Modular transformation and anyonic statistics of multicomponent fractional quantum Hall states

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We investigate the response to modular transformations and the fractional statistics of Abelian multicomponent fractional quantum Hall (FQH) states. In particular, we analytically derive the modular matrices encoding the statistics of anyonic excitations for general Halperin states using the conformal field theories (CFTs). We validate our theory by several microscopic examples, including the spin-singlet state using the anyon condensation picture and the Halperin (221) state in a topological flat-band lattice model using numerical calculations. Our results, uncovering that the modular matrices and associated fractional statistics are solely determined by the K matrix, further strengthens the correspondence between the two-dimensional (2D) CFTs and $(2+1)$D topological orders for multicomponent FQH states.

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I. INTRODUCTION

Fractionalized quasiparticles [1–4] is a defining feature of the topological orders in fractional quantum Hall (FQH) states. These quasiparticles, dubbed anyons, obey exotic fractional statistics [5,6] which emerges only in two spatial dimensions. How to effectively describe and systematically classify the fractional statistics and the associated topological orders is a key problem in the study of the FQH effect. On the one hand, Abelian-type anyonic statistics has been well captured by the Chern-Simons effective-field theories describing the low-energy properties in the bulk of the FQH states [7–9]. On the other hand, the wave functions of many FQH states, including those with quasiparticle excitations, are shown to be conformal blocks of suitable rational conformal field theories (CFTs) [2,10–13]. As a result, the fractional statistics of FQH quasiparticles is formally encoded in the modular matrices [14–16], which contain the information of mutual and self statistics, quantum dimensions, and the fusion rule of quasiparticles. For a specific FQH state, the modular matrices can be defined by the degenerate ground states in response to modular transformations (see Fig. 1). Indeed, a large class of Abelian [17–19] and non-Abelian [20] FQH states has been successfully classified by their modular matrices.

So far, the identification of FQH states via the modular matrices is mostly limited to single-component systems [17–20]. When particles possess more internal degrees of freedom (e.g., spins, valleys, layers), the scope of FQH physics further expands to the multicomponent case [21,22]. Due to the great variety and tunability of effective interactions, multicomponent FQH states provide a playground for realizing emergent topological orders that have no analog in single-component systems. In this context, how to generalize the modular-matrix approach to multicomponent FQH states is largely unexplored. Previous attempt [23] on the response of the Halperin model wave function to the modular transformations found an unexpected size-dependent phase factor appearing in the modular matrices. The reason of this unphysical phase factor is not well justified. It is essential to distinguish our work from this previous approach. In contrast, our study undertakes a fresh perspective by adopting the tools of conformal field theory (CFT), thereby addressing and resolving this perplexing issue of the unphysical phase factor. Furthermore, the K-matrix formalism plays a crucial role in the bulk description of Abelian multicomponent FQH states [7–9]. It is thus natural to ask whether the CFT-FQH correspondence [12,24] can be supported by relating modular matrices extracted from the CFT to the K matrix for multicomponent states.

In this paper, we address these questions critically. First, we analytically derive the modular matrices of general Abelian multicomponent Halperin states [22] directly from the underlying CFT. Remarkably, the result establishes an explicit relation between the modular matrices and the K matrix in the corresponding Chern-Simons theory. Moreover, compared with model wave-function approach [23], the unphysical size-dependent phase factor does not appear from the perspective of CFT, demonstrating the superiority of our CFT method. Our result thus provides compelling evidence for the correspondence between the two-dimensional (2D) CFTs and $(2+1)$D bulk topological orders for multicomponent FQH states. We support our theory by independently deriving modular matrices of the spin-singlet Halperin states using the anyon condensation picture and using extensive exact diagonalization in a microscopic lattice model.

II. MODULAR MATRICES FROM CFT

Modular matrices capture the quasiparticle statistics of topologically ordered states [7,15,25,26]. To be specific, the

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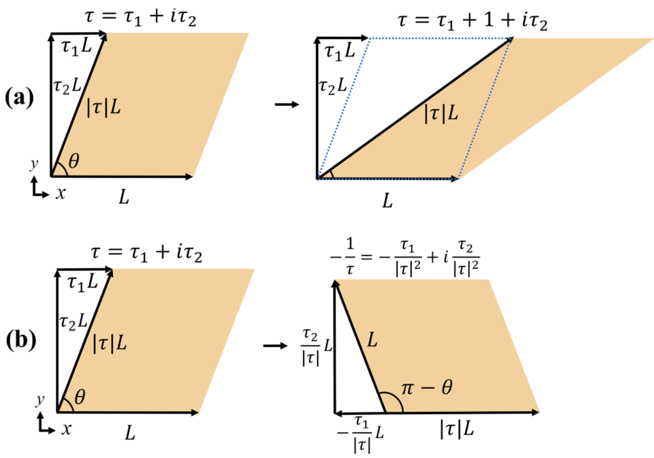

FIG. 1. The torus geometry is defined by two fundamental vectors $\vec{L}_2 = L\vec{\tau}$ and $\vec{L}_1 = L\vec{e}_x$, and the twist angle is θ. (a) The $\mathcal{T}$ transformation sends $\vec{\tau} = \tau_1\vec{e}_x + \tau_2\vec{e}_y$ to its equivalent geometry $\vec{\tau} + \vec{e}_x$, thus leaving the torus geometry unchanged. (b) The modular $\mathcal{S}$ transformation generates a counterclockwise rotation and transforms the torus spanned by $L\vec{e}_x$ and $\vec{\tau}L$ to a torus spanned by $|\tau|L\vec{e}_x$ and $-|\tau|L/\vec{\tau}$.

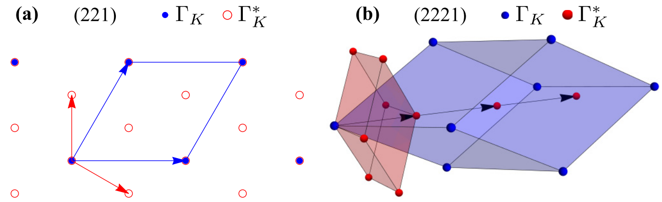

FIG. 2. Schematic plot of the K lattice and its dual lattice. The blue dots and red dots respectively denote the Γ_K lattice spanned by $\{\boldsymbol{e}_J\}$ and the Γ_K^* lattice spanned by $\{\boldsymbol{e}_J^*\}$. The coset Γ_K^*/Γ_K is the parallelogram spanned by $\{\boldsymbol{e}_J\}$ (shaded by light blue). Here we draw two examples of Γ_K and Γ_K^* for (a) the Halperin (221) state with $K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and (b) the Halperin (2221) state.

modular $\mathcal{S}$-matrix contains braiding statistics and the fusion rules of anyons, and the modular $\mathcal{T}$-matrix contains the self statistics of quasiparticles. These two matrices are respectively related to the modular $\mathcal{S}$ and $\mathcal{T}$ transformation on a torus (Fig. 1) [17].

Our goal in this section is to calculate the modular $\mathcal{S}$ and $\mathcal{T}$ matrices for Abelian multicomponent FQH states using CFT, which was rarely studied before to our best knowledge. For simplicity we only consider the bosonic states throughout this paper. Recall that the single-component bosonic Laughlin state at filling $\nu = 1/(2m)$ is described by the compactified boson which manifests the $\widehat{u(1)}_{2m}$ CFT (see Appendix A 1) [2,9,15,27–29]. To incorporate Abelian multicomponent FQH states, one can generalize the $\widehat{u(1)}$ theory to $\widehat{u(1)}_{\kappa,K}$ [29,30], where the level is no longer a single number but a $\kappa \times \kappa$ positive-definite integer matrix K. It has been proposed that many physical quantities (e.g., fractional charge, Hall conductance) of Abelian multicomponent FQH states can be well captured by this K matrix [9,29].

In this context, the partition function of the multicomponent state can be written as summation of the character of various topological sectors $\boldsymbol{a}$:

$$Z(K) = \sum_{\boldsymbol{a}\in\Gamma_K^*/\Gamma_K} |\chi_{\boldsymbol{a}}(\tau)|^2, \tag{1}$$

where the character χ is expressed by q expansion as

$$\chi_{\boldsymbol{a}}(\tau) = \frac{1}{\eta(\tau)^\kappa} \sum_{\boldsymbol{n}\in\Gamma_K} q^{\frac{1}{2}(\boldsymbol{n}+\boldsymbol{a})\cdot(\boldsymbol{n}+\boldsymbol{a})}, \tag{2}$$

with $q = e^{2\pi\tau i}$. Here the vector $\vec{\tau}$ parametrizes the torus (Fig. 1) and η is Dedekind's function. We formulate different topological sectors by the so-called K lattice Γ_K and its dual lattice Γ_K^*, as shown in Fig. 2. Γ_K denotes a set of vectors $\{\boldsymbol{n} = \sum_{I=1}^{\kappa} n_I\boldsymbol{e}_I | n_I \in \mathbb{Z}\}$, with the basis satisfying $\boldsymbol{e}_I \cdot \boldsymbol{e}_J = K_{IJ}$ (K_{IJ} the element of K matrix). The corresponding dual lattice Γ_K^* is spanned by the dual basis $\boldsymbol{e}_I^*$ satisfying the relation $\boldsymbol{e}_I \cdot \boldsymbol{e}_J^* = \delta_{IJ}$. The linear combination of dual basis $\sum_J K_{IJ}\boldsymbol{e}_J^*$ is exactly the basis of Γ_K: $\boldsymbol{e}_I \cdot \boldsymbol{e}_J = \sum_N K_{IN}\boldsymbol{e}_N^* \cdot \boldsymbol{e}_J = K_{IJ}$. With the help of this lattice representation, it is straightforward to express the inequivalent topological sectors as the coset of $\boldsymbol{a} \in \Gamma_K^*/\Gamma_K$ [10].

The modular $\mathcal{S}$ and $\mathcal{T}$ matrices are determined by the changes of the character under respective modular transformation. For the modular $\mathcal{S}$ transformation $\tau \to -1/\tau$ as shown in Fig. 1(b), the character becomes (see Appendix A 2 for details)

$$\begin{aligned}\chi_{\boldsymbol{a}}\left(-\frac{1}{\tau}\right) &= \frac{1}{(-i\tau)^{\kappa/2}\eta(\tau)^\kappa} \sum_{\boldsymbol{n}\in\Gamma_K} e^{-\frac{\pi i}{\tau}(\boldsymbol{n}\cdot\boldsymbol{n}+2\boldsymbol{n}\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{a})} \\ &= \frac{1}{\sqrt{|\det K|}} \frac{1}{\eta(\tau)^\kappa} \sum_{\boldsymbol{m}\in\Gamma_K^*} e^{\pi i\tau\boldsymbol{m}\cdot\boldsymbol{m}-2\pi i\boldsymbol{m}\cdot\boldsymbol{a}},\end{aligned} \tag{3}$$

where we have used the generalized Poisson's resummation formula

$$\sum_{\boldsymbol{q}\in\Gamma_K} e^{-\pi a q^2+\boldsymbol{q}\cdot\boldsymbol{b}} = \frac{1}{\sqrt{|\det K|}a^{\frac{\kappa}{2}}} \sum_{\boldsymbol{p}\in\Gamma_K^*} e^{-\frac{\pi}{a}(\boldsymbol{p}+\frac{\boldsymbol{b}}{2\pi i})^2}.$$

As any vector in Γ_K^* can be expressed as $\boldsymbol{m} = \sum_I m_I\boldsymbol{e}_I^* = \sum_I(\sum_J n_J K_{JI} + b_I)\boldsymbol{e}_I^* = \boldsymbol{n} + \boldsymbol{b}$, with $\boldsymbol{n} = \sum_I n_I\boldsymbol{e}_I \in \Gamma_K$ and $\boldsymbol{b} = \sum_I b_I\boldsymbol{e}_I^* \in \Gamma_K^*/\Gamma_K$, we can rewrite the summation over Γ_K^* in Eq. (3) and obtain

$$\chi_{\boldsymbol{a}}\left(-\frac{1}{\tau}\right) = \sum_{\boldsymbol{b}\in\Gamma_K^*/\Gamma_K} \frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}\chi_{\boldsymbol{b}}(\tau) \equiv \sum_{\boldsymbol{b}\in\Gamma_K^*/\Gamma_K} \mathcal{S}_{ab}\chi_{\boldsymbol{b}}(\tau),$$

which gives the modular $\mathcal{S}$ matrix as

$$\mathcal{S} = \frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}. \tag{4}$$

Similarly, the character changes as

$$\chi_{\boldsymbol{a}}(\tau+1) = e^{2\pi i\left(\frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{a}-\frac{\kappa}{24}\right)}\chi_{\boldsymbol{a}}(\tau) \tag{5}$$

under the modular $\mathcal{T}$ transformation $\tau \to \tau + 1$ [Fig. 1(a)], leading to modular $\mathcal{T}$ matrix

$$\mathcal{T} = e^{2\pi i\left(\frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{a}-\frac{\kappa}{24}\right)}. \tag{6}$$

Equations (4) and (6) are the main results of this work. They give the exact forms of modular matrices of general multicomponent bosonic Halperin states, which are universal and capture the global statistical features of anyons in the $(2+1)$D topological orders. In particular, the form of the $\mathcal{S}$ matrix demonstrates a clear relation with the K matrix. First, the prefactor is proportional to $\sqrt{|\det K|}$, reflecting the Abelian nature of the state. Second, the braiding phases of anyons are uniquely determined by $e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}$, where vectors $\boldsymbol{a}$, $\boldsymbol{b}$ belong to coset lattice defined by the K matrix. Since the K matrix plays a crucial role in the Chern-Simons theory of Halperin states, our result strengthens the correspondence between CFT and Chern-Simons descriptions of multicomponent FQH states.

Concrete examples

Based on the results in Eqs. (4) and (6), we now present the modular matrices for some typical Halperin states. First, let us consider the two-component spin-singlet Halperin $(m, m, m-1)$ state at filling $\nu = \frac{2}{2m-1}$. In this case, the K matrix is

$$\begin{pmatrix} m & m-1 \\ m-1 & m \end{pmatrix}$$

and the coset Γ_K^*/Γ_K contains $|\det K| = 2m-1$ independent vectors $\boldsymbol{a} = \{\frac{1}{2m-1}(a\boldsymbol{e}_1 + a\boldsymbol{e}_2)|a = 0, 1, \dots, 2m-2\}$. These $\boldsymbol{a}$ vectors form a one-dimensional lattice [see Fig. 2(a) for $m=2$]. Accordingly, we have

$$\mathcal{S}_{ab} = \frac{1}{\sqrt{2m-1}} \exp\left(-2\pi i \frac{2ab}{2m-1}\right), \tag{7}$$

and

$$\mathcal{T}_{ab} = \delta_{ab} e^{\frac{2\pi i}{12}} e^{2\pi i \frac{a^2}{2m-1}}, \tag{8}$$

where $a, b = 0, 1, \dots, 2m-2$ take integer numbers. Our theory also applies to two-component states that are not spin singlets. In Appendix D, we give the modular matrices for the Halperin (441) state, in which case the dimension of the modular matrices is 15 and the $\boldsymbol{a}$ vectors in the coset Γ_K^*/Γ_K form a two-dimensional lattice.

Moreover, our theory is not limited to the two-component case. For instance, for the three-component $(m, m, m, m-1)$ state with

$$K = \begin{pmatrix} m & m-1 & m-1 \\ m-1 & m & m-1 \\ m-1 & m-1 & m \end{pmatrix},$$

the coset lattice is $\Gamma_K^*/\Gamma_K = \{\frac{1}{3m-2}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)|a = 0, 1, \cdots 3m-3\}$ [Fig. 2(b) depicts the case of $m=2$], and the corresponding modular matrices are

$$\mathcal{S}_{ab} = \frac{1}{3m-2} e^{-6\pi i \frac{ab}{3m-2}}, \quad \mathcal{T}_{aa} = e^{2\pi i (\frac{3a^2}{6m-4} - \frac{3}{24})}. \tag{9}$$

1. Anyon condensation for spin-singlet states

The central results in Eqs. (4) and (6) can be further examined by several parallel methods. As an example, for the two-component spin-singlet Halperin $(m, m, m-1)$ state, we do not need to resort to the K matrix when deriving the modular matrices. Instead we can rely on the CFT of the $\widehat{su(2)}_1$ Wess-Zumino-Witten (WZW) model with a $\widehat{u(1)}_{4m-2}$ boson which describes the $(m, m, m-1)$ state [2,31,32]. There are $8m-4$ primary fields in the CFT of the WZW model, whose topological spin and fusion rule are [10]

$$h_{(\lambda,a)} = \frac{\lambda(\lambda+2)}{12} + \frac{a^2}{8m-4}, \tag{10}$$

$$(\lambda, a) \otimes (\mu, b) = ((\lambda+\mu)_{\text{mod } 2}, (a+b)_{\text{mod } 4m-2}), \tag{11}$$

with $\lambda, \mu = 0, 1$ and $a, b = 0, 1, \dots, 4m-3$. Interestingly, there exists a special primary field [denoted $J = (1, 2m-1)$] with integer topological spin $h_{(1,2m-1)} = \frac{1}{4} + \frac{(2m-1)^2}{4(2m-1)} = \frac{m}{2} \in \mathbb{Z}$. In addition to the vacuum $\mathbf{1}$, the existence of such a bosonic field with integer spin points to an extended chiral algebra [10], i.e., akin to that one cannot distinguish (λ, a) and $(\lambda, a) \otimes \mathbf{1}^l$ $(l \in \mathbb{N})$, one cannot distinguish (λ, a) with $(\lambda, a) \otimes J^l$ $(l \in \mathbb{N})$. After condensation of the bosonic fields, the $8m-4$ primary fields are mapped into $4m-2$ distinguishable primary fields $\{[(0, a)]_J|a = 0, 1, \dots, 4m-3\}$, where $[\cdots]_J$ denotes a equivalent class under the fusion with J field $[(0, a)]_J : (0, a) \sim (0, a) \otimes J$. Moreover, we notice that the difference between the topological spin of a and $a \otimes J$ (in the sense of modulo 1) is

$$h_{a\otimes J} - h_a = \frac{1}{4} + \frac{(a+2m-1)}{8m-4} - \frac{a^2}{8m-4} = \frac{a}{2}.$$

Thus only half anyons with even a are deconfined [33], which form a set $\{a \equiv [(0, 2a)]_J|a = 0, 1, \dots, 2m-2\}$ with the topological spin $h_a = \frac{a^2}{2m-1}$ and fusion rule $\mathcal{N}_{ab}^{\ c}$: $a \otimes b = (a+b)_{\text{mod } 2m-1}$. The braiding statistics of these deconfined anyons can be described by the $\mathcal{S}$ matrix

$$\mathcal{S}_{ab} = \frac{1}{\sqrt{2m-1}} \exp\left(-2\pi i \frac{2ab}{2m-1}\right), \tag{12}$$

where we have used the formula $\mathcal{S}_{ab} = \frac{1}{\mathcal{D}} \sum_c \mathcal{N}_{a\bar{b}}^{\ c} \exp[2\pi i(h_c - h_a - h_b)] \mathrm{d}_c$ [33]. Here $\mathrm{d}_c = 1$ is the quantum dimension of Abelian anyon c, and $\mathcal{D} = \sqrt{2m-1}$ is the total quantum dimension. Thus, we conclude that, using the anyon condensation picture we can obtain exactly the same modular matrices as in Eq. (7), without resorting to the explicit K matrix. This result also matches that from the extended chiral algebra of spin-singlet Halperin states [32] (see Appendix C for more technical details).

2. Numerical simulation for the halperin (221) state

So far our discussion are solely based on analytical derivations. Now we turn to numerical simulations in microscopic models to further validate our analytical results. A well-developed numerical recipe for the extraction of modular matrices is the "minimal-entanglement state" (MES) scheme [17–20], which requires bipartition of the whole system in real space. To conveniently realize such bipartition, we consider two-component hardcore bosons in a checkerboard lattice model [34–38]. The system's Hamiltonian takes the form of

$$H = \sum_\sigma \Bigg[-t \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} e^{i\phi_{\mathbf{r}'\mathbf{r}}} b_{\mathbf{r}',\sigma}^\dagger b_{\mathbf{r},\sigma} - \sum_{\langle\langle \mathbf{r},\mathbf{r}'\rangle\rangle} t'_{\mathbf{r},\mathbf{r}'} b_{\mathbf{r}',\sigma}^\dagger b_{\mathbf{r},\sigma} - t'' \sum_{\langle\langle\langle \mathbf{r},\mathbf{r}'\rangle\rangle\rangle} b_{\mathbf{r}',\sigma}^\dagger b_{\mathbf{r},\sigma} + \text{H.c.} \Bigg] + U \sum_{\mathbf{r}} n_{\mathbf{r},\uparrow} n_{\mathbf{r},\downarrow}. \tag{13}$$

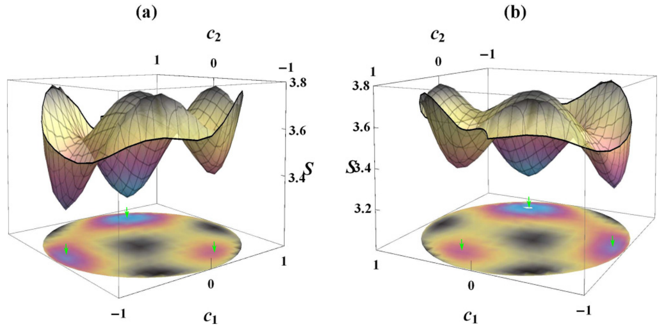

FIG. 3. Surface and contour plots of the entanglement entropy S of $|\Psi(c_1, c_2, \phi_2, \phi_3)\rangle$ versus c_1 and c_2 [$c_3 = (1 - c_1^2 - c_2^2)^{1/2}$]. Here we fix ϕ_2 and ϕ_3 at their optimal values $\phi_2 = 0.5\pi$, $\phi_3 = 0.25\pi$. The entanglement bipartition is along (a) the x direction and (b) the y direction. The calculation is performed on a 3×3 checkerboard lattice at $\nu = 2/3$.

Here $b^\dagger_{\mathbf{r},\sigma}$ creates a boson with pseudospin $\sigma = \uparrow, \downarrow$ at site $\mathbf{r}$, and $\langle \ldots \rangle$, $\langle\langle \ldots \rangle\rangle$, and $\langle\langle\langle \ldots \rangle\rangle\rangle$ denote the nearest-neighbor, the next-nearest-neighbor, and the next-next-nearest-neighbor pairs of sites, respectively. In what follows, we choose the parameters as $t' = 0.3t$, $t'' = -0.2t$, $\phi = \pi/4$ [34], and $U = 4.0$. A robust Halperin (221) state with threefold nearly degenerate ground states ($|\xi_{i=1,2,3}\rangle$) can be stabilized in this system at the total filling $\nu = 2/3$ [39].

According to the MES scheme, we bipartite the lattice along x and y directions, respectively, and search for the corresponding MESs over all superpositions $|\Psi(c_1, c_2, \phi_2, \phi_3)\rangle = c_1|\xi_1\rangle + c_2 e^{i\phi_2}|\xi_2\rangle + c_3 e^{i\phi_3}|\xi_3\rangle$ within the ground-state manifold, where c_1, c_2, c_3, ϕ_2, ϕ_3 are real parameters. For each bipartition, by minimizing the entanglement entropy over the ranges of $c_i \in [0, 1]$ and $\phi_i \in [0, 2\pi]$, we can identify three entropy valleys in the parameter space, each of which gives a global MES $|\Xi^{x,y}_{i=1,2,3}\rangle$ (Fig. 3). Then the overlap between the MESs along the two noncontractible bipartition directions gives the modular matrix $\mathcal{S} = \langle \Xi^y | \Xi^x \rangle$ [17,18]. Our numerical calculation returns

$$\mathcal{S} = \begin{pmatrix} 0.587 & 0.572 & 0.572 \\ 0.572 & 0.581 e^{-i0.67\pi} & 0.581 e^{i0.67\pi} \\ 0.572 & 0.581 e^{i0.67\pi} & 0.581 e^{-i0.67\pi} \end{pmatrix}. \tag{14}$$

This result is quite close to the theoretical prediction

$$\mathcal{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-i2\pi/3} & e^{i2\pi/3} \\ 1 & e^{i2\pi/3} & e^{-i2\pi/3} \end{pmatrix}$$

for the Halperin (221) state, indicating the unit quasiparticle quantum dimension and the Z_3 mutual statistics between quasiparticles. Additionally, the statistics phases are in units of $2\pi/3$, which clearly signals the fractional charge $e/3$ of quasiparticles [40].

III. SUMMARY AND DISCUSSION

We have investigated the modular transformation and fractional statistics for Abelian multicomponent fractional quantum Hall states. Crucially, we derive a universal formula for the modular matrices by utilizing the conformal field theory. We establish that the modular matrices of multicomponent Halperin states can be fully formulated in terms of the K matrix within the CFT framework, without resorting to the Chern-Simons field theory or other empirical knowledge, further strengthening the CFT-FQH correspondence. Moreover, we illustrate several examples to validate our theory. One is the spin-singlet Halperin $(m, m, m-1)$ state obtained by the anyon condensation picture, and the other example is the Halperin (221) state living on a microscopic lattice model, where the modular matrix are numerically computed via the minimal entangled states. In both examples, the extracted modular matrices coincide with our theory.

Several remarks are given in order. Compared with some examples on calculating modular matrices for special Halperin states [41], our results exhaust all Abelian multicomponent FQH states. Apart from the field-theory methods, the modular matrices can also be directly calculated from the trial wave functions of FQH states. By explicitly applying the modular $\mathcal{S}$ and $\mathcal{T}$ transformations on the trial wave function, we recover the CFT results of the modular matrices in Eqs. (4) and (6) up to a particle-number-dependent phase factor (see Appendix B for details). In this regard, modular matrices derived by the CFT can be understood via the trial wave function from a different perspective. The discrepancy between the CFT-base calculation and the wave-function-based calculation may come from the choice of gauge fixing scheme in the calculation of the Berry phase, which calls for a more careful treatment of the gauge field for the future Halperin wave function in the adiabatic evolution (see Appendix B for details).

Our work opens several directions for the future study. First, hopefully our results can be extended to other FQH states possessing K matrices, such as those described by the affine Lie algebra conformal field theories [42], the hierarchy FQH states [43] and the chiral spin liquids [31,44,45]. It would be interesting to probe the relation between modular matrices and the K matrix in those cases also. Second, investigation of modular matrices for lattice FQH states in Bloch bands with Chern number $C > 1$ may be helpful to unveil the potential difference of these states from the usual C-component Halperin states [46–50]. Finally, one can study the modular matrices for FQH states beyond the K-matrix description, such as non-Abelian multicomponent FQH states [51,52].

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APPENDIX A: CONFORMAL FIELD THEORY METHOD

In the Appendix, we show more details to support the discussion in the main text. In Appendix A, we show the details of the derivation of modular matrices by the conformal field theory (CFT) that describes the edge. In Appendix B, we calculate the modular matrices based on the trial wave function of the general Halperin state. In Appendix C, we compute the modular matrices of the Halperin $(m, m, m-1)$ state, through an extended chiral algebra. In Appendix D, we present more examples that can be described by our theory.

1. Review of conformal field theory for the Laughlin state

In this section, we first review the CFT which describes the edge of the single-component $\nu = 1/(2q)$ Laughlin state. The simplest example in CFT is the free boson model $S = \frac{m}{4\pi}\int dx^2 \partial^\mu \phi(x)\partial_\mu \phi(x)$. We put this model on a torus (space-time manifold) and impose the following boundary condition:

$$\phi(z + k\omega_1 + k'\omega_2) = \phi(z) + 2\pi R(km + k'm'),\\ \times\ k, k', m, m' \in \mathbb{Z},$$

where z is the coordinate in the torus, ω_1 and ω_2 are the periods of the torus (i.e., $z \sim z + \omega_1 \sim z + \omega_2$). The doublet (m, m') counts the winding number of ϕ when it goes around the torus in the periodic directions. Under this condition, ϕ is call the compactified boson on a circle with radius R. When R^2 is a rational number $R^2 = 2q/p$ with p, q two coprime positive numbers, this theory is called the rational CFT (RCFT). RCFT means that the Virasoro primary fields can be reorganized into finite number of extended blocks, and they are linearly covariant under the action of the modular group [10]. Here we set $p = 1$, the partition function is

$$Z(R = \sqrt{2q}) = \frac{1}{|\eta(\tau)|^2}\sum_{e,m\in\mathbb{Z}} q^{\frac{(e+qm)^2}{4q}}\bar{q}^{\frac{(e-qm)^2}{4q}}, \tag{A1}$$

with topological spins of primary fields

$$h_{e,m} = \frac{(e+qm)^2}{4q}, \quad \bar{h}_{e,m} = \frac{(e-qm)^2}{4q}.$$

In Eq. (A1), $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the τ-dependent Dedekind's η-function with $q = e^{2\pi\tau i}$, and $\tau = \omega_2/\omega_1$ is the parameter of the torus (see Fig. 1 in the main text). Those fields can be reorganized into a finite number of extended blocks:

$$\begin{aligned} Z(\sqrt{2q}) &= \frac{1}{|\eta(\tau)|^2}\sum_{e,m\in\mathbb{Z}} q^{\frac{(e+qm)^2}{4q}}\bar{q}^{\frac{(e-qm)^2}{4q}} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{n^2}{4q}}\bar{q}^{\frac{(n-2qm)^2}{4q}} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{a=0}^{2q-1}\sum_{k,m\in\mathbb{Z}} q^{\frac{(a+2kq)^2}{4q}}\bar{q}^{\frac{(a+2(k-m)q)^2}{4q}} \\ &= \frac{1}{|\eta(\tau)|^2}\sum_{a=0}^{2q-1}\sum_{k\in\mathbb{Z}} q^{\frac{(a+2qk)^2}{4q}}\sum_{\bar{k}\in\mathbb{Z}}\bar{q}^{\frac{(a+2q\bar{k})^2}{4q}} \\ &= \sum_{a=0}^{2q-1}\left|K_a^{(2q)}(\tau)\right|^2, \end{aligned}$$

where the extended character of block a is

$$K_a^{(N)}(\tau) = \frac{1}{\eta(\tau)}\sum_{n\in\mathbb{Z}} q^{\frac{(Nn+a)^2}{2N}}. \tag{A2}$$

We are interested in the transformation of $K_a^{(2q)}(\tau)$ under modular transformations. The modular transformation $\tau \to \tau + 1$ is easily seen to be

$$K_a^{(2q)}(\tau+1) = e^{2\pi i\left(\frac{a^2}{4q}-\frac{1}{24}\right)}K_a^{(2q)}(\tau), \tag{A3}$$

where $\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau)$ and $e^{2\pi i\frac{(2qn+a)^2}{4q}} = e^{2\pi i\frac{a^2}{4q}}$. Another transformation $\tau \to -\frac{1}{\tau}$ is more involved:

$$\begin{aligned} K_a^{(2q)}(-1/\tau) &= \frac{1}{\sqrt{-i\tau}\,\eta(\tau)}\sum_n \exp\left[-\frac{2\pi i}{\tau}\frac{(2qn+a)^2}{4q}\right] \\ &= \frac{1}{\sqrt{-i\tau}\,\eta(\tau)}\sum_n \\ &\quad\times \exp\left[-\frac{\pi i}{\tau}\left(2qn^2 + 2na + \frac{a^2}{2q}\right)\right] \\ &= \frac{1}{\sqrt{2q}}\sum_k \exp\left[\pi i\tau\frac{(k-a/\tau)^2}{2q} - \frac{\pi i}{\tau}\frac{a^2}{2q}\right] \\ &= \sum_k \frac{e^{-2\pi i\frac{ak}{2q}}}{\sqrt{2q}}\exp\left(2\pi i\tau\frac{k^2}{4q}\right) \\ &= \sum_{b=0}^{2q-1}\frac{e^{-2\pi i\frac{ab}{2q}}}{\sqrt{2q}}K_b^{(2q)}(\tau). \end{aligned} \tag{A4}$$

Here we have used the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}}\exp(-\pi an^2 + bn) = \frac{1}{\sqrt{a}}\sum_{k\in\mathbb{Z}}\exp\left[-\frac{\pi}{a}\left(k + \frac{b}{2\pi i}\right)^2\right]$$

in the third line. Therefore, the extended characters are linearly covariant under the modular transformations, and these results give the modular matrices

$$\mathcal{T}_{ab} = e^{2\pi i(\frac{a^2}{4q}-\frac{1}{24})}\delta_{ab}, \quad \mathcal{S}_{ab} = \frac{e^{-2\pi i\frac{ab}{2q}}}{\sqrt{2q}}. \tag{A5}$$

The unitary of $\mathcal{S}$ does ensure the modular invariance of the partition function $Z(\sqrt{2q}) = \sum_{a=0}^{2q-1}|K_a^{(2q)}(\tau)|^2$. In additional, the quantum dimensions of the fields are $\mathrm{d}_a = \frac{\mathcal{S}_{a0}}{\mathcal{S}_{00}} = 1$. Using the Verlinde formula, we can derive the fusion rules of the extended fields

$$\mathcal{N}_{ab}{}^{c} = \sum_m \frac{\mathcal{S}_{am}\mathcal{S}_{bm}\bar{\mathcal{S}}_{mc}}{\mathcal{S}_{0m}} = \delta_{a+b,c}^{\mathrm{mod}\ 2q}, \tag{A6}$$

where $\delta_{a,b}^{\mathrm{mod}\ 2q}$ means $a = b \mod 2q$. This theory is called the $\widehat{u(1)}_{2q}$ CFT [2,9,15,27–29], where the level is square of the radius R. The $\widehat{u(1)}_{2q}$ CFT is the edge theory of bosonic Laughlin $\nu = 1/(2q)$ state and is also the physical realization of the $\mathbb{Z}_{2q}^{(\frac{1}{2})}$ anyon model [53].

2. Multicomponent Abelian fractional quantum Hall states

In this section, we extend the discussion of the CFT to general multicomponent Abelian Halperin states. One way to deal with the Halperin state is to generalize the $\widehat{u(1)}$ theory to $\widehat{u(1)}^{\oplus_\kappa}$ [29,30], where κ is the number of components. The level of $\widehat{u(1)}^{\oplus_\kappa}$ is no longer a single number. Instead the level becomes a $\kappa \times \kappa$ positive-definite integer matrix K, which we denote as $\widehat{u(1)}_{\kappa,K}$. The simplest case is a diagonal K where K_{ii} are positive integers, thus this theory is just a simple direct sum of each $\widehat{u(1)}_{K_{ii}}$. Here we consider the bosonic states, thus the diagonal of K is even. (Throughout this work we focus on bosonic states, and the extension to fermionic cases should be straightforward.)

In this context, the partition function of the multicomponent state is written as

$$\begin{aligned} Z(K) &= \sum_{\boldsymbol{a}\in\Gamma_K^*/\Gamma_K} |\chi_{\boldsymbol{a}}(\tau)|^2, \\ \chi_{\boldsymbol{a}}(\tau) &= \frac{1}{\eta(\tau)^\kappa}\sum_{\boldsymbol{n}\in\Gamma_K} q^{\frac{1}{2}(\boldsymbol{n}+\boldsymbol{a})\cdot(\boldsymbol{n}+\boldsymbol{a})}. \end{aligned} \tag{A7}$$

Here we use an intuitive way to define the topological sectors. That is, we define the so-called a K lattice $\Gamma_K = \{\boldsymbol{n} = \sum_I n_I \boldsymbol{e}_I | n_I \in \mathbb{Z}, \boldsymbol{e}_I\cdot\boldsymbol{e}_J = K_{IJ}\}$. And its dual lattice Γ_K^* is spanned by the dual basis $\boldsymbol{e}_I^*$ satisfying the relation $\boldsymbol{e}_I\cdot\boldsymbol{e}_J^* = \delta_{IJ}$. Γ_K^*/Γ_K is the coset (see Fig. 1 in the main text). The transformation $\tau\to\tau+1$ can be easily evaluated with the assumption that K_{ii} is even:

$$\chi_{\boldsymbol{a}}(\tau+1) = e^{2\pi i(\frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{a}-\frac{\kappa}{24})}\chi_{\boldsymbol{a}}(\tau). \tag{A8}$$

So we reach

$$\mathcal{T} = e^{2\pi i(\frac{1}{2}\boldsymbol{a}\cdot\boldsymbol{a}-\frac{\kappa}{24})}. \tag{A9}$$

To consider the $\mathcal{S}$ transformation $\tau\to-\frac{1}{\tau}$, we utilize the generalized Poisson's resummation formula

$$\sum_{\boldsymbol{q}\in\Gamma_K} e^{-\pi a\boldsymbol{q}^2+\boldsymbol{q}\cdot\boldsymbol{b}} = \frac{1}{\sqrt{|\det K|}a^{\frac{\kappa}{2}}}\sum_{\boldsymbol{p}\in\Gamma_K^*} e^{-\frac{\pi}{a}(\boldsymbol{p}+\frac{\boldsymbol{b}}{2\pi i})^2},$$

and apply it to the character. Then we have

$$\begin{aligned} \chi_{\boldsymbol{a}}(-1/\tau) &= \frac{1}{(-i\tau)^{\kappa/2}\eta(\tau)^\kappa}\sum_{\boldsymbol{n}\in\Gamma_K} e^{-\frac{\pi i}{\tau}(\boldsymbol{n}\cdot\boldsymbol{n}+2\boldsymbol{n}\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{a})} \\ &= \frac{1}{\sqrt{|\det K|}}\frac{1}{\eta(\tau)^\kappa}\sum_{\boldsymbol{m}\in\Gamma_K^*} e^{\pi i\tau\boldsymbol{m}\cdot\boldsymbol{m}-2\pi i\boldsymbol{m}\cdot\boldsymbol{a}}. \end{aligned} \tag{A10}$$

The finial step is to rewrite the summation over Γ_K^*, by noticing the vector in Γ_K^* can be written as

$$\boldsymbol{m} = \sum_I m_I\boldsymbol{e}_I^* = \sum_I\left(\sum_J n_J K_{JI} + b_I\right)\boldsymbol{e}_I^* = \boldsymbol{n}+\boldsymbol{b},$$

with $\boldsymbol{n} = \sum_I n_I\boldsymbol{e}_I \in \Gamma_K$, $\boldsymbol{b} = \sum_I b_I\boldsymbol{e}_I^* \in \Gamma_K^*/\Gamma_K$, and the linear combination of dual basis $\sum_J K_{IJ}\boldsymbol{e}_J^*$ is exactly the basis of Γ_K: $\boldsymbol{e}_I\cdot\boldsymbol{e}_J = \sum_N K_{IN}\boldsymbol{e}_N^*\cdot\boldsymbol{e}_J = K_{IJ}$. Thus, we can rewrite Eq. (A10) as

$$\begin{aligned} \chi_{\boldsymbol{a}}(-1/\tau) &= \frac{1}{\sqrt{|\det K|}}\frac{1}{\eta(\tau)^\kappa}\sum_{\boldsymbol{m}\in\Gamma_K^*} e^{\pi i\tau\boldsymbol{m}\cdot\boldsymbol{m}-2\pi i\boldsymbol{m}\cdot\boldsymbol{a}} \\ &= \sum_{\boldsymbol{b}\in\Gamma_K^*/\Gamma_K}\frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}\chi_{\boldsymbol{b}}(\tau). \end{aligned} \tag{A11}$$

The prefactor is just the modular $\mathcal{S}$ matrix

$$\mathcal{S} = \frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}. \tag{A12}$$

This is the result shown in the main text.

At last, as an example, we consider the $(m, m, m-1)$ state, for which the K matrix is

$$\begin{pmatrix} m & m-1 \\ m-1 & m \end{pmatrix}$$

and the coset Γ_K^*/Γ_K contains $|\det K| = 2m-1$ independent vectors $\boldsymbol{a} = \{\frac{1}{2m-1}(a\boldsymbol{e}_1+a\boldsymbol{e}_2)|a=0,1,\ldots,2m-2\}$ ($\boldsymbol{b}$ is defined similarly). We can write the $\mathcal{T}$ matrix as

$$\mathcal{T}_{ab} = \delta_{ab}e^{\frac{2\pi i}{12}}e^{2\pi i\frac{a^2}{(2m-1)^2}} \tag{A13}$$

and the $\mathcal{S}$ matrix as

$$\mathcal{S}_{ab} = \frac{1}{\sqrt{2m-1}}\exp\left(-2\pi i\frac{2ab}{2m-1}\right). \tag{A14}$$

For more examples please see Appendix D.

APPENDIX B: TRIAL WAVE FUNCTION METHOD

In this section, we introduce another method for deriving the modular matrices. This method is based on the K-matrix related trial wave function. This is method is independent of the CFT, which can be viewed as a complementary method to the CFT. Part of results have overlaps with an unpublished work [23].

1. Gauge transformation

Let us consider the Hamiltonian of a charged electron on the torus spanned by $\vec{L}_1 = L\vec{e}_x$ and $\vec{L}_2 = L\vec{\tau}$ with a uniform perpendicular magnetic field [54]

$$H_0(\mathbf{A},\tau) = \tfrac{1}{2}g^{ab}(\tau)D_a(\mathbf{A})D_b(\mathbf{A}), \tag{B1}$$

where $D_a(\mathbf{A}) = -i\hbar\partial/\partial X^a + |e|A_a$ and $\mathbf{A} = (-\tau_2L^2BX^2, 0)$ are the covariant derivative and vector potential, respectively. $g(\tau)$ is the τ-dependent metric,

$$g(\tau) = \frac{1}{S\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}, \tag{B2}$$

where $S = |\vec{L}_1\times\vec{L}_2| = \tau_2L^2$ is the area of the torus which is invariant under any modular transformation. The ground state is the lowest Landau level (LLL)

$$\Psi_m(X^1, X^2|\tau) = \frac{1}{\sqrt{\pi^{1/2}L\ell}}e^{i\pi N_\phi\tau[X^2]^2}\theta_{\frac{m}{N_\phi}}(N_\phi z/L|N_\phi\tau), \tag{B3}$$

with $m = 0, 1, \ldots, N_\phi - 1$, where $N_\phi = \tau_2L^2/2\pi\ell^2$ is the total flux through the torus, $\ell = \sqrt{\hbar/|e|B}$ is the magnetic length,

$z = x + iy = L(X^1 + \tau X^2)$ is the complex coordinate of the electron, and $\theta_m(z|\tau)$ is the theta function, which is defined as

$$\theta_\alpha(z|\tau) = \sum_{n\in Z} \exp\left[i\pi\tau(n+\alpha)^2 + i2\pi(n+\alpha)z\right]. \quad \text{(B4)}$$

The modular S transformation makes a $\pi/2$ rotation in (X^1, X^2) space

$$\begin{pmatrix} X'^1 \\ X'^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \quad \text{(B5)}$$

and transforms τ to $-1/\tau$. Since the exchange of the two sides of the parallelogram (torus), the torus is spanned by $\vec{L}'_1 = |\tau|L\vec{e}_x$, $\vec{L}'_2 = L\frac{(-\tau_1,\tau_2)}{|\tau|}$ and the complex coordinate should be expressed as $z' = |\tau|L(X'^1 - \frac{1}{\tau}X'^2) = \frac{|\tau|}{\tau}z = \frac{\tau^*}{|\tau|}z$. After the S transformation, the single-particle Hamiltonian becomes

$$H_0(\mathbf{A}', -1/\tau) = \tfrac{1}{2}g^{ab}(-1/\tau)D'_a(\mathbf{A}')D'_b(\mathbf{A}'), \quad \text{(B6)}$$

with

$$g(-1/\tau) = \frac{1}{S\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad \text{(B7)}$$

and the LLL wave function becomes

$$\Psi_m(X'^1, X'^2|-1/\tau) = \frac{1}{\sqrt{\pi^{1/2}L\ell}} e^{i\pi N_\phi \frac{1}{-\tau}[X'^2]^2} \times \theta_{\frac{m}{N_\phi}}[N_\phi z'/L|N_\phi(-1/\tau)], \quad \text{(B8)}$$

where $D'_a(\mathbf{A}') = -i\hbar\partial/\partial X'^a + |e|A'_a$ and $\mathbf{A}' = (-\tau_2 L^2 B X'^2, 0)$ are the covariant derivative and vector potential in (X'^1, X'^2) space, respectively. To compare Eq. (B34) with Eq. (B3), we need to write them in the same coordinate frame. Therefore, $H_0(\mathbf{A}', \tau+1)$ should be rewritten in (X^1, X^2) by using the relation

$$g(-1/\tau) = M_S^\dagger g(\tau) M_S \quad \text{(B9)}$$

with

$$M_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad \text{(B10)}$$

Thus, Eq. (B32) can be written as

$$H_0(\mathbf{A}', -1/\tau) = \tfrac{1}{2}[M_S\boldsymbol{D}'(\mathbf{A}')]_a g^{ab}(\tau)[M_S\boldsymbol{D}'(\mathbf{A}')]_b. \quad \text{(B11)}$$

After denoting

$$\begin{pmatrix} D_1(\tilde{\mathbf{A}}) \\ D_2(\tilde{\mathbf{A}}) \end{pmatrix} = M_S\boldsymbol{D}'(\mathbf{A}') = M_S\begin{pmatrix} D'_1(\mathbf{A}') \\ D'_2(\mathbf{A}') \end{pmatrix}, \quad \text{(B12)}$$

with $\tilde{\mathbf{A}} = (0, \tau_2 L^2 B X'^2) = (0, \tau_2 L^2 B X^1)$, we find

$$H_0(\mathbf{A}', -1/\tau) = H_0(\tilde{\mathbf{A}}, \tau) = \tfrac{1}{2}g^{ab}(\tau)D_a(\tilde{\mathbf{A}})D_b(\tilde{\mathbf{A}}), \quad \text{(B13)}$$

with $D_a(\tilde{\mathbf{A}}) = -i\hbar\partial/\partial X^a + |e|\tilde{A}_a$. Since $D_a(\mathbf{A})$ and $D_a(\tilde{\mathbf{A}})$ can be related by a gauge transformation $\hat{\mathcal{U}} = e^{-2\pi i N_\phi X^1 X^2}$, i.e.,

$$D_a(\mathbf{A}) = \hat{\mathcal{U}}^\dagger D_a(\tilde{\mathbf{A}})\hat{\mathcal{U}}, \quad \text{(B14)}$$

we get

$$H_0(\mathbf{A}, \tau) = \hat{\mathcal{U}}^\dagger H_0(\mathbf{A}', -1/\tau)\hat{\mathcal{U}}. \quad \text{(B15)}$$

Now we can see that the LLL wave function after this transformation is

$$\begin{aligned} &\hat{\mathcal{U}}\Psi_m(X'^1, X'^2|-1/\tau) \\ &= \frac{e^{i\pi N_\phi\frac{1}{-\tau}[X'^2]^2 - 2\pi i N_\phi X^1 X^2}}{\sqrt{\pi}L\ell}\theta_{\frac{m}{N_\phi}}\left(N_\phi z'\middle| -\frac{N_\phi}{\tau}\right) \\ &= \frac{e^{-i\pi N_\phi\frac{z^2}{\tau L^2} - i\pi\tau N_\phi[X^2]^2}}{\sqrt{\pi}L\ell}\theta_{\frac{m}{N_\phi}}\left(N_\phi\frac{|\tau|z}{\tau}\middle| -\frac{N_\phi}{\tau}\right). \end{aligned}$$

The Poisson's resummation formula gives us

$$\begin{aligned} &\theta_{\frac{m}{N_\phi}}\left(N_\phi\frac{|\tau|z}{\tau}\middle| -\frac{N_\phi}{\tau}\right) \\ &= \sqrt{\frac{-i\tau}{N_\phi}}\sum_{n=0}^{N_\phi-1}\theta_{\frac{n}{N_\phi}}\left(N_\phi\frac{z}{L}\middle| N_\phi\tau\right)e^{-2\pi i\frac{mn}{N_\phi} + i\pi N_\phi\frac{z^2}{\tau L^2}}. \end{aligned}$$

This suggests us to redefine the wave function $\Psi(X^1, X^2|\tau)$ as $\Psi(X^1, X^2|\tau)/\eta(\tau)$. The newly defined wave function satisfies the modular covariance

$$\hat{\mathcal{U}}\frac{\Psi_m(X'^1, X'^2|-1/\tau)}{\eta(-1/\tau)} = \sum_{n=0}^{N_\phi-1}\mathcal{S}_{mn}\frac{\Psi_n(X^1, X^2|\tau)}{\eta(\tau)}, \quad \text{(B16)}$$

where $\mathcal{S}_{mn} = \frac{1}{\sqrt{N_\phi}}\exp(-2\pi i\frac{mn}{N_\phi})$. One may ask why the single-orbital wave function under an S transformation cannot be directly related by $\hat{\mathcal{U}}\Psi_m(-1/\tau) = \Psi_m(\tau)$. The reason is degeneracy. We denote the space of degenerate LLL wave functions by $\mathcal{M}(\tau)$, for which

$$\mathcal{M}(\tau) = \hat{\mathcal{U}}\mathcal{M}(-1/\tau) = \hat{\mathcal{U}}\hat{\mathcal{S}}\mathcal{M}(\tau).$$

Thus, the action of S transformation will induce a unitary transformation on single state:

$$\hat{\mathcal{U}}\hat{\mathcal{S}}\Psi_m(\tau) = \hat{\mathcal{U}}\Psi_m(-1/\tau) = \mathcal{S}_{mn}\Psi_n(\tau),$$

where $\mathcal{S}_{mn}$ on the right-hand side of above equation is the representation of modular S transformation in space $\mathcal{M}(\tau)$:

$$\langle\Psi_n; \tau|\hat{\mathcal{S}}|\Psi_m; \tau\rangle = \mathcal{S}_{mn}.$$

Here the gauge transformation are implicitly included in the Dirac notation since we cannot compare wave functions in different gauges:

$$\begin{aligned} &\langle\Psi_n; \tau|\hat{\mathcal{S}}|\Psi_m; \tau\rangle \\ &= \int dX^1 dX^2 \Psi^*(X^1, X^2|\tau)\hat{\mathcal{U}}\Psi(X'^1, X'^2|-1/\tau). \end{aligned} \quad \text{(B17)}$$

2. Fractional quantum Hall wave function and modular $\mathcal{S}$ matrix

On the torus geometry, the general Halperin state can be expressed in terms of the theta function [7,55–57]:

$$\Psi^{\boldsymbol{a}}(\{z_i^I\}|\tau) = \mathcal{N}(\tau)f_c^{\boldsymbol{a}}(\{Z^I\})f_r(\{z_i^I\}) \times \exp\left\{i\pi\tau N_\phi\sum_{I,i}\left(\frac{y_i^I}{L\tau_2}\right)^2\right\}, \quad \text{(B18)}$$

where I is the index of component, $\boldsymbol{a} \in \Gamma_K^*/\Gamma_K$ is the vector labeling degenerate states, $z_i^I = L(X_i^{I1} + \tau X_i^{I2})$ is the coordinate of the ith particle in the Ith component, and $Z^I = \sum_i z_i^I$ is the center-of-mass coordinate of the Ith component. f_r and f_c are the relative part and center-of-mass part of the wave function:

$$f_r(\{z_i^I\}|\tau) = \left\{\prod_{I<J}\prod_{i,j}\eta^{-K_{IJ}}(\tau)\theta_{11}^{K_{IJ}}(z_i^I/L - z_j^J/L|\tau)\right\} \times \left\{\prod_{\mathrm{I}}\prod_{i<j}\eta^{-K_{II}}(\tau)\theta_{11}^{K_{II}}(z_i^I/L - z_j^I/L|\tau)\right\},$$
$$f_c^{\boldsymbol{a}}(\{Z^I\}|\tau) = \eta^{-\kappa}(\tau) f^{(\boldsymbol{a})}(\mathbf{Z}/L|\tau), \tag{B19}$$

where K_{IJ} is the underlying K matrix with dimension $\dim(K) = \kappa$ and diagonal elements $\boldsymbol{\kappa} = (K_{11}, K_{22}, \dots, K_{\kappa\kappa})^T$,

$$\theta_{11}(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left\{i\pi\tau\left(n+\frac{1}{2}\right)^2 + i2\pi\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right\},$$
$$f^{(\boldsymbol{a})}(\mathbf{Z}|\tau) = \sum_{\boldsymbol{n}\in\Gamma_K} \exp\{i\pi\tau(\boldsymbol{n}+\boldsymbol{a})^2 + 2\pi i(\boldsymbol{n}+\boldsymbol{a})\cdot\mathbf{Z}\}, \tag{B20}$$

and $\mathbf{Z} = \sum_I Z^I \boldsymbol{e}_I$ with $\boldsymbol{e}_I$ the basis of Γ_K (i.e., $\boldsymbol{e}_I \cdot \boldsymbol{e}_J = K_{IJ}$). The normalization factor is

$$\mathcal{N}(\tau) = N_0[\sqrt{\tau_2}\eta(\tau)^2]^{\frac{1}{2}\sum_I \kappa_I N^I}, \tag{B21}$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)|_{q=e^{i2\pi\tau}}$ is the τ-dependent Dedekind's η function, and N_0 is an area-dependent constant. There are more detailed discussions of the FQH wave function on the torus in Ref. [55].

Now, we can extract the modular $\mathcal{S}$ matrix by considering

$$\mathcal{S}_{\boldsymbol{ba}} = \langle\Psi^{\boldsymbol{b}};\tau|\hat{\mathcal{S}}|\Psi^{\boldsymbol{a}};\tau\rangle = \int\prod_{I,i} dz_i^I\,\Psi^{\boldsymbol{b}*}(\{z_i^I\}|\tau)\hat{\mathcal{U}}_g\Psi^{\boldsymbol{a}}\left(\{z_i'^I\}\middle|\frac{1}{\tau}\right), \tag{B22}$$

where $\hat{\mathcal{U}}_g$ the gauge transformation

$$\hat{\mathcal{U}}_g = \exp\left\{-2\pi i N_\phi \sum_{I,i} X_i^{1,I} X_i^{2,I}\right\} \tag{B23}$$

derived above, and $z' = |\tau|z/\tau$ is the coordinate after the modular S transformation. Before the tedious derivation, we list some useful relations (here we assume the total flux N_ϕ through the torus is even):

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau),$$
$$\theta_{11}(z/\tau|-1/\tau) = -\sqrt{-i\tau}\,e^{\frac{i\pi z^2}{\tau}}\theta_{11}(z|\tau),$$
$$f^{(\boldsymbol{a})}(\mathbf{Z}/\tau|-1/\tau) = (-i\tau)^{\kappa/2}e^{i\pi\mathbf{Z}^2/\tau}\sum_{\boldsymbol{b}\in\Gamma_K^*/\Gamma_K}\frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}f^{(\boldsymbol{b})}(\mathbf{Z}|\tau).$$

Next we deal with the transformation term by term.

a. Derivation of $\mathcal{N}(-1/\tau)$

$\mathcal{N}(-1/\tau)$ is derived as follows:

$$\mathcal{N}(-1/\tau) = N_0\left[\frac{\sqrt{\tau}_2}{|\tau|}\eta\left(\frac{1}{-\tau}\right)^2\right]^{\frac{1}{2}\sum_I\kappa_I N^I} = \left(\frac{-i\tau}{|\tau|}\right)^{\frac{1}{2}\sum_I\kappa_I N^I}\mathcal{N}(\tau), \tag{B24}$$

where we have used the first relation in Eq. (B24).

b. Center-of-mass part $f_c(\{Z^I\}|-1/\tau)$

The center-of-mass part $f_c(\{Z^I\}|-1/\tau)$ is

$$f_c^{\boldsymbol{a}}(\{Z'^I\}|-1/\tau) = \eta^{-\kappa}(-1/\tau)f^{(\boldsymbol{a})}\left(\frac{\mathbf{Z}}{\tau L}\middle|\frac{1}{-\tau}\right) = e^{i\pi\frac{Z^2}{\tau L^2}}\sum_{\boldsymbol{b}\in\Gamma_K^*/\Gamma_K}\frac{e^{-2\pi i\boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}f_c^{\boldsymbol{b}}(\{Z^I\}|\tau), \tag{B25}$$

where we have used the first and third relations in Eq. (B24).

c. Relative part $f_r(\{z_i^I\}|\tau)$

The relative part $f_r(\{z_i^I\}|\tau)$ is

$$f_r(\{z_i'^I\}|-1/\tau) = e^{\frac{i\pi}{\tau L^2}\left[\sum_{I<J}\sum_{i\in I,j\in J}K_{IJ}(z_{ij}^{IJ})^2 + \sum_{\mathrm{I}}\sum_{i<j}K_{II}(z_{ij}^{II})^2\right]} \times (-1)^{\sum_{I<J}\sum_{i\in Ij\in J}K_{IJ} + \sum_{\mathrm{I}}\sum_{i<j}K_{II}}f_r(\{z_i^I\}|\tau), \tag{B26}$$

where $z_{ij}^{IJ} = z_i^I - z_j^J$ and the first two relations in Eq. (B24) have been used. The phase factor can be simplified. The constant phase is

$$\sum_{I<J}\sum_{i\in Ij\in J}K_{IJ} + \sum_{\mathrm{I}}\sum_{i<j}K_{II} = \sum_{I<J}N^I K_{IJ}N^J + \sum_{\mathrm{I}}\frac{1}{2}(N^I-1)N^I K_{II} = \frac{1}{2}\left(\boldsymbol{N}^2 - \sum_I \kappa_I N^I\right), \tag{B27}$$

where $\boldsymbol{N} = (N_1, N_2, \dots)$ is the particle numbers. The coordinate-dependent phase needs a careful treatment, since they should be fully canceled when putting all parts together:

$$\frac{i\pi}{\tau L^2}\left(\sum_{I<J}\sum_{i\in I,j\in J}K_{IJ}(z_{ij}^{IJ})^2 + \sum_{\mathrm{I}}\sum_{i<j}K_{II}(z_{ij}^{II})^2\right) = \frac{i\pi}{2\tau L^2}\sum_{I,J}\sum_{i\in I,j\in J}K_{IJ}(z_{ij}^{IJ})^2 = \frac{i\pi}{\tau L^2}\sum_{I,J}\sum_{i\in I,j\in J}(z_i^I K_{IJ}z_i^I - z_i^I K_{IJ}z_j^J) = -\frac{i\pi}{\tau L^2}\mathbf{Z}^2 + \frac{i\pi}{\tau L^2}\sum_{I,i\in I}z_i^I z_i^I\sum_J K_{IJ}N^J = -i\pi\frac{\mathbf{Z}^2}{\tau L^2} + \frac{i\pi N_\phi}{\tau L^2}\sum_{I,i}(z_i^I)^2. \tag{B28}$$

d. Exponential and gauge transformation

Since $y_i^I/(\tau_2 L) = X_i^{I,2}$, we have

$$\hat{\mathcal{U}}_g \exp\left\{i\pi \frac{1}{-\tau} N_\phi \sum_{I,i} \left(X_i^{\prime I,2}\right)^2\right\}$$
$$= \hat{\mathcal{U}}_g \exp\left\{-\frac{i\pi N_\phi}{\tau} \sum_{I,i} \left(X_i^{I,1}\right)^2\right\}$$
$$= \exp\left\{-\frac{i\pi N_\phi}{\tau} \sum_{I,i} \left(z_i^I\right)^2\right\} \exp\left\{i\pi\tau N_\phi \sum_{I,i} \left(X_i^{I,2}\right)^2\right\}. \tag{B29}$$

Substituting Eq. (B18) and Eqs. (B24)–(B29) into Eq. (B22), we finally have

$$\langle \Psi^{\boldsymbol{b}}; \tau | \mathcal{S} | \Psi^{\boldsymbol{a}}; \tau \rangle = \left(\frac{i\tau}{|\tau|}\right)^{\frac{1}{2}\sum_I \kappa_I N^I} (-1)^{\frac{1}{2}N^2} \frac{e^{-2\pi i \boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}, \tag{B30}$$

where $\boldsymbol{N} = \sum_I N^I \boldsymbol{e}_I$ is a vector enclosing the particle number in each component N^I. We expect that the intrinsic properties such as braiding statistics should be independent of the particle number $\boldsymbol{N}$. If we neglect the particle number dependent $U(1)$ factor [58], we have derived the modular $\mathcal{S}$ matrix from the trial wave function:

$$\boldsymbol{S} = \frac{e^{-2\pi i \boldsymbol{a}\cdot\boldsymbol{b}}}{\sqrt{|\det K|}}. \tag{B31}$$

The above result is consistent with Eq. (A12) and Eq. (4) in the main text.

3. Modular $\mathcal{T}$ transformation

Let us first consider how the LLL wave functions evolve under the Dehn-twist transformation $\tau \to \tau + 1$. After the Dehn twist, the coordinate $z = L(X^1 + \tau X^2)$ can be rewritten as $z = L[X'^1 + (\tau+1)X'^2]$. Thus we express the single-particle Hamiltonian in terms of (X'^1, X'^2) as

$$H_0(\mathbf{A}', \tau+1) = \tfrac{1}{2} g^{ab}(\tau+1) D'_a(\mathbf{A}') D'_b(\mathbf{A}'), \tag{B32}$$

with

$$g(\tau+1) = \frac{1}{L^2\tau_2^2}\begin{pmatrix} |\tau+1|^2 & -\tau_1 - 1 \\ -\tau_1 - 1 & 1 \end{pmatrix}, \tag{B33}$$

and the LLL wave function takes the form

$$\Psi_m(X'^1, X'^2 | \tau+1) = \frac{1}{\sqrt{\pi^{1/2} L \ell}} e^{i\pi N_\phi(\tau+1)[X'^2]^2} \times \theta_{\frac{m}{N_\phi}}[N_\phi z/L | N_\phi(\tau+1)], \tag{B34}$$

where $D'_a(\mathbf{A}') = -i\hbar\partial/\partial X'^a + |e| A'_a$ and $\mathbf{A}' = (-\tau_2 L^2 B X'^2, 0)$ are the covariant derivative and vector potential in (X'^1, X'^2), respectively. To compare Eq. (B34) with Eq. (B3), we need to write them in the same coordinate frame. Therefore, $H_0(\mathbf{A}', \tau+1)$ should be rewritten in (X^1, X^2). By using relations

$$g(\tau+1) = \frac{1}{L^2\tau_2^2}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} g(\tau) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{B35}$$

and

$$\begin{pmatrix} D_1(\tilde{\mathbf{A}}) \\ D_2(\tilde{\mathbf{A}}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} D'_1(\mathbf{A}') \\ D'_2(\mathbf{A}') \end{pmatrix}, \tag{B36}$$

with $\tilde{\mathbf{A}} = (-\tau_2 L^2 B X'^2, \tau_2 L^2 B X'^2) = (-\tau_2 L^2 B X^2, \tau_2 L^2 B X^2)$, we find

$$H_0(\mathbf{A}', \tau+1) = H_0(\tilde{\mathbf{A}}, \tau) = \tfrac{1}{2} g^{ab}(\tau) D_a(\tilde{\mathbf{A}}) D_b(\tilde{\mathbf{A}}), \tag{B37}$$

with $D_a(\tilde{\mathbf{A}}) = -i\hbar\partial/\partial X^a + |e|\tilde{A}_a$. Because $D_a(\mathbf{A})$ and $D_a(\tilde{\mathbf{A}})$ can be related by a gauge transformation $\hat{\mathcal{U}} = e^{-i\pi N_\phi [X^2]^2}$, i.e.,

$$D_a(\mathbf{A}) = \hat{\mathcal{U}}^\dagger D_a(\tilde{\mathbf{A}}) \hat{\mathcal{U}}, \tag{B38}$$

we get

$$H_0(\mathbf{A}, \tau) = \hat{\mathcal{U}}^\dagger H_0(\tilde{\mathbf{A}}, \tau) \hat{\mathcal{U}}. \tag{B39}$$

Now we can see that the LLL wave function after the Dehn twist, when written in (X^1, X^2), is

$$\hat{\mathcal{U}} \Psi_m(X'^1, X'^2 | \tau+1) = e^{i\pi \frac{m^2}{N_\phi}} \Psi_m(X^1, X^2 | \tau), \tag{B40}$$

where we have used

$$\theta_{\frac{m}{N_\phi}}[N_\phi z | N_\phi(\tau+1)]$$
$$= \sum_n e^{i\pi N_\phi \tau (n + \frac{m}{N_\phi})^2 + i2\pi(n+\frac{m}{N_\phi})N_\phi z + i\pi N_\phi (n+\frac{m}{N_\phi})^2}$$
$$= e^{i\pi \frac{m^2}{N_\phi}} \sum_n (-1)^{nN_\phi} e^{i\pi N_\phi \tau (n+\frac{m}{N_\phi})^2 + i2\pi(n+\frac{m}{N_\phi})N_\phi z}$$
$$= e^{i\pi \frac{m^2}{N_\phi}} \theta_{\frac{m}{N_\phi}}(N_\phi z | N_\phi \tau). \tag{B41}$$

Therefore, a phase factor $\exp(i\pi \frac{m^2}{N_\phi})$ is gained in the mth LLL orbital after the Dehn twist.

Similar to the $\mathcal{S}$ transformation, we introduce a many-body gauge transformation

$$\hat{\mathcal{U}}_g = \exp\left\{i\pi N_\phi \sum_{I,i} \left(\frac{y_i^I}{L\tau_2}\right)^2\right\} \tag{B42}$$

to relate the many-body wave functions before and after the modular transformation $\{\mathcal{T} : \tau \to \tau+1\}$. Using Eqs. (B18)–(B21), we can get

$$\hat{\mathcal{U}}_g \Psi^{\boldsymbol{a}}\left(\{z_i^I\} | \tau+1\right) = \hat{\mathcal{U}}_g \mathcal{N}(\tau+1) f_c^{\boldsymbol{a}}(\{Z^I\} | \tau+1) f_r(\{z_i^I\} | \tau+1) e^{-i\pi N_\phi(\tau+1)\sum_{I,i}(y_i^I/L\tau_2)^2}$$
$$= \mathcal{N}(\tau) f_c^{\boldsymbol{a}}(\{Z^I\} | \tau) f_r\left(\{z_i^I\} | \tau\right) e^{i\pi N_\phi \tau \sum_{I,i}(y_i^I/L\tau_2)^2} e^{\frac{1}{12} i\pi (N^T \cdot N - \kappa)} e^{i2\pi h_{\boldsymbol{a}}}$$
$$= \Psi^{\boldsymbol{a}}\left(\{X_{I,i}^1, X_{I,i}^2\} | \tau\right) e^{\frac{1}{12} i\pi (N^T \cdot N - \kappa)} e^{i2\pi h_{\boldsymbol{a}}}, \tag{B43}$$

where $\boldsymbol{N} = \sum_I N^I \boldsymbol{e}_I$ is a vector enclosing the particle number in each component N^I and we have used the following useful relations (here we assume the total flux N_ϕ through the torus is even):

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau),$$
$$\theta_{11}(z|\tau+1) = e^{i\pi/4}\theta_{11}(z|\tau),$$
$$f^{(\boldsymbol{a})}(\boldsymbol{Z}|\tau+1) = e^{i\pi\boldsymbol{a}^T\cdot\boldsymbol{a}} f^{(\boldsymbol{a})}(\boldsymbol{Z}|\tau).$$

Equation (B43) immediately gives the matrix representation of the $\mathcal{T}$ transform as

$$\mathcal{T}^{\boldsymbol{ab}} = \langle\Psi^{\boldsymbol{b}};\tau|\mathcal{T}|\Psi^{\boldsymbol{a}};\tau\rangle = \delta_{\boldsymbol{ab}} e^{i2\pi\left(h_{\boldsymbol{a}}-\frac{c}{24}\right)} e^{\frac{1}{12}i\pi\boldsymbol{N}^T\cdot\boldsymbol{N}}, \quad \text{(B44)}$$

where $c = \kappa$ is the chiral central charge of the underlying edge CFT and $h_{\boldsymbol{a}}$ is the topological spin of the topological sector $\boldsymbol{a}$ satisfying

$$h_{\boldsymbol{a}} = \tfrac{1}{2}\boldsymbol{a}^T\cdot\boldsymbol{a} \ (\text{mod } 1). \quad \text{(B45)}$$

Again, we find that the result of $\mathcal{T}$ matrix is consistent with the calculation based on the CFT [Eq. (6) in the main text], except for a particle number related phase factor.

APPENDIX C: MODULAR MATRICES FOR THE HALPERIN $(m, m, m-1)$ STATE

In this section, we explicitly calculate the modular matrices for the Halperin $(m, m, m-1)$ state, using the method of anyon condensation and extended chiral algebra, which serves as an example to validate our theory. We see that these direct calculations for the Halperin $(m, m, m-1)$ state give consistent results with those obtained in the previous sections. The difference is that the calculation shown in this section is specific to $(m, m, m-1)$ state, while the CFT and trial wave-function methods shown above are more general for any Halperin state.

1. Anyon condensation method for Halperin $(m, m, m-1)$ state

Apart from the single-component Laughlin state, we consider another special class of FQH states called the Halperin $(m, m, m-1)$ state at filling $\nu = \frac{2}{2m-1}$. The Halperin state lives on a double-layered system, where the particles are labeled by the pseudospin index $\{\uparrow,\downarrow\}$ and its wave function is (on the disk) [22]

$$\Psi(\{z^\uparrow, z^\downarrow\}) = \prod_{i<j}(z_i^\uparrow - z_i^\uparrow)^m \prod_{i<j}(z_i^\downarrow - z_i^\downarrow)^m \prod_{i,j}(z_i^\uparrow - z_i^\downarrow)^{m-1} \times \exp\left[-\frac{1}{4}\sum_i(|z_i^\uparrow|^2 + |z_i^\downarrow|^2)\right]. \quad \text{(C1)}$$

The bosonic Halperin $(m, m, m-1)$ state (m is even) is described by the $\widehat{su(2)}_1$ WZW model with a $\widehat{u(1)}_{4m-2}$ boson [2]. The central charge, topological spin and fusion rules are given by [10]

$$c = 2, \quad h_{(\lambda,a)} = \frac{\lambda(\lambda+2)}{12} + \frac{a^2}{8m-4}, \quad \text{(C2)}$$

$$(\lambda, a)\otimes(\mu, b) = ((\lambda+\mu)_{\text{mod } 2}, (a+b)_{\text{mod } 4m-2}), \quad \text{(C3)}$$

where $\lambda, \mu = 0, 1$ and $a, b = 0, 1, \ldots, 4m-3$. Both fields have quantum dimension $\mathrm{d}_{(\lambda,a)} = \mathrm{d}_\lambda \mathrm{d}_a = 1$.

Here, the number of primary fields $2\times(4m-2) = 8m-4$ is greater than the ground-state degeneracy $2m-1$ of the Halperin $(m, m, m-1)$ state on the torus. The reason can be understood by noticing a special primary field with integer topological spin $h_{(1,2m-1)} = \frac{1}{4} + \frac{(2m-1)^2}{4(2m-1)} = \frac{m}{2} \in \mathbb{Z}$. The existence of such a bosonic field (with integer spin) beside vacuum indicates the phenomenon Bose-condensation [33]. Let us denote this field as $J = (1, 2m-1)$. The period of J is 2, since its fusion rule is $J^2 = (0, 0)$. J should behave like the identity $\mathbf{1} = (0, 0)$. The reason is, similar to the case that we cannot distinguish (λ, a) from $(\lambda, a)\otimes\mathbf{1}^m$ ($m\in\mathbb{N}$), we cannot distinguish (λ, a) from $(\lambda, a)\otimes J^m$ ($m\in\mathbb{N}$) either. After condensation, the $8m-4$ primary fields are mapped into $4m-2$ distinguishable primary fields $\{[(0, a)]_J | a = 0, 1, \ldots, 4m-3\}$, where $[\cdots]$ denotes a equivalence class under the fusion with J field $[(0, a)]_J : (0, a) \sim (0, a)\otimes J$. For the notation simplicity, we will abbreviate $[(0, a)]_J$ to a in the following. Moreover, not all fields in $[(0, a)]_J$ are available, since some of corresponding anyons are confined unless all fields in the same equivalence class have the same topological spin. The terminology deconfined means the anyons can exist in the bulk. To find the deconfined fields, we need to consider the difference between the topological spin of a and $a\otimes J$ (in the sense of modulo 1):

$$h_{a\otimes J} - h_a = \frac{1}{4} + \frac{(a+2m-1)}{8m-4} - \frac{a^2}{8m-4} = \frac{a}{2}.$$

Clearly, for even a two anyons only differ by an integer spin. Thus only half anyons in set $[(0, a)]_J$ are deconfined and form the final theory

$$\{a = [(0, 2a)]_J | a = 0, 1, \ldots, 2m-2\}, \quad \text{(C4)}$$

with the topological spins

$$h_a = \frac{a^2}{2m-1}, \quad a\otimes b = (a+b)_{\text{mod}2m-1}, \quad \text{(C5)}$$

which gives the modular $\mathcal{T}$ matrix as

$$\mathcal{T}_{ab} = \delta_{ab} e^{2\pi i \frac{a^2}{2m-1}}. \quad \text{(C6)}$$

Using the formula

$$\mathcal{S}_{ab} = \frac{1}{\mathcal{D}}\sum_c \mathcal{N}_{a\bar{b}}{}^c \exp\left[2\pi i(h_c - h_a - h_b)\right]\mathrm{d}_c, \quad \text{(C7)}$$

the $\mathcal{S}$ matrix of the final theory is

$$\mathcal{S}_{ab} = \frac{1}{\sqrt{2m-1}}\exp\left(-2\pi i\frac{2ab}{2m-1}\right). \quad \text{(C8)}$$

The modular matrix obtained here exactly matches the results from the CFT and trial wave function.

2. Extended chiral algebra method for Halperin $(m, m, m-1)$ state

Furthermore, there is another way to construct the modular matrices, as we illustrate below. The existence of an extra bosonic field $h_a = 0 \mod 1$ signals the possibility of a "block-diagonal modular invariant" partition function with a "extended vacuum block" [10]:

$$Z = |\chi_0 + \chi_{n_1} + \cdots + \chi_{n_i}|^2 + \cdots.$$

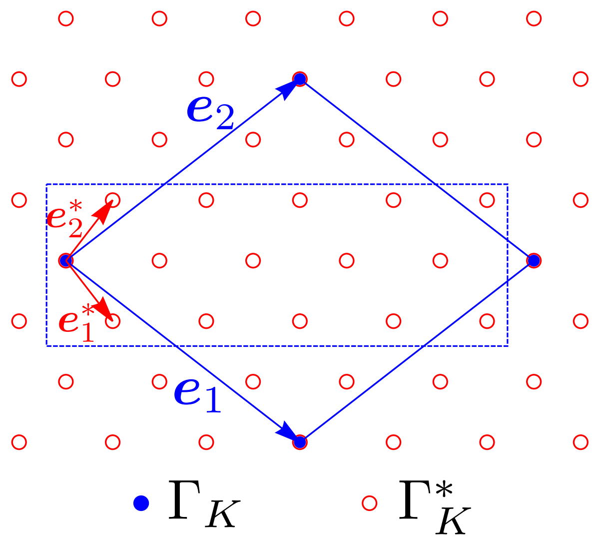

FIG. 4. The Γ_K and Γ_K^* lattices for the (441) state. The solid blue parallelogram is the coset of (441) states. An equivalent representation of coset is shown by the dashed blue rectangular.

In our example $\mathcal{A} = \widehat{su(2)}_1 \oplus \widehat{u(1)}_{2m-1}$, the characters of all primary fields are

$$\chi_{(\lambda,a)}(\tau) = \chi_\lambda^{\widehat{su(2)}_1}(\tau) K_a^{(4m-2)}(\tau), \tag{C9}$$

where $\chi_\lambda^{\widehat{su(2)}_1}(\tau) = \frac{1}{\eta(\tau)} \sum_{m\in\mathbb{Z}+\lambda/2} q^{m^2} = K_\lambda^{(2)}(\tau)$ is the character of $\widehat{su(2)}_1$ and is equivalent to $\widehat{u(1)}_2$. The bosonic field $h_{(1,2m-1)} = 1$ extends the vacuum sector to "vacuum block" $C_0(\tau) = \chi_{(0,0)}(\tau) + \chi_{(1,2m-1)}(\tau)$, and other "blocks" contain fields with topological spin differing by integers

$$C_p(\tau) = \chi_{(0,2p)}(\tau) + \chi_{(1,2p+2m-1)}(\tau).$$

In terms of the block character $C_p(\tau)$, the block-diagonal modular invariant partition function can be written as

$$Z = \sum_{p=0}^{2m-2} |C_p(\tau)|^2.$$

The integer difference of primary fields ensures the modular-$\mathcal{T}$ invariant, and the $\mathcal{S}$ transformation of block characters is [see Eqs. (A4) and (C9)]

$$C_p(-1/\tau) = \frac{1}{\sqrt{2m-1}} \sum_{q=0}^{2m-2} [e^{-2\pi i \frac{2pq}{2m-1}} \chi_{0,2q}(\tau) + e^{-2\pi i \frac{2p(2q+1)}{2m-1}} \chi_{1,2q+1}(\tau)t]$$

$$= \sum_{q=0}^{2m-2} \frac{e^{-2\pi i \frac{2pq}{2m-1}}}{\sqrt{2m-1}} [\chi_{0,2q}(\tau) + \chi_{1,2q+2m-1}(\tau)]$$

$$= \sum_{q=0}^{2m-2} \frac{e^{-2\pi i \frac{2pq}{2m-1}}}{\sqrt{2m-1}} C_q(\tau).$$

Therefore, the $\mathcal{S}$ matrix of the block characters is

$$\mathcal{S}_{pq} = \frac{e^{-2\pi i \frac{2pq}{2m-1}}}{\sqrt{2m-1}}, \tag{C10}$$

which is consistent with Eq. (C8). The unitary $\mathcal{S}^\dagger \mathcal{S} = 1$ gives the modular-$\mathcal{S}$ invariant of the partition function Z. Conclusively, the bosonic field gives another type of modular invariant, the block-diagonal modular invariant, which can be constructed by grouping the fields with integer difference topological spin into blocks. After that, the modular $\mathcal{S}$ matrix of the block characters is the final result that we are looking for. Meanwhile, the structure of the vacuum block indicates a symmetry enhancement or an extended chiral algebra [31,41,59,60], which is the illustration of anyon condensation.

APPENDIX D: MORE EXAMPLES

Our theory is general, which has broad applications. In this section, we present some examples of modular matrices of multicomponent states that are beyond two-component spin-singlet states.

1. Halperin (441) state

This section gives an example of a double-layer (441) state whose K matrix is

$$K = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}. \tag{D1}$$

This filling of this state is $\nu = 2/5$, and the degeneracy $D = |\det K| = 15 = 5 \times 3$. Numerically, this state has been found recently [61]. The K lattice and its dual lattice are shown in Fig. 4. The coset is written as $\Gamma_K^*/\Gamma_K = \{\Gamma_0 + a\boldsymbol{e}_1^* + a\boldsymbol{e}_2^* | a = 0, 1, 2, 3, 4\}$ where $\Gamma_0 = \{(0, 0), \boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$. According to the results of the modular matrix [Eqs. (4) and (6)

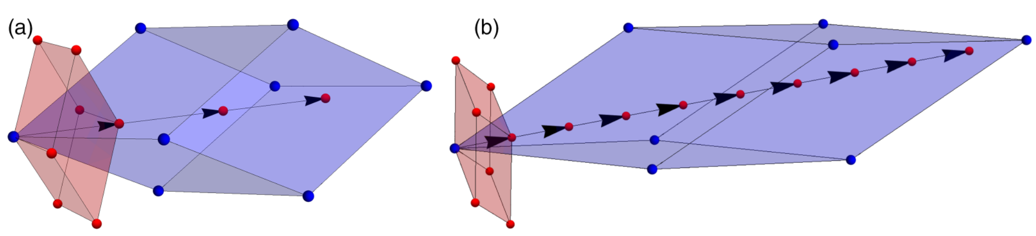

FIG. 5. The Γ_K and Γ_K^* lattices for (a) (2221) and (b) (4443) states. The blue and red spheres represent the lattice points of Γ_K and Γ_K^*, respectively. The red spheres inside the parallelepiped are the coset lattice of Γ_K^*/Γ_K.

in the main text], we obtain

$$\mathcal{T} = e^{-2\pi(\frac{2}{24})}\mathrm{diag}\left\{0, \frac{2}{15}, \frac{2}{15}, \frac{1}{5}, \frac{8}{15}, \frac{8}{15}, \frac{4}{5}, \frac{1}{3}, \frac{1}{3}, \frac{4}{5}, \frac{8}{15}, \frac{8}{15}, \frac{1}{5}, \frac{2}{15}, \frac{2}{15}\right\} \tag{D2}$$

and

$$\mathcal{S} = \frac{1}{\sqrt{15}}\left(\begin{matrix}
\begin{pmatrix}1 & 1 & 1\\ 1 & \kappa^4 & \frac{1}{\kappa}\\ 1 & \frac{1}{\kappa} & \kappa^4\end{pmatrix} &
\begin{pmatrix}1 & 1 & 1\\ \kappa^3 & \kappa^7 & \kappa^2\\ \kappa^3 & \kappa^2 & \kappa^7\end{pmatrix} &
\begin{pmatrix}1 & 1 & 1\\ \kappa^6 & \kappa^{10} & \kappa^5\\ \kappa^6 & \kappa^5 & \kappa^{10}\end{pmatrix} &
\begin{pmatrix}1 & 1 & 1\\ \kappa^9 & \kappa^{13} & \kappa^8\\ \kappa^9 & \kappa^8 & \kappa^{13}\end{pmatrix} &
\begin{pmatrix}1 & 1 & 1\\ \kappa^{12} & \kappa^{16} & \kappa^{11}\\ \kappa^{12} & \kappa^{11} & \kappa^{16}\end{pmatrix}\\
\begin{pmatrix}1 & \kappa^3 & \kappa^3\\ 1 & \kappa^7 & \kappa^2\\ 1 & \kappa^2 & \kappa^7\end{pmatrix} &
\begin{pmatrix}\kappa^6 & \kappa^9 & \kappa^9\\ \kappa^9 & \kappa^{16} & \kappa^{11}\\ \kappa^9 & \kappa^{11} & \kappa^{16}\end{pmatrix} &
\begin{pmatrix}\kappa^{12} & \kappa^{15} & \kappa^{15}\\ \kappa^{18} & \kappa^{25} & \kappa^{20}\\ \kappa^{18} & \kappa^{20} & \kappa^{25}\end{pmatrix} &
\begin{pmatrix}\kappa^{18} & \kappa^{21} & \kappa^{21}\\ \kappa^{27} & \kappa^{34} & \kappa^{29}\\ \kappa^{27} & \kappa^{29} & \kappa^{34}\end{pmatrix} &
\begin{pmatrix}\kappa^{24} & \kappa^{27} & \kappa^{27}\\ \kappa^{36} & \kappa^{43} & \kappa^{38}\\ \kappa^{36} & \kappa^{38} & \kappa^{43}\end{pmatrix}\\
\begin{pmatrix}1 & \kappa^6 & \kappa^6\\ 1 & \kappa^{10} & \kappa^5\\ 1 & \kappa^5 & \kappa^{10}\end{pmatrix} &
\begin{pmatrix}\kappa^{12} & \kappa^{18} & \kappa^{18}\\ \kappa^{15} & \kappa^{25} & \kappa^{20}\\ \kappa^{15} & \kappa^{20} & \kappa^{25}\end{pmatrix} &
\begin{pmatrix}\kappa^{24} & \kappa^{30} & \kappa^{30}\\ \kappa^{30} & \kappa^{40} & \kappa^{35}\\ \kappa^{30} & \kappa^{35} & \kappa^{40}\end{pmatrix} &
\begin{pmatrix}\kappa^{36} & \kappa^{42} & \kappa^{42}\\ \kappa^{45} & \kappa^{55} & \kappa^{50}\\ \kappa^{45} & \kappa^{50} & \kappa^{55}\end{pmatrix} &
\begin{pmatrix}\kappa^{48} & \kappa^{54} & \kappa^{54}\\ \kappa^{60} & \kappa^{70} & \kappa^{65}\\ \kappa^{60} & \kappa^{65} & \kappa^{70}\end{pmatrix}\\
\begin{pmatrix}1 & \kappa^9 & \kappa^9\\ 1 & \kappa^{13} & \kappa^8\\ 1 & \kappa^8 & \kappa^{13}\end{pmatrix} &
\begin{pmatrix}\kappa^{18} & \kappa^{27} & \kappa^{27}\\ \kappa^{21} & \kappa^{34} & \kappa^{29}\\ \kappa^{21} & \kappa^{29} & \kappa^{34}\end{pmatrix} &
\begin{pmatrix}\kappa^{36} & \kappa^{45} & \kappa^{45}\\ \kappa^{42} & \kappa^{55} & \kappa^{50}\\ \kappa^{42} & \kappa^{50} & \kappa^{55}\end{pmatrix} &
\begin{pmatrix}\kappa^{54} & \kappa^{63} & \kappa^{63}\\ \kappa^{63} & \kappa^{76} & \kappa^{71}\\ \kappa^{63} & \kappa^{71} & \kappa^{76}\end{pmatrix} &
\begin{pmatrix}\kappa^{72} & \kappa^{81} & \kappa^{81}\\ \kappa^{84} & \kappa^{97} & \kappa^{92}\\ \kappa^{84} & \kappa^{92} & \kappa^{97}\end{pmatrix}\\
\begin{pmatrix}1 & \kappa^{12} & \kappa^{12}\\ 1 & \kappa^{16} & \kappa^{11}\\ 1 & \kappa^{11} & \kappa^{16}\end{pmatrix} &
\begin{pmatrix}\kappa^{24} & \kappa^{36} & \kappa^{36}\\ \kappa^{27} & \kappa^{43} & \kappa^{38}\\ \kappa^{27} & \kappa^{38} & \kappa^{43}\end{pmatrix} &
\begin{pmatrix}\kappa^{48} & \kappa^{60} & \kappa^{60}\\ \kappa^{54} & \kappa^{70} & \kappa^{65}\\ \kappa^{54} & \kappa^{65} & \kappa^{70}\end{pmatrix} &
\begin{pmatrix}\kappa^{72} & \kappa^{84} & \kappa^{84}\\ \kappa^{81} & \kappa^{97} & \kappa^{92}\\ \kappa^{81} & \kappa^{92} & \kappa^{97}\end{pmatrix} &
\begin{pmatrix}\kappa^{96} & \kappa^{108} & \kappa^{108}\\ \kappa^{108} & \kappa^{124} & \kappa^{119}\\ \kappa^{108} & \kappa^{119} & \kappa^{124}\end{pmatrix}
\end{matrix}\right), \tag{D3}$$

with $\kappa = e^{-2\pi i \frac{1}{15}}$. Each small 3×3 matrix corresponds to the three points in $\Gamma_0 + a(\boldsymbol{e}_1 + \boldsymbol{e}_2) = \{a\boldsymbol{e}_1 + a\boldsymbol{e}_2, (a+1)\boldsymbol{e}_1 + a\boldsymbol{e}_2, a\boldsymbol{e}_1 + (a+1)\boldsymbol{e}_2\}$.

We would like to stress that, here the representation of the coset Γ_K^*/Γ_K of the Halperin (441) state is two dimensional, which is different from the one-dimensional representation of the spin-singlet state (see Fig. 1 in the main text).

2. Halperin $(m, m, m, m-1)$ state

In this section we consider the three-component Halperin $(m, m, m, m-1)$ state. The K matrix of $(m, m, m, m-1)$ state is defined as

$$K = \begin{pmatrix} m & m-1 & m-1\\ m-1 & m & m-1\\ m-1 & m-1 & m\end{pmatrix}, \tag{D4}$$

with filling $\nu = 3/(3m-2)$ and degeneracy $D = |\det K| = 3m-2$. Two examples of the K lattice and its dual lattice are shown in Fig. 5. The coset lattice is $\Gamma_K^*/\Gamma_K = \{\frac{1}{3m-2}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)|a = 0, 1, \dots, 3m-3\}$, and we can write down their modular matrices accordingly:

$$\mathcal{T}_{aa} = e^{2\pi i\left(\frac{3a^2}{6m-4} - \frac{3}{24}\right)}, \quad \mathcal{S}_{ab} = \frac{1}{\sqrt{3m-2}} e^{-6\pi i \frac{ab}{3m-2}}. \tag{D5}$$

As an example, if we choose $m = 2$, the modular matrices are

$$\mathcal{T} = e^{-\frac{\pi i}{4}}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & e^{2\pi i \frac{3}{8}} & 0 & 0\\ 0 & 0 & e^{2\pi i \frac{1}{2}} & 0\\ 0 & 0 & 0 & e^{2\pi i \frac{3}{8}}\end{pmatrix}, \tag{D6}$$

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{pmatrix}. \tag{D7}$$

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