Polarity of the fermionic condensation in the *p*-wave Kitaev model on a square lattice

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In a p-wave Kitaev model, the nearest-neighbor pairing term results in the formation of the BCS pair in the ground state. In this work, we study the fermionic condensation of real-space pairs in a p-wave Kitaev model on a square lattice with a uniform phase gradient pairing term along both directions. The exact solution shows that the ground state can be expressed in a coherent-state-like form, indicating the condensation of a collective pairing mode, which is the superposition of different configurations of pairs in real space. The amplitudes of each configuration depend not only on the size but also on the orientation of the pair. We employ three quantities to characterize the ground state in the thermodynamic limit. (i) A BCS-pair order parameter is introduced to characterize the phase diagram, consisting of gapful and topological gapless phases. (ii) The particle-particle correlation length is obtained to reveal the polarity of the pair condensation. (iii) In addition, a pair-pair correlator is analytically derived to indicate the presence of off-diagonal long-range order. Our work proposes an alternative method for understanding fermionic condensation.

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I. INTRODUCTION

The concept of fermionic condensates has received much attention over a long period of time, from the first discovery of superconductivity in metal to the first atomic fermionic condensate created by using ^{40}K atoms [1]. A fermionic condensate is a superfluid phase formed by fermionic particles at low temperatures. It can describe the state of electrons in a superconductor or the analog for fermionic atoms. For the exact results on the condensation of fermionic pairs in a realistic model Hamiltonian, we can look back for much earlier investigations of excited η-pairing eigenstates in the Hubbard model for electrons, which possess off-diagonal long-range order (ODLRO) [2–4]. Although a fermionic condensate is closely related to the Bose-Einstein condensate, the fermion pair has its own intrinsic structure, for instance, the size and orientation of the two-fermion dimer. In the Hubbard model, two electrons with opposite spins can form an on-site pair $c_{\mathbf{r},\uparrow}^{\dagger}c_{\mathbf{r},\downarrow}^{\dagger}|0\rangle$, known as a doublon, which acts as a hard-core boson. In this context, a many-body η-pairing state $(\eta^{+})^{n}|0\rangle$ describes the condensation of n doublons in a collective mode $\eta^{+}=\sum_{\mathbf{r}}(-1)^{\mathbf{r}}c_{\mathbf{r},\uparrow}^{\dagger}c_{\mathbf{r},\downarrow}^{\dagger}$, which is a superposition of single-doublon states at different lattice sites. On the other hand, an on-site pair is forbidden for a triplet-pairing mechanism [5,6], where the fundamental building block is the spinless fermion, and then the real-space pair cannot be regarded as a point particle. Interesting questions include whether a similar collective pair mode can exist in a spinless fermionic system and whether condensation will occur in the ground state.

In this paper, we present some exact results for a p-wave Kitaev model on a square lattice. As is well known, the nearest-neighbor pairing term results in the formation of a BCS pair in the ground state. In this work, we study the ground state from the perspective of fermionic condensation of real-space pairs. We consider the p-wave Kitaev model on a square lattice with a uniform phase gradient pairing term along both directions. The phase gradient provides another dimension in the phase diagram. We employ three quantities to characterize the ground state in the thermodynamic limit. (i) A BCS-pair order parameter is introduced to characterize the phase diagram, consisting of gapful and topological gapless phases. (ii) The particle-particle correlation length is obtained to reveal the polarity of the pair condensation. (iii) In addition, a pair-pair correlator is analytically derived to indicate the existence of ODLRO. It is a four-operator correlator to measure the long-range correlation of two pairs, and its magnitude is determined not only by the density of the pairs but also by the correlation between two pairs. In this sense, the present work proposes an alternative method for understanding fermionic condensation. The underlying mechanism is that the ground state can be expressed in a coherent-state-like form, which is a standard formalism for the condensation of a collective pairing mode. Such a pair mode is the superposition of different configurations of pairs in real space, which can be regarded as an extension of the η-pairing mode. In contrast to the uniform amplitude in the η-pairing mode, the amplitudes of each p-wave pairing configuration depend not only on the size but also on the orientation of the pair.

There are several realistic proposals to realize effective spinless superconductivity [7–9], and two recent papers [10,11] refer to a special class of long-range Kitaev models (called Kitaev tie models) that can be realized in condensed matter. In addition, a recent work claimed the realization of the two-dimensional (2D) Kitaev model in experiments [12].

It is well known that a spinless Kitaev model can become a spin model via the Jordan-Winger transformation in low dimension. The newly developed cold-atom technology may

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provide another avenue to realize the Kitaev model in low dimension. The spin model can be stimulated by cold atoms in experiments [13–16], which provides another promising method to realize the Kitaev model.

The organization of this paper is as follows. We begin Sec. II by discussing the exact solution of the two-dimensional Kitaev model under periodic boundary conditions, and we present the phase diagram. In Sec. III the BCS-like order parameter is introduced to characterize the pairing condition of the ground state, and we investigate the nonanalytical behavior of the parameter. The properties of the ground state and the form of pairs in coordinate space are discussed in Sec. IV. In Sec. V we study particle-particle correlation in the ground state, which shows that correlation intension is direction dependent in thermodynamic limit. In Sec. VI we derive the correlator of two pairs, which indicates that ODLRO exists in the ground state. In Sec. VII, we draw the conclusions. Some necessary proofs and derivations are given in the Appendix.

II. MODEL AND PHASE DIAGRAM

We consider the Kitaev model on a square lattice, which is employed to depict 2D p-wave superconductors. The Hamiltonian of the tight-binding model on a square $N \times N$ lattice takes the following form:

$$H = -t \sum_{\mathbf{r},\mathbf{a}} e^{i\phi} c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}+\mathbf{a}} - \Delta \sum_{\mathbf{r},\mathbf{a}} c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}+\mathbf{a}}^{\dagger} + \text{H.c.} + \mu \sum_{\mathbf{r}} (2c_{\mathbf{r}}^{\dagger} c_{\mathbf{r}} - 1), \quad (1)$$

where $\mathbf{r} = (x, y)$ are the coordinates of lattice sites and $c_{\mathbf{r}}$ are the fermion annihilation operators at site $\mathbf{r}$. Vectors $\mathbf{a} = (a_x, a_y)$ are the lattice vectors in the x and y directions. t is the hopping amplitude between neighboring sites, and the real number Δ is the strength of the pair operators. The last term gives the chemical potential. When the periodic boundary is taken, we define $(N + x, y) \to (x, y)$ and $(x, N + y) \to (x, y)$. There are several papers about the 2D square lattice Kitaev model, but with zero phase ϕ; two of them focused on the gapless phase [17,18], and another two of them focused on Majorana edge states [19,20]. In addition, there is a paper [21] considering the spinful fermion. The phase ϕ can be caused by the supercurrent flowing along the x and y directions; the supercurrent can lead to a spatial phase gradient of the pairing potential [22,23], and it can be transformed to hopping terms by gauge transformation. Therefore, ϕ meets the constraint $N\phi = 2\pi n$ ($n \in Z$) due to the periodicity of c_j. The Kitaev model is known to have a topological gapless phase for zero ϕ [23–27] . In this work, we consider only the influence of ϕ and μ on the ground state for the Hamiltonian with a fixed $\Delta > 0$.

Imposing periodic boundary conditions in both directions, the Hamiltonian can be exactly diagonalized. Applying the Fourier transformation

$$c_{\mathbf{r}} = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{k}},$$

$$k_x, k_y = \frac{2\pi m}{N}, \quad m \in Z. \quad (2)$$

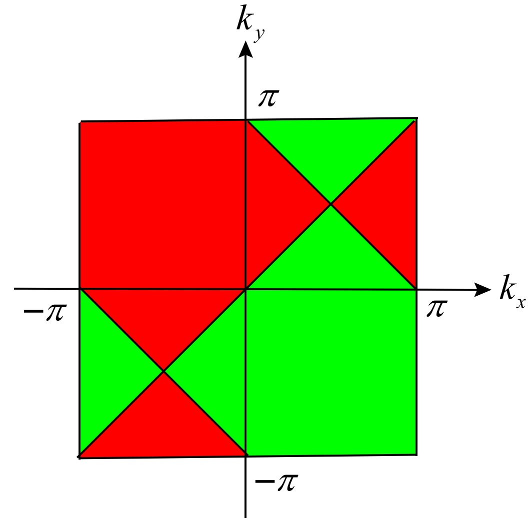

FIG. 1. Schematic diagram of the Brillouin zone in momentum space. $\mathbf{k} = (k_x, k_y)$, and $k_x, k_y \in (-\pi, \pi]$; the whole region can be divided into two parts according to the sign of $(\sin k_x - \sin k_y)$. $\mathbf{k}$ in the red region satisfies $\sin k_x - \sin k_y < 0$, which is called region A in this paper.

The Hamiltonian can be written in the form

$$H = 2 \sum_{\mathbf{k}\in A} \left[(c_{\mathbf{k}}^{\dagger} \quad c_{-\mathbf{k}}) h_{\mathbf{k}} \begin{pmatrix} c_{\mathbf{k}} \\ c_{-\mathbf{k}}^{\dagger} \end{pmatrix} - 2t \cos k_{+} \cos(\phi - k_{-}) \right], \quad (3)$$

where the domain of the summation is $A = \{\mathbf{k}, |\sin k_x - \sin k_y < 0\}$ and is illustrated in Fig. 1. There are multiple choices for A, which ensures the simplicity of the form of

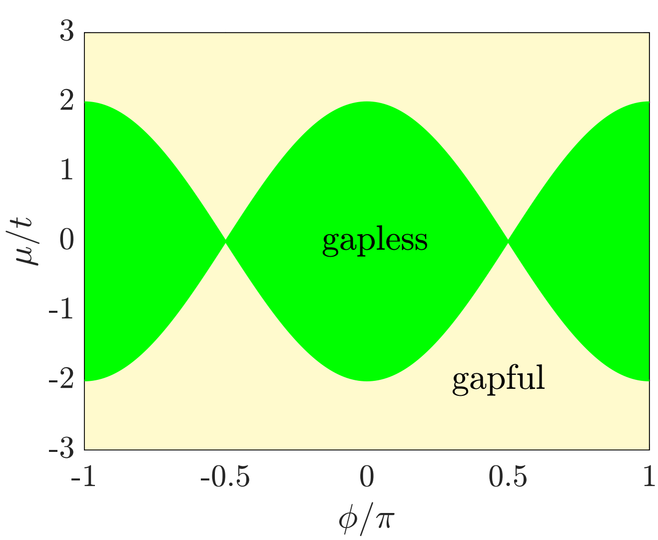

FIG. 2. Phase diagram of the Hamiltonian in Eq. (1) on the parameter ϕ-μ/t plane. Different colored regions represent different phases that are distinguished by whether an energy gap exists between two eigenstates with opposite $\mathbf{k}$. The boundaries are determined by the equation $|\mu/(2t)| = |\cos\phi|$. The green region is a nontrivial phase, in which the topological number is nonzero and the derivative of the eigenenergy is discontinuous.

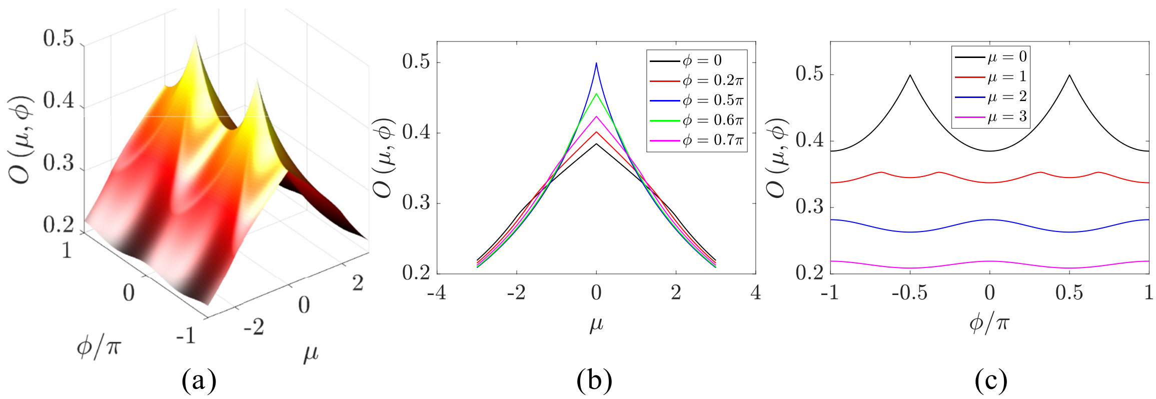

FIG. 3. (a) Color contour plots of the numerical results of order parameter $O_g(\mu, \phi)$ defined in Eq. (19). (b) Plots of $O_g(\mu, \phi_0)$ for several representative values of ϕ_0. (c) Plots of $O_g(\mu_0, \phi)$ for several different μ_0. The parameters are $N = 2000$, $t = 1$, and $\Delta = 2$. (a) shows that the order parameter reaches its maximum of 1/2 at the triple critical points $(\mu, \phi) = (0, \pm\pi/2)$. (b) and (c) indicate that O is smooth if the parameters are in the gapful region but has nonanalytic points in the gapless region.

the ground state. Here, the matrix $h_{\mathbf{k}}$ can be expressed as

$$h_{\mathbf{k}} = \sum_{i=0}^{3} B_i\sigma_i = \begin{pmatrix} \mu - 2t\cos k_+ \cos(\phi + k_-) & -2i\Delta \cos k_+ \sin k_- \\ 2i\Delta\cos k_+ \sin k_- & 2t\cos k_+ \cos(\phi - k_-) - \mu \end{pmatrix}; \quad (4)$$

according to the discussion in [28], our model is the $p_x - p_y$ superconductivity. We define modified wave vectors $k_\pm = (k_x \pm k_y)/2$ for the sake of simplicity, where $\{\sigma_i\}$ are Pauli matrices in the form

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

Here, the energy basis $B_0 = 2t\sin\phi\cos k_+ \sin k_-$, and the components of the auxiliary field $\mathbf{B}(\mathbf{k}) = (B_1, B_2, B_3)$ are

$$B_1 = 0,$$
$$B_2 = 2\Delta\cos k_+ \sin k_-,$$
$$B_3 = \mu - 2t\cos\phi\cos k_+ \cos k_-. \quad (6)$$

The Hamiltonian can be diagonalized in the form

$$H = \sum_{\mathbf{k}\in \mathrm{BZ}} [2(\varepsilon_{\mathbf{k}} + B_0)\left(\gamma_{\mathbf{k}}^\dagger \gamma_{\mathbf{k}} - \frac{1}{2}\right) + B_0] \quad (7)$$

by introducing the Bogoliubov operator

$$\gamma_{\mathbf{k}} = \cos\frac{\theta_{\mathbf{k}}}{2} c_{\mathbf{k}} + \sin\frac{\theta_{\mathbf{k}}}{2} e^{-i\varphi} c_{-\mathbf{k}}^\dagger, \quad (8)$$

which satisfies the commutation relations of the fermion within region A,

$$\{\gamma_{\mathbf{k}}, \gamma_{\mathbf{q}}^\dagger\} = \delta_{\mathbf{k},\mathbf{q}}, \ \{\gamma_{\mathbf{k}}, \gamma_{\mathbf{q}}\} = \{\gamma_{\mathbf{k}}^\dagger, \gamma_{\mathbf{q}}^\dagger\} = 0. \quad (9)$$

Here, the reduced dispersion relation is

$$\varepsilon_{\mathbf{k}} = \sqrt{B_2^2 + B_3^2}, \quad (10)$$

where $\theta_{\mathbf{k}}$ and φ are determined by

$$\tan\theta_{\mathbf{k}} = \frac{2\Delta|\cos k_+ \sin k_-|}{\mu - 2t\cos\phi\cos k_+ \cos k_-}, \quad (11)$$
$$\sin\varphi = \mathrm{sgn}(\cos k_+ \sin k_-). \quad (12)$$

If $\Delta > t > 0$, the ground state can be constructed as

$$|G(\mu,\phi)\rangle = \prod_{\mathbf{k}} \gamma_{\mathbf{k}}|0\rangle, \quad (13)$$

with the ground state energy

$$E_g = -\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}, \quad (14)$$

where $|0\rangle$ is the vacuum state of $c_{\mathbf{k}}$, satisfying $c_{\mathbf{k}}|0\rangle = 0$ for all $\mathbf{k}$. We note that the gapless ground state appears when $\varepsilon_{\mathbf{k}}$ has a zero point or band touching point of a single $\gamma_{\mathbf{k}}$-particle spectrum.

The band degenerate point $\mathbf{k}_0 = (k_{0+}, k_{0-})$ is determined by

$$\cos k_+ \sin k_- = 0,$$
$$\mu - 2t\cos\phi\cos k_+ \cos k_- = 0. \quad (15)$$

Two phases, gapless and gapful demonstrated in Fig. 2, with the boundary at the curve $|\mu/(2t)| = |\cos\phi|$ exist, i.e.,

$$|\mu| < |2t\cos\phi| \text{ (gapless)},$$
$$|\mu| > |2t\cos\phi| \text{ (gapful)}. \quad (16)$$

In the gapless phase, the degenerate points are isolated and can be classified as topologically trivial and nontrivial using topological numbers or invariants [18,29,30].

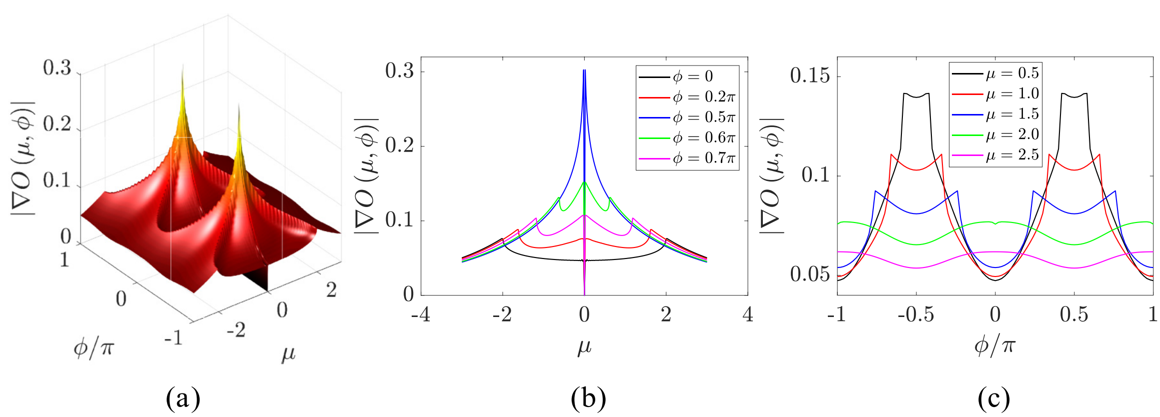

FIG. 4. (a) Color contour plots of numerical results of the absolute value of the gradient of the order parameter $|\nabla O_g(\mu,\phi)|$ defined in Eq. (23). (b) Plots of $|\nabla O_g(\mu,\phi_0)|$ for several representative values of ϕ_0. (c) Plots of $|\nabla O_g(\mu_0,\phi)|$ for several different μ_0. The parameters are $N=2000$, $t=1$, and $\Delta=2$. We can see that there are boundaries between different regions in (a), which are phase boundaries in Fig. 2, and $(\mu,\phi)=(0,\pm\pi/2)$ are nonanalytic points. (b) shows that phase boundaries are determined by the equations $|\mu|=|2t\cos\phi|$, and (c) indicates $|\nabla O_g(\mu,\phi)|$ has nonanalytic points at the boundaries if the parameters are in the gapless region.

III. BCS-PAIR ORDER PARAMETER

In this section, we investigate the phase diagram from another point of view. We introduce a BCS-pair order parameter, which is a quantity to describe the rate of the transition from free fermions to BCS pairs. It is related to a set of pseudospin operators, which is defined as

$$s_{\mathbf{k}}^{+}=(s_{\mathbf{k}}^{-})^{\dagger}=c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger},$$
$$s_{\mathbf{k}}^{z}=\tfrac{1}{2}(c_{-\mathbf{k}}^{\dagger}c_{-\mathbf{k}}+c_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}-1), \tag{17}$$

satisfying the Lie algebra commutation relations

$$[s_{\mathbf{k}}^{+},s_{\mathbf{k}}^{-}]=2s_{\mathbf{k}}^{z},\quad [s_{\mathbf{k}}^{z},s_{\mathbf{k}}^{\pm}]=\pm s_{\mathbf{k}}^{\pm}. \tag{18}$$

Obviously, $s_{\mathbf{k}}^{\pm}$ is the BCS-pair creation or annihilation operator in the $\mathbf{k}$ channel, while $s_{\mathbf{k}}^{z}$ relates to the pair number.

We introduce an order parameter

$$O_g=\frac{2}{N^2}\sum_{\mathbf{k}\in A}|\langle G(\mu,\phi)|s_{\mathbf{k}}^{+}|G(\mu,\phi)\rangle|, \tag{19}$$

which is the sum of the expectation absolute value of $c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger}$ for the ground state, characterizing the pairing progress. The exact expression of the ground state results in

$$O_g=\frac{\Delta}{N^2}\sum_{\mathbf{k}\in A}\frac{\sin k_y-\sin k_x}{\varepsilon_{\mathbf{k}}}, \tag{20}$$

which indicates that O_g is a quantity intimately related to the spectrum $\varepsilon_{\mathbf{k}}$. It is expected that the analytic behavior of O_g should demonstrate the phase diagram. To this end, we plot O_g in Fig. 3 and the corresponding derivatives,

$$\frac{\partial O_g}{\partial\mu}=\frac{1}{N^2}\sum_{\mathbf{k}\in A}\frac{B_2B_3}{\varepsilon_{\mathbf{k}}^3}, \tag{21}$$

$$\frac{\partial O_g}{\partial\phi}=\frac{1}{N^2}\sum_{\mathbf{k}\in A}\frac{J\sin\phi(\cos k_x+\cos k_y)B_2B_3}{\varepsilon_{\mathbf{k}}^3}, \tag{22}$$

and

$$|\nabla O_g(\mu,\phi)|=\sqrt{\left(\frac{\partial O_g}{\partial\mu}\right)^2+\left(\frac{\partial O_g}{\partial\phi}\right)^2}, \tag{23}$$

in Fig. 4. We find that O_g attains its maximum at the point $(0,\pi/2)$, and there is an evident jump at the quantum phase boundary.

The nonzero order parameter indicates only that there are some pairs of electrons with zero momentum in the ground state; when local pairs in a set are spatially separated from each other, it is only a pair state rather than a superconducting state. However, a k-space pair is a superposition of pairs with both long- and short-range corrections, which is different from a single short-range corrected real-space pair. Therefore, it is easy to condense pairs separated by long distances. In the following, we study the features of condensation from the perspective of real space.

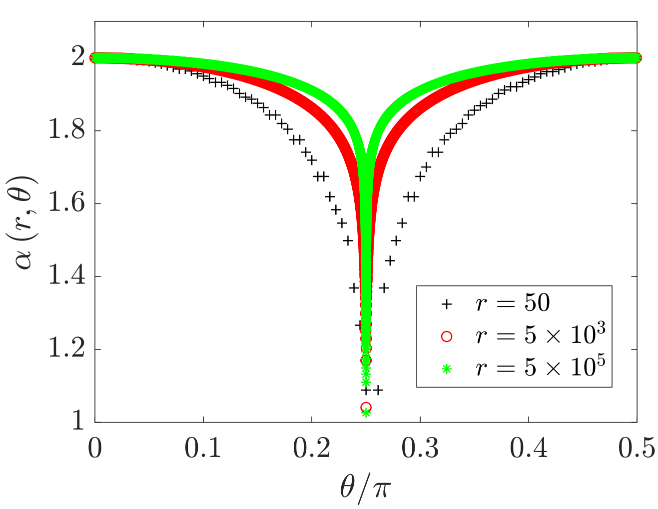

FIG. 5. The numerical results of polarity defined in Eq. (41). The results show that α is 2 for $\theta=0$ or $\theta=\pi/2$ but approaches 1 for $\theta=\pi/4$ as r increases, and there is a drastic change in the vicinity of $\pi/4$. This indicates that the correlation intension of pairs is different in different directions.

IV. COHERENT-STATE-LIKE GROUND STATE

We note that the expression of the order parameter O_g in Eq. (19) can also be written in the form

$$O_g = \frac{2}{N^2}|\langle G(\mu,\phi)|s^+|G(\mu,\phi)\rangle| \tag{24}$$

due to the fact that $\sin k_y - \sin k_x$ is always positive within region A. Here, a set of operators

$$s^{\pm} = \sum_{\mathbf{k}\in A} s_{\mathbf{k}}^{\pm},\, s^z = \sum_{\mathbf{k}\in A} s_{\mathbf{k}}^z \tag{25}$$

is still pseudospin operators, satisfying the Lie algebra commutation relations

$$[s^+, s^-] = 2s^z, \quad [s^z, s^{\pm}] = \pm s^{\pm}. \tag{26}$$

The finite value of O_g indicates that the ground state $|G(\mu,\phi)\rangle$ is probably the superposition of the set of states

$$|\psi_n\rangle = \frac{1}{\Omega_n}(s^+)^n|0\rangle, \quad n \in [0, N^2/2], \tag{27}$$

in some cases. In fact, the ground state can be expressed explicitly in terms of $\theta_{\mathbf{k}}$,

$$|G(\mu,\phi)\rangle = \prod_{\mathbf{k}\in A}\left(i\cos\frac{\theta_{\mathbf{k}}}{2} + \sin\frac{\theta_{\mathbf{k}}}{2}c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}\right)|0\rangle_{\mathbf{k}}|0\rangle_{-\mathbf{k}}, \tag{28}$$

which can be expressed as a tensor product state of a pair in real space. We consider an arbitrary tensor product state that has the form as follows

$$|\phi(\beta)\rangle = \prod_{l=1}^{m}[-ie^{i\alpha l}\sin(\beta/2)s_l^+ + \cos(\beta/2)]|\downarrow\rangle_l, \tag{29}$$

where the set of spin (or hard-core boson) operators $\{s_l^+\}$ of number m has actions $s_l^+|\downarrow\rangle_l = |\uparrow\rangle_l$, $s_l^+|\uparrow\rangle_l = 0$, and $(s_l^+)^{\dagger}|\downarrow\rangle_l = 0$. The parameters α and β are two arbitrary angles to determine the profile of the state. Direct derivation shows that it can also be written in the form

$$|\phi(\beta)\rangle = \sum_{n=1}^{m} d_n|\psi_n\rangle, \tag{30}$$

where a set of states $\{|\psi_n\rangle, n \in [0, m]\}$ can be constructed as

$$|\psi_n\rangle = \frac{1}{(n!)\sqrt{C_m^n}}\left(\sum_{l=1}^{m} e^{i\alpha l}s_l^+\right)^n \prod_{l=1}^{m}|\downarrow\rangle_l, \tag{31}$$

where $|\psi_n\rangle$ is defined as

$$d_n = \sqrt{C_m^n}(-i)^n \sin^n(\beta/2)\cos^{(m-n)}(\beta/2). \tag{32}$$

Note that $|\phi(\beta)\rangle$ is in the form of a coherent-state-like state, which always exhibits the feature of condensation in $\mathbf{k}$ space and ODLRD in real space. One can apply the above result to span part of the product state of the ground state,

$$\prod_{\mathbf{k}\in C}\left(i\cos\frac{\theta_{\mathbf{k}}}{2} + \sin\frac{\theta_{\mathbf{k}}}{2}c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}\right)|0\rangle_{\mathbf{k}}|0\rangle_{-\mathbf{k}}, \tag{33}$$

with $A \supseteq C$, in which $\theta_{\mathbf{k}}$ is $\mathbf{k}$ independent. For instance, taking $\mu = 0$, we have

$$\tan\theta_{\mathbf{k}} = \frac{\Delta}{t\cos\phi}\tan k_-, \tag{34}$$

which ensures the constancy of $\theta_{\mathbf{k}}$ if the set C corresponds to the line with fixed k_- in the Brillouin zone. In this work, we focus on an extreme case with $\phi = \pi/2$, which ensures $A = C$, and then

$$\begin{aligned}|G(0,\pi/2)\rangle &= \prod_{\mathbf{k}\in A}\frac{1 - ic_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}}{\sqrt{2}}|0\rangle_{\mathbf{k}}|0\rangle_{-\mathbf{k}} \\ &= \sum_{n=0}^{N^2/2}\frac{i^n 2^{-N^2/4}}{n!}(s^+)^n|0\rangle.\end{aligned} \tag{35}$$

It is obvious that $|G(0,\pi/2)\rangle$ represents the condensates of a collective BCS-like pair, which is a superposition of all possible pairs $c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger}$ in $\mathbf{k}$ space. Importantly, it can also be regarded as the condensate of a collective pair in real space due to the following fact. In the thermodynamic limit, a direct derivation from the Appendix shows

$$s^+ = \frac{2i}{\pi^2}\sum_{\mathbf{r},\mathbf{r}'}\frac{\sin[\pi(\Delta x + \Delta y)/2]}{(\Delta y)^2 - (\Delta x)^2}c_{\mathbf{r}}^{\dagger}c_{\mathbf{r}'}^{\dagger}, \tag{36}$$

where the displacement $\mathbf{r}' - \mathbf{r}$ has two components $(\Delta x, \Delta y)$. It is a superposition of all possible pairing configurations with $(\Delta x, \Delta y)$-dependent amplitudes and is then referred to as the collective pair operator. Notably, the amplitudes are zero when $\Delta x + \Delta y$ is even. The nonzero amplitudes decay as a power law with an exponent of 2 in the x and y directions and have strong polarity. In the following section, we study this feature by means of a correlation function.

V. CORRELATION LENGTH AND POLARITY

The exact expression for the ground state in Eq. (36) allows us to investigate the details of the fermion condensates. In this section, we calculate the particle-particle correlation function between two sites $(0, 0)$ and (x, y),

$$g(r,\theta) = |\langle G(0,\pi/2)|c_{(0,0)}^{\dagger}c_{(x,y)}^{\dagger}|G(0,\pi/2)\rangle|, \tag{37}$$

which represents the correlation intension of the pairs with distance $r = \sqrt{x^2 + y^2}$ in the θ direction. The coherent-state-like ground state yields

$$g(r,\theta) = \frac{1}{N^2}\left|\sum_{\mathbf{k}\in A}\sin(k_x x + k_y y)\right|, \tag{38}$$

which becomes

$$g(r,\theta) = \frac{1}{4\pi^2}\left|\int_{\mathbf{k}\in A}\sin(k_x x + k_y y)\mathrm{d}k_x\mathrm{d}k_y\right|, \tag{39}$$

in the thermodynamic limit. A straightforward derivation leads to the following result:

$$\begin{aligned}g(r,\theta) &= \frac{2}{\pi^2|y^2 - x^2|} \quad (x + y = \text{odd}) \\ &= \frac{2}{\pi^2 r^2|\cos(2\theta)|},\end{aligned} \tag{40}$$

a power law decay, indicating an infinite correlation length. This agrees with the fact that $(\mu,\phi) = (0,\pi/2)$ is the critical point with a gapless spectrum in the phase diagram. Nevertheless, the relative correlation intension of the pairs in different directions, i.e., the exponent, is polarized and can be obtained from

$$\alpha(\infty,\theta) = -\lim_{r\longrightarrow\infty} \frac{\ln g(r,\theta)+\ln(\pi^2/2)}{\ln r}. \tag{41}$$

We have

$$\begin{aligned}\alpha(\infty,0) &= \alpha(\infty,\pi/2) = 2,\\ \alpha(\infty,\pi/4) &= 1.\end{aligned} \tag{42}$$

We plot the exponent in Eq. (41) in Fig. 5. We find that α is approximately 2 but only experiences a sharp dip at $\theta = \pi/4$.

In general, the particle-particle correlation measures the response of a state to the external field, and a local pair can be regarded as a quasiboson with an intrinsic structure. In this sense, the response should be anisotropic. Intuitively, the existence of polarity of the pairing may induce the anisotropic current density.

VI. CORRELATOR AND ODLRO

It is clear that state $|G\rangle$ is a coherent state of zero-momentum-pair condensation. In the following, we will show that such a state literally possesses ODLRO. To get more details about condensation, we introduce the correlator

$$C_{\mathbf{RR}'} = |\langle G(0,\pi/2)|(c_{\mathbf{r}}c_{\mathbf{r}+\Delta\mathbf{r}})^{\dagger}c_{\mathbf{r}'}c_{\mathbf{r}'+\Delta\mathbf{r}'}|G(0,\pi/2)\rangle|, \tag{43}$$

where $\mathbf{R} = \mathbf{r}+\Delta\mathbf{r}$ and $\mathbf{R}' = \mathbf{r}'+\Delta\mathbf{r}'$. It measures the correlation of two pairs and characterizes the condensation of pairs. Under the condition $|\mathbf{r}'-\mathbf{r}| \gg |\Delta\mathbf{r}|, |\Delta\mathbf{r}'|$, two pairs can be regarded as bosons located at $\mathbf{r}$ and $\mathbf{r}'$, i.e., $(c_{\mathbf{r}}c_{\mathbf{r}+\Delta\mathbf{r}})^{\dagger} \to b_{\mathbf{r}}^{\dagger}$ and $c_{\mathbf{r}'}c_{\mathbf{r}'+\Delta\mathbf{r}'} \longrightarrow b_{\mathbf{r}'}$. Then, for a boson condensate state

$$|b_{\mathbf{k}},n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{N}\sum_{\mathbf{r}} e^{i\mathbf{k}\cdot\mathbf{r}} b_{\mathbf{r}}^{\dagger}\right)^{n}|0\rangle, \tag{44}$$

we have

$$|\langle b_{\mathbf{k}},n|b_{\mathbf{r}}^{\dagger}b_{\mathbf{r}'}|b_{\mathbf{k}},n\rangle| \sim \frac{n}{N^2}, \tag{45}$$

which identifies the ODLRO of the state $|b,n\rangle$. In parallel, we can employ the correlator $C_{\mathbf{RR}'}$ to identify the condensation of fermion pairs. In the thermodynamic limit, the direct derivation of $C_{\mathbf{RR}'}$ in the Appendix shows that

$$C_{\mathbf{RR}'} = g(\Delta r,\theta)g(\Delta r',\theta') \tag{46}$$

for $|\mathbf{r}'-\mathbf{r}| \gg |\Delta\mathbf{r}|, |\Delta\mathbf{r}'|$. Then, we conclude that the ground state at the triple critical point possesses the exact ODLRO. In comparison with the real bosons, the values of $C_{\mathbf{RR}'}$ depend on the parity of $(\Delta r, \Delta r')$ and the orientation of (θ,θ') since the fermion pair is not a point particle and has its own intrinsic structure. In this sense, the correlator $C_{\mathbf{RR}'}$ not only can identify condensation but can also disclose the relation between pairing and condensation of p-wave superconductivity.

Obviously, the magnitude of $C_{\mathbf{RR}'}$ is determined by two factors: the density of the p-wave pairs and the correlation of the two pairs. In this sense, nonzero $C_{\mathbf{RR}'}$ indicates the nonzero p-wave pair density. In contrast, zero $C_{\mathbf{RR}'}$ cannot rule out the existence of p-wave pairs.

VII. SUMMARY

In summary, we have provided a concrete example to demonstrate the fermionic condensate in a spinless system, which is different from the η-pairing mechanism in a spinful system, such as the Hubbard model. We have shown that the ground state can be expressed in two forms: (i) a BCS-like state, which consists of two fermions with opposite momenta, and (ii) a coherent-state-like form, which is a standard formalism for the condensation of a collective pairing mode. In this sense, such a pair mode can be regarded as an extension of the η-pairing mode. We also employed three quantities to characterize the ground state in the thermodynamic limit. The BCS-pair order parameter measures the average number of pairs with zero momentum and can be used to identify the phase diagram; however, the order parameter cannot provide condensation information. The two-operator correlation function characterizes the features of pairing of p-wave superconductivity and reveals the components and the polarity of the pairing. The four-operator correlator not only confirms the ODLRO as expected but also supplies us with more complete information about the condensation of pairs in real space. In our model, the fact that the correlator of two pairs is the product of the two two-operator correlations is interesting, which may stimulate further study. Taken together, these results unveil the properties of the p-wave superconductivity state, including the pairing mechanism and fermionic condensation.

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APPENDIX

In this Appendix, we present a derivation of the expression of s^+ in coordinate space, Eq. (25) in the main text, on the basis of which we obtained the two correlation functions in Eqs. (37) and (43).

Applying the Fourier transformation

$$c_{\mathbf{k}} = \frac{1}{N}\sum_{\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{r}}, \tag{A1}$$

we have

$$\begin{aligned} s^+ &= \sum_{\mathbf{k}\in A} c_{-\mathbf{k}}^{\dagger}c_{\mathbf{k}}^{\dagger} = \frac{1}{N^2}\sum_{\mathbf{k}\in A}\sum_{\mathbf{r},\mathbf{r}'} e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}c_{\mathbf{r}}^{\dagger}c_{\mathbf{r}'}^{\dagger}\\ &= \frac{i}{N^2}\sum_{\mathbf{k}\in A}\sum_{\mathbf{r},\mathbf{r}'} \sin[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]c_{\mathbf{r}}^{\dagger}c_{\mathbf{r}'}^{\dagger}\\ &= i\sum_{\mathbf{r},\mathbf{r}'} \Gamma(\mathbf{r}'-\mathbf{r})c_{\mathbf{r}}^{\dagger}c_{\mathbf{r}'}^{\dagger}, \end{aligned} \tag{A2}$$

where $\Gamma(\mathbf{r}'-\mathbf{r})$ is only a function of $\mathbf{r}'-\mathbf{r} = (\Delta x, \Delta y)$,

$$\Gamma(\mathbf{r}'-\mathbf{r}) = \frac{1}{N^2}\sum_{\mathbf{k}\in A}\sin[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]. \tag{A3}$$

In the thermodynamic limit, it can be integrated as a function of $(\Delta x, \Delta y)$,

$$\Gamma(\mathbf{r}'-\mathbf{r}) = \frac{1}{4\pi^2}\int_{\mathbf{k}\in A} \sin[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]\mathrm{d}^2\mathbf{k} = \frac{2}{\pi^2}\frac{\sin[\pi(\Delta x+\Delta y)/2]}{(\Delta y)^2-(\Delta x)^2}, \tag{A4}$$

and then we have

$$s^+ = \frac{2i}{\pi^2}\sum_{\mathbf{r},\mathbf{r}'}\frac{\sin[\pi(\Delta x+\Delta y)/2]}{(\Delta y)^2-(\Delta x)^2}c^\dagger_{\mathbf{r}}c^\dagger_{\mathbf{r}'}. \tag{A5}$$

On the other hand, inversely, we have

$$c_{\mathbf{r}}c_{\mathbf{r}'} = \frac{1}{N^2}\sum_{\mathbf{k},\mathbf{k}'} e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}c_{\mathbf{k}}c_{\mathbf{k}'} = \frac{1}{N^2}\sum_{\mathbf{k},\mathbf{k}'\in A,(\mathbf{k}'\neq\mathbf{k})}\{[e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{k}'\cdot\mathbf{r}'}-(\mathbf{k}\leftrightarrows-\mathbf{k}')]c_{\mathbf{k}}c_{-\mathbf{k}'}$$
$$+ e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}c_{\mathbf{k}}c_{\mathbf{k}'} + (\mathbf{k},\mathbf{k}'\to-\mathbf{k},-\mathbf{k}')\} + \frac{2i}{N^2}\sum_{\mathbf{k}\in A}\sin[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]c_{-\mathbf{k}}c_{\mathbf{k}}, \tag{A6}$$

and then

$$c_{\mathbf{r}}c_{\mathbf{r}'} = \frac{1}{N^2}\sum_{\mathbf{k},\mathbf{k}'\in A,(k'_x>k_x)}\{[e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}'\cdot\mathbf{r}'}-(\mathbf{k}\leftrightarrows\mathbf{k}')]c_{\mathbf{k}}c_{\mathbf{k}'} + [e^{-i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{k}'\cdot\mathbf{r}'}-(\mathbf{k}\leftrightarrows\mathbf{k}')]c_{-\mathbf{k}}c_{-\mathbf{k}'}$$
$$+ [e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\mathbf{k}'\cdot\mathbf{r}'}-(\mathbf{k}\leftrightarrows-\mathbf{k}')]c_{\mathbf{k}}c_{-\mathbf{k}'} + [e^{i\mathbf{k}'\cdot\mathbf{r}}e^{-i\mathbf{k}\cdot\mathbf{r}'}-(\mathbf{k}\leftrightarrows-\mathbf{k}')]c_{\mathbf{k}'}c_{-\mathbf{k}}\} + \frac{2i}{N^2}\sum_{\mathbf{k}\in A}\sin[\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})]c_{-\mathbf{k}}c_{\mathbf{k}}. \tag{A7}$$

Taking $\mathbf{r}=0$ and $\mathbf{r}'=\mathbf{r}$, this relation directly results in

$$g(r,\theta) = |\langle G(0,\pi/2)|c^\dagger_0 c^\dagger_{\mathbf{r}}|G(0,\pi/2)\rangle| = \left|\frac{2i}{N^2}\sum_{\mathbf{k}\in A}\langle G(0,\pi/2)|\sin(\mathbf{k}\cdot\mathbf{r})c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}}|G(0,\pi/2)\rangle\right|$$
$$= \frac{2}{\pi^2|y^2-x^2|} = \frac{2}{\pi^2 r^2|\cos(2\theta)|} \quad (x+y=\text{odd}). \tag{A8}$$

In addition, applying the relation to four operators

$$(c_{\mathbf{r}}c_{\mathbf{r}+\Delta\mathbf{r}})^\dagger c_{\mathbf{r}'}c_{\mathbf{r}'+\Delta\mathbf{r}'} = \frac{1}{N^4}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} e^{-i\mathbf{k}_1\cdot(\mathbf{r}+\Delta\mathbf{r})}e^{-i\mathbf{k}_2\cdot\mathbf{r}}e^{i\mathbf{k}_3\cdot\mathbf{r}'}e^{i\mathbf{k}_4\cdot(\mathbf{r}'+\Delta\mathbf{r}')}c^\dagger_{\mathbf{k}_1}c^\dagger_{\mathbf{k}_2}c_{\mathbf{k}_3}c_{\mathbf{k}_4} \tag{A9}$$

results in

$$\langle G(0,\pi/2)|c^\dagger_{\mathbf{k}_1}c^\dagger_{\mathbf{k}_2}c_{\mathbf{k}_3}c_{\mathbf{k}_4}|G(0,\pi/2)\rangle = \frac{1}{4}\begin{cases}\delta_{\mathbf{k}_1,\mathbf{k}_4}\delta_{\mathbf{k}_2,\mathbf{k}_3}, (|k_{1x}|>|k_{2x}|, |k_{4x}|>|k_{3x}|),\\ -\delta_{\mathbf{k}_3,\mathbf{k}_1}\delta_{\mathbf{k}_2,\mathbf{k}_4}, (|k_{1x}|<|k_{2x}|, |k_{4x}|>|k_{3x}|),\\ -\delta_{\mathbf{k}_3,\mathbf{k}_1}\delta_{\mathbf{k}_2,\mathbf{k}_4}, (|k_{1x}|>|k_{2x}|, |k_{4x}|<|k_{3x}|),\\ \delta_{\mathbf{k}_1,\mathbf{k}_4}\delta_{\mathbf{k}_2,\mathbf{k}_3}, (|k_{1x}|<|k_{2x}|, |k_{4x}|<|k_{3x}|),\\ 2\delta_{\mathbf{k}_3,-\mathbf{k}_4}\delta_{\mathbf{k}_1,-\mathbf{k}_2}\delta_{\mathbf{k}_2,\mathbf{k}_3},\\ -2\delta_{\mathbf{k}_3,-\mathbf{k}_4}\delta_{\mathbf{k}_1,-\mathbf{k}_2}\delta_{\mathbf{k}_2,-\mathbf{k}_3},\\ S(\mathbf{k}_3)S(\mathbf{k}_2)\delta_{\mathbf{k}_3,-\mathbf{k}_4}\delta_{\mathbf{k}_1,-\mathbf{k}_2}, (\mathbf{k}_3\neq\pm\mathbf{k}_2),\end{cases} \tag{A10}$$

where S is defined as

$$S(\mathbf{k}) = \mathrm{sgn}(\sin k_x - \sin k_y). \tag{A11}$$

Thus, we have

$$C_{\mathbf{R}\mathbf{R}'} = |\frac{1}{N^4}\sum_{\mathbf{k}_2,\mathbf{k}_3\in A,(\mathbf{k}_3\neq\mathbf{k}_2)}\sin(\mathbf{k}_2\cdot\Delta\mathbf{r})\sin(\mathbf{k}_3\cdot\Delta\mathbf{r}') + \frac{1}{4N^4}\sum_{|k_{1x}|\neq|k_{2x}|}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{r}'-\mathbf{r})}[e^{i\mathbf{k}_1\cdot(\Delta\mathbf{r}'-\Delta\mathbf{r})}-e^{i\mathbf{k}_2\cdot\Delta\mathbf{r}'}e^{-i\mathbf{k}_1\cdot\Delta\mathbf{r}}]|, \tag{A12}$$

where $\mathbf{R}=\mathbf{r}+\Delta\mathbf{r}$ and $\mathbf{R}'=\mathbf{r}'+\Delta\mathbf{r}'$. In the thermodynamic limit, it can be integrated as

$$C_{\mathbf{R}\mathbf{R}'} = |\Gamma(\Delta\mathbf{r}')\Gamma(\Delta\mathbf{r}) + \tfrac{1}{4}(\delta_{\mathbf{r}',\mathbf{r}}\delta_{\Delta\mathbf{r},\Delta\mathbf{r}'} - \delta_{\mathbf{r}',\mathbf{r}+\Delta\mathbf{r}}\delta_{\mathbf{r},\mathbf{r}'+\Delta\mathbf{r}'})|. \tag{A13}$$

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