use the three-propagator result in Eq. (C11). We find

$$M(k;p) = \int \frac{d^3q}{(2\pi)^3} \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$= \int \frac{d^3q}{(2\pi)^3}(\boldsymbol{k}\cdot\boldsymbol{q})\left(\frac{1}{q^2(q+k)^2(q+p)^2} - \frac{1}{q^2(q+k)^2(q+k+p)^2}\right.$$
$$\left. + \frac{1}{(q+k)^2(q+k+p)^2(q+p)^2} - \frac{1}{q^2(q+k+p)^2(q+p)^2}\right)\frac{1}{2(k\cdot p)}$$
$$= \frac{1}{2(k\cdot p)}(L(\boldsymbol{k},k,p) - L(\boldsymbol{k},k,k+p) + 0 - L(\boldsymbol{k},k+p,p)) + \frac{1}{2(k\cdot p)}\int \frac{d^3\ell}{(2\pi)^3}\frac{(\ell-k)\cdot\boldsymbol{k}}{\ell^2(\ell+p)^2(\ell-k+p)^2}$$
$$= \frac{1}{2(k\cdot p)}\left(L(\boldsymbol{k},k,p) - L(\boldsymbol{k},k,k+p) + L(\boldsymbol{k},p,p-k) - \frac{\boldsymbol{k}^2}{8|p|\,|p-k|\,|k|} - L(\boldsymbol{k},k+p,p)\right). \tag{C13}$$

A "quadratic in q integral" can be obtained from the above building blocks,

$$\int \frac{d^3q}{(2\pi)^3} \frac{(k\cdot q)\,(\boldsymbol{k}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2} = \int \frac{d^3q}{(2\pi)^3} \frac{\frac{1}{2}[(q+k)^2-k^2-q^2]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\,(\boldsymbol{k}\cdot\boldsymbol{q}) \tag{C14}$$
$$= \frac{1}{2}\left(L(\boldsymbol{k},p,k+p) - k^2\,M(k;p) - L(\boldsymbol{k},p,p-k) + \boldsymbol{k}^2\frac{1}{8|p|\,|p-k|\,|k|}\right). \tag{C15}$$

b. More complicated integral formulas

The momenta $p_5 = (p_5^0, p_5^x, p_5^y)$ and $p_5 = (q_6^0, q_6^x, q_6^y)$ appearing below are arbitrary external three-momenta that are independent of any integral momentum variables. We list the following integrals:

$$N(p_5; p, k)|_{p\neq k} \equiv \int \frac{d^3q}{(2\pi)^3} \frac{q\cdot p_5}{(q^2)\,(q^2)\,(q+p)^2(q+k)^2} = (-1)\frac{|p|^2(p_5\cdot k) + |k|^2(p_5\cdot p)}{16\,|k|^3\,|p|^3\,|k-p|}, \tag{C16}$$

$$R_1(p_5; k) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{q\cdot p_5}{q^2(q+k)^2q^2} = (-1)\frac{p_5\cdot k}{16\,|k|^3}, \tag{C17}$$

$$R_2(p_5, p_6; k) \equiv \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)\,(q\cdot p_6)}{q^2(q+k)^2q^2} = \frac{k^2(p_5\cdot p_6) + (p_5\cdot k)(p_6\cdot k)}{32\,|k|^3}, \tag{C18}$$

$$\begin{aligned} S(p_5; k, p) &\equiv \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)^2}{q^2(q+k)^2\,(q+p)^2} \\ &= \frac{|p|\,(\,|k| + 2|p| + |k-p|\,)\,(p_5\cdot k)^2 + |k|\,(\,|p| + 2|k| + |k-p|\,)\,(p_5\cdot p)^2}{16\,|k-p|\,|k|\,|p|\,(\,|k| + |p| + |k-p|\,)^2} \\ &\quad + \frac{|k-p|\,|k|\,|p|(\,|k| + |p| + |k-p|\,)\,p_5^2 + 2|k|\,|p|\,(p_5\cdot k)\,(p_5\cdot p)}{16\,|k-p|\,|k|\,|p|\,(\,|k| + |p| + |k-p|\,)^2}, \end{aligned} \tag{C19}$$

$$\begin{aligned} R_3(p_5; k, p) &\equiv \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)^2}{q^2(q+k)^2q^2(q+p)^2} \\ &= \frac{p^2\,|k-p|\,k^2\,p_5^2\,(|k-p| + |k| + |p|) + 2k^2\,p^2\,(p_5\cdot k)\,(p_5\cdot p)}{16\,|k-p|\,|k|^3\,|p|^3\,(\,|k| + |p| + |k-p|\,)^2} \\ &\quad + \frac{k^3(|k-p| + |k| + 2|p|)\,(p_5\cdot p)^2 + p^3(|k-p| + 2|k| + |p|)\,(p_5\cdot k)^2}{16\,|k-p|\,|k|^3\,|p|^3\,(\,|k| + |p| + |k-p|\,)^2}. \end{aligned} \tag{C20}$$

Next we consider

$$\begin{aligned} M_2(p_5; k, p) &\equiv \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2} \\ &= \int \frac{d^3q}{(2\pi)^3} (q\cdot p_5)^2 \Big(\frac{1}{q^2(q+k)^2(q+p)^2} - \frac{1}{q^2(q+k)^2(q+k+p)^2} \\ &\quad + \frac{1}{(q+k)^2(q+k+p)^2(q+p)^2} - \frac{1}{q^2(q+k+p)^2(q+p)^2} \Big) \frac{1}{2(k\cdot p)} \\ &= \frac{1}{2(k\cdot p)} \Big[S(p_5; k, p) - S(p_5; k, k+p) \\ &\quad + \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)^2}{(q+k)^2(q+k+p)^2(q+p)^2} - S(p_5; k+p, p) \Big]. \end{aligned} \tag{C21}$$

To further evaluate, we can shift the momentum,

$$\begin{aligned} \int \frac{d^3q}{(2\pi)^3} \frac{(q\cdot p_5)^2}{(q+k)^2(q+k+p)^2(q+p)^2} &= \int \frac{d^3\ell}{(2\pi)^3} \frac{[(\ell-k)\cdot p_5]^2}{\ell^2(\ell+p)^2(\ell-k+p)^2} \\ &= \int \frac{d^3\ell}{(2\pi)^3} \frac{(\ell\cdot p_5)^2 - 2(\ell\cdot p_5)(k\cdot p_5) + (k\cdot p_5)^2}{\ell^2(\ell+p)^2(\ell+p-k)^2} \\ &= S(p_5; p, p-k) - 2(k\cdot p_5)\,L(p_5; p, p-k) + (k\cdot p_5)^2 \frac{1}{8|p|\,|p-k|\,|k|}. \end{aligned} \tag{C22}$$

Thus, we have reduced M_2 to a linear combination of integrals we have already computed,

$$\begin{aligned} M_2(p_5; k, p) = \frac{1}{2(k\cdot p)} \Big[& S(p_5; k, p) - S(p_5; k, k+p) - S(p_5; k+p, p) \\ & + S(p_5; p, p-k) - 2(k\cdot p_5)\,L(p_5; p, p-k) + (k\cdot p_5)^2 \frac{1}{8|p|\,|p-k|\,|k|} \Big]. \end{aligned} \tag{C23}$$

2. Anti symmetric component of $\Pi^{(b+c)}$

The antisymmetric part of the gauge field propagator is

$$D^{\text{Anti}}_{\alpha\beta}(k) = \frac{\kappa}{\kappa^2 + g_X^4 + \frac{w_x g_X^2\,|\boldsymbol{k}|}{|k|}}\left(\frac{\epsilon_{\alpha\beta\lambda}k_\lambda\delta_{\lambda j}}{\boldsymbol{k}^2}\right) \equiv \tilde{C}_A(k)\,\frac{\epsilon_{\alpha\beta\lambda}k_\lambda\delta_{\lambda j}}{\boldsymbol{k}^2}. \tag{C24}$$

It is important to keep in mind that $\tilde{C}_A$ is a function of the momentum k carried by the gauge field.

We want to evaluate $\Pi^{(b+c)}_{(\text{A})}$,

$$\Pi^{(b+c)}_{(\text{A})}(p) = \frac{p_\lambda\epsilon_{\mu\nu\lambda}}{2|p|}\Pi^{(b+c)}_{\mu\nu}(p)$$
$$= \frac{p_\lambda\epsilon_{\mu\nu\lambda}}{2|p|}\times 2\frac{(-1)g^4}{N_f}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\text{Tr}\left[\gamma_\mu\frac{i\,q^\rho\gamma_\rho}{q^2}\gamma_\alpha\frac{i\,(k+q)^\tau\gamma_\tau}{(k+q)^2}\gamma_\beta\frac{i\,q^\sigma\gamma_\sigma}{q^2}\gamma_\nu\frac{i\,(p+q)^\kappa\gamma_\kappa}{(p+q)^2}\right]D^{\text{Anti}}_{\alpha\beta}(k). \tag{C25}$$

The following identity is useful:

$$\epsilon_{ijk}\epsilon_{abc} = \text{Det}\begin{pmatrix}\delta_{ia} & \delta_{ib} & \delta_{ic}\\ \delta_{ja} & \delta_{jb} & \delta_{jc}\\ \delta_{ka} & \delta_{kb} & \delta_{kc}\end{pmatrix}. \tag{C26}$$

Consider the following terms in the integrand in Eq. (C25):

$$\text{Tr}\left[\gamma_\mu\frac{i\,q^\rho\gamma_\rho}{q^2}\gamma_\alpha\frac{i\,(k+q)^\tau\gamma_\tau}{(k+q)^2}\gamma_\beta\frac{i\,q^\sigma\gamma_\sigma}{q^2}\gamma_\nu\frac{i\,(p+q)^\kappa\gamma_\kappa}{(p+q)^2}\right]\,(p_\lambda\epsilon_{\mu\nu\lambda})\,\frac{(\epsilon_{\alpha\beta\eta}\,k_\eta\delta_{\eta j})}{\boldsymbol{k}^2} \tag{C27}$$

$$= \frac{1}{q^2(k+q)^2q^2(p+q)^2}\text{Tr}[\gamma_\mu\gamma_\rho\gamma_\alpha\gamma_\tau\gamma_\beta\gamma_\sigma\gamma_\nu\gamma_\kappa]\,q^\rho(k+q)^\tau q^\sigma(p+q)^\kappa\,(p_\lambda\epsilon_{\mu\nu\lambda})\,\frac{(\epsilon_{\alpha\beta\eta}\,k_\eta\delta_{\eta j})}{\boldsymbol{k}^2} \tag{C28}$$

$$= \frac{1}{q^2(k+q)^2q^2(p+q)^2}\times 8\,([q^2(p\cdot q)+p^2q^2]\,\boldsymbol{k}^2 + [(p\cdot q)\,q^2 + p^2q^2]\,\boldsymbol{k}\cdot q)\frac{1}{\boldsymbol{k}^2}. \tag{C29}$$

We perform the convergent q integral first,

$$\int\frac{d^3q}{(2\pi)^3}(\text{Eq. (C29)}) = \int\frac{d^3q}{(2\pi)^3}\frac{1}{(k+q)^2q^2(p+q)^2}8\,([(p\cdot q)+p^2]\,\boldsymbol{k}^2 + [(p\cdot q)\;+p^2]\,\boldsymbol{k}\cdot q)\frac{1}{\boldsymbol{k}^2}. \tag{C30}$$

Look at

$$\int\frac{d^3q}{(2\pi)^3}\frac{p\cdot q+p^2}{(k+q)^2q^2(p+q)^2} = \int\frac{d^3q}{(2\pi)^3}\frac{\frac{1}{2}[(p+q)^2-p^2-q^2]+p^2}{(k+q)^2q^2(p+q)^2}$$
$$= \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{(k+q)^2q^2} - \frac{1}{(k+q)^2(p+q)^2} + \frac{p^2}{(k+q)^2q^2(p+q)^2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{8|k|} - \frac{1}{8|p-k|} + \frac{p^2}{8|p||k||p-k|}\right), \tag{C31}$$

where we used a formula in Appendix C 1 a. Next consider

$$\int\frac{d^3q}{(2\pi)^3}\frac{(p\cdot q+p^2)\,\boldsymbol{k}\cdot q}{(k+q)^2q^2(p+q)^2} = \frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\left(\frac{\boldsymbol{k}\cdot q}{(k+q)^2q^2} - \frac{\boldsymbol{k}\cdot q}{(k+q)^2(p+q)^2} + \frac{p^2\boldsymbol{k}\cdot q}{(k+q)^2q^2(p+q)^2}\right)$$
$$= \frac{1}{2}\left(\frac{-\boldsymbol{k}^2}{16|k|} + \frac{\boldsymbol{k}^2+\boldsymbol{k}\cdot\boldsymbol{p}}{16|k-p|} + p^2\,L(\boldsymbol{k},p,k)\right) \tag{C32}$$

where we used Eq. (C11). Therefore,

$$\int\frac{d^3q}{(2\pi)^3}(\,\text{Eq. (C29)}) = \frac{8}{2}\left(\frac{1}{8|k|} - \frac{1}{8|p-k|} + \frac{p^2}{8|p||k||p-k|}\right) + \frac{8}{\boldsymbol{k}^2}\frac{1}{2}\left(\frac{-\boldsymbol{k}^2}{16|k|} + \frac{\boldsymbol{k}^2+\boldsymbol{k}\cdot\boldsymbol{p}}{16|k-p|} + p^2\,L(\boldsymbol{k},p,k)\right) \tag{C33}$$

$$= \frac{1}{2}\left(\frac{1}{|k|} - \frac{1}{|p-k|} + \frac{p^2}{|p||k||p-k|}\right) + \frac{1}{2}\left(\frac{-1}{2|k|} + \frac{1+\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{\boldsymbol{k}^2}}{2|k-p|}\right) + \frac{4\,p^2}{\boldsymbol{k}^2}\,L(\boldsymbol{k},p,k) \tag{C34}$$

$$= \frac{1}{4}\left(\frac{1}{|k|} + \frac{-1+\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{\boldsymbol{k}^2}}{|k-p|}\right) + \frac{p^2}{2\,|p||k||p-k|} + \frac{4\,p^2}{\boldsymbol{k}^2}\,L(\boldsymbol{k},p,k), \tag{C35}$$

where we used

$$\int \frac{d^3q}{(2\pi)^3}\frac{\boldsymbol{k}\cdot q}{(k+q)^2q^2} = \frac{-\boldsymbol{k}^2}{16|k|}, \tag{C36}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{\boldsymbol{k}\cdot q}{(q+k)^2(q+p)^2} = -\frac{\boldsymbol{k}^2+\boldsymbol{k}\cdot\boldsymbol{p}}{16|k-p|}. \tag{C37}$$

Summarizing, we find that Eq. (C25) equals

$$\begin{aligned}\Pi^{(b+c)}_{(\mathrm{A})}(p) &= \frac{1}{2|p|}\,2\frac{(-1)g^4}{N_f}\left(\int \tilde{C}_A(k)\frac{dk_0d^2\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{4}\left(\frac{1}{|k|}+\frac{-1+\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{\boldsymbol{k}^2}}{|k-p|}\right)+\frac{p^2}{2\,|p||k||p-k|}+\frac{4\,p^2}{\boldsymbol{k}^2}L(\boldsymbol{k},p,k)\right)\\ &= \frac{1}{2|p|}\,2\frac{(-1)g^4}{N_f}\left(\int \tilde{C}_A(k)\frac{dk_0d^2\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{4}\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{|k-p|\,\boldsymbol{k}^2}+\frac{p^2}{2\,|p||k||p-k|}+\frac{4\,p^2}{\boldsymbol{k}^2}L(\boldsymbol{k},p,k)+\frac{1}{4|k|}-\frac{1}{4|k-p|}\right).\end{aligned} \tag{C38}$$

3. Anti-symmetric component of $\Pi^{(a)}$

We aim to evaluate

$$\Pi^{(a)}_{(\mathrm{A})}(p) = \frac{p_\lambda\epsilon_{\mu\nu\lambda}}{2|p|}\Pi^{(a)}_{\mu\nu}(p) = \frac{p_\lambda\epsilon_{\mu\nu\lambda}}{2|p|}\frac{(-1)g^4}{N_f}\int_k\int_q \mathrm{Tr}[\gamma_\mu G(q)\gamma_\alpha G(q+k)\gamma_\nu G(q+k+p)\gamma_\beta G(p+q)]\frac{\tilde{C}_A(k)\,\epsilon_{\alpha\beta\lambda}k_\eta\delta_{\eta j}}{\boldsymbol{k}^2} \tag{C39}$$

$$\begin{aligned}&= \frac{1}{2|p|}\frac{(-1)g^4}{N_f}\int_k \tilde{C}_A(k)\int_q[-8(k\cdot p)\,(p\cdot q)k_\eta - 8(p\cdot q)^2k_\eta + 8(k\cdot q)p^2k_\eta + 4(p\cdot q)p^2k_\eta\\ &\quad + 12p^2q^2k_\eta + 8(k\cdot q)(p\cdot q)p_\eta + 4k^2(p\cdot q)p_\eta - 4(k\cdot p)q^2p_\eta - 4(k\cdot p)p^2q_\eta - 8(k\cdot q)p^2q_\eta - 4k^2p^2q_\eta]\\ &\quad\times\frac{1}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\times\frac{k_\eta\delta_{\eta j}}{\boldsymbol{k}^2}.\end{aligned} \tag{C40}$$

The integral over q in Eq. (C40) can be decomposed as

$$\mathcal{I_C} \equiv \int_q \frac{C_1(p\cdot q)+C_2(k\cdot q)+C_3\,q^2+C_4(k\cdot q)(p\cdot q)+C_5(\boldsymbol{k}\cdot\boldsymbol{q})+C_6(k\cdot q)(\boldsymbol{k}\cdot\boldsymbol{q})+C_7\,(p\cdot q)^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2}, \tag{C41}$$

for some C_i, independent of q. Using the building block integrals from Appendix C 1 and writing the norms $|k|=k$, and $|p|=p$, we find

$$\begin{aligned}\mathcal{I_C} &= \left(\frac{-1}{2|k|}+\frac{1}{4|k-p|}+\frac{1}{4|k+p|}\right)+\left(\frac{2p}{4|k|\,|k-p|}+\frac{3p}{4|k|\,|k+p|}\right)+\frac{-k^2\,p}{2|k|\,|k-p|\,(k\cdot p)}\\ &+\frac{-p^3}{2|k|\,|k-p|\,(k\cdot p)}+\frac{k^2\,p}{2|k|\,|k+p|\,(k\cdot p)}+\frac{p^3}{2|k|\,|k+p|\,(k\cdot p)}+\frac{-p^2}{8|k|\,(|p|\,|k|+k\cdot p)}\\ &+\frac{p^2}{8|k-p|\,(|p|\,|k|+k\cdot p)}+\frac{p^3}{8\,|k|\,|k-p|\,(|p|\,|k|+k\cdot p)}+\frac{-p}{4(k^2+|k|\,|k+p|+k\cdot p)}\\ &+\frac{p^2}{8k\,(k^2+|k|\,|k+p|+k\cdot p)}+\frac{p^2}{8|k+p|\,(k^2+|k|\,|k+p|+k\cdot p)}\\ &+\frac{-|p|\,|k|}{8|k+p|\,(k^2+|k|\,|k+p|+k\cdot p)}+\frac{-p\,|k+p|}{8|k|\,(k^2+|k|\,|k+p|+k\cdot p)}\\ &+\frac{p^2}{8|k|\,(p^2+|p|\,|k-p|-k\cdot p)}+\frac{-p^2}{8|k-p|\,(p^2+|p|\,|k-p|-k\cdot p)}\\ &+\frac{p^3}{8|k|\,|k-p|\,(p^2+|p|\,|k-p|-k\cdot p)}+\frac{p^2}{8|k|\,(p^2+|p|\,|k+p|\,+k\cdot p)}\\ &+\frac{-p^2}{8|k+p|\,(p^2+|p|\,|k+p|\,+k\cdot p)}+\frac{1}{8|k|\,|k+p|\,(p^2+|p|\,|k+p|+k\cdot p)}.\end{aligned} \tag{C42}$$

Note that, throughout the calculation, we have kept p general. Many terms above are individually divergent when integrated over k. We cut off the divergent k integrals with the cutoff Λ: For instance,

$$\int_{-1}^{1} d\cos\theta \int_0^{\Lambda} dk \frac{2\pi k^2}{(2\pi)^3}\frac{1}{k} = \frac{\Lambda^2}{4\pi^2}, \tag{C43}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{|k-p|} = \frac{\Lambda^2}{4\pi^2} + \frac{-p^2}{12\pi^2}, \tag{C44}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{-k^2\,p}{2|k|\,|k-p|\,(k\cdot p)} = \frac{p^2(12+\pi^2)}{64\pi^2} + 0\cdot\log\Lambda + \frac{-p\Lambda}{4\pi^2}. \tag{C45}$$

We are careful not to shift the momenta arbitrarily in any divergent integral; otherwise, we are liable to obtain an incorrect result.

4. Combining the anti-symmetric components of $\Pi^{(a)}$ and $\Pi^{(b+c)}$

We now add together the antisymmetric components of $\Pi^{(a)}$ and $\Pi^{(b+c)}$ in (C40) and (C38) to find

$$\Pi^{(a+b+c)}_{(\mathrm{A})}(p) = -\frac{1}{2|p|}\frac{g^4}{N_f}\int_k \tilde{C}_A(k)\bigg[2\times\bigg(\frac{1}{4}\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{|k-p|\,\boldsymbol{k}^2} + \frac{p^2}{2\,|p||k||p-k|} + \frac{4\,p^2}{\boldsymbol{k}^2}L(\boldsymbol{k},p,k) + \frac{1}{4|k|} - \frac{1}{4|k-p|}\bigg) + \mathcal{I}_C\bigg]. \tag{C46}$$

To perform the integral over k, we take p to lie along the k_τ axis, with $k\cdot p = |k||p|\cos\theta$, so that $\mathbf{k}\cdot\mathbf{p} = 0$ and

$$\int \frac{d^3k}{(2\pi)^3}(\dots) = \int_{-1}^{1} d\cos\theta \int_0^{\Lambda} dk \frac{2\pi k^2}{(2\pi)^3}(\dots). \tag{C47}$$

After performing the $\int_0^{\Lambda} dk$ integral, we perform a $\frac{1}{\Lambda}$ expansion, and then do the $\int_{-1}^{1} d\cos\theta$ integral. The $\log\Lambda$ divergent terms vanish after the angular integration. Letting $\cos\theta = z$, we symmetrize with respect to z to get rid of terms that are odd in z and should therefore vanish after performing the z integral. The result is

$$\Pi^{(a+b+c)}_{(\mathrm{A})}(p) = \frac{1}{2|p|}\frac{(-1)g^4}{N_f}\int_{-1}^{1} dz\,|p|^2\,\frac{-z+(-1+z^2)\mathrm{ArcTanh}[z]}{8\pi^2 z}\frac{1}{A_X + B_X\sqrt{1-z^2}}, \tag{C48}$$

$$\sigma^{\psi}_{xy} = \frac{i}{\omega}\Pi^{(b+c+a)}(\boldsymbol{p}=0, |p_0|\to i|\omega|), \tag{C49}$$

with $A_X \equiv \frac{\kappa^2+g_X^4}{\kappa}$, $B_X \equiv \frac{w_x g_X^2}{\kappa}$.

As a consistency check: When $w_x = 0$ (vanishing Coulomb interaction),

$$\Pi^{(a+b+c)}_{(\mathrm{A})}(p; w_x=0) = \frac{1}{2|p|}\frac{(-1)g^4}{N_f}\frac{\kappa}{\kappa^2+g_X^4}\bigg(-p^2\,\frac{4+\pi^2}{32\pi^2} + 0\cdot\log\Lambda + \frac{p}{4\pi^2}\,\Lambda\bigg), \tag{C50}$$

which agrees with [38]. We drop the linear divergence since it is an artifact of the (gauge-noninvariant) hard cutoff.

5. Symmetric component of $\Pi^{(b+c)}$

Now we consider the symmetric component of $\Pi^{(b+c)}$. We will need the symmetric part of the gauge field propagator,

$$D^{\mathrm{Sym}}_{00}(k_0,\boldsymbol{k}) = \frac{|k|}{\boldsymbol{k}^2}\times\frac{1}{\frac{\kappa^2+g_X^4}{g_X^2} + w_x\sqrt{1-\cos^2\theta}}\bigg(1+\frac{w_x}{g_X^2}\sqrt{1-\cos^2\theta}\bigg), \tag{C51}$$

$$D^{\mathrm{Sym}}_{ij}(k_0,\boldsymbol{k}) = \frac{1}{|k|}\bigg(\delta_{ij} - \frac{k_ik_j}{\boldsymbol{k}^2}\bigg)\times\frac{1}{\frac{\kappa^2+g_X^4}{g_X^2} + w_x\sqrt{1-\cos^2\theta}}. \tag{C52}$$

Here we are denoting $\sqrt{1-\cos^2\theta} = |\mathbf{k}|/|k|$. We parametrize the gauge field propagator as

$$D^{\mathrm{Sym}}_{\alpha\beta}(k) = (\delta_{\alpha0}\delta_{\beta0}\,f_A(k) + \delta_{\alpha i}\delta_{\beta j}\,\delta_{ij}f_B(k) + \delta_{\alpha i}\delta_{\beta j}\,k_ik_jf_C(k)), \tag{C53}$$

where f_A, f_B, f_C are constants. Using the gauge field propagator above, the symmetric component of $\Pi^{(b+c)}$ is

$$\Pi^{(b+c)}_{(\mathrm{S})}(p) = \frac{1}{2}\delta_{\mu\nu}\Pi^{(b+c)}_{\mu\nu}(p) \tag{C54}$$

$$= \frac{\delta_{\mu\nu}}{2}\times 2\frac{(-1)g^4}{N_f}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}\bigg[\gamma_\mu\frac{i\,q^\rho\gamma_\rho}{q^2}\gamma_\alpha\frac{i\,(k+q)^\tau\gamma_\tau}{(k+q)^2}\gamma_\beta\frac{i\,q^\sigma\gamma_\sigma}{q^2}\gamma_\nu\frac{i\,(p+q)^\kappa\gamma_\kappa}{(p+q)^2}\bigg]D^{\mathrm{Sym}}_{\alpha\beta}(k). \tag{C55}$$

The trace $\mathrm{Tr}[\ldots]_{\mu\nu}$ in the integrand evaluates to

$$\mathrm{Tr}[\gamma_\mu \frac{i\,q^\rho \gamma_\rho}{q^2}\gamma_\alpha \frac{i\,(k+q)^\tau \gamma_\tau}{(k+q)^2}\gamma_\beta \frac{i\,q^\sigma \gamma_\sigma}{q^2}\gamma_\nu \frac{i\,(p+q)^\kappa \gamma_\kappa}{(p+q)^2}]\ \left(\frac{\delta_{\mu\nu}}{2}\right) D^{\mathrm{Sym}}_{\alpha\beta}(k) \tag{C56}$$

$$\begin{aligned}
&= \frac{1}{q^2(k+q)^2q^2(p+q)^2}(\delta_{\alpha\beta}[2(k\cdot q)\,(p\cdot q) - (k\cdot p)\,q^2 + (k\cdot q)\,q^2 + (p\cdot q)\,q^2 + q^4\,] \\
&\quad + (p_\alpha k_\beta + p_\beta k_\alpha)\,q^2 + (q_\alpha k_\beta + q_\beta k_\alpha)\,[-2(p\cdot q) - q^2] + (q_\alpha p_\beta + q_\beta p_\alpha)\,q^2 \\
&\quad + (q_\alpha q_\beta)\,[-4(p\cdot q) - 2q^2])\,D^{\mathrm{Sym}}_{\alpha\beta}(k) \\
&= \frac{1}{q^2(k+q)^2q^2(p+q)^2}(\delta_{\alpha\beta}[2(k\cdot q)\,(p\cdot q) - (k\cdot p)\,q^2 + (k\cdot q)\,q^2 + (p\cdot q)\,q^2 + q^4\,] \\
&\quad + (p_\alpha k_\beta + p_\beta k_\alpha)\,q^2 + (q_\alpha k_\beta + q_\beta k_\alpha)\,[-2(p\cdot q) - q^2] + (q_\alpha p_\beta + q_\beta p_\alpha)\,q^2 \\
&\quad + (q_\alpha q_\beta)\,[-4(p\cdot q) - 2q^2])\,D^{\mathrm{Sym}}_{\alpha\beta}(k).
\end{aligned} \tag{C57}$$

Next, we decompose the terms in the integrand with different q dependencies into various partial fractions,

$$\mathrm{Tr}[\ldots]_{\mu\nu}\left(\frac{\delta_{\mu\nu}}{2}\right) D^{\mathrm{Sym}}_{\alpha\beta}(k) \tag{C58}$$

$$\begin{aligned}
&= \delta_{\alpha\beta} D^{\mathrm{Sym}}_{\alpha\beta}(k)\Bigg[\frac{1}{2(q+k)^2(q+p)^2} + \frac{1}{2q^2\,q^2} + \frac{-k^2}{2(q+k)^2q^4} + \frac{-p^2}{2(q+p)^2q^4} \\
&\quad + \frac{k^2\,p^2}{2(q+k)^2(q+p)^2\,q^4} + \frac{-k\cdot p}{(q+k)^2(q+p)^2\,q^2}\Bigg] + (p_\alpha k_\beta + p_\beta k_\alpha) D^{\mathrm{Sym}}_{\alpha\beta}(k)\frac{1}{q^2(q+k)^2(q+p)^2} \\
&\quad + \left(\frac{-2f_B(\boldsymbol{k}\cdot\boldsymbol{q}) - 2f_C\,\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{q}) - 2f_A\,k_0\,q_0}{(q+k)^2q^4} + \frac{2f_B\,p^2(\boldsymbol{k}\cdot\boldsymbol{q}) + 2f_C\,p^2\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{q}) + 2f_A\,p^2\,k_0\,q_0}{(q+k)^2(q+p)^2\,q^4}\right) \\
&\quad + \frac{2f_B(\boldsymbol{p}\cdot\boldsymbol{q}) + 2f_A\,p_0q_0 + 2f_C\,(\boldsymbol{k}\cdot\boldsymbol{p})\,(\boldsymbol{k}\cdot\boldsymbol{q})}{(q+k)^2\,(q+p)^2\,q^2} + (-2)\frac{f_B\boldsymbol{q}^2 + f_C\,(\boldsymbol{k}\cdot\boldsymbol{q})^2 + f_A\,q_0^2}{(q+k)^2\,q^4} \\
&\quad + (+2)\frac{f_B\,p^2\boldsymbol{q}^2 + f_C\,p^2\,(\boldsymbol{k}\cdot\boldsymbol{q})^2 + f_A\,p^2\,q_0^2}{(q+k)^2(q+p)^2\,q^4}.
\end{aligned} \tag{C59}$$

Each of the terms in the first few lines diverge as $|q| \to 0^+$ in the IR, however, their sum is IR finite. Combining some of these terms with one another, we perform the following q integral:

$$\int \frac{d^3q}{(2\pi)^3}\frac{1}{2}\left(\frac{1}{q^2\,q^2} + \frac{-k^2}{(q+k)^2q^4} + \frac{-p^2}{(q+p)^2q^4} + \frac{k^2\,p^2}{(q+k)^2(q+p)^2\,q^4}\right) \tag{C60}$$

$$= \frac{1}{2}\left[\int \frac{d^3q}{(2\pi)^3}\left(\frac{1}{q^2\,q^2} + \frac{-p^2}{(q+p)^2q^4}\right) + \int \frac{d^3q}{(2\pi)^3}\left(\frac{-k^2}{(q+k)^2q^4} + \frac{k^2\,p^2}{(q+k)^2(q+p)^2\,q^4}\right)\right] \tag{C61}$$

$$= \frac{1}{2}\left[\int \frac{d^3q}{(2\pi)^3}\frac{q^2 + 2(q\cdot p)}{(q+p)^2q^4} + \int \frac{d^3q}{(2\pi)^3}\frac{-k^2q^2 - 2k^2(q\cdot p)}{(q+k)^2(q+p)^2\,q^4}\right] \tag{C62}$$

$$= \frac{1}{2}\left[\frac{1}{8|p|} - \frac{p\cdot p}{8p^3} + \frac{-k^2}{8}\frac{1}{|p||k||p-k|} + (-2k^2)(-1)\frac{p^2(p\cdot k) + k^2(p\cdot p)}{16k^3p^3|k-p|}\right] \tag{C63}$$

$$= \frac{1}{2}\frac{p\cdot k}{8|k|\,|p|\,|k-p|}, \tag{C64}$$

where we have used the "dot p_5" formulas in Appendix C 1 b, with $p_5 = p$ The remaining integrals over q are straightforwardly performed using formulas we have already given.

Next, we recall the following list of SO(3) noninvariant integrals that we evaluated in Appendix C 1 b,

$$\int \frac{d^3q}{(2\pi)^3}\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{(q+k)^2q^4} = R_1(\boldsymbol{k};k), \quad p_5 = (0,\boldsymbol{k}), \tag{C65}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{k_0\,q_0}{(q+k)^2q^4} = R_1(k_0;k), \quad p_5 = (k_0,0,0), \tag{C66}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{(\boldsymbol{k}\cdot\boldsymbol{q})}{(q+k)^2(q+p)^2\,q^4} = N(\boldsymbol{k};p,k),\ \int \frac{d^3q}{(2\pi)^3}\frac{k_0\,q_0}{(q+k)^2(q+p)^2\,q^4} = N(k_0;p,k), \tag{C67}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{(\boldsymbol{p}\cdot\boldsymbol{q})}{(q+k)^2\,(q+p)^2\,q^2} = L(\boldsymbol{p};,k,p),\ \int \frac{d^3q}{(2\pi)^3}\frac{p_0q_0}{(q+k)^2\,(q+p)^2\,q^2} = L(p_0;,k,p), \tag{C68}$$

and

$$\int \frac{d^3q}{(2\pi)^3}\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{(q+k)^2\,q^4} = R_2(\boldsymbol{k},\boldsymbol{k};k), \tag{C69}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{\boldsymbol{q}^2}{(q+k)^2\,q^4} = \int \frac{d^3q}{(2\pi)^3}\frac{(q_x+iq_y)(q_x-iq_y)}{(q+k)^2\,q^4} = \int \frac{d^3q}{(2\pi)^3}\frac{(q\cdot(0,1,i))\,(q\cdot(0,1,-i))}{(q+k)^2\,q^4}$$
$$= R_2((0,1,i),(0,1,-i);k) = \frac{2k^2+\boldsymbol{k}^2}{32\,k^3}, \tag{C70}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{q_0^2}{(q+k)^2\,q^4} = \int \frac{d^3q}{(2\pi)^3}\frac{\big(q\cdot(1,0,0)\,\big)^2}{(q+k)^2\,q^4} = R_2((1,0,0),(1,0,0);k) = \frac{k^2+k_0^2}{32k^3}. \tag{C71}$$

Finally, we will need

$$\int \frac{d^3q}{(2\pi)^3}\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{(q+k)^2(q+p)^2\,q^4} = R_3(\boldsymbol{k};k,p), \tag{C72}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{q_0^2}{(q+k)^2(q+p)^2\,q^4} = R_3((1,0,0);k,p), \tag{C73}$$

$$\int \frac{d^3q}{(2\pi)^3}\frac{\boldsymbol{q}^2}{(q+k)^2(q+p)^2\,q^4} = \int \frac{d^3q}{(2\pi)^3}\frac{q^2-q_0^2}{(q+k)^2(q+p)^2\,q^4} = \frac{1}{8|k|\,|p|\,|k-p|} - R_3((1,0,0);k,p). \tag{C74}$$

Plugging these in, we find the q integral equals

$$\begin{aligned}
I_{bc} &\equiv \int \frac{d^3q}{(2\pi)^3}\mathrm{Tr}[\ldots]_{\mu\nu}\left(\frac{\delta_{\mu\nu}}{2}\right)D^{\mathrm{Sym}}_{\alpha\beta}(k)\\
&= (f_A+2f_B+f_C\,\boldsymbol{k^2})\left[\frac{J_2(p,k)}{2}+(-p\cdot k)\frac{J_3(p,k)}{2}\right] + [2f_B\boldsymbol{k}\cdot\boldsymbol{p}+2f_C\boldsymbol{k^2}\boldsymbol{k}\cdot\boldsymbol{p}+2f_Ak_0p_0]J_3(p,k)\\
&\quad+\frac{f_A\,k_0^2+f_B\boldsymbol{k}^2+f_C\boldsymbol{k^4}}{8k^3} + [2f_Bp^2N(\boldsymbol{k};p,k)+2f_C\boldsymbol{k^2}p^2\,N(\boldsymbol{k};p,k)+2f_A\,p^2\,N(k_0;p,k)]\\
&\quad+[2f_B\,L(\boldsymbol{p};k,p)+2f_A\,L(p_0;k,p)+2f_C(\boldsymbol{k}\cdot\boldsymbol{p})\,L(\boldsymbol{k};k,p)]\\
&\quad+(-1)\left[f_B\frac{2k^2+\boldsymbol{k^2}}{16k^3}+f_C\boldsymbol{k^2}\frac{k^2+\boldsymbol{k^2}}{16k^3}+f_A\frac{k^2+k_0^2}{16k^3}\right]\\
&\quad+2p^2\,(f_B\ [J_3(k,p)-R_3((1,0,0);k,p)\,]+f_C\,R_3(\boldsymbol{k};k,p)+f_A\,R_3((1,0,0);k,p))
\end{aligned} \tag{C75}$$

Note that in the definition of I_{bc} above, we do not include the factor 2 which counts the contribution from diagrams b and c.

6. Symmetric component of $\Pi^{(a)}$

The symmetric component of $\Pi^{(a)}$ is

$$\begin{aligned}
\Pi^{(a)}_{(\mathrm{S})}(p) &= \frac{1}{2}\delta_{\mu\nu}\Pi^{(a)}_{\mu\nu}(p)\\
&= \frac{\delta_{\mu\nu}}{2}\times\frac{(-1)g^4}{N_f}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}[\gamma_\mu G(q)\gamma_\alpha G(q+k)\gamma_\nu G(q+k+p)\gamma_\beta G(p+q)]\,D^{\mathrm{Sym}}_{\alpha\beta}(k)\\
&= \frac{(-1)g^4}{N_f}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}(\delta_{\alpha\beta}[-4(k\cdot p)(k\cdot q)-4(k\cdot q)^2-2(k\cdot q)(p\cdot q)+3k^2(p\cdot q)-p^2\,(k\cdot q)-q^2(k\cdot p)\\
&\quad-2q^2(k\cdot q)+3k^2\,q^2\ -2q^2(p\cdot q)-p^2\,q^2-q^4\,]+k_\alpha k_\beta[-4q^2-4(p\cdot q)]\\
&\quad+q_\alpha q_\beta[2p^2-4k^2]+p_\alpha p_\beta[2q^2+2(k\cdot q)]\\
&\quad+p_\alpha q_\beta[2(k\cdot q)-3k^2+2q^2]+p_\beta q_\alpha[-2(k\cdot p)-2(k\cdot q)-k^2-4(p\cdot q)-2q^2]
\end{aligned}$$

$$+k_\alpha q_\beta[4(k\cdot p)+4(k\cdot q)+p^2]+q_\alpha k_\beta[4(k\cdot q)+p^2]+p_\alpha k_\beta[q^2+4(k\cdot q)]+k_\alpha p_\beta[-2(p\cdot q)-q^2]) \times \frac{1}{q^2(q+k)^2(q+k+p)^2(q+p)^2}D^{\mathrm{Sym}}_{\alpha\beta}(k). \tag{C76}$$

Above, we have used the parametrization of the symmetric part of the gauge field propagator in Eq. (C53).

We rewrite the first part of the integrand using partial fractions,

$$D^{\mathrm{sym}}_{\alpha\beta}(k)\delta_{\alpha\beta}\bigg(\frac{2(k\cdot p)k^2-k^4-\frac{3}{2}k^2p^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{2k^2}{q^2\,(q+k+p)^2(q+k)^2} + \frac{-2(k\cdot p)+2k^2-(q+k)^2}{q^2\,(q+k+p)^2(q+p)^2}+\frac{\frac{-1}{2}}{(q+k+p)^2(q+k)^2}+\frac{\frac{3}{2}}{(q+k+p)^2(q+p)^2}+\frac{\frac{-1}{2}}{(q+k+p)^2q^2} + \frac{(k\cdot p)-\frac{q^2}{2}}{(q+k+p)^2(q+k)^2(q+p)^2}\bigg). \tag{C77}$$

Integrating this over q gives

$$D^{\mathrm{sym}}_{\alpha\beta}(k)\delta_{\alpha\beta}\bigg([2(k\cdot p)k^2-k^4-\frac{3}{2}k^2p^2]J_4(k,p)+2k^2J_3(k+p,k) + [-2(k\cdot p)+2k^2]J_3(k+p,p)-I(p,-k)+\frac{-1}{2}J_2(k+p,k)+\frac{3}{2}J_2(k+p,p)+\frac{-1}{2}J_2(k+p,0) + (k\cdot p)J_3(k,k-p)+\frac{-1}{2}I(k,p)\bigg) \tag{C78}$$

$$= D^{\mathrm{sym}}_{\alpha\beta}(k)\delta_{\alpha\beta}\bigg([-k^4-\frac{3}{2}k^2p^2]J_4(k,p)+\frac{|k|}{4p|k-p|}-\frac{|k|}{4p|k+p|}+\frac{-1}{4p}+\frac{1}{16|k-p|}+\frac{1}{16|k+p|}+\frac{|k|}{2p\,|k+p|}\bigg) \tag{C79}$$

$$= D^{\mathrm{sym}}_{\alpha\beta}(k)\delta_{\alpha\beta}\bigg([-k^4-\frac{3}{2}k^2p^2]J_4(k,p)+0+\frac{-1}{4p}+\frac{1}{16|k-p|}+\frac{1}{16|k+p|}+\frac{|k|}{2p\,|k+p|}\bigg). \tag{C80}$$

Next, we use partial fractions to reexpress

$$D^{\mathrm{sym}}_{\alpha\beta}(k)k_\alpha k_\beta\frac{-4q^2-4(p\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2} = \big(D^{\mathrm{sym}}_{\alpha\beta}(k)k_\alpha k_\beta\big)\bigg(\frac{2p^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2}-\frac{2}{(q+k)^2(q+k+p)^2(q+p)^2}-\frac{2}{(q+k)^2(q+k+p)^2q^2}\bigg). \tag{C81}$$

Integrating this over q gives

$$\big(D^{\mathrm{sym}}_{\alpha\beta}(k)k_\alpha k_\beta\big)(2p^2J_4(k,p)-2J_3(p,p-k)-2J_3(k,k+p)) = \big(D^{\mathrm{sym}}_{\alpha\beta}(k)k_\alpha k_\beta\big)\bigg(\frac{p}{4|k|\,|k-p|(k\cdot p)}+\frac{-p}{4|k|\,|k+p|(k\cdot p)}+\frac{-1}{4|k|\,|p|\,|k-p|}+\frac{-1}{4|k|\,|p|\,|k+p|}\bigg). \tag{C82}$$

The next term is

$$D^{\mathrm{sym}}_{\alpha\beta}(k)q_\alpha q_\beta\frac{(2p^2-4k^2)}{q^2(q+k)^2(q+k+p)^2(q+p)^2} = (2p^2-4k^2)\frac{f_C(\boldsymbol{k}\cdot q)^2+f_Aq_0^2+f_B\boldsymbol{q}^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2} = (2p^2-4k^2)\frac{f_C(\boldsymbol{k}\cdot q)^2+(f_A-f_B)q_0^2+f_Bq^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2}. \tag{C83}$$

Integrating this over q gives

$$(2p^2-4k^2)(f_C\,M_2(\boldsymbol{k};k,p)+(f_A-f_B)\,M_2((1,0,0);k,p)+f_B\,J_3(k-p,k)). \tag{C84}$$

Moving on to the next term

$$D^{\mathrm{sym}}_{\alpha\beta}(k)p_\alpha p_\beta\frac{2q^2+2(k\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2} = D^{\mathrm{sym}}_{\alpha\beta}(k)p_\alpha p_\beta\bigg(\frac{-k^2}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{1}{(q+k+p)^2(q+k)^2(q+p)^2}+\frac{1}{(q+k+p)^2(q+p)^2\,q^2}\bigg). \tag{C85}$$

Integrating over q gives

$$D^{\rm sym}_{\alpha\beta}(k)p_\alpha p_\beta(-k^2 J_4(k,p)+J_3(k,k-p)+J_3(p,k+p)) \tag{C86}$$

$$= \left[f_C(k\cdot\boldsymbol{p})^2+f_A\,p_0^2+f_B\,\boldsymbol{p^2}\right]\left(\frac{-|k|}{8|p|\,|k-p|\,(k\cdot p)}+\frac{|k|}{8|p|\,|k+p|\,(k\cdot p)}+\frac{1}{8|k|\,|p|\,|k-p|}+\frac{1}{8|k|\,|p|\,|k+p|}\right). \tag{C87}$$

Next, we consider (arranging the terms by their power of q)

$$\begin{aligned}
&D^{\rm sym}_{\alpha\beta}(k)\frac{p_\alpha q_\beta[\ldots]+q_\alpha p_\beta[\ldots]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\frac{-4f_C(\boldsymbol{k}\cdot\boldsymbol{p})\,(p\cdot q)\,(\boldsymbol{k}\cdot q)-4f_B\,(p\cdot q)\,(\boldsymbol{p}\cdot\boldsymbol{q})-4f_A\,(p\cdot q)\,(p_0q_0)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{[-2f_B(k\cdot p)-4f_B\,k^2](\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&\quad+\frac{[-2f_C(k\cdot p)(\boldsymbol{k}\cdot\boldsymbol{p})-4f_C\,k^2\,(\boldsymbol{k}\cdot\boldsymbol{p})]\,(\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{[-2f_A(k\cdot p)-4f_Ak^2](p_0\,q_0)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}.
\end{aligned} \tag{C88}$$

Integrating over q gives

$$\begin{aligned}
&-4f_C(\boldsymbol{k}\cdot\boldsymbol{p})\,F_1-4f_B\,F_2-4f_A\,F_3+[-2f_B(k\cdot p)-4f_B\,k^2]\,M(p;k)+[-2f_C(k\cdot p)(\boldsymbol{k}\cdot\boldsymbol{p})-4f_C\,k^2\,(\boldsymbol{k}\cdot\boldsymbol{p})]\,M(k;p)\\
&+[-2f_A(k\cdot p)-4f_Ak^2]\left[\frac{1}{2}J_3(k,k+p)-\frac{1}{2}J_3(k-p,k)-\frac{p^2}{2}J_4(k,p)-M(p;k)\right],
\end{aligned} \tag{C89}$$

where F_1, F_2, and F_3 are defined below:

$$\begin{aligned}
F_1&\equiv\int_q\frac{(p\cdot q)\,(\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\int_q\frac{1}{2}\frac{[(q+p)^2-p^2-q^2](\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\frac{1}{2}\int_q\left(\frac{(\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2}-\frac{p^2(\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}-\frac{(\boldsymbol{k}\cdot q)}{(q+k)^2(q+k+p)^2(q+p)^2}\right)\\
&=\frac{1}{2}(L(\boldsymbol{k};k,k+p)-p^2\,M(k;p)-L(\boldsymbol{k};p,p-k)+\boldsymbol{k}^2\,J_3(p,p-k)),
\end{aligned} \tag{C90}$$

$$\begin{aligned}
F_2&\equiv\int_q\frac{(p\cdot q)\,(\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\frac{1}{2}\int_q\frac{[(q+p)^2-p^2-q^2]\,(\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\frac{1}{2}\left(\int_q\frac{(\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2}-\int_q\frac{p^2\,(\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}-\int_q\frac{(\boldsymbol{p}\cdot\boldsymbol{q})}{(q+k)^2(q+k+p)^2(q+p)^2}\right)\\
&=\frac{1}{2}(L(\boldsymbol{p};k,k+p)-p^2\,M(p;k)-L(\boldsymbol{p};k,k-p)+\boldsymbol{p}^2\,J_3(k,k-p)),
\end{aligned} \tag{C91}$$

$$\begin{aligned}
F_3&\equiv\int_q\frac{(p\cdot q)\,(p_0q_0)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}\\
&=\frac{1}{2}\left(\int_q\frac{(p_0q_0)}{q^2(q+k)^2(q+k+p)^2}-\int_q\frac{p^2\,(p_0q_0)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}-\int_q\frac{(p_0q_0)}{(q+k)^2(q+k+p)^2(q+p)^2}\right)\\
&=\frac{1}{2}\left(L(\vec{p}_0;k,k+p)-\int_q\frac{p^2\,(p\cdot q-\boldsymbol{p}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}-L(\vec{p}_0;k,k-p)+p_0^2\,J_3(k,k-p)\right)\\
&=\frac{1}{2}\left(L(\vec{p}_0;k,k+p)-\left[\frac{p^2}{2}J_3(k,k+p)-\frac{p^4}{2}\,J_4(k,p)-\frac{p^2}{2}J_3(k,k-p)-p^2\,M(p;k)\right]\right.\\
&\qquad\left.-L(\vec{p}_0;k,k-p)+p_0^2\,J_3(k,k-p)\right).
\end{aligned} \tag{C92}$$

Note that $M(p;k)$ is not symmetric with respect to the exchange of its arguments.

Next, we consider

$$D^{\rm sym}_{\alpha\beta}(k)\frac{k_\alpha q_\beta[\ldots]+q_\alpha k_\beta[\ldots]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$=\frac{8(k\cdot q)[f_B(\boldsymbol{k}\cdot q)+f_C\,(\boldsymbol{k^2})(\boldsymbol{k}\cdot q)+f_A\,k_0\,q_0]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{(\boldsymbol{k}\cdot q)[4f_B(k\cdot p)+4f_C\,(\boldsymbol{k^2})(k\cdot p)+2f_B\,p^2+2f_C(\boldsymbol{k^2})\,p^2]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$+\frac{[4f_A(k\cdot p)+2f_A\,p^2]\,k_0q_0}{q^2(q+k)^2(q+k+p)^2(q+p)^2}. \tag{C93}$$

Integrating over q gives

$$[8f_B+8f_C\,\boldsymbol{k^2}]\,F_4+8f_A\,[M_2(k;k,p)-F_4]+[4f_B(k\cdot p)+4f_C\,(\boldsymbol{k^2})(k\cdot p)+2f_B\,p^2+2f_C(\boldsymbol{k^2})\,p^2]\,M(k;p)$$
$$+[4f_A(k\cdot p)+2f_A\,p^2]\left[\frac{1}{2}J_3(k+p,p)-\frac{k^2}{2}J_4(k,p)-\frac{1}{2}J_3(k,k-p)-M(k,p)\right], \tag{C94}$$

where, using (C15),

$$F_4\equiv\int_q\frac{(k\cdot q)(\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$=\frac{1}{2}\int_q\frac{[(q+k)^2-k^2-q^2](\boldsymbol{k}\cdot q)}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$=\frac{1}{2}\left(L(\boldsymbol{k},p,k+p)-k^2\,M(k;p)-L(\boldsymbol{k},p,p-k)+\boldsymbol{k}^2\frac{1}{8|p|\,|p-k|\,|k|}\right). \tag{C95}$$

We also note the integral relation,

$$F_5\equiv\int_q\frac{(k\cdot q)\,k_0\,q_0]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}=\int_q\frac{(k\cdot q)\,(k\cdot q-\boldsymbol{k}\cdot\boldsymbol{q})}{q^2(q+k)^2(q+k+p)^2(q+p)^2}=M_2(k;k,p)-F_4, \tag{C96}$$

where the definition in Eq. (C23) for M_2 was used. In fact, $M_2(k;k,p)$ can be simplified further in terms of J_2, J_3, J_4, I.

Finally, we consider

$$D^{\rm sym}_{\alpha\beta}(k)\frac{+p_\alpha k_\beta[\ldots]+k_\alpha p_\beta[\ldots]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}$$
$$=\frac{(k\cdot q)\,[4f_B(\boldsymbol{k}\cdot\boldsymbol{p})+4f_C\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{p})+4f_A\,k_0p_0]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}+\frac{(p\cdot q)\,[-2f_B(\boldsymbol{k}\cdot\boldsymbol{p})-2f_C\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{p})-2f_A\,k_0p_0]}{q^2(q+k)^2(q+k+p)^2(q+p)^2}. \tag{C97}$$

Integrating over q gives

$$[4f_B(\boldsymbol{k}\cdot\boldsymbol{p})+4f_C\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{p})+4f_A\,k_0p_0]\,\times\tfrac{1}{2}[J_3(k+p,p)-k^2J_4(p,k)-J_3(k,k-p)]$$
$$+[-2f_B(\boldsymbol{k}\cdot\boldsymbol{p})-2f_C\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{p})-2f_A\,k_0p_0]\times\tfrac{1}{2}[J_3(k+p,k)-p^2J_4(p,k)-J_3(k,k-p)]. \tag{C98}$$

Putting these results together produces I_a (schematically),

$$I_a\equiv\int\frac{d^3q}{(2\pi)^3}{\rm Tr}[...]_{\mu\nu}\left(\frac{\delta_{\mu\nu}}{2}\right)D^{\rm Sym}_{\alpha\beta}(k). \tag{C99}$$

The result is a lengthy expression that we do not write out here.

7. Combining the symmetric components of $\Pi^{(a)}$ and $\Pi^{(b+c)}$

We add together (C99) and (C75) to find [including multiplying (C75) by a factor of 2] the symmetric component,

$$\Pi^{(a+b+c)}_{\rm (S)}(p)=\frac{1}{2}\delta_{\mu\nu}\Pi^{(a+b+c)}_{\mu\nu}(p)=\frac{(-1)g^4}{N_f}\int\frac{d^3k}{(2\pi)^3}(2\,I_{bc}+I_a)$$
$$=\frac{(-1)g^4}{N_f}\int_{-1}^{1}d\cos\theta\int_0^\Lambda dk\frac{2\pi k^2}{(2\pi)^3}(2\,I_{bc}+I_a). \tag{C100}$$

As before, we take the external momentum p to lie along the k_τ direction so that $k\cdot p=|k||p|\cos\theta$. This choice allows us to set all $\boldsymbol{k}\cdot\boldsymbol{p}=0$ in $2\,I_{bc}+I_a$ expression. We plug in the propagator in (C53), perform the dk radial direction integral with cut off Λ, and perform a $\frac{1}{\Lambda}$ expansion to find

$$\Pi^{(a+b+c)}_{(\mathrm{S})}(p) = -\frac{g^4}{N_f}|p|\int_{-1}^{1} d\cos\theta\left(f_0^{w_x}(\cos\theta) + f_L^{w_x}(\cos\theta)\,\log\left[\frac{\Lambda}{|p|}\right] + f_1^{w_x}(\cos\theta)\,\Lambda\right). \tag{C101}$$

$f_L^{w_x}$, $f_1^{w_x}$ are odd functions of $\cos\theta$ so they both vanish after performing the angular integration. Symmetrizing $f_0^{w_x}(\cos\theta)$ to remove any antisymmetric part, we find

$$\frac{f_0^{w_x}(z) + f_0^{w_x}(-z)}{2} \equiv \frac{(-5C_Y\,z(-1+z^2) + z\sqrt{1-z^2}(9+z^2) + 6(-1+z^2)(C_Y - C_Y\,z^2 + 2\sqrt{1-z^2})\mathrm{arctanh}[z])}{192\pi^2\,z\,[\,A_Y\sqrt{1-z^2} + B_Y(1-z^2)]}, \tag{C102}$$

where $z = \cos\theta$. Thus, we have

$$\sigma_{xx}^{\text{2loop-gauge}} = \frac{i}{\omega}\Pi^{(a+b+c)}_{(\mathrm{S})}(\boldsymbol{p} = 0, |p_0| \to i\omega) = \frac{(-1)g^4}{N_f}\int_{-1}^{1} dz \frac{f_0^{w_x}(z) + f_0^{w_x}(-z)}{2}. \tag{C103}$$

This agrees with Eq. (3.14) in the main text.

As a consistency check, we note that our result agrees with [39,40] when $w_x = 0$,

$$\frac{f_0^{w_x=0}(z) + f_0^{w_x=0}(-z)}{2} = \frac{g_X^2}{\kappa^2 + g_X^4}\frac{1}{192\pi^2\,z}(9z + z^3 + 12(z^2-1)\mathrm{arctanh}[z]) \tag{C104}$$

and so

$$\int_{-1}^{1} d\cos\theta\, f_0^{w_x=0}(\cos\theta)\Big|_{\kappa=0,g_X^2=\frac{1}{16},g=1} \approx 0.00893191 = \frac{1}{16}\times 0.14291062. \tag{C105}$$

APPENDIX D: THREE-LOOP CORRECTIONS TO THE COMPOSITE FERMION CONDUCTIVITY

1. Triangle subdiagram

We begin by evaluating Eq. (3.19),

$$C_{\alpha\mu}(Q,p) = \int \frac{d^3k}{(2\pi)^3}(i^3)\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\tau\gamma_\mu\gamma_\rho]\frac{(k-Q)_\sigma k_\tau (k-p)_\rho}{(k-Q)^2\,k^2\,(k-p)^2}. \tag{D1}$$

First, we expand the trace,

$$\begin{aligned}\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\tau\gamma_\mu\gamma_\rho] = {}& (\delta_{\rho\mu}\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\tau] - \delta_{\rho\tau}\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\mu] + \delta_{\rho\alpha}\mathrm{Tr}[\gamma_\sigma\gamma_\tau\gamma_\mu] - \delta_{\rho\sigma}\mathrm{Tr}[\gamma_\alpha\gamma_\tau\gamma_\mu]) + (\delta_{\tau\mu}\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\rho] \\ & - \delta_{\alpha\mu}\mathrm{Tr}[\gamma_\sigma\gamma_\tau\gamma_\rho] + \delta_{\sigma\mu}\mathrm{Tr}[\gamma_\alpha\gamma_\tau\gamma_\rho]) + (\delta_{\tau\alpha}\mathrm{Tr}[\gamma_\sigma\gamma_\mu\gamma_\rho] - \delta_{\tau\sigma}\mathrm{Tr}[\gamma_\alpha\gamma_\mu\gamma_\rho]) + \delta_{\sigma\alpha}\mathrm{Tr}[\gamma_\tau\gamma_\mu\gamma_\rho],\end{aligned} \tag{D2}$$

where $\mathrm{Tr}[\gamma_a\gamma_b\gamma_c] = 2i\epsilon_{abc}$. Contracting this trace with the momenta gives

$$\mathrm{Tr}[\gamma_\sigma\gamma_\alpha\gamma_\tau\gamma_\mu\gamma_\rho](k-Q)_\sigma k_\tau(k-p)_\rho \tag{D3}$$

$$\begin{aligned}&= -2ik^2\epsilon_{\sigma\alpha\mu}k_\sigma + 2ik^2\epsilon_{\alpha\mu\rho}p_\rho - 4i\epsilon_{\eta\mu\rho}k_\alpha k_\eta p_\rho + 2ik^2\epsilon_{\sigma\alpha\mu}Q_\sigma + 4i\epsilon_{\eta\alpha\sigma}k_\mu k_\eta Q_\sigma \\ &\quad + 2i(p\cdot Q)\epsilon_{\eta\alpha\mu}k_\eta - 2i(k\cdot Q)\epsilon_{\eta\alpha\mu}p_\eta + 2i\epsilon_{\mu\rho\eta}k_\eta p_\rho Q_\alpha + 2i\epsilon_{\eta\rho\alpha}k_\eta p_\rho Q_\mu - 2i(k\cdot p)\epsilon_{\alpha\mu\eta}Q_\eta \\ &\quad + 2i\epsilon_{\tau\mu\eta}k_\tau p_\alpha Q_\eta + 2i\epsilon_{\alpha\tau\eta}k_\tau p_\mu Q_\eta + 2i\epsilon_{\mu\rho\eta}k_\alpha p_\rho Q_\eta + 2i\epsilon_{\alpha\rho\eta}k_\mu p_\rho Q_\eta - 2i\delta_{\alpha\mu}\epsilon_{\tau\rho\eta}k_\tau p_\rho Q_\eta.\end{aligned} \tag{D4}$$

The k dependence of the integrand is of one of the following types: $k^2\,k_\sigma$, k^2, $k_\alpha k_\beta$, k_σ.

To evaluate the integrals over each type, we will use the following integrals:

$$I_\sigma \equiv \int \frac{d^3k}{(2\pi)^3}\frac{k^2\,k_\sigma}{(k-Q)^2\,k^2\,(k-p)^2} = \frac{1}{16}\frac{p_\sigma + Q_\sigma}{|p-Q|}, \tag{D5}$$

$$I \equiv \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{(k-Q)^2\,k^2\,(k-p)^2} = \frac{1}{8}\frac{1}{|p-Q|}, \tag{D6}$$

$$J_\sigma \equiv \int \frac{d^3k}{(2\pi)^3}\frac{k_\sigma}{(k-Q)^2\,k^2\,(k-p)^2} = \frac{(|Q| + |p| - |Q-p|)\,(|Q|\,p_\sigma + |p|\,Q_\sigma)}{16|Q|\,|p|\,|Q-p|\,(Q\cdot p + |Q|\,|p|)}, \tag{D7}$$

$$J_{\alpha\beta} \equiv \int \frac{d^3k}{(2\pi)^3}\frac{k_\alpha k_\beta}{(k-Q)^2\,k^2\,(k-p)^2} \equiv A p_\alpha p_\beta + B Q_\alpha Q_\beta + C p_\alpha Q_\beta + D Q_\alpha p_\beta + E\delta_{\alpha\beta}, \tag{D8}$$

where the coefficients are

$$A = \frac{2p^2|Q| + |Q|^3 - |p-Q|(p\cdot Q + Q^2) + |p|\,(p\cdot Q - 2|p-Q|\,|Q| + 2Q^2)}{32|p|\,|p-Q|(p\cdot Q + |p|\,|Q|)^2}, \tag{D9}$$

$$B = \frac{2Q^2|p| + |p|^3 - |p-Q|(p\cdot Q + p^2) + |Q|\,(p\cdot Q + 2p^2 - 2|p|\,|p-Q|)}{32|Q|\,|p-Q|(p\cdot Q + |p|\,|Q|)^2}, \tag{D10}$$

$$C = D = \frac{-p\cdot Q + p^2 - |p-Q|\,|Q| + Q^2 + |p|\,(|Q| - |p-Q|)}{32|p-Q|\,(p\cdot Q + |p|\,|Q|)^2}, \tag{D11}$$

$$E = \frac{2p\cdot Q - p^2 + |p-Q||p| + |Q| - Q^2}{32|p-Q|\,(p\cdot Q + |p|\,|Q|)}. \tag{D12}$$

Equation (D8) is not easy to derive directly from the usual Feynman parametrization. To derive it, we contract both sides of the equation with $p_\alpha p_\beta$, $Q_\alpha Q_\beta$, $p_\alpha Q_\beta$, $Q_\alpha p_\beta$, $\delta_{\alpha\beta}$. This gives a set of linear equations that we solve for A, B, C, D, E (note that some conditions may not be linearly independent in this procedure).

Plugging this into the expression for $C_{\alpha\mu}(Q, p)$ gives

$$\begin{aligned} C_{\alpha\mu}(Q,p) = (i^3)\times i(&-2\epsilon_{\sigma\alpha\mu} I_\sigma + 2\epsilon_{\alpha\mu\rho} p_\rho I - 4\epsilon_{\eta\mu\rho} p_\rho J_{\alpha\eta} + 2\epsilon_{\sigma\alpha\mu} Q_\sigma I + 4\epsilon_{\eta\alpha\sigma} Q_\sigma J_{\mu\eta} \\ &+ 2(p\cdot Q)\epsilon_{\eta\alpha\mu} J_\eta - 2\epsilon_{\eta\alpha\mu} p_\eta (J_\sigma\cdot Q_\sigma) + 2\epsilon_{\mu\rho\eta} p_\rho Q_\alpha J_\eta + 2\epsilon_{\eta\rho\alpha} p_\rho Q_\mu J_\eta - 2\epsilon_{\alpha\mu\eta} Q_\eta (J_\sigma\cdot p_\sigma) \\ &+ 2\epsilon_{\tau\mu\eta} p_\alpha Q_\eta J_\tau + 2\epsilon_{\alpha\tau\eta} p_\mu Q_\eta J_\tau + 2\epsilon_{\mu\rho\eta} p_\rho Q_\eta J_\alpha + 2\epsilon_{\alpha\rho\eta} p_\rho Q_\eta J_\mu - 2\delta_{\alpha\mu}\epsilon_{\tau\rho\eta} p_\rho Q_\eta J_\tau). \end{aligned} \tag{D13}$$

Note that $I_\sigma, I, J_\sigma, J_{\alpha\beta}$ are functions of p and Q.

2. The complete 3-loop diagrams $\tilde{\Pi}_{ij}$

Equipped with Eq. (D13), we can compute the full 3-loop diagrams in Eq. (3.18). Consider the integral over Q_μ. The delta function $\delta(-Q_0 + p_0)$ arising from the disorder line allows us to set $Q_0 = p_0$. It remains to integrate over Q_x and Q_y. Define $Q_r \equiv \sqrt{Q_x^2 + Q_y^2}$. The gauge propagator is

$$D^{\text{Anti}}_{\alpha\beta}(Q) = \frac{\kappa}{\kappa^2 + g_X^4 + w_x g_X^2 \frac{Q_r}{\sqrt{p_0^2+Q_r^2}}}\left(\frac{\epsilon_{\alpha\beta\lambda}\, Q_\lambda\, \delta_{\lambda j}}{Q_r^2}\right), \tag{D14}$$

$$D_{00}(Q) = \frac{\sqrt{p_0^2 + Q_r^2}}{Q_r^2} \times \frac{1}{\frac{\kappa^2+g_X^4}{g_X^2} + w_x \frac{Q_r}{\sqrt{p_0^2+Q_r^2}}}\left(1 + \frac{w_x}{g_X^2}\frac{Q_r}{\sqrt{p_0^2+Q_r^2}}\right), \tag{D15}$$

and

$$D_{ij}(Q) = \frac{1}{\sqrt{p_0^2 + Q_r^2}} \times \frac{1}{\frac{\kappa^2+g_X^4}{g_X^2} + w_x \frac{Q_r}{\sqrt{p_0^2+Q_r^2}}}\left(\delta_{ij} - \frac{Q_i Q_j}{Q_r^2}\right). \tag{D16}$$

Parameterize $(Q_x, Q_y) = Q_r(\cos\phi, \sin\phi)$ and rewrite Eq. (3.18) as

$$\tilde{\Pi}_{\mu\nu}(p) = 4\times g_m \int_0^\infty \frac{dQ_r\, Q_r}{(2\pi)^2}\int_0^{2\pi} d\phi\, D_{\alpha\beta}(Q) C_{\alpha\mu}(Q,p)\ \times (-1) C_{\beta\nu}(Q,p), \tag{D17}$$

where it is understood that $(Q_0, Q_x, Q_y) = (p_0, Q_r\cos\phi, Q_r\sin\phi)$. To simplify further, we set the external momentum $p = (p_0, p_x, p_y) = (p_0, 0, 0)$, and change to the dimensionless variable z, defined by $z \equiv Q_r/|p_0|$. Integrating over $d\phi$, we obtain Eqs. (3.21) and (3.22).

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