


Synchronization and spacetime vortices in one-dimensional driven-dissipative condensates and coupled oscillator models

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Driven-dissipative condensates, such as those formed from polaritons, expose how the coherence of Bose-Einstein condensates evolves far from equilibrium. We consider the phase and frequency ordering in the steady-states of a one-dimensional lattice of condensates, described by a coupled oscillator model with nonodd couplings, including both time-dependent noise and a static random potential. We present numerical results for the phase and frequency distributions, and discuss them in terms of the Kardar-Parisi-Zhang equation and the physics of spacetime vortices. We find that the nucleation of spacetime vortices causes the breakdown of the single-frequency steady state and produces a variation in the frequency with position. Such variation would provide an experimental signature of spacetime vortices. More generally, our results expose the nature of synchronization in oscillator chains with nonodd couplings, random frequencies, and noise.

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I. INTRODUCTION

In equilibrium statics and dynamics are related through the fluctuation-dissipation theorem. Nonequilibrium systems, however, break this relationship and allow new forms of ordering [1,2]. A topical example is provided by driven-dissipative Bose-Einstein condensates of exciton polaritons [3–5] in one-dimensional lattices [6,7]. The phase correlations in such condensates decay exponentially in space [8], as they do in equilibrium, but are stretched exponentials in time. The correlation functions found in a recent experiment [6] agree with those predicted by the Kardar-Parisi-Zhang (KPZ) equation, which is obeyed by the phase of a driven-dissipative condensate [9–14].

Static disorder is ubiquitous in condensed-matter systems, and often plays a decisive role in their collective behavior. It can destroy the ordered states present in the clean limit, for example in low-dimensional magnets [15], and lead to glassy states, for example in Bose gases [16]. Although disorder could be expected to play a similarly important role for driven-dissipative condensates, it has typically been neglected. It has been considered by some of the present authors [17], among others [18–21], but we neglected the time-dependent noise that is important for the coherence properties of the condensate. In this paper, we evaluate the combined effect of disorder of strength σ , and noise of strength D , in a one-dimensional driven-dissipative condensate. We find when both the noise and disorder are nonzero, $\sigma, D \neq 0$, the characteristic single-frequency steady state is destroyed, and the condensate acquires small variations in frequency with position. This is because the combination of disorder and noise leads to a net generation of spacetime vorticity. The inhomogeneous frequency—or equivalently chemical potential—of this nonequilibrium condensate is a fundamental difference compared with equilibrium.

The physics of driven-dissipative condensates is one of three closely related problems, with the others being the synchronization of coupled oscillators [22] and the growth of interfaces [23]. The driven-dissipative Gross-Pitaevskii equation for the macroscopic wave function in a condensate implies that the phase degree of freedom obeys the KPZ equation [8,9,24,25], originally introduced to describe a growing interface [26]. The stochastic nature of the gain and loss process in the condensate gives rise to noise in the Gross-Pitaevskii equation, which becomes the standard spacetime noise term in the KPZ equation, while a random potential gives rise to so-called columnar disorder in the KPZ equation [27–29], i.e., an interface growth rate which is random in space but fixed in time. For a lattice of condensates, the Gross-Pitaevskii equation can be mapped to a coupled-oscillator model with, in general, both noise and random frequencies. Unlike the better known and studied Kuramoto model, this model includes a nonodd (cosine) coupling term, which plays an important role and allows global frequency synchronization in large systems [17,30]. Such Kuramoto-Sakaguchi models were introduced in Ref. [31] which, along with several more recent works [17,32–34], uses their connection to the KPZ equation. Here, we go beyond our analysis [17] of synchronization in a Kuramoto-Sakaguchi model with random frequencies to study the impact of noise in the synchronized state. We find that it generates a small but nonzero range of time-averaged frequencies, with an unusual activated dependence related to localization effects. Our results apply not just to driven-dissipative condensates, but to the many other systems described by Kuramoto-Sakaguchi models.

The mechanism behind the breakdown of the single-frequency steady state is the nucleation of spacetime vortices by noise. Spacetime vortices are the topological defects of a one-dimensional phase field $\theta(x, t)$, in which the phase winds by a multiple of 2π around a closed loop in spacetime. They

are not described by the KPZ equation, which is for a noncompact variable and so ignores topological defects [35,36]. They have been considered previously [33,37] in the absence of disorder, shown to modify the KPZ scaling of the correlation functions at long times, and produce a vortex turbulence phase which has yet to be observed. In these cases vortices and antivortices are equally likely, and the resulting states have no net vorticity on large scales. In contrast, the steady-states we find have large-scale vorticity patterns, corresponding to their inhomogeneous frequency profile. Measurements of an inhomogeneous frequency profile would provide evidence of spacetime vortices and condensate physics beyond the KPZ equation.

II. MODEL

We consider a one-dimensional lattice of polariton condensates, in which the condensate phase on the k th lattice site is $\theta_k(t)$, with dynamics given by the Kuramoto-Sakaguchi model

$$\frac{d\theta_j}{dt} = \sum_k J_{jk} \left[\frac{1}{\alpha} \sin(\theta_k - \theta_j) - \cos(\theta_k - \theta_j) \right] + \epsilon_j + \eta_j(t). \quad (1)$$

This model can be derived from the Gross-Pitaevskii equation describing a lattice of condensates formed in the wells of a potential [6,17,38], in which case J_{jk} is the real-valued Josephson coupling strength between sites j and k , ϵ_j is the energy per particle of the condensate on site j , and $\alpha = \Gamma/U$ is the gain saturation parameter divided by the interaction strength. Equation (1) is valid provided the density fluctuations are fast as well as small [17]. Including a dissipative part to the coupling [39] produces a model of the same form with a redefinition of the parameters. The dissipative couplings are small for condensates trapped in the wells of a potential, for which the tight-binding wave functions can be taken as real, but can be large for untrapped condensates [40,41]. Such condensates also allow the realization of time-delayed couplings [42], which are beyond the scope of Eq. (1).

We consider the case of nearest-neighbor couplings, which we take to be uniform and positive, $J > 0$. This is appropriate for a regular lattice where the fluctuations in the coupling will be small compared with its average value. The predominant source of disorder will then be in the site-energies, ϵ_j , which we suppose have standard deviation σ . The time-dependent random driving η , which arises physically from the gain and loss processes, is Gaussian white noise with strength D , so that $\langle \eta_i(t) \eta_j(t') \rangle = 2D \delta(t - t') \delta_{ij}$.

While the most direct application of Eq. (1) is to lattices of coupled condensates, it can also be understood as a generalization of the KPZ equation for a single extended condensate [8,9,24,25] that incorporates the compactness of the phase θ [37]. In this case the lattice can be viewed as a formal device that allows vortices to be treated within a phase-only theory. Conversely, if the phase differences are small, we may expand the trigonometric functions in Eq. (1) and take the continuum limit to obtain the KPZ equation with an additional time-independent random term from the disorder,

$$\frac{\partial \theta(x, t)}{\partial t} = \frac{Ja^2}{\alpha} \frac{\partial^2 \theta}{\partial x^2} + Ja^2 \left(\frac{\partial \theta}{\partial x} \right)^2 + \epsilon(x) + \eta(x, t). \quad (2)$$

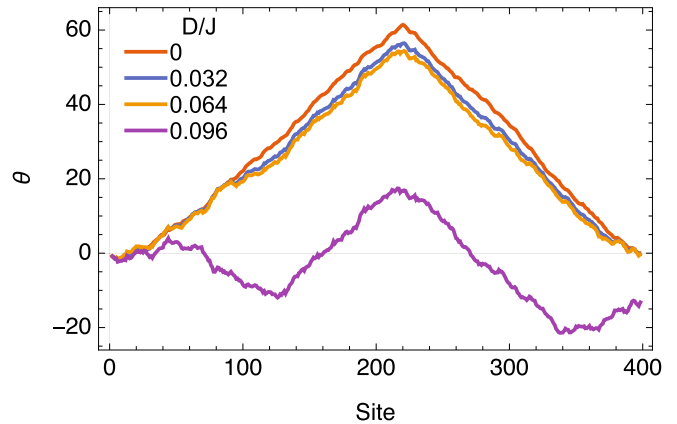


FIG. 1. Phases in a chain of 400 oscillators at the time $tJ = 6000$ for various noise strengths. $\alpha = 1$, $\sigma/J = 0.2$, and noise strengths $D/J = 0$ (red), 0.032 (blue), 0.064 (yellow), 0.128 (purple). Noise increases from top to bottom in the center of the figure. The zero of phase is chosen to be that of the first oscillator.

Here a is the lattice constant, which will be set to one in the following. Using the Cole-Hopf transform $Z = e^{a\theta}$ we can rewrite this as the imaginary-time Schrödinger equation for a particle in a static and a dynamic random potential

$$\begin{aligned} \frac{\partial Z}{\partial t} &= \frac{J}{\alpha} \frac{\partial^2 Z}{\partial x^2} + \alpha \epsilon(x)Z + \alpha \eta(x, t)Z \\ &= -H_0 Z + \alpha \eta(x, t)Z. \end{aligned} \quad (3)$$

Before discussing the general case of Eq. (1), we recall some previous results when only one type of disorder is present. We consider, here and in the remainder of this work, only the regime $\alpha \lesssim 1$, which is appropriate for polariton condensates. In the opposite limit, $\alpha \gtrsim 1$, lattice effects dominate [17] and there is a first-order transition to a disordered state [33,37].

Without the noise term, Eq. (1) is the Kuramoto-Sakaguchi model for a one-dimensional system of coupled self-sustained oscillators with random natural frequencies [31]. In contrast to the Kuramoto model, the coupling is a nonodd function of the relative phases. This allows for a globally synchronized state in which all the oscillators have a single frequency [17,30], even in the limit of large numbers of oscillators. The synchronized state occurs for $\sigma < \sigma_c$, at which point there is a transition to a desynchronized state.

The nature of the synchronized states and the form of the phase boundary [17,30–32] can be understood using the mapping to the imaginary-time Schrödinger Eq. (3). Its solution is $Z(x, t) = \sum_n c_n e^{-E_n t} \phi_n(x)$, where ϕ_n and E_n are the eigenfunctions and energies of the effective Hamiltonian H_0 . In the long-time limit Z approaches the ground state of the effective random potential $-\alpha \epsilon(x)$, which is a localized state $\phi \sim e^{-|x-x_0|/\zeta}$, at some position x_0 , with localization length [29] $\zeta \sim (J/\alpha^2 \sigma)^{2/3}$. This implies that the state is synchronized in the long-time limit—the phase increases at the same rate at every point in space—and that the phase profile is a triangular function of position. An example can be seen in the topmost curve of Fig. 1.

The approach to the steady-state can also be understood in this way, because at late times Z will comprise a few low-energy localized states, giving rise to a phase profile comprised of a set of triangular peaks. These grow at slightly different rates, until eventually only the fastest-growing peak, corresponding to the ground state of H_0 , remains. (We take, without loss of generality, the ground state energy to be negative).

We can obtain the phase boundary for synchronization [17] by noting that if the gradients become too large $|\partial_x \theta| \gtrsim 1$, the compactness of the phase becomes relevant, and Eq. (1) cannot be approximated by Eq. (2). Since the synchronized state has $|\partial_x \theta| \sim 1/\alpha \zeta \sim (\alpha \sigma^2/J^2)^{1/3}$, it occurs only below a critical disorder strength, $\sigma < \sigma_c \sim J\alpha^{-1/2}$.

In the case where there is no static disorder, $\sigma = 0$, and Eq. (2), which approximates Eq. (1), becomes the standard one-dimensional KPZ equation for a growing interface, with θ playing the role of the interface position (height). The interface is rough [23], with correlation function $C(x, t) = \langle (\theta(x, t) - \theta(0, 0))^2 \rangle \sim t^{2/3} f(|x|/|t|^{2/3})$, where the scaling function $f(y)$ is a nonzero constant at $y \rightarrow 0$, and behaves as $f(y) \sim |y|$ as $y \rightarrow \infty$. Since the width of the interface—corresponding to the range of phases—grows as $\Delta\theta = \sqrt{C(0, t)} \sim t^{1/3}$, the range of time-averaged frequencies decays to zero in the long-time limit: $\Delta\omega = \Delta\theta/t \sim t^{-2/3}$. The broadening of the interface by noise does not occur fast enough to give rise to different time-averaged frequencies [33]. This conclusion remains unchanged on considering spacetime vortices, nucleation of which is expected to lead to diffusive behavior for the phase differences at long times [33,37], $\Delta\theta = \sqrt{C(0, t)} \sim t^{1/2}$, so that $\Delta\omega \sim t^{-1/2}$.

III. RESULTS

A. Phase ordering and first-order coherence

Figure 1 shows the phases $\theta_j(t)$ in a chain of oscillators, obtained by integrating Eq. (1) using a stochastic Runge-Kutta method [43]. Discontinuities in the resulting phase profiles, where neighboring phases differ by multiples of 2π , have been removed to produce a smooth curve. The highest (red) curve is a typical result obtained with disorder but without noise. The disorder strength $\sigma = 0.2J < \sigma_c$ is such that the long-time solution is synchronized, and takes the form of a triangular phase profile as discussed above. There are smaller variations around this overall profile, due to the residual effects of disorder [27]. This profile is unchanged in time, apart from an overall shift.

The remaining curves show the effects of introducing increasing amounts of noise on the phase profiles. Qualitatively, the effect of weak noise is to add time-dependent fluctuations about the phase profile produced by the disorder. For the strongest noise shown, the situation is slightly more complex, with the presence of two large-scale peaks in the solution, rather than one. This corresponds to the presence of both the ground state and the first excited state of the effective Hamiltonian H_0 in the solution at this time.

We can use these observations to obtain the behavior of the first-order coherence function of a lattice of condensates with both disorder and noise. In polariton condensates, the coherence function $g^{(1)}(\Delta x = x - x', \Delta t = t - t')$ is deter-

mined by interfering the light emitted from one position in the lattice at one time, (x, t) , with that from another position at another, (x', t') [6]. $|g^{(1)}|$ quantifies the coherence of the condensates separated by Δx and time Δt .

We consider the synchronized state in the regime where we may take the continuum limit, so that there is a phase field $\theta(x, t)$ with dynamics given by Eq. (2). Neglecting intensity fluctuations, in line with our assumptions, we have $g^{(1)}(\Delta x = x - x', \Delta t = t - t') = \langle e^{i\theta(x, t) - i\theta(x', t')} \rangle$, where $\langle \rangle$ denotes an average.

To study the decay of first-order coherence, we write

$$\theta(x, t) = \theta_0(x, t) + \phi(x, t), \quad (4)$$

where $\theta_0(x, t)$ is the steady-state solution in the absence of noise. We consider experiments done with only one realization of the random potential, which is appropriate if the static disorder arises from imperfections in the structure and only one structure is used. We also suppose that the measurement is done after the steady-state is reached. We then have $g^{(1)} = e^{i\theta_0(x, t) - i\theta_0(x', t')} \langle e^{i\phi(x, t) - i\phi(x', t')} \rangle$, where the relevant average is over noise or time but not disorder. The first factor does not fluctuate, and so does not lead to a decay of $|g^{(1)}|$, which is entirely due to the second factor. From Eq. (2), $\phi(x, t)$ obeys

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{J}{\alpha} \frac{\partial^2 \phi}{\partial x^2} - 2J \left(\frac{\partial \theta_0}{\partial x} \right) \left(\frac{\partial \phi}{\partial x} \right) + J \left(\frac{\partial \phi}{\partial x} \right)^2 + \eta(x, t) \\ &\approx \frac{J}{\alpha} \frac{\partial^2 \phi}{\partial x^2} - 2Jc \left(\frac{\partial \phi}{\partial x} \right) + J \left(\frac{\partial \phi}{\partial x} \right)^2 + \eta(x, t). \end{aligned} \quad (5)$$

In the second line we have used the fact that the steady-state solution θ_0 consists of large regions where the slope is approximately constant, $\partial_x \theta_0 \approx c$, and considered one such region. Equation (5) is then the standard KPZ equation with a tilted substrate [23], and the second term on the right-hand side, the tilt, can be eliminated by a Galilean transformation $\tilde{x} = x - 2Jct$, $\tilde{t} = t$. Thus, over each region in the solution θ_0 , defined by an approximately constant slope, the statistics of $\phi(\tilde{x}, \tilde{t})$ and hence, the decay of first-order coherence, is related to that of the standard KPZ equation.

B. Desynchronization by noise

We now turn to consider the frequencies in the steady-state of Eq. (1). The frequency of the k th oscillator, averaged over some long time T , is

$$\omega_k = [\theta_k(T + t) - \theta_k(t)]/T. \quad (6)$$

This gives a frequency profile, for each realization of the disorder, whose width may be characterized by the standard deviation of the ω_k , $\Delta\omega$. Figure 2 shows the results of numerical calculations of the disorder averaged width, $\langle \Delta\omega \rangle$. These results are obtained for a chain of $N = 400$ oscillators, with $J = 1$ and $\alpha = 1$. We use a constant initial condition, and evolve to a time $t = 1500$ to allow for the transients, computing the time-averaged frequencies from the phases a time $T = 1500$ later.

The results along the two axes, $\sigma = 0$ [8,10,33,37] and $D = 0$ [17] are expected from previous works. The state is frequency synchronized, $\Delta\omega = 0$, for any D when $\sigma = 0$, and for $\sigma < \sigma_c \approx 0.4$ when $D = 0$. However, when both noise

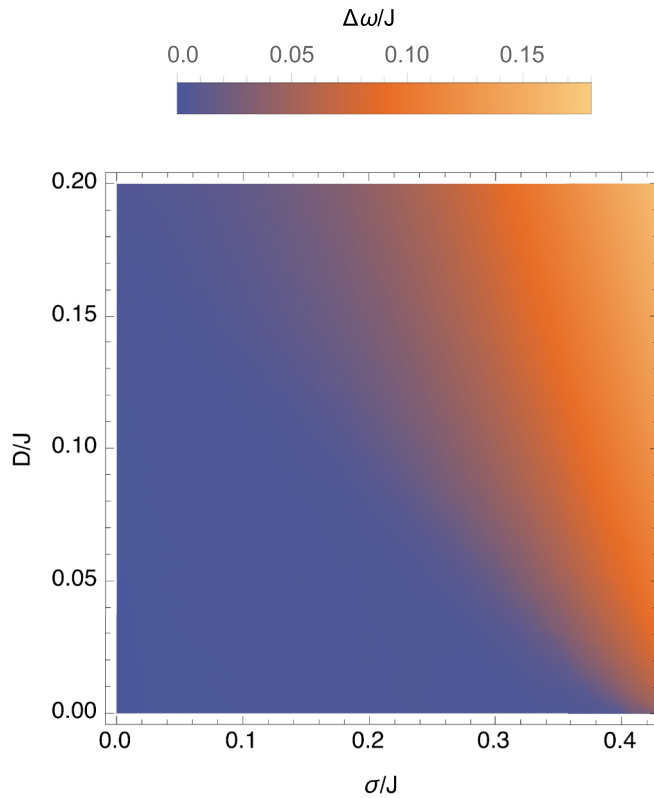


FIG. 2. Standard deviation of the time-averaged frequencies, $\Delta\omega$, for chains of 400 oscillators with static disorder σ , noise strength D , and $\alpha = 1$. The frequencies are computed over a time interval $T = 1500/J$. Each point is an average of the frequency width over 32 realizations of the disorder and noise.

and disorder are present we find that $\Delta\omega \neq 0$, and a range of time-averaged frequencies emerges in the solution.

The presence of multiple time-averaged frequencies in the solution can be related to the presence of spacetime vorticity. We express the phase change of a given site, $\theta_i(t+T) - \theta_i(t)$, as the integral of the derivative $d\theta_i/dt$. The frequency difference between two sites i and j , with $i > j$, can then be expressed as a line integral around a closed path,

$$\begin{aligned} \omega_i - \omega_j &= \frac{1}{T} \left(\int_t^{t+T} \frac{d\theta_i(\tau)}{d\tau} d\tau - \int_t^{t+T} \frac{d\theta_j(\tau)}{d\tau} d\tau \right) \\ &= \frac{1}{T} \oint ds \frac{d\theta}{ds} = \frac{2\pi}{T} n_v, \end{aligned} \quad (7)$$

which counts the enclosed vorticity, n_v . Thus, a nonvanishing frequency difference is equivalent to a nonvanishing density of spacetime vorticity. The path in Eq. (7) is a rectangle starting at (x_i, t) and going in the direction of increasing time to $(x_i, t+T)$ then, in order, to $(x_j, t+T)$, (x_j, t) , and back to (x_i, t) . The integrals along the parts of the path in the time direction are explicit in the first line of Eq. (7), and we have chosen a gauge such that the integrals along the space direction are zero [37]. The integrals and derivatives represent sums and differences where they refer to a discrete coordinate.

Figure 3 illustrates the relationship between frequency variations and vortices. It shows the time dependence of the

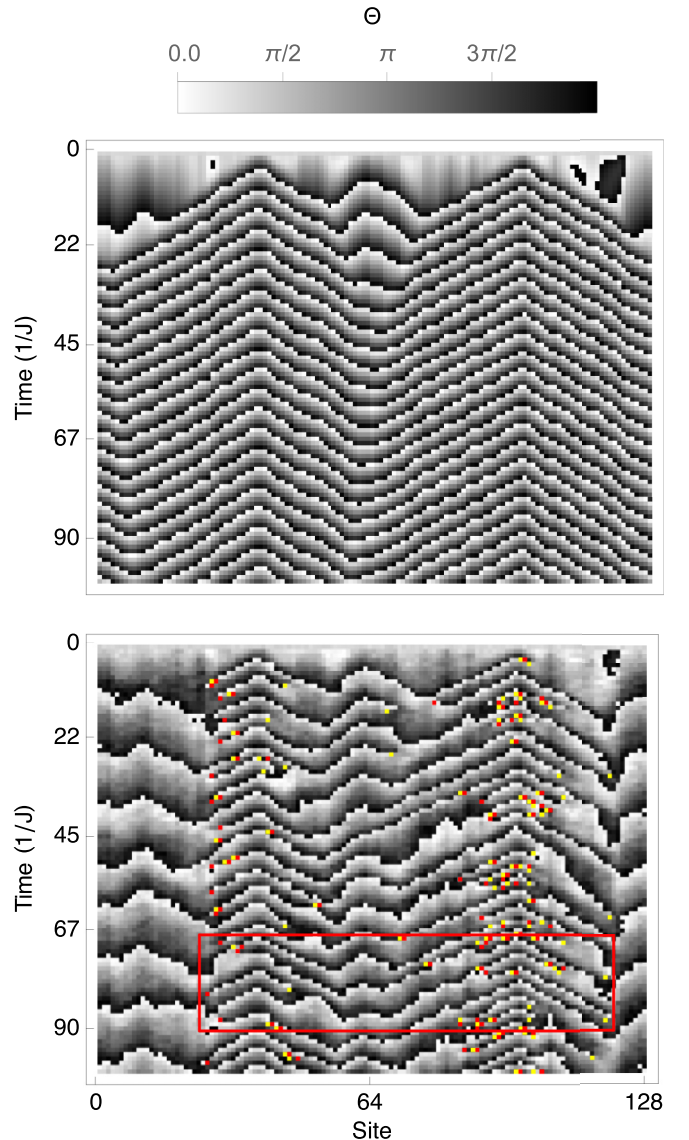


FIG. 3. Phases in a chain of $N = 128$ oscillators, without noise (top panel) and with noise (center panel). Position is along the horizontal axis, with time along the vertical axes, increasing from top to bottom. $\alpha = 1$, $\sigma/J = 0.4$, and $D = 0$ (top) and $D/J = 0.07$ (center). The colored points in the center panel mark spacetime vorticity $+1$ (red) or -1 (yellow). The bottom panel is an enlargement showing the vorticity in the region of the center panel marked with the red box, with vorticity $+1$ and -1 in black and white.

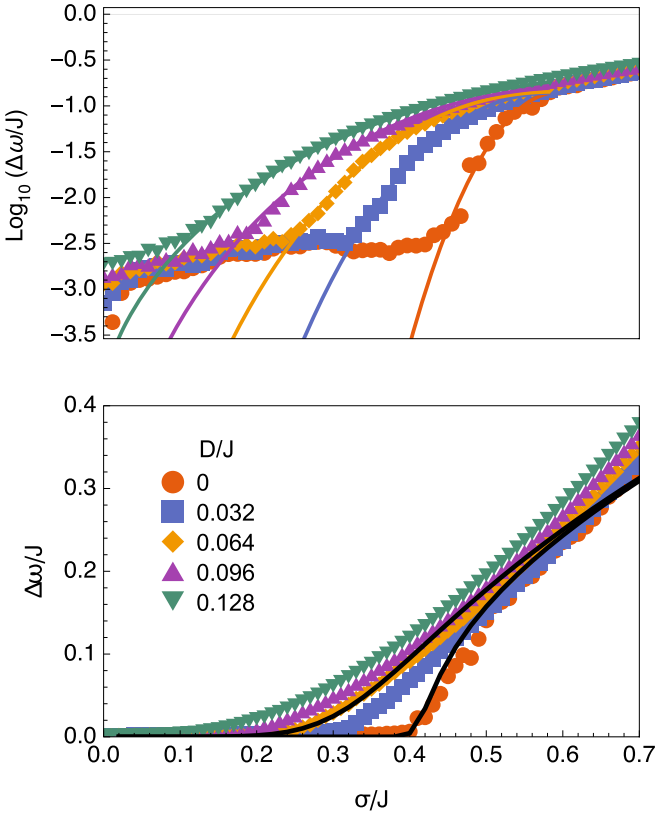


FIG. 4. Dependence of the frequency width $\Delta\omega$ on disorder strength σ for a chain of $N = 400$ oscillators with $\alpha = 1$ (colored points). The same data is shown on a logarithmic (top) and linear (bottom) scale. Colors indicate the noise strengths D/J of 0.128 (green), 0.096 (purple), 0.064 (yellow), 0.032 (blue), and 0 (red). The curves are fits to Eq. (9) (top panel) and Eq. (12) (bottom panel), as described in the text.

phases with disorder alone (top), and with both disorder and noise (center). The phase is shown over a single interval of length 2π using a grayscale, and spacetime vortices appear as dislocations in the pattern visible in the center panel. The introduction of noise leads to a state with a range of time-averaged frequencies, in this case a noticeably higher frequency in the center of the chain than at the edge. The colored dots in the center panel mark spacetime vortices with positive and negative charges shown as different colors, and the frequency variation along the chain can be seen to arise, as it must, from the presence of regions with unbalanced vorticity.

C. Theory of vortex nucleation

Figure 4 shows in more detail the computed frequency width, as a function of the disorder strength σ , for several values of the noise D . We now consider the form of these curves, in the regime $\sigma < \sigma_c$, and suggest how it can be understood in terms of vortex nucleation.

For a first approach, we recall that the continuum description, Eq. (2), is based on an expansion of the trigonometric functions in Eq. (1), and hence becomes invalid when the phase gradients are too large. This leads us to suggest that

vortices will be generated where the magnitude of the phase gradient fluctuates to reach a critical value, $|\partial_x\theta| = k_c$. Furthermore, we suggest that the sign of the slope at this point corresponds to the charge of the resulting vortex. This is consistent with Fig. 3, where we see that the positive (negative) vortices tend to occur predominantly in the regions where there is an overall positive (negative) slope of the phase profile.

As noted above, the phase profile can be decomposed as $\theta = \theta_0 + \phi$, and in a region where the slope of the background is approximately constant, $\partial_x\theta_0 \approx c$, the fluctuations ϕ obey the tilted KPZ equation, and hence the standard KPZ equation after a Galilean transformation. Thus, the equal-time statistics of ϕ are identical to those of the KPZ equation, which are known to be unaffected by the nonlinear term and hence, Gaussian [23]. More specifically, the steady-state distribution of ϕ is a Gaussian with zero mean, $P[\phi] \propto \exp[-\frac{J}{2D} \int dx (\partial_x\phi)^2]$. Since the slopes $\partial_x\theta = \partial_x\theta_0 + \partial_x\phi \approx c + \partial_x\phi$, their distribution is this same Gaussian, shifted by c . The probability of a fluctuation causing the magnitude of the slope to exceed the critical value k_c is then

$$P(|\partial_x\theta| > k_c) \propto \text{erfc}\left(\frac{k_c \mp c}{\sqrt{2D/J}}\right) \sim e^{-(k_c \mp c)^2 J/D}, \quad (8)$$

where the minus and plus signs in the arguments are for the cases $\partial_x\theta > k_c$ and $\partial_x\theta < -k_c$, respectively. In a region with positive (negative) background slope, the first (second) of these will be exponentially more likely than the other in the regime of weak noise, and positive (negative) charged vortices will predominate. This leads us to expect that a region of average slope c will have a frequency width proportional to the vortex generation rate

$$G \sim e^{(2k_c - |c|)J/D}. \quad (9)$$

Since the average slope of the background scales as $c \sim 1/\alpha\zeta \sim (\alpha\sigma^2/J^2)^{1/3}$, the exponent is a sum of terms proportional to $\sigma^{2/3}$ and $\sigma^{4/3}$. The curves in the top panel of Fig. 4 are fits to a dependence of this form, which can be seen to give a good account of the data.

We have, in addition, developed a heuristic argument for the form of the noise-induced frequency width based on known results for two coupled oscillators [44]. For a two-site chain, Eq. (1) gives

$$\frac{d}{dt}(\theta_2 - \theta_1) = -\frac{2J}{\alpha} \sin(\theta_2 - \theta_1) + (\epsilon_2 - \epsilon_1) + (\eta_2 - \eta_1). \quad (10)$$

This is identical to the case of Kuramoto oscillators, since the cosine term cancels. Equation (10) has a synchronized steady state for detunings $\delta = |\epsilon_2 - \epsilon_1| < 2J/\alpha$ in the absence of noise. The noise term nucleates phase slips in this state and introduces a frequency difference, which can be computed from the solution of the Fokker-Planck equation [44]. In the present notation it is,

$$\delta\omega(\delta) = \delta \frac{\sinh(\pi\mu)}{\pi\mu} |I_{i\mu}(J/\alpha D)|^{-2}, \quad (11)$$

where $\mu = \delta/2D$, and $I_{i\mu}$ is a Bessel function of imaginary order.

To apply this result to the non-Kuramoto chain, we recall the approximately triangular form of the background solution θ_0 , corresponding to the localized ground state of the effective Hamiltonian H_0 . This state would also be obtained for a Δ -function potential of strength related to the localization length, $\epsilon(x) = -(2J/\alpha^2\xi)\delta(x - x_0)$. More generally, a solution θ_0 of saw-tooth form, arising from several low-energy states of the random potential, would be obtained from a set of such δ -function potentials. This suggests associating the frequency difference δ in the coupled oscillator model with the strength of these potentials, $\delta \sim (2J/\alpha^2\xi) \sim \alpha^{-2/3}J^{1/3}\sigma^{2/3}$. We therefore propose that the frequency width in the chain is of the form

$$\Delta\omega = C_1\delta\omega(C_2\alpha^{-2/3}J^{1/3}\sigma^{2/3}), \quad (12)$$

with $\delta\omega(\delta)$ given by Eq. (11). The fitting parameter C_1 is introduced to account for the number of sites in the chain where vortex nucleation occurs. The factor C_2 accounts for the details of the relationship between the localization length and the other parameters.

The lower panel of Fig. 4 shows a comparison between Eq. (12) and our simulation results. We have chosen the parameters C_1 and C_2 such that this form is close to the data in the absence of noise. We have then used these values to plot the result for $D = 0.096J$. This can be seen to produce a curve close to the data (purple triangles) for that noise strength. Both curves deviate from the data in the region well above the transition, which is expected as we have neglected vortex-vortex interactions and changes in the number of sites where vortex nucleation occurs.

IV. CONCLUSIONS

In summary, we have studied the combined effects of noise and disorder in a one-dimensional chain of driven-dissipative condensates, described by a Kuramoto-Sakaguchi oscillator model. The phase profiles, in the regimes of weak disorder and noise, consist of triangular forms produced by the disorder, with additional time-dependent fluctuations due to

the noise. When spacetime vortices can be neglected, these time-dependent fluctuations will be described by a tilted KPZ equation. The steady-state contains a single frequency, and the first-order coherence functions are related by a Galilean transformation to those obtained in the absence of disorder. More dramatic effects appear when spacetime vortices are considered, which lead to the breakdown of the single-frequency steady-state and the appearance of small variations in the frequency along the chain. This is due to the creation of spacetime vortices by the noise, which is biased by the currents that are induced by the disorder potential. The resulting frequency width has an unusual form: an exponential involving fractional powers of the disorder strength, reflecting the localization length of a quantum particle.

One implication of our work is that measurements of an inhomogeneous frequency profile would provide a signature of spacetime vortices. While we have focused on the case where such frequency variations appear due to the presence of disorder, we would expect similar effects in other potentials, since these too will induce currents in the driven-dissipative condensate that will lead to unbalanced vorticity generation in different regions of the sample. A straightforward example would be a lattice with a single site at a different frequency, corresponding to a δ -function potential in the continuum model. The mechanism we propose would also be expected to occur if a supercurrent is generated, in the absence of a potential, by imposing a phase difference between the ends of the lattice [21]. In this case the drop in frequency along the chain corresponds to dissipation in the supercurrent due to spacetime vortices.

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