# Exact emergent higher-form symmetries in bosonic lattice models

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Although condensed matter systems usually do not have higher-form symmetries, we show that, unlike 0-form symmetry, higher-form symmetries can emerge as exact symmetries at low energies and long distances. In particular, emergent higher-form symmetries at zero temperature are robust to arbitrary local UV perturbations in the thermodynamic limit. This result is true for both invertible and noninvertible higher-form symmetries. Therefore emergent higher-form symmetries are *exact emergent symmetries*: they are not UV symmetries but constrain low-energy dynamics as if they were. Since phases of matter are defined in the thermodynamic limit, this implies that a UV theory without higher-form symmetries can have phases characterized by exact emergent higher-form symmetries. We demonstrate this in three lattice models, the quantum clock model and emergent $\mathbb{Z}_N$ and U(1) $p$-gauge theory, finding regions of parameter space with exact emergent (anomalous) higher-form symmetries. Furthermore, we perform a generalized Landau analysis of a 2+1D lattice model that gives rise to $\mathbb{Z}_2$ gauge theory. Using exact emergent 1-form symmetries accompanied by their own energy/length scales, we show that the transition between the deconfined and Higgs/confined phases is continuous and equivalent to the spontaneous symmetry-breaking transition of a $\mathbb{Z}_2$ symmetry, even though the lattice model has no symmetry. Also, we show that this transition line must *always* contain two parts separated by multi-critical points or other phase transitions. We discuss the physical consequences of exact emergent higher-form symmetries and contrast them to emergent 0-form symmetries. Lastly, we show that emergent 1-form symmetries are no longer exact at finite temperatures, but emergent $p$-form symmetries with $p \geqslant 2$ are.

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## I. INTRODUCTION

A longstanding pillar for understanding strongly interacting quantum many-body systems is to identify and understand their symmetries. Indeed, symmetries provide powerful constraints and universal characterizations of a system's dynamics and phases. This point of view has become increasingly fruitful with modern generalizations of symmetry [1–19] (see Refs. [20,21] for recent reviews). For instance, topological order [22], which provided the first indication that conventional symmetries [23,24] are not all-powerful, can now be understood in a symmetry framework [2,8,12,20,25–29]. These generalizations open up an exciting frontier for the discovery of new phases of quantum matter and the conceptual organization/systematic understanding [16] of quantum phases.

One of the simplest generalizations of symmetry is called higher-form symmetry. For ordinary symmetries, charged operators act on a point in space and the unitary operator that generates the symmetry transformation acts on all of space. Higher-form symmetries generalize this by allowing charged operators to be extended [1–3]. For a $p$-form symmetry, the charged operators act on $p$-dimensional subspaces and the unitary generating the transformation acts on a closed $(d-p)$-dimensional subspace of $d$-dimensional space. So, an ordinary symmetry is just a 0-form symmetry.

Most things 0-form symmetries can do, higher-form symmetries can also do. For example, higher-form symmetries can spontaneously break, giving rise to a topological ground state degeneracy (gapless Goldstone bosons) when discrete (continuous) [3,30–36]. Indeed, Abelian topological orders reflect anomalous discrete 1-form symmetries spontaneously breaking, and photons in a Coulomb phase arise from U(1) 1-form symmetries spontaneously breaking. A higher-form symmetry can also have a 't Hooft anomaly, providing powerful constraints on the IR through generalized Lieb-Schultz-Mattis-Oshikawa-Hastings theorems and introducing higher-form symmetry-protected topological phases [4,25,26,37–47].

These applications of higher-form symmetries make them a powerful tool in studying quantum many-body systems. However, unfortunately, models with exact higher-form symmetries are rather special and, in a sense, fine-tuned. So, it is natural to wonder if they play a role in more typical, physically relevant models.

One possibility is that while they may not be exact microscopic symmetries, they could still arise as emergent symmetries. However, experience with emergent ordinary (0-form) symmetries causes apprehension since their consequences are typically approximate since they can be violated by irrelevant operators.[1] In other words, explicitly breaking 0-form symmetries creates $\mathcal{O}(E^\gamma)$ errors at energy scale $E$, even in the thermodynamic limit. Amazingly, common folklore suggests that this 0-form symmetry-based intuition does not carry over to higher-form symmetries. They can constrain a system exactly even as emergent symmetries, as discussed

[1] A counterexample to this usual rule are emanant symmetries [48].

 

in the context of gauge theories [49–51] in early days, and in the context of higher-form symmetry [20,25,36,48,52–56].

Here we investigate this robustness of higher-form symmetries in detail and from a lattice perspective, considering bosonic lattice Hamiltonian models without higher-form symmetries. These UV-complete theories are simple, well-defined, and relevant to condensed matter physics.

For this class of models, we demonstrate how higher-form symmetries can emerge and, when they do, why they constrain the IR exactly. We find that at energies with the emergent higher-form symmetry, the dynamics of states are affected by the emergent higher-form symmetry as if it were an exact UV symmetry. More precisely, any errors coming from the higher-form symmetries being emergent below a finite energy scale $E$ are of order $\mathcal{O}(\mathrm{e}^{-L^\gamma})$, where $L$ is the system size measured by lattice constant, and thus vanish in the thermodynamic limit. Our arguments apply to both invertible and noninvertible higher-form symmetries.

Therefore phases of microscopic models without exact higher-form symmetries can be exactly characterized by emergent higher-form symmetries. To emphasize this, we refer to emergent higher-form symmetries as exact emergent symmetries. Consequently, to understand how emergent higher-form symmetries characterize phases, one should first partition parameter space by the theory's exact *and* exact emergent symmetries. These partitions then lay a foundation for the system's phases to be labeled and characterized using generalized symmetries.

The rest of this paper goes as follows. In Sec. II, we show that emergent higher-form symmetries in lattice models are exact. We use the point of view that symmetries are described by algebras of local symmetric operators [57], but also develop low-energy effective Hamiltonians with the emergent symmetries. In Sec. III, we consider three examples with exact emergent higher-form symmetries: the $\mathbb{Z}_N$ quantum clock model and models of emergent $\mathbb{Z}_N$ and U(1)$p$-gauge theory. In Sec. IV, we use the concept of exact emergent symmetry to perform a complete generalized Landau analysis of the Fradkin-Shenker model with periodic boundary conditions. We recover known results regarding the universality classes of the phase transitions and make new predictions about the phase diagram's general structure. In Sec. V, we discuss general physical consequences of emergent higher-form symmetries being exact. In particular, how exact emergent higher-form symmetries can characterize phases of systems, both when they are, and are not, spontaneously broken in the bulk. In Sec. VI, we consider emergent higher-form symmetries at finite temperature, showing that only emergent $p$-form symmetries with $p > 1$ are exact. Then, in Sec. VII, we conclude and discuss some open questions arising from this work.

## II. SCALES HIERARCHIES AND EMERGENT SYMMETRIES

Consider a lattice bosonic quantum system described by the local Hamiltonian $H$ and whose total Hilbert space $\mathcal{V}$ is tensor product decomposable: $\mathcal{V} = \bigotimes_i \mathcal{V}_i$. Since $H$ includes the exact interactions at the microscopic scale and describes the system at all energies throughout the entire parameter space, we refer to it as the UV Hamiltonian, adopting the language used in field theory. While, in theory, the system's physical properties can be extracted from $H$, this proves much too difficult in practice [58].

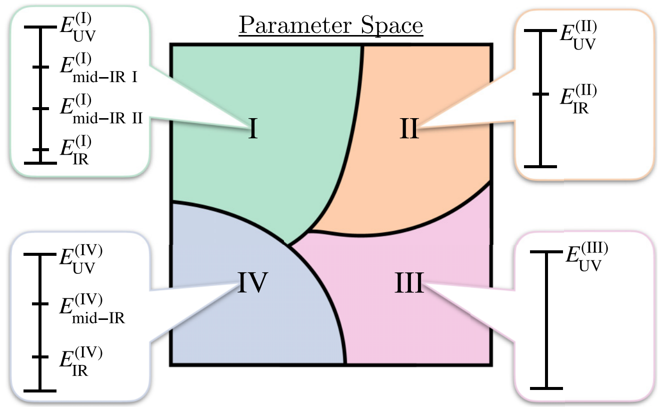


FIG. 1. The parameter space of a many-body Hamiltonian can be partitioned by its differing hierarchies of energy scales. A schematic depiction of this is shown here. The parameter space is partitioned into four regions, labeled I, II, III, and IV, with their differing energy scale hierarchies shown.

A guiding principle to overcome this daunting problem is the separation of energy scales (assuming no UV/IR mixing [10,59–62]). We will always denote the lowest energy scale as $E_{\mathrm{IR}}$ and refer to the sub-Hilbert space $\mathcal{V}_{\mathrm{IR}} = \mathrm{span}\{|E_n\rangle \mid E_n < E_{\mathrm{IR}}\}$, where $|E_n\rangle$ is an energy eigenstate, as the IR. Furthermore, we will always denote the largest possible energy value as $E_{\mathrm{UV}}$. However, there can be other interesting energy scales between the IR and the UV scales, which we will call mid-IR energies $E_{\mathrm{mid-IR}}$ and refer to the sub-Hilbert space $\mathcal{V}_{\mathrm{mid-IR}} = \mathrm{span}\{|E_n\rangle \mid E_n < E_{\mathrm{mid-IR}}\}$ as the mid-IR. Generally, there can be multiple of these mid-IR scales, and as demonstrated in Fig. 1, different regions of parameter space will have a different hierarchy of energy scales.

### A. Exact emergent generalized symmetries

Low-energy eigenstates will often have additional structures absent from high-energy eigenstates. For example, these additional structures could reflect the presence of new, emergent symmetries, as depicted in Fig. 2.

It is nontrivial to systematically identify the scale hierarchies of a general UV Hamiltonian. Here we will specialize to a typical situation where the UV Hamiltonian can be written as

$$H = H_0 + H_1, \tag{1}$$

where a mid-IR scale $E_{\mathrm{mid-IR}}$ of $H_0$ is known (e.g., an energy gap of a quasiparticle) and both $H_0$ and $H_1$ are translation-invariant. We assume $H_0$ is not pathological in the mid-IR and that the qualitative features of $H$ resemble those of $H_0$. For example, $H_0$ cannot have perfectly flat bands in the mid-IR since they'd introduce exponential amounts of degeneracies in the spectra, which will not arise in $H$ since a generic $H_1$ will lift the degeneracy. Furthermore, we assume there is a collection of mutually commuting local projection operators $\{\mathcal{P}_i\}$ acting only on degrees of freedom near site $i$ such

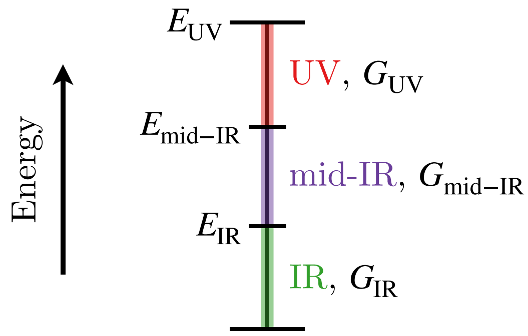


FIG. 2. The symmetries of a quantum many-body system generally depend on the energy scale of an observer. In particular, there can be emergent symmetries at low energies absent from the microscopic (UV) symmetries $G_{\mathrm{UV}}$. These can be generalized symmetries and can be anomalous.

that $\mathcal{V}_{\mathrm{mid-IR}}^{(H_0)}$ is spanned by energy eigenstates of $H_0$ satisfying $\langle\mathcal{P}_i\rangle = 0$. In other words, $|\psi_{\mathrm{mid-IR}}\rangle \in \mathcal{V}_{\mathrm{mid-IR}}^{(H_0)}$ satisfy $\mathcal{P}_i|\psi_{\mathrm{mid-IR}}\rangle = 0$, while $\mathcal{P}_i|\psi\rangle \neq 0$ for $|\psi\rangle \notin \mathcal{V}_{\mathrm{mid-IR}}^{(H_0)}$. Features of $H_0$ that emerge at $E < E_{\mathrm{mid-IR}}$ are determined by the constraint $\mathcal{P}_i = 0$. In fact, as we will soon argue, these local projection operators also determine the emergent symmetries.

We will assume $H_1$ includes terms that mix states with $\langle\mathcal{P}_i\rangle = 0$ and states with $\langle\mathcal{P}_i\rangle \neq 0$. Because of $H_1$, energy eigenstates of $H$ are a superposition of states with $\langle\mathcal{P}_i\rangle = 0$ and states with $\langle\mathcal{P}_i\rangle \neq 0$ and, therefore, the sub-Hilbert space spanned by states satisfying $\langle\mathcal{P}_i\rangle = 0$ is not a mid-IR of $H$. Consequently, it appears that any emergent structures arising from $\mathcal{P}_i = 0$ (such as symmetry) are destroyed by the $H_1$ term.

On the other hand, if all parameters in $H_1$ are much smaller than those in $H_0$, it is tempting to think that the mid-IR of $H$ is typically closely related to the $\langle\mathcal{P}_i\rangle = 0$ states. This intuition motivates one to introduce the parameter $s \in [0, 1]$ and family of Hamiltonians

$$H(s) = H_0 + s\,H_1, \tag{2}$$

from which the mid-IR of $H$ can be constructed from the mid-IR of $H_0$ by slowly tuning $s = 0$ to $s = 1$ [50]. Indeed, let us denote the $n$th many-body energy eigenstate of $H(s)$ as $|\psi_n^{(s)}\rangle$ and define the unitary operator $V_s = \sum_n |\psi_n^{(s)}\rangle\langle\psi_n^{(0)}|$ which satisfies $V_s|\psi_n^{(0)}\rangle = |\psi_n^{(s)}\rangle$ and

$$\langle\psi_n^{(0)}|A|\psi_n^{(0)}\rangle = \langle\psi_n^{(s)}|V_s A V_s^\dagger|\psi_n^{(s)}\rangle \tag{3}$$

for any operator $A$. Therefore the $\langle\mathcal{P}_i\rangle = 0$ sector of $H_0$ is related to the $\langle V_1\mathcal{P}_i V_1^\dagger\rangle = 0$ sector of $H$. However, this definition of $V_s$ is unphysical since $V_s\mathcal{P}V_s^\dagger$ is likely to be nonlocal even if $\mathcal{P}$ is local. Reference [50] found a local unitary operator $U_{\mathrm{LU}}$ that approximates $V_s$ very well while ensuring local operators remain local when dressed. An explicit form of $U_{\mathrm{LU}}$ is [63]

$$\begin{aligned} U_{\mathrm{LU}} &= \mathcal{S}'\left\{\exp\left[\mathrm{i}\int_0^1 \mathrm{d}s'\,\mathcal{D}_{s'}\right]\right\},\\ \mathcal{D}_s &\equiv \mathrm{i}\int \mathrm{d}t\,F(t)\mathrm{e}^{\mathrm{i}H(s)t}\partial_s H(s)\mathrm{e}^{-\mathrm{i}H(s)t}, \end{aligned} \tag{4}$$

where $\mathcal{S}'$ denotes $s'$-ordering and $F(t)$ is a function satisfying a particular set of requirements, such as $F(t) = -F(-t)$ which ensures $\mathcal{D}_s$ is anti-Hermitian.

Motivated by those results, here we assume that there exists a proper local unitary operator $U_{\mathrm{LU}}$ with the following properties:

(1) it maps a local operator $O_i$ to a local operator $\widetilde{O}_i \equiv U_{\mathrm{LU}} O_i U_{\mathrm{LU}}^\dagger$ that acts on degrees of freedom near $i$ (the operator is fattened);

(2) it maps the $n$th eigenstate of $H(0)$ to a superposition of some eigenstates $|\psi_{n'}^{(1)}\rangle$ of $H(1)$ with energy $E_n^{(1)} - \delta < E_{n'}^{(1)} < E_n^{(1)} + \delta$, where $E_n^{(1)}$ is the energy of the $n$th eigenstate of $H(1)$ and $\delta \ll E_{\mathrm{min-IR}}$;

(3) it does not break the symmetries of $H_0$ and $H_1$.

If such a unitary operator satisfying these properties does not exist for a particular $H$ in parameter space, it means that the mid-IR does not exist at that point of parameter space (e.g., due to the gapped quasiparticles defining $E_{\mathrm{mid-IR}}$ condensing). The existence of $U_{\mathrm{LU}}$ is a conjecture, and we will obtain our results based on this conjecture. It would be interesting to see if one could apply the mathematical techniques and proofs developed in Ref. [64] to construct $U_{\mathrm{LU}}$ rigorously.

Consider an eigenstate $|\psi_n^{(0)}\rangle$ of $H(0)$ with energy $E_n^{(0)} \ll E_{\mathrm{mid-IR}}$, thus satisfying $\mathcal{P}_i|\psi_n^{(0)}\rangle = 0$. $U_{\mathrm{LU}}$ will map it to some eigenstates of $H(1)$ with eigenvalues much less then $E_{\mathrm{mid-IR}}$, which satisfy $\widetilde{\mathcal{P}}_i|\psi_n^{(1)}\rangle = 0$, where $\widetilde{\mathcal{P}}_i = U_{\mathrm{LU}}\mathcal{P}_i U_{\mathrm{LU}}^\dagger$ is also a set of mutually commuting local projectors. This is true for all mid-IR eigenstates of $H(0)$, so the mid-IR of $H$ can be identified as the sub-Hilbert space spanned by the mid-IR eigenstates of $H(0)$ transformed by $U_{\mathrm{LU}}$. In other words, the mid-IR states of $H$ which span $\mathcal{V}_{\mathrm{mid-IR}}^{(H)}$ satisfy $\widetilde{\mathcal{P}}_i = 0$. Therefore any emergent low-energy structures of $H_0$ specified by the projectors $\mathcal{P}_i$ become emergent low-energy structures of $H$ specified by the projectors $\widetilde{\mathcal{P}}_i$. Since $\{\mathcal{P}_i\}$ and $\{\widetilde{\mathcal{P}}_i\}$ are related by a local unitary transformation $U_{\mathrm{LU}}$, we expect the two exact low-energy structures to be equivalent.

This result was obtained in Ref. [50] for emergent $\mathbb{Z}_2$ and U(1) gauge symmetry, and proved rigorously for the $\mathbb{Z}_2$ case. In that case, the low-energy subspace satisfies the modified Gauss law $\widetilde{\rho}_i = U_{\mathrm{LU}}\rho_i U_{\mathrm{LU}}^\dagger = 0$ exactly and is exactly gauge invariant. We believe such a result remains valid for more general situations.

Having identified a mid-IR of $H$, we now identify its emergent symmetries at $E < E_{\mathrm{mid-IR}}$. It is useful to adopt the perspective that a symmetry is described/defined by an algebra of local symmetric operators [57,65,66]. For instance, if the UV symmetries are generated by the unitaries $\{U_g\}$, i.e., $[U_g, H_0] = [U_g, H_1] = 0$, then the associated algebra of local symmetric operators is

$$\mathcal{A}_{\mathrm{UV}} = \{O_{\mathrm{UV}}\,|\,O_{\mathrm{UV}}U_g = U_g O_{\mathrm{UV}}\,\forall\,g\}, \tag{5}$$

where $O_{\mathrm{UV}}$ is a local operator acting on the full Hilbert space $\mathcal{V}$. Indeed, given $\mathcal{A}$, one can recover the symmetry transformation operators by finding all operators that commute with its elements.

In the mid-IR, operators that violate the $\langle\widetilde{\mathcal{P}}_i\rangle = 0$ constraint are not allowed. Therefore the mid-IR symmetries are described by the algebra of local symmetric operators

$$\begin{aligned} \mathcal{A}_{\mathrm{mid-IR}} = \{&O_{\mathrm{mid-IR}}\,|\,O_{\mathrm{mid-IR}}\widetilde{\mathcal{P}}_i = \widetilde{\mathcal{P}}_i O_{\mathrm{mid-IR}}\,\forall\,i,\\ &\forall\,O_{\mathrm{mid-IR}} \in \mathcal{A}_{\mathrm{UV}}\}. \end{aligned} \tag{6}$$

The symmetry transformations are then all operators that commute with the elements of $\mathcal{A}_{\text{mid}-\text{IR}}$ as well as the projectors $\widetilde{\mathcal{P}}_i$. These will include the UV symmetry operators but could include additional emergent symmetries, reflecting the possibility depicted in Fig. 2. We refer to emergent symmetries identified by $\mathcal{A}_{\text{mid}-\text{IR}}$ as *exact emergent symmetries* to emphasize that at $E < E_{\text{mid}-\text{IR}}$, they are equally impactful as UV symmetries. Indeed, given only $\mathcal{A}_{\text{mid}-\text{IR}}$, one cannot distinguish between UV symmetries and exact emergent symmetries. Importantly, exact emergent symmetries are not approximate symmetries.

The exact emergent symmetries found using $\mathcal{A}_{\text{mid}-\text{IR}}$ arise from the commuting projectors $\widetilde{\mathcal{P}}_i$. Since these projectors are local, 0-form symmetries will typically not appear as exact emergent symmetries. Indeed, since the charged operators of 0-form symmetries are local, these projectors would likely have to be nonlocal to forbid them from appearing in $\mathcal{A}_{\text{mid}-\text{IR}}$. In other words, even weakly breaking a 0-form symmetry in the UV theory, $\mathcal{A}_{\text{mid}-\text{IR}}$ will typically include terms charged under the symmetry, explicitly breaking the symmetry at all scales. This recovers how emergent 0-form symmetries generally are not exact and instead approximate symmetries.

However, this description of symmetry is capable of describing all generalizations of symmetries. So, the exact emergent symmetries can be higher-form symmetries.

The charged operators of higher-form symmetries are nonlocal, winding around nontrivial cycles of the lattice, and thus will never appear in $\mathcal{A}_{\text{mid}-\text{IR}}$. Therefore emergent higher-form symmetries are always exact symmetries in the thermodynamic limit since they cannot be broken by low-energy local operators. In other words, emergent higher-form symmetries are robust against translation-invariant local perturbations[2] of the UV theory and, thus, are topologically robust. All of this is true for both invertible and noninvertible higher-form symmetries.[3]

To build some intuition for why this is true, we consider a space-time picture [75,76]. Suppose $H_0$ has a 1-form symmetry, so loops carrying the symmetry charge cannot be cut open by unitary time evolution, in $(3+1)D$. In space-time, it means that its worldsheet will not have any holes. Turning on $H_1$ and explicitly breaking the 1-form symmetry, these loops can now be cut open so their worldsheets will have holes. When the perturbation $H_1$ is small, these holes are also small and can be coarse-grained away to yield a low-energy subspace with only closed worldsheets and an exact emergent 1-form symmetry. However, when the perturbation is large, these holes are larger than the worldsheets themselves, disintegrating them by the Higgs mechanism and preventing a 1-form symmetry from emerging. It is straightforward to generalize this to a general higher-form symmetry, and it would be interesting to study this space-time picture further using the renormalization group.

Furthermore, an exact emergent $p$-form symmetry below an energy scale $E$ in $d$ dimensional space implies the exact emergence of its dual symmetry below the same energy scale $E$, which is a $(d-p-1)$-form symmetry when the $p$-form symmetry is discrete. When $d-p-1=0$, this leads to an exact emergent dual 0-form symmetry.

For lattices with vacancy defects, there can be nontrivial $p$-cycles involving a finite number of $p$-cells. Then, operators charged under an emergent $p$-form symmetry could appear in $\mathcal{A}_{\text{mid}-\text{IR}}$, making it an approximate symmetry. However, if the vacancy defect density is small, these nontrivial $p$-cycles are also small. Therefore they will disappear under coarse-graining, and the emergent $p$-form symmetry will become exact.

Emergent higher-form symmetries, therefore, have an associated energy *and* length scale. The former is $E_{\text{mid}-\text{IR}}$, designating at which energies the symmetry emerges. The latter is the length scale that the symmetry's operators are fattened by $U_{\text{LU}}$. If this energy scale goes to zero (this length scale goes to infinity), the symmetry is explicitly broken at all scales (is no longer well defined) and cannot emerge.

One can possibly discover all of the emergent symmetries of a system in a Hamiltonian-independent way by constructing $\mathcal{A}$ at all energy scales for each energy hierarchy in parameter space. However, it is desirable to have a Hamiltonian description reflecting the emergent symmetries. The symmetries that emerge at $E < E_{\text{mid}-\text{IR}}$ are hidden from the UV Hamiltonian $H$ since it describes the dynamics of both states with $\widetilde{\mathcal{P}}_i = 0$ and $\widetilde{\mathcal{P}}_i = 1$. Therefore, to make the emergent symmetries manifest, we will develop an effective mid-IR theory $H_{\text{mid}-\text{IR}}$ that describes only the dynamics of states with $\widetilde{\mathcal{P}}_i = 0$. Since $H$ is a sum of only operators in $\mathcal{A}_{\text{UV}}$, the effective mid-IR theory $H_{\text{mid}-\text{IR}}$ should be a sum of only operators in $\mathcal{A}_{\text{mid}-\text{IR}}$. Therefore symmetries of $H_{\text{mid}-\text{IR}}$ will include the UV symmetries but also include additional ones, which we will identify as exact emergent symmetries.

The effective mid-IR Hamiltonian should act only on the mid-IR Hilbert space $\mathcal{V}_{\text{mid}-\text{IR}}$. The most general form for it is

$$H_{\text{mid}-\text{IR}} = \sum_{\mathcal{O}\in\mathcal{A}_{\text{mid}-\text{IR}}} (C_{\mathcal{O}}\,\mathcal{O} + \text{H.c.}). \tag{7}$$

The constants $\{C_{\mathcal{O}}\}$ are renormalized versions of the UV parameters. One can in principle determine $\{C_{\mathcal{O}}\}$ by requiring the spectra and correlation functions of $H_{\text{mid}-\text{IR}}$ to match with the UV theory's for $|\psi\rangle \in \mathcal{V}_{\text{mid}-\text{IR}}$. Since UV theory is local, we require the effective theory to be a local Hamiltonian. Therefore the greater number of operators involved in $\mathcal{O}$ or the larger the region of the lattice $\mathcal{O}$ acts on, the smaller $C_{\mathcal{O}}$ is. We view the ability to define a local effective mid-IR Hamiltonian as a requirement for the mid-IR itself to be well defined.

The mid-IR having an exact emergent symmetry means there is a transformation that leaves the $H_{\text{mid}-\text{IR}}$ unchanged. This implies the existence of an emergent conservation law obeyed at $E < E_{\text{mid}-\text{IR}}$ which cannot be broken by local operators. The existence of this exact emergent conservation law

[2]More precisely, any translation-invariant $k$-local perturbation where $k$ is much smaller than the linear system size measured in units of lattice spacing.

[3]An exact emergent $p$-form symmetry below an energy scale implies $q$-form symmetries ($q < p$) below that same energy scale generated by "condensation defects" [8,67–71]. While these are $q$-form symmetries, they transform $p$-dimensional operators. Therefore they too are exact emergent symmetries even when $q = 0$. These $p$-dimensional objects carry generalized charges of the $q$-form symmetry [72–74].

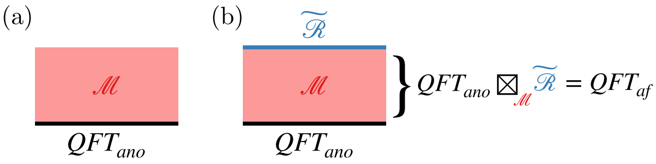


FIG. 3. (a) The low-energy properties of $QFT_{ano}$ ($QFT_{af}$ restricted to the $\mathcal{R}$-symmetric sub-Hilbert space) can be exactly simulated by a boundary of a topological order $\mathcal{M}$ in one higher dimension. $QFT_{ano}$ uniquely determines $\mathcal{M}$. (b) The low-energy properties of $QFT_{af}$ throughout all $\mathcal{R}$ sectors can be simulated by including the boundary $\widetilde{\mathcal{R}}$.

can be used as a definition of the existence of the exact emergent symmetry. For $p$-form symmetries, this conservation law means that at $E < E_{\text{mid}-\text{IR}}$, there are only closed $p$-branes excitations.

It is important to note that this effective Hamiltonian is different than those found using, for instance, Brillouin-Wigner perturbation theory [77] or Schrieffer-Wolff transformations [78]. Indeed, these effective Hamiltonians describe the mid-IR of $H_0$ (the $\langle \mathcal{P}_i \rangle = 0$ subspace), where as the effective Hamiltonian Eq. (7) describes the genuine mid-IR of $H$ (the $\langle \widetilde{\mathcal{P}}_i \rangle = 0$ subspace).

The philosophy behind $H_{\text{mid}-\text{IR}}$ is similar to that of effective field theory, where one writes down an effective action that includes all allowed terms. Its definition is physically reasonable but not rigorously derived. We will not present a rigorous justification or proof of $H_{\text{mid}-\text{IR}}$. Here, we state Eq. (7) with the restrictions on $C_{\mathcal{O}}$ as the conjectured form of $H_{\text{mid}-\text{IR}}$, and we will examine the consequences of this conjecture.

### B. A holographic picture

An algebra of local symmetric operators [e.g., Eqs. (5) and (6)] determines a nondegenerate braided fusion (higher) category $\mathcal{M}$ [57,79]. To emphasize its utility in describing a symmetry, $\mathcal{M}$ is called a *symmetry category* or *symmetry topological order*.[4] For a finite symmetry, its symmetry category $\mathcal{M}$ describes a topological order in one higher dimension, which we also denote by $\mathcal{M}$. References [16,81] proposed a unified description of all types of symmetries (including their higher-form, higher-group, anomaly, and noninvertibility properties) using such topological order in one higher dimension, which leads to a symmetry/topological order (Symm/TO) correspondence [79,80]. The bulk topological order (i.e., the symmetry topological-order) can be realized by a topological field theory (TFT). To emphasize its utility in describing symmetry, this bulk TFT is called symmetry TFT [82].

More precisely, given an anomaly-free system[5] $QFT_{af}$ with a symmetry, its low energy properties within the symmetric sub-Hilbert space are exactly simulated by a boundary of a topological order $\mathcal{M}$ in one higher dimension. Indeed, $QFT_{af}$ restricted within the symmetric sub-Hilbert space can be viewed as a new system $QFT_{ano}$ with a noninvertible gravitational anomaly [80,83], which corresponds to $\mathcal{M}$ in one higher dimension [84,85] [see Fig. 3(a)].

Fully simulating $QFT_{af}$ requires the boundary of $\mathcal{M}$ to also capture states outside the symmetric subspace. This can be achieved by adding an additional gapped boundary $\widetilde{\mathcal{R}}$ of $\mathcal{M}$ [16,19], as shown in Fig. 3(b). Provided that the topological order $\mathcal{M}$ and the boundary $\widetilde{\mathcal{R}}$ have an infinite energy gap, the low-energy properties of $QFT_{af}$ are described by the composition of the topological order $\mathcal{M}$ with two boundaries $QFT_{ano}$ and $\widetilde{\mathcal{R}}$, which we denote as $QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$.

If the spatial dimension of the boundary $\widetilde{\mathcal{R}}$ is $d$, its gapped excitations are described by a fusion $d$-category, which we will also denote as $\widetilde{\mathcal{R}}$. On the other hand, the excitations of $\mathcal{M}$ are described by a braided fusion $d$-category denoted as $\mathcal{M}$. The boundary fusion $d$-category $\widetilde{\mathcal{R}}$ uniquely determines the bulk braided fusion $d$-category $\mathcal{M}$, and are related to one another by

$$\mathcal{M} = \mathcal{Z}(\widetilde{\mathcal{R}}), \tag{8}$$

where $\mathcal{Z}(\widetilde{\mathcal{R}})$ is the center of $\widetilde{\mathcal{R}}$. When $d = 1$, $\mathcal{Z}(\widetilde{\mathcal{R}})$ is the Drinfeld center of $\widetilde{\mathcal{R}}$.

We say $QFT_{af}$ is described by the symmetry category $\mathcal{M}$ if admits a decomposition $QFT_{af} = QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$. The decomposition also implies that $QFT_{af}$ has a symmetry whose symmetry defects (i.e., symmetry transformations) are described by fusion $d$-category $\widetilde{\mathcal{R}}$. We will call such a symmetry as $\widetilde{\mathcal{R}}$-symmetry. For example, $d-\mathcal{V}\text{ec}_G$-symmetry is the ordinary global symmetry described by a group $G$.

If $\widetilde{\mathcal{R}}$ is a local[6] fusion $d$-category, then the $\widetilde{\mathcal{R}}$-symmetry is an anomaly-free symmetry. The symmetry charges of an anomaly-free $\widetilde{\mathcal{R}}$-symmetry are described by a fusion $d$-category $\mathcal{R}$, which is the dual of $\widetilde{\mathcal{R}}$. However, if $\widetilde{\mathcal{R}}$ is not local, the $\widetilde{\mathcal{R}}$-symmetry is anomalous. We note that in Ref. [19], the pair $(\widetilde{\mathcal{R}}, \mathcal{M})$ is regarded as a generalized symmetry regardless if $\widetilde{\mathcal{R}}$ is local or not. Thus such a description includes both anomaly-free and anomalous symmetries.

Using $QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$ to describe the symmetries of $QFT_{af}$ is very general and provides a unifying formalism capable of describing all generalizations of symmetry. As we will now argue, $QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$ is also able to describe the exact emergent symmetries of $QFT_{af}$ discussed in the previous subsection.

Recall from the previous subsection the general Hamiltonian (1), where a known mid-IR of $H_0$ was spanned by states satisfying $\mathcal{P}_i|\psi\rangle = 0$ for local commuting projectors $\{\mathcal{P}_i\}$. Any exact emergent symmetries in the mid-IR are determined

[4]Symmetry category was called *categorical symmetry* in Refs. [79,80]. Since the term categorical symmetry is now commonly used to mean noninvertible/algebraic symmetry, we rename *categorical symmetry* to *symmetry category* to avoid confusion.

[5]An anomaly-free system is one with a lattice UV completion.

[6]A fusion $d$-category $\widetilde{\mathcal{R}}$ is *local* if there exists a fusion $d$-category $\mathcal{R}$ such that $\mathcal{Z}(\mathcal{R}) = \mathcal{Z}(\widetilde{\mathcal{R}})$ and $\mathcal{R} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}} = n\mathcal{V}\text{ec}$, where $d\mathcal{V}\text{ec}$ is the braided fusion $d$-category describing excitations in a trivial topological order (i.e., above a trivial product state). Such $\mathcal{R}$ is called the dual of $\widetilde{\mathcal{R}}$.

entirely by $\{\mathcal{P}_i\}$, and the exact emergent symmetries of $H$ could be found by dressing $\{\mathcal{P}_i\}$ with $U_{\text{LU}}$. Here for simplicity, we will just consider $H_0$ since our results will hold even after we include an arbitrary translation-invariant perturbation $H_1$, as we discussed in the last subsection. Without a loss of generality, we will assume that $H_0$ has the form

$$H_0 = \sum_i O_i + E_{\text{mid-IR}} \sum_i \mathcal{P}_i, \tag{9}$$

where $O_i$ are local operators and $[O_i, \mathcal{P}_j] = 0$ is required since $O_i$ is assumed not to mix the $\mathcal{P}_i = 0$ and $\mathcal{P}_i = 1$ states. Thus the local energy dynamics controlled by $O_i$ are constrained by $\mathcal{P}_i$, and consequently, the local projectors $\{\mathcal{P}_i\}$ can give rise to an emergent symmetry.

It is believed that a topological order with a gapped boundary can be realized by a commuting projector model. Therefore the $\widetilde{\mathcal{R}}$-boundary and the $\mathcal{M}$-bulk in Fig. 3(b) can be realized by a commuting projector model. To have $H_0$ as the Hamiltonian described by the slab in Fig. 3(b), we take $\mathcal{P}_i$ in $H_0$ to be those commuting projectors. $O_i$'s in $H_0$ are the boundary Hamiltonian terms describing the $QFT_{ano}$ boundary in Fig. 3(b).

We also enlarge $\{O_i\}$ to include local operators in the bulk $\mathcal{M}$, and require that they still commute with $\mathcal{P}_i$. Since the thickness of the bulk is finite, $\{O_i\}$ can include operators that connect the $QFT_{ano}$ and $\widetilde{\mathcal{R}}$ boundaries, which we will refer to as interboundary operators. There are also intraboundary operators, which are those that act only on the degrees of freedom near the boundary $QFT_{ano}$. To explicitly break a symmetry on $QFT_{ano}$, $\{O_i\}$ must include an interboundary operator that transfers symmetry charge from $\widetilde{\mathcal{R}}$ to $QFT_{ano}$. Doing this for a $p$-form symmetry requires a $(p+1)$-dimensional operator.

For a 0-form symmetry, such an operator would include a finite number of local operators acting in a line from $\widetilde{\mathcal{R}}$ to $QFT_{ano}$. This is a local operator and thus allowed in $H_0$. It is unlikely local projectors $\mathcal{P}_i$ can forbid such an operator, and therefore they cannot produce exact emergent 0-form symmetries. For emergent higher-form symmetries, any interboundary operators that transfer symmetry charge are noncontractible extended operators, acting on the whole system. These operators are not local and are not included in the set of local operators $\{O_i\}$. Thus emergent higher-form symmetries are exact.

The discussion so far has been pretty general, so let us consider an example of a model with an exact emergent $\mathbb{Z}_2^{(1)}$ symmetry in 1+1D. We consider $\mathcal{M} = \mathcal{Z}(\mathcal{V}\text{ec}_{\mathbb{Z}_2})$, which corresponds to the $\mathbb{Z}_2$ toric code model, and $\widetilde{\mathcal{R}}$ the $\mathbb{Z}_2$-charge condensed boundary (the rough boundary [86]). The slab $QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$ in Fig. 3(b) becomes the lattice shown in Fig. 4(a), with qubits residing on the links.

The toric code model contains two types of projectors: star terms $\mathcal{P}_i^{vert} = (1 - Z_1Z_2Z_3Z_4)/2$ acting on the qubits of the four links touching each vertex and plaquette terms $\mathcal{P}_i^{plaq} = (1 - X_1X_2X_3X_4)/2$ acting on the qubits of the four links around each square. The $\widetilde{\mathcal{R}}$ boundary has truncated plaquette terms $\mathcal{P}_i^{bdry} = \frac{1}{2}(1 - X_1X_2X_3)$, as shown in Fig. 4(a).

Since the bulk is topological, let us take a thin slab limit of Fig. 4(a), which gives us Fig. 4(b) where we label vertical

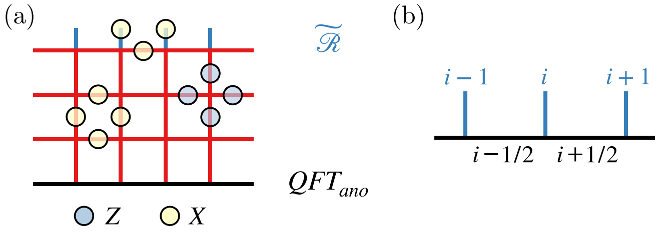


FIG. 4. (a) A lattice realization of Fig. 3(b), where qubits live on the links. The star and plaquette terms of the toric code model are shown both in the bulk and on the $\widetilde{\mathcal{R}}$ boundary. (b) Since the bulk is topological, one can take the thin slab limit.

(horizontal) links by $i$ $(i + \frac{1}{2})$. The only remain projector surviving this limit is $\mathcal{P}_i^{bdry} = \frac{1}{2}(1 - X_iX_{i+\frac{1}{2}}X_{i+1})$, and thus all allowed $O_i$'s are generated by products of $X_i$, $X_{i+\frac{1}{2}}$, and $Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}}$. Notice that while $X_{i+\frac{1}{2}}$, $X_iX_{i+\frac{1}{2}}X_{i+1}$, $Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}}$ are intraboundary operators, the $X_i$'s are interboundary operators.

Let us first restrict ourselves to only the intraboundary operators. We will consider the full setup, where $O_i$ includes intra and interboundary operators, after. The intraboundary operators form an algebra of the local symmetric operators generated by

$$\mathcal{A}_{\text{mid}-\text{IR}}^{(\text{intraonly})} = \{X_{i+\frac{1}{2}}, X_iX_{i+\frac{1}{2}}X_{i+1}, Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}}\}. \tag{10}$$

The operators (local or nonlocal) that commute with $\mathcal{A}_{\text{mid}-\text{IR}}^{(\text{intraonly})}$ are generated by

$$\mathcal{T}_{\text{mid}-\text{IR}}^{(\text{intraonly})} = \left\{X_iX_{i+\frac{1}{2}}X_{i+1},\ \prod_i (Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}})\right\} \tag{11}$$

and give rise to all symmetry transformations. Therefore, when restricted to the intraboundary operators, there are two mid-IR symmetries arising from $\mathcal{P}_i$. $X_iX_{i+\frac{1}{2}}X_{i+1}$ acts on loops and corresponds to a $\mathbb{Z}_2^{(1)}$ symmetry, while $\prod_i(Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}})$ acts on the entire lattice and corresponds to a $\mathbb{Z}_2^{(0)}$ symmetry. When the (2 + 1)D system $QFT_{ano} \boxtimes_{\mathcal{M}} \widetilde{\mathcal{R}}$ is mapped to the (1 + 1)D system $QFT_{af}$, the $\mathbb{Z}_2^{(0)}$ symmetry still acts on the entire lattice while the $\mathbb{Z}_2^{(1)}$ symmetry now acts on a single lattice site.

Let us now include the interboundary operators. Doing so, the algebra of local symmetric operators becomes

$$\mathcal{A}_{\text{mid-IR}} = \{X_{i+\frac{1}{2}},\ X_i,\ Z_{i-\frac{1}{2}}Z_iZ_{i+\frac{1}{2}}\}, \tag{12}$$

and the symmetry transformations are now generated by

$$\mathcal{T}_{\text{mid-IR}} = \{X_iX_{i+\frac{1}{2}}X_{i+1}\}. \tag{13}$$

The $\mathbb{Z}_2^{(1)}$ symmetry is still present, but the $\mathbb{Z}_2^{(0)}$ symmetry is now gone. This is because the allowed interboundary operators $X_i$ transfer the charges of the $\mathbb{Z}_2^{(0)}$ symmetry between the two boundaries, breaking the $\mathbb{Z}_2^{(0)}$ symmetry. The operators that could transform the $\mathbb{Z}_2^{(1)}$ symmetry charges between the two boundaries were not included in $\mathcal{A}_{\text{mid-IR}}$ since they are not local operators, and as a result, the $\mathbb{Z}_2^{(1)}$ symmetry is still a mid-IR symmetry.

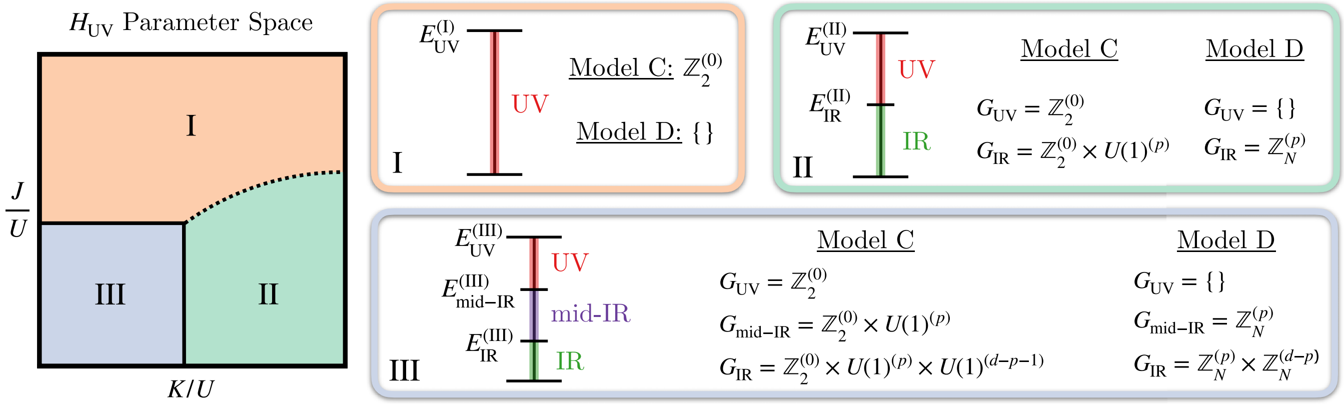


FIG. 5. (left) The parameter space of models C (41) and D (26) can be partitioned into three regions, I, II, and III, corresponding to its different energy scale hierarchies. These regions are not necessarily distinct phases of the models. Here, solid lines indicate a phase transition while dashed lines do not. (right) In these regions, we identify the exact emergent symmetries at each energy scale. Here $p$ is a positive integer and $d$ is the dimension of space such that $d > p + 1$ ($d > p$) for model C (D). The emergent $\mathrm{U}(1)^{(d-p-1)}$ symmetry in region III for model C is only nontrivial in the continuum limit. When $J/U = 0$ ($K/U = 0$), the exact emergent $\mathrm{U}(1)^{(p)}$ and $\mathbb{Z}_N^{(p)}$ ($\mathrm{U}(1)^{(d-p-1)}$ and $\mathbb{Z}_N^{(d-p)}$) symmetries of models C and D, respectively, are exact UV symmetries.

## III. EXAMPLES OF EXACT EMERGENT HIGHER-FORM SYMMETRIES

In this section, using the point of view discussed in Sec. II A, we go through examples of lattice models without higher-form symmetries that have exact emergent higher-form symmetries. These models are all described by Hamiltonians governing degrees of freedom on a $d$-dimensional cubic spatial lattice. We extensively use discrete differential geometry notation, which we review in Appendix A. The remainder of this section is organized as follows: each subsection dedicated to a single model and entirely self-contained.

In Sec. III A, we consider the quantum clock model [Eq. (17)]. When its UV $\mathbb{Z}_N$ symmetry is spontaneously broken, we find there is an exact emergent $\mathbb{Z}_N^{(d)}$ symmetry at energies below the domain wall gap. The IR symmetry operators form a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N^{(d)}$, signaling the presence of a mixed 't Hooft anomaly that protects the ground state degeneracy in the SSB phase.

In Sec. III B, we consider a model of emergent $\mathbb{Z}_N$ $p$-gauge theory called model D [Eq. (26)], and in Sec. III C a model of emergent $\mathrm{U}(1)$ $p$-gauge theory called model C [Eq. (41)]. The exact emergent symmetries and energy scale hierarchies of these models are summarized in Fig. 5. The left panel of Fig. 5 is a schematic depiction, and we do not investigate the precise shapes of the regions nor the nature of their boundaries. Region III corresponds to the deconfined phase of the emergent gauge theory. Region II corresponds to the confined "phase," where the gauge charges are still well-defined gapped excitations and confined. Figure 5 and our discussion throughout Secs. III B and III C portray the possibility that region II exists for $J \neq 0$. Lastly, region I is where the gauge charges are condensed/are no longer well-defined gapped excitations.

Model D with $p = 1$, $d = 2$, and $N = 2$ is emergent (2 + 1)D $\mathbb{Z}_2$ lattice gauge theory. In Eq. (26), the $J$ term creates $\mathbb{Z}_2$ charge fluctuations and the $K$ term creates $\mathbb{Z}_2$ flux fluctuations, both breaking UV $\mathbb{Z}_2^{(1)}$ symmetries. Region III corresponds to the deconfined phase of the $\mathbb{Z}_2$ gauge theory, and in the IR, there is a spontaneously broken exact emergent anomalous $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ symmetry. This mixed anomaly is characterized by the SPT [see Eq. (B48)]

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] = \mathrm{e}^{\mathrm{i}\pi \int_{M^4} \mathcal{A} \cup \hat{\mathcal{A}}}, \tag{14}$$

where $\mathcal{A}$ and $\hat{\mathcal{A}}$ are $\mathbb{Z}_2$-valued 2-cocycles, and protects the ground state degeneracy on a torus. Large $J$ drives a $\mathbb{Z}_2$-charge condensation transition (i.e., a Higgs transition) and large $K$ drives a $\mathbb{Z}_2$-flux condensation transition (i.e., a confinement transition). When the $\mathbb{Z}_2$ gauge charges have an energy gap, the exact emergent $\mathbb{Z}_2^{(1)}$ symmetry is present on both sides of the confinement transition, II $\leftrightarrow$ III, controlling the transition, and its unphysical part corresponds to the exact emergent $\mathbb{Z}_2$ gauge redundancy [50].

Model C with $p = 1$ and $d = 3$ is emergent (3 + 1)D U(1) lattice gauge theory. In Eq. (41), the $J$ creates U(1)-charge fluctuations and the $K$ term creates magnetic monopole fluctuations. Region III corresponds to the deconfined phase of the U(1) gauge theory, and in the continuum limit of the IR, there is a spontaneously broken exact emergent $\mathrm{U}(1)^{(1)} \times \mathrm{U}(1)^{(1)}$ symmetry. This mixed 't Hooft anomaly is characterized by the SPT [see Eq. (C59)]

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] = \mathrm{e}^{\mathrm{i}2\pi \int_{M^5} \mathcal{A} \wedge \mathrm{d}\hat{\mathcal{A}}}, \tag{15}$$

where $\mathcal{A}$ and $\hat{\mathcal{A}}$ are $\mathbb{R}/\mathbb{Z}$-valued 2-form fields, and protects the gaplessness of the photon [87,88]. Large $J$ drives a U(1)-charge condensation transition (i.e., a Higgs transition) and large $K$ drives a magnetic monopole condensation transition (i.e., a confinement transition). When the U(1) charges have an energy gap, the exact emergent $\mathrm{U}(1)^{(1)}$ symmetry is present on both sides of confinement transition II $\leftrightarrow$ III, controlling the transition, and its unphysical part corresponds to the exact emergent U(1) gauge redundancy [50].

### A. Quantum clock model

Consider $\mathbb{Z}_N$ quantum rotors residing on the 0-cells (sites) of the spatial $d$-dimensional cubic lattice at zero temperature. These are described by the clock operators $X_{c_0}$ and $Z_{c_0}$, which are unitary operators satisfying

$$Z_{\widetilde{c}_0} X_{c_0} = \omega^{\delta_{c_0,\widetilde{c}_0}} X_{c_0} Z_{\widetilde{c}_0}, \quad X_{c_0}^N = Z_{c_0}^N = \mathbf{1}, \tag{16}$$

where $\omega \equiv e^{i2\pi/N}$. They are $N$-dimensional generalizations of the Pauli matrices and have eigenvalues $\{1, \omega, \omega^2, \cdots, \omega^{N-1}\}$. The Hamiltonian of the quantum clock model is

$$H = -\frac{J}{2}\sum_{c_1}\prod_{c_0\in\partial c_1} X_{c_0} - \frac{K}{2}\sum_{c_0} Z_{c_0} + \text{H.c.}, \tag{17}$$

where $X_{-c_0} \equiv X_{c_0}^\dagger$, the first sum is over 1-cells, and the second sum is over all 0-cells. This theory has an exact $\mathbb{Z}_N$ 0-form—$\mathbb{Z}_N^{(0)}$—symmetry, which is generated by

$$U = \prod_{c_0} Z_{c_0}. \tag{18}$$

The charged operator of this symmetry is $X_{c_0}$, which from the clock operator algebra transform as $X_{c_0} \to e^{2\pi i/N} X_{c_0}$. Therefore the algebra of local symmetric operators is generated by

$$\mathcal{A}_{\text{UV}} = \left\{ Z_{c_0}, \prod_{c_0\in\partial c_1} X_{c_0} \right\}. \tag{19}$$

### 1. An exact emergent $\mathbb{Z}_N^{(d)}$ symmetry and mixed 't Hooft anomaly

When $K/J \ll 1$, the quantum clock model lies in a $\mathbb{Z}_N^{(0)}$ spontaneous symmetry broken (SSB) phase. Indeed, in the tractable $K/J \to 0$ limit, the ground state satisfies $X_{c_0}^\dagger X_{\widetilde{c}_0}|\text{vac}\rangle = |\text{vac}\rangle$ for all neighboring 0-cells, and thus $\langle X_{c_0}^\dagger X_{c_0'}\rangle = 1$ for any 0-cells $c_0$ and $c_0'$. In this phase, there are gapped domain walls carrying $\mathbb{Z}_N$ topological charge. Indeed, in the $K/J \to 0$ limit, the domain-wall density $\hat{\rho}$ for a state $|\psi\rangle$ is defined by

$$\prod_{c_0\in\partial c_1} X_{c_0}|\psi\rangle = e^{\frac{2\pi i}{N}(*\hat{\rho})_{c_1}}|\psi\rangle, \tag{20}$$

where $(*\hat{\rho})_{c_1} \equiv \hat{\rho}_{*c_1}$. Therefore the operator $(*Z)_{\hat{c}_d}$ excites a domain wall on $\partial\hat{c}_d$.

The domain wall gap $J$ provides a candidate energy scale below which new symmetries may emerge. Let us call this energy scale the IR. However, when $K/J \neq 0$, there no longer exists a low-energy sub-Hilbert space spanned by states satisfying $\langle\hat{\rho}_{\hat{c}_{d-1}}\rangle = 0 \mod N$. This is because the $K$ term in $H$ causes the $\langle\hat{\rho}_{\hat{c}_{d-1}}\rangle = 0$ and $\langle\hat{\rho}_{\hat{c}_{d-1}}\rangle \neq 0$ states to mix. This does not necessarily mean the domain walls no longer exist when $K > 0$, just that their operators depend on $K$. Indeed, a corresponding low-energy sub-Hilbert space can be identified using $U_{\text{LU}}$ from Sec. II A. Therefore there exists a low-energy sub-Hilbert space for $K/J \ll 1$ in the SSB phase spanned by states satisfying $\langle\widetilde{\hat{\rho}}_{\hat{c}_{d-1}}\rangle \equiv \langle U_{\text{LU}}\hat{\rho}_{\hat{c}_{d-1}}U_{\text{LU}}^\dagger\rangle = 0$. By the definition of $U_{\text{LU}}$, the $\mathbb{Z}_N^{(0)}$ symmetry operator satisfies

$$U = \widetilde{U} = \prod_{c_0}\widetilde{Z}_0. \tag{21}$$

Having identified the IR of the SSB phase, we would now like to find an effective IR theory. This IR satisfies the constraint $\widetilde{\hat{\rho}}_{\hat{c}_{d-1}} = 0$, or equivalently $\prod_{c_0\in\partial c_1}\widetilde{X}_{c_0} = 1$. Due to the constraint, the $\mathbb{Z}_N^{(0)}$ symmetric IR operators must be constructed from only $\widetilde{Z}_0$. Only one such operator commutes with the constraint: Eq. (21). Therefore, for a finite-size system, the algebra of local symmetric IR operators is

$$\mathcal{A}_{\text{IR}}^{\text{finite}-L} = \left\{\prod_{c_0}\widetilde{Z}_0\right\} = \{U\}. \tag{22}$$

The corresponding effective IR Hamiltonian is

$$H_{\text{IR}}^{\text{finite}-L} = -J\kappa^{N_0}U, \tag{23}$$

where $\kappa \sim K/J$ and $N_0$ is the total number of 0-cells. The $K/J$ dependence of $\kappa$ comes from the fact that $H = 0$ in the IR when $K/J \to 0$.

In the thermodynamic limit, $U$ is a nonlocal operator, so the algebra of local symmetric IR operators becomes

$$\mathcal{A}_{\text{IR}} = \{\}. \tag{24}$$

Indeed, since $K/J \ll 1$, in the thermodynamic limit the effective IR Hamiltonian becomes $H_{\text{IR}} = 0$. Since $\mathcal{A}_{\text{IR}}$ is the empty set (or equivalently, since $H_{\text{IR}}$ is zero), any IR operator commutes with the local symmetric IR operators and thus corresponds to a symmetry. This includes $U$, and thus the UV $\mathbb{Z}_N^{(0)}$ symmetry, as expected. However, the operator $\widetilde{X}_{c_0}$ is also allowed and it generates the transformation

$$U \to e^{\frac{2\pi i}{N}}U. \tag{25}$$

Since the charged object is supported on $d$-cycles and transforms by an element of $\mathbb{Z}_N$, the operator $\widetilde{X}_{c_0}$ generates a $\mathbb{Z}_N^{(d)}$ symmetry (which is always a higher-form symmetry).

This emergent $\mathbb{Z}_N^{(d)}$ symmetry has been noted previously throughout the literature [46,89,90]. Indeed, it is an emergent symmetry since it does not commute with $H$. Here we find it is an exact emergent symmetry since it exactly commutes with the IR effective Hamiltonian. Therefore the ground state subspace of the $\mathbb{Z}_N$ SSB phase has an exact $\mathbb{Z}_N^{(0)} \times \mathbb{Z}_N^{(d)}$ symmetry. Furthermore, this IR symmetry is anomalous, which can be noticed from the fact the symmetry operator of the $\mathbb{Z}_N^{(0)}$ symmetry is charged under the $\mathbb{Z}_N^{(d)}$ symmetry. This mixed 't Hooft anomaly protects the ground state degeneracy of the SSB phase. The only way to eliminate it is to prevent the $\mathbb{Z}_N^{(d)}$ symmetry from emerging by condensing domain walls.

## B. Emergent $\mathbb{Z}_N$ p-gauge theory

In this section, we consider a model for emergent $\mathbb{Z}_N$ $p$-gauge theory, which we call model D. When $p = 1$, this is just typical $\mathbb{Z}_N$ gauge theory. Consider $\mathbb{Z}_N$ quantum rotors residing on the $p$-cells of the spatial $d$-dimensional cubic lattice with $p > 0$. A $\mathbb{Z}_N$ quantum rotor is an $N$-level system described by clock operators $X_{c_p}$ and $Z_{c_p}$ which satisfy Eq. (16). Model D is described by the Hamiltonian

$$H_{\text{UV}} = -\frac{U}{2}\sum_{c_{p-1}}\tau^z_{c_{p-1}} - U\sum_{c_{p+1}}W^\dagger_{c_{p+1}} - \frac{K}{2}\sum_{c_p}Z_{c_p} - \frac{J}{2}\sum_{c_p}X_{c_p} + \text{H.c.},$$

$$\tau^z_{c_{p-1}} \equiv \prod_{c_p\in\delta c_{p-1}} Z_{c_p}, \quad W^\dagger_{c_{p+1}} = \prod_{c_p\in\partial c_{p+1}} X_{c_p} \tag{26}$$

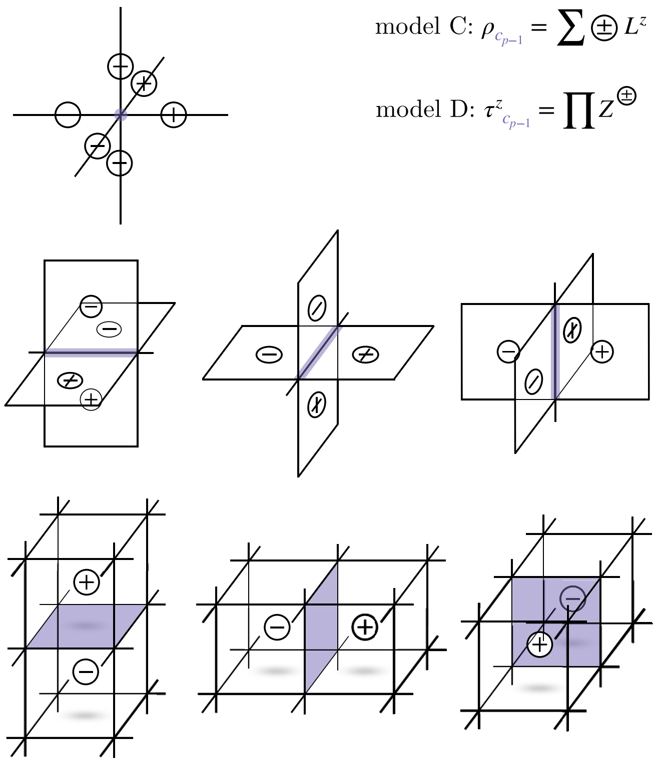


FIG. 6. Graphical representation of $\rho_{c_{p-1}}$ in model C (42) and $\tau^z_{c_{p-1}}$ in model D (26) in 3d space for (first row) $p = 1$, (second row) $p = 2$, and (third row) $p = 3$. The $(p-1)$-cell $c_{p-1}$ is colored purple. The $\pm$ disks on the $p$-cells denote the sign in front of $L^z_{c_p}$ in the sum for $\rho_{c_{p-1}}$, or whether $Z_{c_p}$ is $Z^+ \equiv Z$ or $Z^- \equiv Z^\dagger$ in the product for $\tau^z_{c_{p-1}}$.

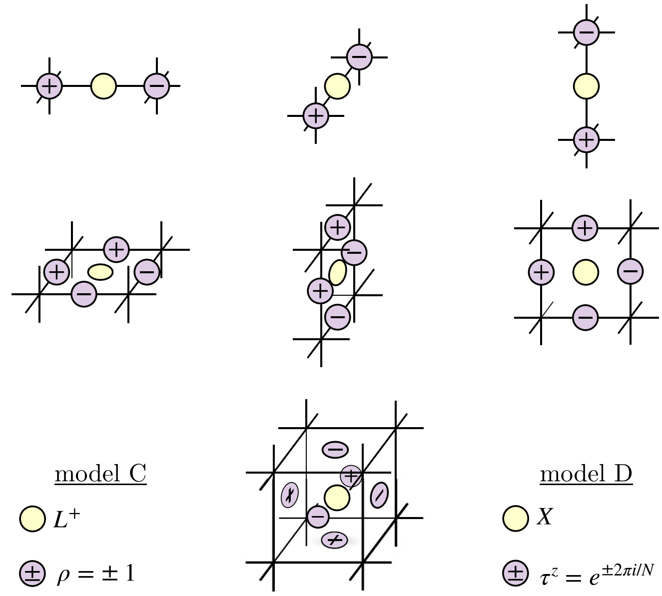


FIG. 7. Graphical representation of the excitation created by $L^+_{c_p}$ [$X_{c_p}$] in model C [D] in 3d space for (first row) $p = 1$, (second row) $p = 2$, and (third row) $p = 3$. The yellow disk represents $L^+_{c_p}$ [$X_{c_p}$]. For model C, the purple $\pm$ disk denotes the sign in $\rho_{c_{p-1}}(L^+_{c_p}|0\rangle) = \pm(L^+_{c_p}|0\rangle)$ for that $(p-1)$-cell. For model D, the $\pm$ disk instead denotes the sign in $\tau^z_{c_{p-1}}(X_{c_p}|0\rangle) = \omega^{\pm 1}(X_{c_p}|0\rangle)$ for that $(p-1)$-cell.

where $\sum_{c_p}$ is over all $p$-cells, $\delta c_{p-1}$ is the coboundary of $c_{p-1}$ [see Eq. (A3)], and$Z_{-c_p} \equiv Z^\dagger_{c_p}$. $\tau^z$ is generally a product of $2(d-p+1)$ operators, examples of which are shown in Fig. 6.

Since $H_{\rm UV}$ has terms linear in $Z_{c_p}$ and $X_{c_p}$, the algebra of local symmetric UV operators is generated by

$$\mathcal{A}_{\rm UV} = \{Z_{c_p}, X_{c_p}\}. \tag{27}$$

Nothing commutes with both $Z_{c_p}$ and $X_{c_p}$, and thus there are no UV symmetries in this theory.

#### 1. An exact emergent $\mathbb{Z}_N^{(p)}$ symmetry

In the limit $J/U \to 0$, there exists a low-energy sub-Hilbert space spanned by states satisfying $\langle \tau^z_{c_{p-1}} \rangle = 1$. Violating this constraint costs energy $U$, and we interpret states that do so in this limit as having a gapped excitation, a segment of which residing on $c_{p-1}$. We'll refer to these bosonic $(p-1)$-dimensional (in space) excitations as "charges" since they are the gauge charges of the emergent $\mathbb{Z}_N$ $p$-gauge theory. From the clock operators' algebra, $X_{c_p}$ excites a charge on $\partial c_p$, examples of which are shown in Fig. 7. Since $X^N_{c_p} = \mathbf{1}$, exciting $N$ charges is the same as not exciting any. Thus the charge number takes values in $\mathbb{Z}_N$.

The charge gap $U$ provides a candidate energy scale below which new symmetries may emerge. However, when $J/U \neq 0$, there no longer exists a low-energy sub-Hilbert space spanned by states satisfying $\langle \tau^z_{c_{p-1}} \rangle = 1$. This is because the $J$ term in $H_{\rm UV}$ causes the $\langle \tau^z_{c_{p-1}} \rangle = 1$ and $\langle \tau^z_{c_{p-1}} \rangle \neq 1$ states to mix. This does not necessarily mean the charges no longer exist when $J > 0$, just that their operators depend on $J$. Indeed, a corresponding low-energy sub-Hilbert space can be identified using the local unitary from Sec. II A, which we denote as $U^{(1)}_{\rm LU}$. Therefore there exists a low-energy sub-Hilbert space spanned by states satisfying $\langle \widetilde{\tau}^z_{c_{p-1}} \rangle \equiv \langle U^{(1)}_{\rm LU} \tau^z_{c_{p-1}} U^{(1)\dagger}_{\rm LU} \rangle = 1$.

We will not find an explicit form for $U^{(1)}_{\rm LU}$ and thus will not precisely know throughout how much of parameter space the dressed (fattened) operators can be defined without violating the assumptions of $U^{(1)}_{\rm LU}$. Instead, we will assume that such an operator exists and can access a greater than measure-zero part of parameter space and will investigate the consequences of this conjecture.

At this point, we cannot tell if the charge gap $\Delta$ is an IR scale or mid-IR I scale or mid-II scale, etc. In Sec. III B 2, we find it is a mid-IR scale in region III, but an IR scale in region II of parameter space (see Fig. 5). For the rest of this section, however, we will adopt the language from the perspective of region III and call the charge gap a mid-IR scale.

Given the mid-IR scale $E_{\text{mid}-\text{IR}} \equiv \Delta$, we would now like to find an effective mid-IR theory describing states at energies $E < E_{\text{mid}-\text{IR}}$. Since $\widetilde{Z}_{c_p}$ commutes with $\widetilde{\tau}^z_{c_{p-1}}$, it does not excite any charges and is an allowed mid-IR operator. The operators $\widetilde{X}_{c_p}$ are not allowed as they excite charges. the allowed operators constructed from $\widetilde{X}_{c_p}$ are

$$\widetilde{W}^\dagger[C_p] = \prod_{c_p \in C_p} \widetilde{X}_{c_p}, \tag{28}$$

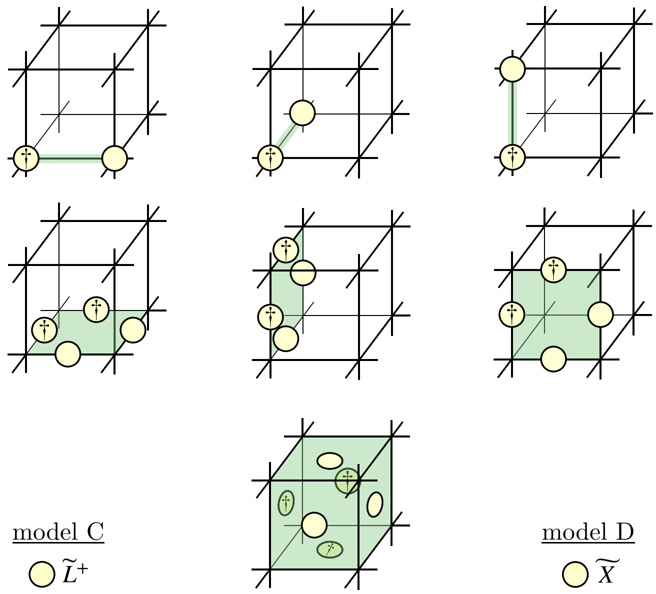


FIG. 8. Graphical representation of $\widetilde{W}^{\dagger}(\partial c_{p+1})$ for model C [Eq. (44)] and D [Eq. (28)] in 3d space for (first row) $p=0$, (second row) $p=1$, and (third row) $p=2$. For model C (D), the yellow colored disks represent $\widetilde{L}^{+}$ ($\widetilde{X}$) operators, the product of which yields $\widetilde{W}^{\dagger}(\partial c_{p+1})$. Discs labeled by † denote the Hermitian conjugate and $c_{p+1}$ is colored green.

which we call the Wilson operator (see Fig. 8). It has the interpretation of exciting a charge, transporting it along a $p$-cycle, and then annihilating it.

While $\widetilde{W}^{\dagger}[C_p]$ does not excite charges, not all $\widetilde{W}^{\dagger}[C_p]$ are mid-IR operators. Indeed, when $K/U \gg 1$ the $p$-brane excitation created by $\widetilde{W}^{\dagger}[C_p]$ costs energy $\sim |C_p|K$, where $|C_p|$ is the number of $p$-cells $C_p$ is made of. So, roughly, $\widetilde{W}^{\dagger}[C_p]$ is allowed in this limit only if $|C_p| \ll \Delta/K$. On the other hand, when $K/U \ll 1$ this $p$-brane's gap does not increase linearly with $|C_p|$ and all $\widetilde{W}^{\dagger}[C_p]$ are mid-IR allowed operators. We will denote the set of $C_p$ for which $\widetilde{W}^{\dagger}[C_p]$ is an allowed Mid-IR operator when acting on $|\mathrm{vac}\rangle$ by $\mathcal{Z}_p$. This set of $p$-cycles depends on the value of $K/U$.

For a finite size system, the algebra of local symmetric mid-IR operators is generated by

$$\mathcal{A}^{\text{finite}-L}_{\text{mid}-\text{IR}} = \{\widetilde{Z}_{c_p}, \widetilde{W}^{\dagger}[C_p] : C_p \in \mathcal{Z}_p\}. \tag{29}$$

Strictly speaking, this is only approximate since $\widetilde{W}[C_p \in \mathcal{Z}_p]$ is only a mid-IR operator when acting on low-energy eigenstates in the mid-IR, not mid-IR states with $E$ close to $E_{\text{mid}-\text{IR}}$. Nevertheless, the symmetries of this should be the same as the exact form of the effective mid-IR theory. The mid-IR Hamiltonian under this approximation is

$$H^{\text{finite}-L}_{\text{mid}-\text{IR}} = -\kappa U \sum_{c_p} \widetilde{Z}_{c_p} - U \sum_{c_{p+1}} \widetilde{W}[\partial c_{p+1}] - U \sum_{C_p \in \mathcal{Z}_p} \varepsilon_{C_p} \widetilde{W}[C_p] + \cdots , \tag{30}$$

where $\kappa \sim K/U$ and $\varepsilon_{C_p} \sim (J/U)^{|C_p|}$.

In the thermodynamic limit, $\widetilde{W}^{\dagger}$ acting on noncontractible $p$-cycles is a nonlocal operator. Denoting the subset of $\mathcal{Z}_p$ with only contractible $p$-cycles as $\mathcal{B}_p$, the algebra of local symmetric mid-IR operators is now generated by

$$\mathcal{A}_{\text{mid}-\text{IR}} = \{\widetilde{Z}_{c_p}, \widetilde{W}^{\dagger}[C_p] : C_p \in \mathcal{B}_p\}. \tag{31}$$

From the effective Hamiltonian point of view, $H^{\text{finite}-L}_{\text{mid}-\text{IR}}$ must be a local Hamiltonian, so the mid-IR theory is only well-defined provided $J/U \ll 1$. Therefore, in the thermodynamic limit, terms with Wilson operators supported on nontrivial $p$-cycles vanish, and the mid-IR theory becomes

$$H_{\text{mid}-\text{IR}} = -\kappa U \sum_{c_p} \widetilde{Z}_{c_p} - U \sum_{c_{p+1}} \widetilde{W}[\partial c_{p+1}] - U \sum_{C_p \in \mathcal{B}_p} \varepsilon_{C_p} \widetilde{W}[C_p] + \cdots , \tag{32}$$

$\mathcal{A}_{\text{mid}-\text{IR}}$ includes an new symmetry absent from $\mathcal{A}_{\text{UV}}$. Indeed, the mid-IR theory is invariant under the transformation

$$\widetilde{X}_{c_p} \to \mathrm{e}^{\mathrm{i}\Gamma_{c_p}} \widetilde{X}_{c_p} \Rightarrow \widetilde{W}[C_p] \to \mathrm{e}^{\mathrm{i}\sum_{c_p \in C_p} \Gamma_{c_p}} \widetilde{W}[C_p], \tag{33}$$

where $(\mathrm{d}\Gamma)_{c_{p+1}} \equiv \sum_{c_p \in \partial c_{p+1}} \Gamma_{c_p} = 0$ and $\Gamma_{c_p} \in 2\pi\mathbb{Z}/N$ for $(X_{c_p})^N = \mathbf{1}$ to be invariant. This is a symmetry because $\sum_{c_p \in C_p} \Gamma_{c_p} = 0$ for $C_p \in \mathcal{B}_p$ since $(\mathrm{d}\Gamma)_{c_{p+1}} = 0$. It is not a gauge symmetry as it transforms Wilson operators on noncontractible $p$-cycles by a nontrivial element of $\mathbb{Z}_N$. Therefore, since the charged operators are supported on a $p$-cycle, the mid-IR has an exact emergent $\mathbb{Z}_N^{(p)}$ symmetry. The operator generating this $\mathbb{Z}_N^{(p)}$ symmetry is

$$\widetilde{U}(\hat{\Sigma}_{d-p}) = \prod_{\hat{c}_{d-p} \in \hat{\Sigma}_{d-p}} (*\widetilde{Z})_{\hat{c}_{d-p}}, \tag{34}$$

where $\hat{\Sigma}_{d-p}$ is a $(d-p)$-cycle of the dual lattice and $(*\widetilde{Z})_{\hat{c}_{d-p}} \equiv \widetilde{Z}_{*\hat{c}_{d-p}}$ (see Fig. 9).

This $\mathbb{Z}_N^{(p)}$ symmetry only emerges in parameter space where the mid-IR—the low-energy regime without gapped charges—exists. Here we associate the mid-IR's existence to the existence of a well-defined effective mid-IR Hamiltonian, so the exact emergent $\mathbb{Z}_N^{(p)}$ symmetry exists only when $H_{\text{mid}-\text{IR}}$ converges. In the approximation scheme used, this requires $(\varepsilon_{C_p})^{1/|C_p|} < 1$ for all $C_p \in \mathcal{B}_p$. The constant of proportionality in $(\varepsilon_{C_p})^{1/|C_p|} \propto J/U$ increases with $|C_p|$ since there are more ways $\widetilde{W}[C_p]$ can be generated from $\widetilde{X}_{c_p}$ for larger $|C_p|$. Therefore it is sufficient to consider only the largest $p$-cycle in $\mathcal{B}_p$. For small enough $K/U$ where $\mathcal{B}_p$ includes all trivial $p$-cycles, the largest $C_p$ does not depend on $K/U$, and so the largest value of $J/U$ with the exact emergent $\mathbb{Z}_N^{(p)}$ symmetry is independent of $K/U$. This value of $J/U$ defines the boundary between regions I and III in Fig. 5. For large enough $K/U$, $\mathcal{B}_p$ does not include all trivial $p$-cycles. The larger $K/U$ is, the smaller the maximum value of $|C_p|$ is, and thus the larger the maximum value of $J/U$ with the exact emergent $\mathbb{Z}_N^{(p)}$ symmetry is. This value of $J/U$ increasing with $K/U$ defines the boundary between regions I and II shown schematically in Fig. 5.

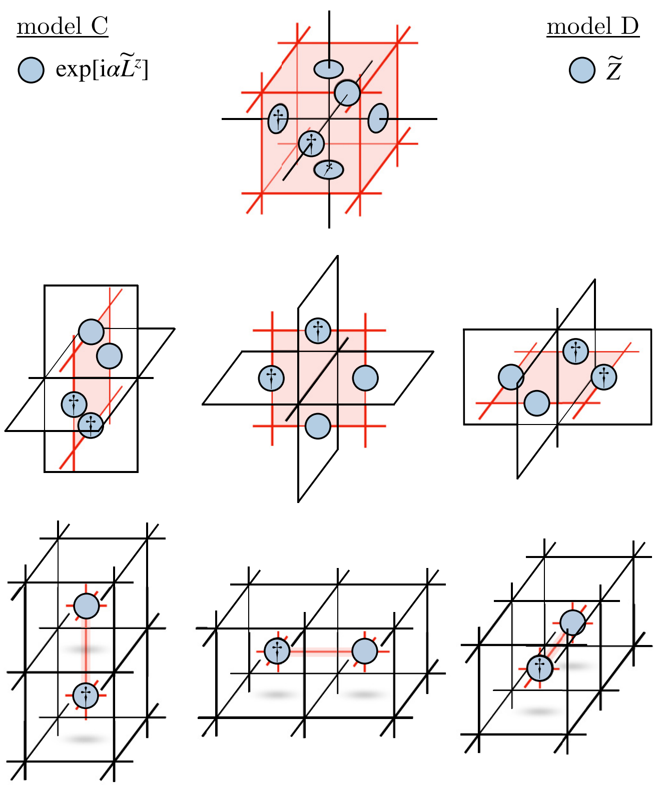


FIG. 9. Graphical representation of the emergent $U(1)^{(p)}$ [$\mathbb{Z}_N^{(p)}$] symmetry operator Eq. (50) [(34)] in model C [D] acting on $\partial\hat{c}_{d-p+1}$ in 3d space for (first row) $p = 1$, (second row) $p = 2$, and (third row) $p = 3$. The blue-colored disks denote the operator $\widetilde{L}^+_{c_p}$ [$\widetilde{X}_{c_p}$], the product of which yields the symmetry operator. Discs labeled by † denote the Hermitian conjugate of the operator and $\hat{c}_{d-p+1}$ is colored red.

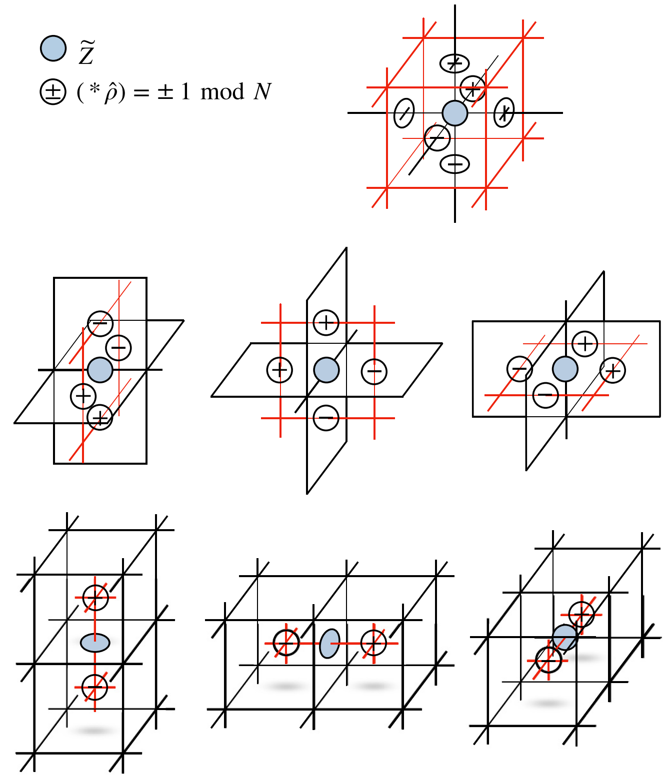


FIG. 10. Graphical representation of the topological defects created by $(*\widetilde{Z})_{\hat{c}_{d-p}}$ in model D for for 3d space and (first row) $p = 0$, (second row) $p = 1$, and (third row) $p = 2$. The blue disk represents the $(*\widetilde{Z})_{\hat{c}_{d-p}}$ operator. The disks labeled by ± represent $(*\hat{\rho})_{c_{p+1}} = \pm 1$ for that $c_{p+1}$ [see Eq. (35)].

### 2. An exact emergent anomalous $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ symmetry

The exact emergent $\mathbb{Z}_N^{(p)}$ symmetry can be spontaneously broken, and the SSB phase corresponds to the deconfined phase of $\mathbb{Z}_N$ $p$-gauge theory. To gain some intuition, we consider two tractable limits of the effective mid-IR theory Eq. (32). When $J/U = 0$ but $K/U \neq 0$ (which is in region II), the ground state satisfies $\widetilde{Z}_{c_p}|\text{vac}\rangle = |\text{vac}\rangle$, and therefore $\langle\widetilde{W}^\dagger[C_p]\rangle = 0$ for all $C_p$. Consequently, this limit lies in a $\mathbb{Z}_N^{(p)}$ symmetric phase. On the other hand, when $K/U = 0$ but $J/U \neq 0$ (which is in region III), the ground state satisfies $\widetilde{W}[C_p \in \mathcal{B}_p]|\text{vac}\rangle = |\text{vac}\rangle$, and consequently $\langle\widetilde{W}^\dagger[C_p]\rangle = 1$ for all trivial $p$-cycles. Therefore the $\mathbb{Z}_N^{(p)}$ symmetry is spontaneously broken in this limit.

A $\mathbb{Z}_N^{(p)}$ symmetry at zero temperature can spontaneously break when $d > p$ [3,31]. Therefore, when $d > p$, we expect the symmetry to be broken even for $K/U \neq 0$ and a stable SSB phase to exist. For small $\kappa$ and $\varepsilon_{C_p}$, a reasonable expectation from Eq. (32) is the SSB phase occurs when $\kappa \lesssim 1$. This determines the boundary between the $\mathbb{Z}_N^{(p)}$ symmetric and $\mathbb{Z}_N^{(p)}$ SSB phases and regions II and III shown in Fig. 5. We leave a more detailed investigation of this phase transition to future work.

Let us now restrict our considerations to the SSB phase. Like 0-form symmetries, breaking higher-form symmetries gives rise to gapped topological defects and, in this case, arise from the nontrivial mappings $Z_p(M_d; \mathbb{Z}_N) \to \mathbb{Z}_N$. In the $K/U \to 0$ limit, the topological defect density $\hat{\rho}$ for a state $|\psi\rangle$ is defined by[7]

$$\prod_{c_p \in \partial c_{p+1}} \widetilde{X}_{c_p}|\psi\rangle = e^{\frac{2\pi i}{N}(*\hat{\rho})_{c_{p+1}}}|\psi\rangle. \tag{35}$$

The topological defects are $(d-p-1)$-dimensional excitations in space, residing on the dual lattice, and carry $\mathbb{Z}_N$ charge. In the familiar $p = 1$ case, they correspond to the $\mathbb{Z}_N$ flux excitations of $\mathbb{Z}_N$ gauge theory. Furthermore, from Eq. (16), the operator $(*\widetilde{Z})_{\hat{c}_{d-p}}$ excites a topological defect on $\partial\hat{c}_{d-p}$ (see Fig. 10).

The topological defect gap $\Delta_{\text{defect}}$ provides a candidate energy scale below which new symmetries may emerge. Let us call this energy scale the IR. However, when $K/U \neq 0$, there no longer exists a low-energy sub-Hilbert space satisfying $\langle\hat{\rho}_{\hat{c}_{d-p-1}}\rangle = 0 \bmod N$. This is because the $\kappa U$ term in $H_{\text{mid-IR}}$ causes the $\langle\hat{\rho}_{\hat{c}_{d-p-1}}\rangle = 0$ and $\langle\hat{\rho}_{\hat{c}_{d-p-1}}\rangle \neq 0$ states to mix. This does not necessarily mean the topological defects no longer exist when $K/U > 0$, just that their operators depend on $K/U$. Indeed, we can use the local unitary discussed in Sec. II A, which we will denote as $U_{\text{LU}}^{(2)}$, to

[7]This is a natural generalization of the $p = 0$ case, where the topological defects are domain walls [see Eq. (20)].

identify the corresponding low-energy sub-Hilbert space. In general, $U_{\rm LU}^{(2)}$ is different than the local unitary $U_{\rm LU}^{(1)}$ used in the previous subsection. Therefore there exists a low-energy sub-Hilbert space in region III spanned by states satisfying $\langle\hat{\rho}'_{\hat{c}_{d-p-1}}\rangle \equiv \langle U_{\rm LU}^{(2)}\hat{\rho}_{\hat{c}_{d-p-1}}U_{\rm LU}^{(2)\dagger}\rangle = 0$. By the definition of $U_{\rm LU}^{(2)}$, the emergent $\mathbb{Z}_N^{(p)}$ symmetry operator satisfies

$$\widetilde{U}(\hat{\Sigma}_{d-p}) = \widetilde{U}'(\hat{\Sigma}_{d-p}) = \prod_{\hat{c}_{d-p}\in\hat{\Sigma}_{d-p}}(*\widetilde{Z}')_{\hat{c}_{d-p}}. \tag{36}$$

Having identified the IR of region III, we would now like to find an effective IR theory. This IR satifies $\hat{\rho}'_{\hat{c}_{d-p-1}} = 0$, or equivalently $\widetilde{W}'[\partial c_{p+1}] = 1$. Due to the constraint, the $\mathbb{Z}_N^{(p)}$ symmetric IR operators must be constructed from only $\widetilde{Z}_{c_p}$. Only one such type of operator commutes with the constraint: Eq. (36). Therefore, for finite-size systems, the algebra of local symmetric IR operators is

$$\mathcal{A}_{\rm IR}^{\text{finite}-L} = \{\widetilde{U}(\hat{\Sigma}_{d-p}) : \hat{\Sigma}_{d-p}\in Z_{d-p}(\hat{M}_d;\mathbb{Z}_N)\}, \tag{37}$$

where $Z_{d-p}(\hat{M}_d;\mathbb{Z}_N)$ is the set of all dual $(d-p)$-cycles with $\mathbb{Z}_N$ coefficients. The corresponding effective IR Hamiltonian is

$$H_{\rm IR}^{\text{finite}-L} = -U\sum_{\hat{\Sigma}_{d-p}\in Z_{d-p}(\hat{M}_d;\mathbb{Z}_N)}\kappa^{|\hat{\Sigma}_{d-p}|}\,\widetilde{U}(\hat{\Sigma}_{d-p}), \tag{38}$$

where $\kappa\sim K/U$ and $|\hat{\Sigma}_{d-p}|$ is the number of $(d-p)$-cells in $\hat{\Sigma}_{d-p}$.

In the thermodynamic limit, $\widetilde{U}$ acting on noncontractible $(d-p)$ cycles are nonlocal operators and Eq. (36) includes only contractible $(d-p)$-cycles. However, such $\widetilde{U}$ can be written as $\widetilde{U}(\partial O) = \prod_{\hat{c}\in O}(*\widetilde{\tau}^z)_{\hat{c}}$, and are thus trivial in the IR since there are no charges. So, in the thermodynamic limit

$$\mathcal{A}_{\rm IR} = \{\}, \tag{39}$$

and $H_{\rm IR} = 0$.

Since $\mathcal{A}_{\rm IR}$ is the empty set, all nontrivial IR operators commute with it and correspond to symmetries. This includes $\widetilde{U}$ supported on nontrivial dual $(d-p)$ cycles, and thus the mid-IR $\mathbb{Z}_N^{(p)}$ symmetry, as expected. However, $\widetilde{W}'$ supported on nontrivial $p$-cycles is also allowed and from Eq. (33), they generate the transformation

$$\widetilde{U}(\hat{\Sigma}_{d-p}) \to e^{i\sum_{\hat{c}_{d-p}\in\hat{\Sigma}_{d-p}}\hat{\Gamma}_{\hat{c}_{d-p}}}\widetilde{U}(\hat{\Sigma}_{d-p}), \tag{40}$$

where $(d\hat{\Gamma})_{\hat{c}_{d-p+1}} = 0$ and $\hat{\Gamma}_{\hat{c}_{d-p}}\in 2\pi\mathbb{Z}/N$. Since the charged operator is supported on $(d-p)$ cycles and transforms by an element of $\mathbb{Z}_N$, $\widetilde{W}'$ generates a $\mathbb{Z}_N^{(d-p)}$ symmetry. This is an exact emergent symmetry because it is not an exact symmetry of the UV but is an exact symmetry of the IR.

So, in the IR of region III—the ground state subspace of the deconfined phase of emergent $\mathbb{Z}_N$-$p$ gauge theory—there is an exact emergent $\mathbb{Z}_N^{(p)}\times\mathbb{Z}_N^{(d-p)}$ symmetry. Furthermore, this IR symmetry is anomalous, which can be noticed from the fact that the symmetry operator of the $\mathbb{Z}_N^{(p)}$ is charged under the $\mathbb{Z}_N^{(d-p)}$ symmetry. This mixed 't Hooft anomaly protects the deconfined phase's topological degeneracy and topological order. The only way to get rid of it is to prevent the $\mathbb{Z}_N^{(p)}\times\mathbb{Z}_N^{(d-p)}$ symmetries from emergent by either condensing the topological defects (to destroy $\mathbb{Z}_N^{(d-p)}$) or the charges (to destroy the entire $\mathbb{Z}_N^{(p)}\times\mathbb{Z}_N^{(d-p)}$).

This emergent anomalous $\mathbb{Z}_N^{(p)}\times\mathbb{Z}_N^{(d-p)}$ symmetry is the same as the exact symmetry of $p$-form BF theory and $p$-form toric code. This is no accident. In Appendix B, we show how the ground states in region III are also the ground states of the $p$-form toric code and that their topological quantum field theory description is $p$-form BF theory.

### C. Emergent U(1) $p$-gauge theory

In this section, we consider a model for emergent U(1) $p$-gauge theory, which we call model C. When $p=1$, this is just typical U(1) gauge theory. Consider U(1) quantum rotors residing on the $p$-cells of the spatial $d$-dimensional cubic lattice with $p>0$. Each rotor can be viewed as a particle on an infinitesimal circle, whose position we denote as the angle $\Theta$, carrying angular momentum $L^z$. The operators $L^z$ and $\Theta$ are Hermitian and satisfy $[\Theta_{c_p}, L^z_{\widetilde{c}_p}] = i\delta_{c_p,\widetilde{c}_p}$, so $L_z = -i\frac{\partial}{\partial\Theta}$. Since the eigenvalue of $\Theta$ is an angle, the eigenvalues of $L^z$ are integers.

Model C is described by the Hamiltonian

$$\begin{aligned} H_{\rm UV} &= \frac{U}{2}\sum_{c_{p-1}}\rho_{c_{p-1}}^2 - U\sum_{c_{p+1}}W_{c_{p+1}}^\dagger\frac{K}{2}\sum_{c_p}\left(L^z_{c_p}\right)^2 \\ &\quad + \frac{J}{2}\sum_{c_p}\left(L^+_{c_p} + \text{H.c.}\right), \\ \rho_{c_{p-1}} &= \sum_{c_p\in\delta c_{p-1}}L^z_{c_p}, \quad W^\dagger_{c_{p+1}} = \prod_{c_p\in\partial c_{p+1}}L^+_{c_p} \end{aligned} \tag{41}$$

where $\sum_{c_p}$ is over all $p$-cells, $\delta c_{p-1}$ is the coboundary of $c_{p-1}$ [see Eq. (A3)], and $L^+_{c_p} = (L^-_{c_p})^\dagger = \exp[i\Theta_{c_p}]$ is the raising operator for $L^z_{c_p}$. Using the definition of $\delta c_p$, $\rho_{c_{p-1}}$ can be written as (see Fig. 6)

$$\rho(\boldsymbol{x})_{\mu_1\cdots\mu_{p-1}} = \sum_\nu L^z(\boldsymbol{x})_{\nu\mu_1\ldots\mu_{p-1}} - L^z(\boldsymbol{x}-\hat{\boldsymbol{\nu}})_{\nu\mu_1\ldots\mu_{p-1}}. \tag{42}$$

The algebra of local symmetric UV operators is generated by

$$\mathcal{A}_{\rm UV} = \left\{\rho^2_{c_{p-1}}, \left(L^z_{c_p}\right)^2, L^+_{c_p}\right\}. \tag{43}$$

This is invariant under the transformation $L^z_{c_p}\to -L^z_{c_p}$, and thus there is a UV $\mathbb{Z}_2^{(0)}$ symmetry.

#### *1. An exact emergent* U(1)$^{(p)}$ *symmetry*

In the limit $J/U\to 0$, there exists a low-energy sub-Hilbert space spanned by states satisfying $\langle\rho_{c_{p-1}}\rangle = 0$. Violating this constraint costs energy $U$, and we interpret states that do so in this limit as having a gapped excitation, a segment of which resides on $c_{p-1}$. We'll refer to these bosonic $(p-1)$-dimensional (in space) excitations as "charges" since they are the gauge charges of the emergent U(1) gauge theory. From the commutation relation satisfied by $L^z$ and $\Theta$, $L^+_{c_p}$ excites a charge on $\partial c_p$, examples of which are shown in Fig. 7.

The charge gap $U$ provides a candidate energy scale below which new symmetries may emerge. However, when

$J/U \neq 0$, there no longer exists a low-energy sub-Hilbert space spanned by states satisfying $\langle \rho_{c_{p-1}} \rangle = 0$. This is because the $J$ term in $H_{\rm UV}$ causes the $\langle \rho_{c_{p-1}} \rangle = 0$ and $\langle \rho_{c_{p-1}} \rangle \neq 0$ states to mix. This does not necessarily mean the charges no longer exist when $J > 0$, just that their operators depend on $J$. Indeed, a corresponding low-energy sub-Hilbert space can be identified using the local unitary from Sec. II A, which we denote as $U_{\rm LU}$. Therefore there exists a low-energy sub-Hilbert space spanned by states satisfying $\langle \widetilde{\rho}_{c_{p-1}} \rangle \equiv \langle U_{\rm LU} \rho_{c_{p-1}} U_{\rm LU}^\dagger \rangle = 0$.

We will not find an explicit form for $U_{\rm LU}$ and thus will not precisely know throughout how much of parameter space the dressed (fattened) operators can be defined without violating the assumptions of $U_{\rm LU}$. Instead, we will assume that such an operator exists and can access a greater than measure-zero part of parameter space and will investigate the consequences of this conjecture.

At this point, we cannot tell if the charge gap $\Delta$ is an IR scale or mid-IR I scale or mid-II scale, etc. In Sec. III C 2, we find it is a mid-IR scale in region III but an IR scale in region II of parameter space (see Fig. 5). For the rest of this section, however, we will adopt the language from the perspective of region III and call the charge gap a mid-IR scale.

Given the mid-IR scale $E_{\rm mid-IR} \equiv \Delta$, we would now like to find an effective mid-IR theory describing states at energies $E < E_{\rm mid-IR}$. Since the UV-symmetric operator $(\widetilde{L}^z_{c_p})^2$ commutes with $\widetilde{\rho}_{c_{p-1}}$, it does not excite any charges and is an allowed mid-IR operator. The operators $\widetilde{L}^+_{c_p}$ are not allowed as they excite charges. the allowed operators constructed from $\widetilde{L}^+_{c_p}$ are

$$\widetilde{W}^\dagger[C_p] = \prod_{c_p \in C_p} \widetilde{L}^+_{c_p}. \tag{44}$$

which we call the Wilson operator (see Fig. 8). It has the interpretation of exciting a charge, transporting it along a $p$-cycle, and then annihilating it.

While $\widetilde{W}^\dagger[C_p]$ does not excite charges, not all $\widetilde{W}^\dagger[C_p]$ are mid-IR operators. Indeed, when $K/U \gg 1$ the $p$-brane excitation created by $\widetilde{W}^\dagger[C_p]$ costs energy $\sim |C_p|K$, where $|C_p|$ is the number of $p$-cells $C_p$ is made of. So, roughly, $\widetilde{W}^\dagger[C_p]$ is allowed in this limit only if $|C_p| \ll \Delta/K$. On the other hand, when $K/U \ll 1$ this $p$-brane's gap does not increase linearly with $|C_p|$ and all $\widetilde{W}^\dagger[C_p]$ are mid-IR allowed operators. We will denote the set of $C_p$ for which $\widetilde{W}^\dagger[C_p]$ is an allowed Mid-IR operator when acting on $|{\rm vac}\rangle$ by $\mathcal{Z}_p$. This set of $p$-cycles depends on the value of $K/U$.

For a finite size system, the algebra of local symmetric mid-IR operators is generated by

$$\mathcal{A}^{\rm finite-L}_{\rm mid-IR} = \{ (\widetilde{L}^z_{c_p})^2, \widetilde{W}^\dagger[C_p] : C_p \in \mathcal{Z}_p \}. \tag{45}$$

Strictly speaking, this is only approximate since $\widetilde{W}[C_p \in \mathcal{Z}_p]$ is only a mid-IR operator when acting on low-energy eigenstates in the mid-IR, not mid-IR states with $E$ close to $E_{\rm mid-IR}$. Nevertheless, the symmetries of this should be the same as the exact form of the effective mid-IR theory. The mid-IR Hamiltonian under this approximation is

$$H^{\rm finite-L}_{\rm mid-IR} = \kappa U \sum_{c_p} (\widetilde{L}^z_{c_p})^2 - U \sum_{c_{p+1}} \widetilde{W}[\partial c_{p+1}] - U \sum_{C_p \in \mathcal{Z}_p} \varepsilon_{C_p} \widetilde{W}[C_p] + \cdots, \tag{46}$$

where $\kappa \sim K/U$ and $\varepsilon_{C_p} \sim (J/U)^{|C_p|}$.

In the thermodynamic limit, $\widetilde{W}^\dagger$ acting on noncontractible $p$-cycles is a nonlocal operator. Denoting the subset of $\mathcal{Z}_p$ with only contractible $p$-cycles as $\mathcal{B}_p$, the algebra of local symmetric mid-IR operators is now generated by

$$\mathcal{A}_{\rm mid-IR} = \{ (\widetilde{L}^z_{c_p})^2, \widetilde{W}^\dagger[C_p] : C_p \in \mathcal{B}_p \}. \tag{47}$$

From the effective Hamiltonian point of view, $H^{\rm finite-L}_{\rm mid-IR}$ must be a local Hamiltonian, so the mid-IR theory is only well-defined provided $J/U \ll 1$.[8] Therefore, in the thermodynamic limit, terms with Wilson operators supported on nontrivial $p$-cycles vanish, and the mid-IR theory becomes

$$H_{\rm mid-IR} = \kappa U \sum_{c_p} (\widetilde{L}^z_{c_p})^2 - U \sum_{c_{p+1}} \widetilde{W}[\partial c_{p+1}] - U \sum_{C_p \in \mathcal{B}_p} \varepsilon_{C_p} \widetilde{W}[C_p] + \cdots, \tag{48}$$

$\mathcal{A}_{\rm mid-IR}$ includes an new symmetry absent from $\mathcal{A}_{\rm UV}$. Indeed, the mid-IR theory is invariant under the transformation

$$\widetilde{L}^+_{c_p} \to e^{i\Gamma_{c_p}} \widetilde{L}^+_{c_p} \Rightarrow \widetilde{W}[C_p] \to e^{i\sum_{c_p \in C_p} \Gamma_{c_p}} \widetilde{W}[C_p], \tag{49}$$

where $(d\Gamma)_{c_{p+1}} \equiv \sum_{c_p \in \partial c_{p+1}} \Gamma_{c_p} = 0$. This is a symmetry because $\sum_{c_p \in C_p} \Gamma_{c_p} = 0$ for $C_p \in \mathcal{B}_p$ since $(d\Gamma)_{c_{p+1}} = 0$. It is not a gauge symmetry as it transforms Wilson operators on noncontractible $p$-cycles by a nontrivial element of U(1). Therefore, since the charged operators are supported on a $p$-cycle, the mid-IR has an exact emergent $U(1)^{(p)}$ symmetry. The symmetry operator of this $U(1)^{(p)}$ symmetry is

$$\widetilde{U}_\alpha(\hat{\Sigma}_{d-p}) = \prod_{\hat{c}_{d-p} \in \hat{\Sigma}_{d-p}} \exp[i\alpha\, (*\widetilde{L}^z)_{\hat{c}_{d-p}}], \tag{50}$$

where $\alpha \in [0, 2\pi)$, $\hat{\Sigma}_{d-p}$ is a $(d-p)$-cycle of the dual lattice, and $(*\widetilde{L}^z)_{\hat{c}_{d-p}} \equiv \widetilde{L}^z_{*\hat{c}_{d-p}}$ (see Fig. 9).

This $U(1)^{(p)}$ symmetry only emerges in parameter space where the mid-IR—the low-energy regime without gapped charges—exists. Here we associate the mid-IR's existence to the existence of a well-defined effective mid-IR Hamiltonian, so the exact emergent $U(1)^{(p)}$ symmetry exists only when $H_{\rm mid-IR}$ converges. In the approximation scheme used, this requires $(\varepsilon_{C_p})^{1/|C_p|} < 1$ for all $C_p \in \mathcal{B}_p$. The constant of proportionality in $(\varepsilon_{C_p})^{1/|C_p|} \propto J/U$ increases with $|C_p|$ since there are more ways $\widetilde{W}[C_p]$ can be generated from $\widetilde{L}^+_{c_p}$ for larger $|C_p|$. Therefore it is sufficient to consider only the

[8] When $p = 1$, Eq. (46) can be thought of as a lattice regularization of the string field theory in Ref. [36]. Here, the suppression of large loops automatically arises from the locality of the UV theory.

largest $p$-cycle in $\mathcal{B}_p$. For small enough $K/U$ where $\mathcal{B}_p$ includes all trivial $p$-cycles, the largest $C_p$ does not depend on $K/U$, and so the largest value of $J/U$ with the exact emergent $\mathrm{U}(1)^{(p)}$ symmetry is independent of $K/U$. This value of $J/U$ defines the boundary between regions I and III in Fig. 5. For large enough $K/U$, $\mathcal{B}_p$ does not include all trivial $p$-cycles. The larger $K/U$ is, the smaller the maximum value of $|C_p|$ is, and thus the larger the maximum value of $J/U$ with the exact emergent $\mathrm{U}(1)^{(p)}$ symmetry is. This value of $J/U$ increasing with $K/U$ defines the boundary between regions I and II shown schematically in Fig. 5.

### 2. An exact emergent anomalous $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ symmetry in the continuum

The exact emergent $\mathrm{U}(1)^{(p)}$ symmetry can be spontaneously broken, and the SSB phase corresponds to the deconfined phase of U(1) $p$-gauge theory. To gain some intuition, we consider two tractable limits of the effective mid-IR theory Eq. (48). When $J/U = 0$ but $K/U \neq 0$ (which is in region II), the ground state satisfies $\widetilde{L}^z_{c_p}|\mathrm{vac}\rangle = 0$, and therefore $\langle \widetilde{W}^\dagger[C_p]\rangle = 0$ for all $C_p$. Consequently, this limit lies in a $\mathrm{U}(1)^{(p)}$ symmetric phase. On the other hand, when $K/U = 0$ but $J/U \neq 0$ (which is in region III), the ground state satisfies $\widetilde{W}[C_p \in \mathcal{B}_p]|\mathrm{vac}\rangle = |\mathrm{vac}\rangle$, and consequently $\langle \widetilde{W}^\dagger[C_p]\rangle = 1$ for all trivial $p$-cycles. Therefore the $\mathrm{U}(1)^{(p)}$ symmetry is spontaneously broken in this limit.

A $\mathrm{U}(1)^{(p)}$ symmetry at zero temperature can spontaneously break when $d > p + 1$ [3,31]. Therefore, when $d > p + 1$, we expect the symmetry to be broken even for $K/U \neq 0$ and a stable SSB phase to exist. For small $\kappa$ and $\varepsilon_{C_p}$, a reasonable expectation from Eq. (48) is the SSB phase occurs when $\kappa \lesssim 1$. This determines the boundary between the $\mathrm{U}(1)^{(p)}$ symmetric and $\mathrm{U}(1)^{(p)}$ SSB phases and regions II and III shown in Fig. 5. We leave a more detailed investigation of this phase transition to future work.

Let us now restrict our considerations to the SSB phase. Like 0-form symmetries, breaking higher-form symmetries gives rise to gapped topological defects and, in this case, arise from the nontrivial mappings $Z_p(M_d;\mathbb{Z}) \to \mathrm{U}(1)$. The topological defects excited in a state $|\psi\rangle$ are probed by repeatedly acting the Wilson operator over a trivial $(p+1)$-cycle $C_{p+1}$:[9]

$$\prod_{c_{p+1}\in C_{p+1}} \widetilde{W}^\dagger[\partial c_{p+1}]|\psi\rangle = \mathrm{e}^{2\pi \mathrm{i}\hat{Q}(C_{p+1})}|\psi\rangle. \tag{51}$$

The eigenvalue $\hat{Q}(C_{p+1})$ is the winding number and yields the net number of topological defects enclosed by $C_{p+1}$. It is given by

$$\hat{Q}(C_{p+1}) = \frac{1}{2\pi}\sum_{c_{p+1}\in C_{p+1}} F_{c_{p+1}}, \tag{52}$$

where $F_{c_{p+1}} = (\mathrm{d}\widetilde{\Theta})_{c_{p+1}} \bmod 2\pi$. Using the identity $x \bmod n = x - n\lfloor x/n\rceil$, where $\lfloor\cdot\rceil$ rounds its input to the nearest integer, $F_{c_{p+1}}$ can be written as

$$F_{c_{p+1}} \equiv (\mathrm{d}\widetilde{\Theta})_{c_{p+1}} + \omega_{c_{p+1}}, \tag{53}$$

where $\omega_{c_{p+1}} \equiv -2\pi\lfloor(\mathrm{d}\widetilde{\Theta})_{c_{p+1}}/(2\pi)\rceil$.

[9]This is a natural generalization of the $p = 0$ case, where the topological defects are vortices.

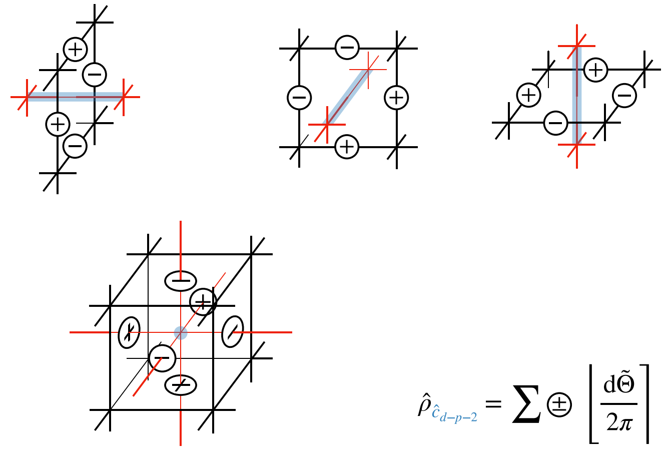


FIG. 11. Graphical representation of $\hat{\rho}_{\hat{c}_{d-p-2}}$ [see Eq. (54)] in 3d space for (first row) $p = 0$ and (second row) $p = 1$. The $\pm$ disks denote the sign in front of that $\lfloor \mathrm{d}\widetilde{\Theta}/(2\pi)\rceil$ in the sum for $\hat{\rho}_{\hat{c}_{d-p-2}}$. The direct lattice is colored in black, the dual lattice is in red, and $\hat{\rho}_{\hat{c}_{d-p-2}}$ is in blue.

The topological defects can be characterized locally by paramerizing $\hat{Q}(C_{p+1} = \partial O_{p+2}) \equiv \sum_{c_{p+2}\in O_{p+2}} (*\,\hat{\rho})_{c_{p+2}}$, where the topological defect density $\hat{\rho}$ is

$$(*\,\hat{\rho})_{c_{p+2}} = \frac{1}{2\pi}(\mathrm{d}F)_{c_{p+2}}. \tag{54}$$

Therefore they are $(d-p-2)$-dimensional excitations in space, residing on the dual lattice, and carry $\mathbb{Z}$ charge (see Fig. 11). In the familiar $p = 1$ case, they correspond to the magnetic monopole excitations of U(1) gauge theory.

However, these topological defects cannot be observed directly in the lattice model.[10] Indeed since $\Theta_{c_p}$ always appears as $L^+_{c_p} = \mathrm{e}^{\mathrm{i}\Theta_{c_p}}$, $(*\,\hat{\rho})_{c_{p+2}}$ too always appears as $\mathrm{e}^{\mathrm{i}2\pi(*\,\hat{\rho})_{c_{p+2}}}$ and so $\hat{\rho}_{\hat{c}_{d-p-2}} \sim \hat{\rho}_{\hat{c}_{d-p-2}} + 1$ on the lattice.

While the topological defects are unobservable on the lattice, their effects emerge in the continuum limit. The general paradigm for lattice models (without UV/IR mixing) is that the effective IR theory deep into a phase of matter is a continuum quantum field theory reflecting that phase's universal properties. Finding the IR effective field theory involves going deep into the $\mathrm{U}(1)^{(p)}$ SSB phase and taking the continuum limit. Deep into the SSB phase, the effective IR hamiltonian Eq. (48) includes only the leading order in $\kappa$ and $\varepsilon_{C_p}$ terms:

$$H_{\text{deep IR}} \approx \frac{\kappa U}{2}\sum_{c_p}\left(\widetilde{L}^z_{c_p}\right)^2 + \frac{U}{2}\sum_{c_{p+1}}\left(F_{c_{p+1}}\right)^2. \tag{55}$$

In the field theory, these higher-order terms could contribute as higher-derivative terms but do not affect the deep IR.

Appendix C shows how we take the continuum limit of $H_{\text{deep IR}}$, doing so carefully to capture the topologically

[10]One could instead consider a Villain type Hamiltonian model for which these topological defects are observable even in the UV/mid-IR [48,91,92]. Nevertheless, these different UV lattice models should have the same IR effective field theory.

nontrivial parts of the quantum fields from the lattice operators. We find that the IR effective field theory is compact $p$-form Maxwell theory, described by the path integral

$$\mathcal{Z}_{\text{deep IR}} = \int \mathcal{D}[a] \sum_{\omega_a \in 2\pi H^{p+1}(X;\mathbb{Z})} \mathrm{e}^{-\frac{\mathrm{i}}{2g^2}\int_X F_a \wedge * F_a}, \tag{56}$$

where $F_a = \mathrm{d}a + \omega_a$, $a$ is a $p$-form in Minkowski space-time $X$, and $H^{p+1}(X;\mathbb{Z})$ is the $(p+1)$th de Rham cohomology group with integral periods. This field theory describes the dynamical fluctuations of the $\frac{(d-1)!}{p!(d-p-1)!}$ $p$-form Goldstone bosons of the $\mathrm{U}(1)^{(p)}$ SSB phase [34] traveling at the "speed of light" $c = U\sqrt{\kappa}$. Furthermore, as reviewed in Appendix Sec. C 1, it has an anomalous $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ symmetry [3]. Therefore deep into the $\mathrm{U}(1)^{(p)}$ SSB phase of the lattice model, a new symmetry emerges in the continuum. So, the IR of region III has an exact emergent $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ symmetry in the continuum.

## IV. GENERALIZED LANDAU PARADIGM IN PRACTICE

As mentioned in the introduction, an exciting prospect of generalized symmetries is the expansion of Landau's original symmetry paradigm [23,24]. Since, as we have argued, emergent higher-form symmetries are exact symmetries, to fully utilize the power of a generalized Landau symmetry paradigm, we must consider both a system's microscopic and exact emergent symmetries. In particular, every emergent symmetry is accompanied by an energy scale or a length scale. As we change parameters, those energy scales may become zero, or the length scales diverge. This modified generalized Landau paradigm can give new results, which is one of the key results of the paper. In this section, we demonstrate how this can be done in practice by studying how the exact emergent 1-form symmetries affect the structure of the phases and phase transitions of a simple concrete model.

### A. Fradkin-Shenker model

We will study $(2+1)$D $\mathbb{Z}_2$ lattice gauge theory with matter. Let us consider the square lattice with periodic boundary conditions and a qubit residing on each link $l$ acted on by the Pauli matrices $X_l$ and $Z_l$. The Hamiltonian is

$$H = -\sum_s Q_s - \sum_p F_p - t_e \sum_l X_l - t_m \sum_l Z_l,$$
$$Q_s = \prod_{l \supset s} Z_l, \quad F_p = \prod_{l \subset p} X_l, \tag{57}$$

where $t_e, t_m \geqslant 0$, $Q_s$ is a product over the four links meeting at site $s$, and $F_p$ is a product over the four links surrounding plaquette $p$. $H$ is the toric code [93] in a magnetic field $(t_e, t_m)$, which is equivalent to the Fradkin-Shenker model [94]. It has been intensely studied [46,52,54,76,95–99] and famous for its phase at $t_e, t_m \ll 1$ with $\mathbb{Z}_2$ topological order [100,101] (see Fig. 12).

This model has exact 1-form symmetries when $t_e$ and/or $t_m$ vanishes. When $t_e = 0$, it enjoys an anomaly-free $\mathbb{Z}_2$ 1-form

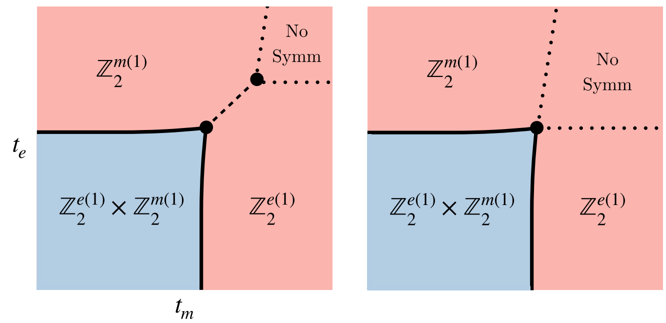


FIG. 12. (Left) The phase diagram of Fradkin-Shenker model (57) labeled by its exact emergent symmetries. The topological phase is colored blue while the trivial phase is red. The solid line corresponds to a continuous transition, the dashed line corresponds to a first-order transition, and the dotted line is not a phase transition. (Right) Another possible phase diagram. The phase transition curve is not smooth and has a singularity, which is a consequence of exact emergent 1-form symmetries.

symmetry generated by

$$U_1^{(e)}(\tilde{\gamma}) = \prod_{l \perp \tilde{\gamma}} Z_l, \tag{58}$$

where the product is over all links crossing the loop $\tilde{\gamma} \in Z_1(\widetilde{M};\mathbb{Z}_2)$ of the dual lattice $\widetilde{M}$. When $t_m = 0$, $H$ has a different anomaly-free $\mathbb{Z}_2$ 1-form symmetry generated by

$$U_1^{(m)}(\gamma) = \prod_{l \subset \gamma} X_l, \tag{59}$$

where the product is over all links in the loop $\gamma \in Z_1(M;\mathbb{Z}_2)$ of the lattice $M$. We distinguish these by calling them the electric and magnetic symmetries, respectively, and denoting them as $\mathbb{Z}_2^{e(1)}$ and $\mathbb{Z}_2^{m(1)}$. When $t_e = t_m = 0$ and $H$ is the toric code model, both symmetries are present and act anomalously. This anomaly ensures that both symmetries are spontaneously broken in any gapped phase [102].

There are additional symmetries besides these two 1-form symmetries. For instance, when $t_e = t_m$, $H$ has a $e$-$m$-exchange symmetry $\mathbb{Z}_2^{em}$ whose action exchanges $X_l \leftrightarrow Z_{l+\hat{x}/2+\hat{y}/2}$. There are other 0-form symmetries at the toric code point as well, which are generated by condensation defects of the two 1-form symmetries. However, these additional symmetries do not appear to play a role in characterizing the model's phase diagram, and we will therefore focus on the two 1-form symmetries from here on

These microscopic symmetries are present in a small subspace of parameter space and thus are inadequate in classifying the model's phases and phase transitions. However, since they include 1-form symmetries, after explicitly breaking them they will survive as exact emergent symmetries at low energies.

For instance, the $\mathbb{Z}_2^{e(1)}$ symmetry at $t_e = 0$ exists as an exact emergent symmetry whenever there exists an $e$ anyon string operator. Thus it emerges at energy scales below twice the $e$ anyon gap when $t_e \neq 0$. When $t_e = 0$, the $e$ anyon's string operator is simply $X_l$, while for $t_e \neq 0$ they are fattened and dressed by a particular local unitary $U_{\text{LU}}$. The emergent symmetry operators will be generated by the fattened loop operators, which is $U_1^{(e)}$ dressed by $U_{\rm LU}$. As an exact emergent symmetry, $\mathbb{Z}_2^{e(1)}$ has a length scale $l_{\rm symm}$ and an energy scale $E_{\rm symm} \sim \mathcal{O}(1-t_e)$. If $t_e$ is too large then $l_{\rm symm} = \infty$ and/or $E_{\rm symm} = 0$ causing the exact emergent higher-form symmetry to disappear.

When $t_e = 0$ and the $\mathbb{Z}_2^{e(1)}$ symmetry is a microscopic symmetry that is spontaneously broken for small $t_m$. The phase transition at $t_m \approx 0.34$ [95] is controlled by $\mathbb{Z}_2^{e(1)}$. Since the dual symmetry of $\mathbb{Z}_2^{e(1)}$ is a $\mathbb{Z}_2$ 0-form symmetry, the transition is described by the gauged Ising—Ising*—conformal field theory. Since $\mathbb{Z}_2^{e(1)}$ is an exact emergent symmetry when $t_e \neq 0$, the symmetry broken phase and phase transition persist away from $t_e = 0$, as shown in Fig. 12, despite the lattice model no longer having a microsopic $\mathbb{Z}_2^{e(1)}$ symmetry.

This discussion applies to $\mathbb{Z}_2^{m(1)}$ as well, just with all electric and magnetic variables exchanged. Therefore the topological phase has an exact emergent anomalous $\mathbb{Z}_2^{e(1)} \times \mathbb{Z}_2^{m(1)}$ symmetry spontaneously broken and a phase transition into the Higgs (confined) regime is driven by restoring the magnetic (electric) 1-form symmetry. This means that the Higgs (confined) regime has an exact emergent unbroken $\mathbb{Z}_2^{m(1)}$ ($\mathbb{Z}_2^{e(1)}$) symmetry. We propose that across the first-order transition line, the exact emergent $\mathbb{Z}_2^{m(1)}$ symmetry switches to the exact emergent $\mathbb{Z}_2^{e(1)}$ symmetry and vice versa.

There should be no region outside of the topological phase with both emergent $\mathbb{Z}_2^{m(1)}$ and $\mathbb{Z}_2^{e(1)}$ symmetries since otherwise, their mixed anomaly would require the gapped phase at the upper right corner of Fig. 12 to be a nonproduct state. This implies that there is a region in the upper right corner of the phase diagram with no emergent symmetry.

Since both the Higgs and confined regimes lie in the trivial phase [94,103], one may wonder how they can have different symmetries. Indeed, Higgs and confined phases can be distinguished when they have different realizations of symmetries (see recent work Refs. [46,47,55,104] on the subject). However, with periodic boundary conditions, both symmetries are realized trivially since the charged states cost infinite energy in the thermodynamic limit. This is why there is no phase transition when the exact emergent unbroken 1-form symmetries disappear across the dotted line in Fig. 12. While the emergent symmetry is not represented faithfully, it is still important to keep track of it since it controls the universality class of the phase transition out of the topological phase.

Since the two continuous transitions out of the topological phase (the Higgsing and confining transitions) have different 1-form symmetries, there must be a singularity (i.e., a multicritical point) where the transition lines meet since their symmetries will switch $\mathbb{Z}_2^{m(1)} \leftrightarrow \mathbb{Z}_2^{e(1)}$. In model (57), this singularity happens to be at the end of the first-order transition line. It would be interesting to study a generalization of the model that has no $e$-$m$-exchange symmetry even along the diagonal line, to see if there are other forms of singularities along the continuous transition line, in particular, if the first-order transition line can shrink to a point [see Fig. 12 (right)]. We predict the existence of singularities along the continuous transition line even for general models.

When transitioning from the topological phase to the Higgs regime, the energy scale for the exact emergent $\mathbb{Z}_2^{e(1)}$ symmetry vanishes which causes the electric symmetry to no longer emerge. We therefore say that the critical point of the transition has a marginal emergent $\mathbb{Z}_2^{e(1)}$ symmetry. As a definition, a system has a marginal emergent symmetryif there exists an infinite sequence of systems approaching the original system, such that each system in the sequence has the exact emergent symmetry. This appears to be a concept that is unique to exact emergent higher-form symmetries. Similarly, the multi-critical point does not have exact emergent $\mathbb{Z}_2^{m(1)} \times \mathbb{Z}_2^{e(1)}$ symmetry, but instead a marginal emergent $\mathbb{Z}_2^{m(1)} \times \mathbb{Z}_2^{e(1)}$ symmetry. An interesting future direction is to investigate the role marginal emergent symmetries play in characterizing phase transitions.

## V. PHYSICAL CONSEQUENCES

Emergent 0-form symmetries are typically not exact, so their consequences are approximate. However, as we have shown, emergent higher-form symmetries are exact emergent symmetries. Therefore their low-energies consequences are exact and equivalently powerful as UV symmetries. Furthermore, since emergent higher-form symmetries are robust against translation-invariant local perturbations, physical properties arising from their existence are also robust. In this section, we summarize the physical consequences of emergent higher-form symmetries being exact. We emphasize their role in characterizing phases of matter, fitting exact emergent symmetries into the generalized Landau classification scheme (see Ref. [20]).

### A. Spontaneous symmetry breaking

Since spontaneous symmetry breaking (SSB) is diagnosed using the ground state, an emergent higher-form symmetry can be spontaneously broken in the same way a UV symmetry can be spontaneously broken. A consequence of this is that a phase with an emergent discrete higher-form symmetry spontaneously broken has an exact ground state degeneracy (GSD) which depends on space-time's topology. Similarly, a phase with an emergent continuous higher-form symmetry spontaneously broken has Goldstone bosons. If the continuous higher-form symmetry emerges at $E < E_{\rm mid-IR}$, these Goldstone bosons are exactly gapless for mid-IR states. However, for states in the sub-Hilbert space spanned by energy eigenstates with $E \geqslant E_{\rm mid-IR}$, the Goldstone bosons acquire a gap.[11]

Since emergent higher-form symmetries are topologically robust, a local translation-invariant UV perturbation does not gap out their Goldstone bosons nor lift the topological GSD. This is very different from 0-form symmetries where even weakly breaking the symmetry in the UV gaps out the Goldstone boson [105] or lifts the GSD.

The SSB phase of an emergent higher $p$-form symmetry has gapped topological defect excitations. When an anomaly-free $\mathrm{U}(1)^{(p)}$ ($\mathbb{Z}_N^{(p)}$) symmetry spontaneously breaks

[11]This is a familiar concept in the $p = 1$ case where electric screening causes the photon to acquire a gap.

in $d$-dimensional space, there are $d - p - 2$ ($d - p - 1$) dimensional topological defects carrying $\mathbb{Z}$ ($\mathbb{Z}_N$) topological charge [5,36,106]. For $p = 1$ and $d = 3$, this is the magnetic monopole of U(1) gauge theory (the flux loop of $\mathbb{Z}_N$ gauge theory). In the trivial symmetric phase, the topological defects are condensed.

As we saw in Sec. III, when the $\mathrm{U}(1)^{(p)}$ ($\mathbb{Z}_N^{(p)}$) topological defect has a gap, there is a low-energy regime with an exact emergent $\mathrm{U}(1)^{(d-p-1)}$ ($\mathbb{Z}_N^{(d-p)}$) symmetry. We can flip this around and define the existence of gapped topological defects by the presence of these exact emergent symmetries. Therefore a $\mathrm{U}(1)^{(p)}$ ($\mathbb{Z}_N^{(p)}$) symmetry can spontaneously break in $d$-dimensional space only when $\mathrm{U}(1)^{(d-p-1)}$ ($\mathbb{Z}_N^{(d-p)}$) can be an exact emergent symmetry. Since only emergent higher-form symmetries are exact at zero temperature (see Sec. VI), a $\mathrm{U}(1)^{(p)}$ ($\mathbb{Z}_N^{(p)}$) symmetry can only spontaneously break when $d > p + 1$ ($d > p$), agreeing with Refs. [3,31].

In the SSB phase of an emergent higher-form symmetry, the low-energy states and observables are sometimes organized into the symmetry's representations. This is not generically true since emergent $p$-form symmetries have nontrivial charged operators only when there are nontrivial $p$-cycles in space. When nontrivial $p$-cycles exist, charged operators create $p$-brane excitations costing finite energy in the SSB phase, so low-energy states fall into representations of the emergent symmetry. This then gives rise to selection rules on the correlation functions of low-energy operators.

When space has no nontrivial $p$-cycles, the emergent $p$-form symmetry is trivialized, and one may be tempted to say there is no emergent symmetry. Nevertheless, it still has a corresponding exact emergent conservation law, and thus there is a transformation that leaves the low-energy effective theory unchanged. Furthermore, the SSB phase still has neutral charges condensed and, in the U(1) case, gapless Goldstone bosons. Therefore the emergent symmetry in this case still has many nontrivial consequences of a symmetry, and we thus interpret it as a symmetry.

### B. 't Hooft anomalies

Exact emergent anomaly-free symmetries can be gauged at the energy scales they exist. Since every symmetry implies a dual symmetry [80] found by gauging [14], this implies that exact emergent higher-form symmetries also have dual symmetries. However, it also implies there can be obstructions to gauging and thus 't Hooft anomalies

An emergent higher-form symmetry can be anomalous with or without spontaneous symmetry breaking and has consequences regardless of space's topology and boundaries (see also Ref. [56]). Such an anomaly can include only exact emergent symmetries or both exact emergent and exact symmetries.

A 't Hooft anomaly prevents the ground state from being a trivial product state due to anomaly matching, providing IR constraints from UV data. For an exact emergent anomalous symmetry, all energy scales below which the emergent anomalous symmetry is present must also respect anomaly matching. Therefore exact emergent anomalous symmetries also obstruct a trivial ground state, thus providing useful IR constraints using mid-IR data.

In the examples from Sec. III, the SSB phases of the models had exact emergent anomalous higher-form symmetries below the topological defect's gap. One can view the ground state degeneracies and gaplessness of Goldstone bosons in these phases as being protected by the 't Hooft anomaly. Another example is a bosonic superfluid. There is an exact emergent higher-form U(1) symmetry below the vortex gap that is not spontaneously broken but whose existence contributes to a 't Hooft anomaly protecting superflow [107]. Thus, if the topological defect's gaps were held at infinity in these examples, the ground state could never become a trivial product state.

### C. Without spontaneous symmetry breaking

When a $p$-form symmetry ($p > 0$) is unbroken, its $p$-brane symmetry excitations are gapped, and their gap grows with their size. Therefore, in the thermodynamic limit of a compact space, charged states costs infinite energy, and all finite-energy states are in the symmetric sector of the emergent higher-form symmetry. Since the emergent symmetry is trivial, one may again be tempted to say there is no emergent symmetry at all. However, even exact symmetries trivialize at low energies when unbroken, and they still have physical consequences, although very subtle. Therefore, as we will discuss, unbroken emergent higher-form symmetries can still have the nontrivial effects of a symmetry, and we thus interpret it as a symmetry.

If an exact emergent higher-form symmetry is anomaly-free and not spontaneously broken in the absence of a boundary, it can characterize nontrivial symmetry-protected topological (SPT) phases [44–47]. Indeed, the existence of an emergent higher-form symmetry implies that there are boundaries with the emergent symmetry. Arbitrary perturbations of such boundaries also have the emergent higher-form symmetry. Therefore a corresponding nontrivial SPT order could exist in the bulk if the emergent higher-form symmetry is realized anomalously on such boundaries. The bulk SPT order cancels the 't Hooft anomaly by anomaly in-flow [108], ensuring the theory remains gauge invariant when background gauge fields are turned on.

Emergent SPT orders have direct physical consequences in the presence of a spatial boundary. Indeed, since the emergent higher-form symmetry is realized anomalously on the boundary, all the physical consequences discussed in the previous two subsections, like symmetry breaking and obstructions to trivial ground states, apply on this boundary.

In the absence of a boundary, the effective IR theory of the SPT will be an invertible topological field theory in terms of the background fields. From a low-energy point of view, this is no different from an SPT protected by a UV symmetry. Indeed, the UV symmetry is trivial in the IR since it is unbroken, but an invertible topological field theory in terms of its background gauge fields characterizes the SPT order [109,110]. Moreover, this invertible theory has physical meaning: it is the effective response theory of the SPT.

This emphasizes an important distinction between SPTs protected by 0-form and higher-form symmetries. 0-form SPTs cannot occur in regions of parameter space where the 0-form symmetry is explicitly broken. However, this is untrue

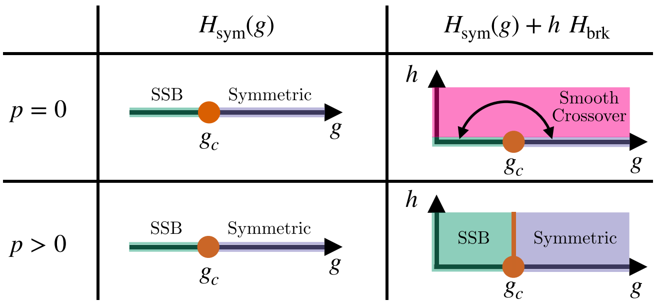


FIG. 13. A Hamiltonian $H_{\rm sym}(g)$ with a $p$-form symmetry spontaneously broken at $g < g_c$ has a qualitatively similar phase diagram whether $p = 0$ or $p > 0$. However, explicitly breaking the symmetry with $hH_{\rm brk}$ ($|h| \ll 1$), the phase diagram of $H_{\rm sym}(g) + hH_{\rm brk}$ differs between $p = 0$ and $p > 0$. For $p = 0$, there is now a smooth crossover between what used to be the SSB phase and the symmetric phase. Shown here is the case where the symmetry-breaking perturbation is relevant. For $p > 0$, $H_{\rm sym}(g) + hH_{\rm brk}$ has an exact emergent $p$-form symmetry, so there is not a smooth crossover between the phases. Starting in the SSB phase and increasing $h$ would lead to an eventual phase transition out of the SSB phase.

of higher-form SPTs since emergent higher-form symmetries are exact. Therefore, to identify higher-form SPT phases, instead of partitioning parameter space by exact higher-form symmetries, one should partition it by the emergent higher-form symmetries.

We note that this perspective of SPTs privileges the characteristic that the protecting symmetry is realized anomalously on a boundary. It then uses anomaly inflow to relate this boundary feature to a bulk property, which can be detected through its topological response at low energy. Whether this is enough to sharply define a bulk property/observable that characterizes a phase of matter is important to further investigate.

Emergent higher-form symmetries can also exactly characterize their SSB phase transitions (with or without spatial boundaries) [36,52], as we saw in Sec. IV. Indeed, transitioning from the SSB phase of an exact emergent symmetry into its symmetric phase, the critical point will have the exact emergent symmetry and can be in its symmetry-breaking pattern universality class (see Fig. 13). An example is the confinement transition of $(2+1)$D $\mathbb{Z}_2$ lattice gauge theory with dynamical matter. The matter fluctuations explicitly break a $\mathbb{Z}_2^{(1)}$ symmetry, but the transition is still in the Ising universality class since there is an exact emergent $\mathbb{Z}_2^{(1)}$ symmetry [52].

Emergent invertible higher-form symmetries can interact nontrivially with exact symmetries, forming an emergent higher-group symmetry [6,9]. For example, when an $A^{(1)}$ symmetry emerges in the presence of an exact 2-group symmetry, the total low-energy symmetry is described by a 2-group $\mathbb{G}^{(2)} = (G, A, \rho, [\beta])$, where $G$ is the IR 0-form symmetry, $\rho : G \to \mathrm{Aut}(A)$, and $[\beta] \in H^3_\rho(\mathcal{B}G; A)$. The 1-form symmetry, even without spontaneous symmetry breaking, can be nontrivial if the Postnikov class $[\beta]$ is nontrivial.

### *1. A dynamical effect*

An exact emergent higher-form symmetry constrains the $p$-brane symmetry excitations dynamics. For simplicity, let

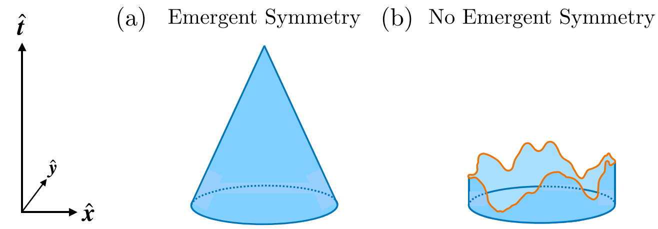


FIG. 14. Shows a cartoon of the different decay processes of 1-form symmetry excitations in $2+1$d space-time. The world sheet of the 1-brane excitation is colored blue, while the worldline of gauge charge excitations is orange.

us set $p = 1$ and consider a state with symmetry excitation excited on a contractible 1-cycle $C_1$.

We first assume that symmetry excitations have a fixed energy $\epsilon$ per lattice edge and that open string ends (e.g., gauge charges) have an energy gap $\Delta$. The 1-form symmetry exists only at energies $E < \Delta$ and affects symmetry excitations with $|C_1|\epsilon \ll \Delta$. For symmetry excitations with $|C_1| \ll \Delta/\epsilon$, due to the emergent 1-form symmetry, the only way for them to decay is by contracting to a point [see Fig. 14(a)]. So, their life time $\tau$ grows with their size $|C_1|$. Symmetry excitations with $|C_1| > \Delta/\epsilon$ are not affected by the 1-form symmetry and can, therefore, decay by quantum tunneling to states with open string ends [see Fig. 14(b)]. Therefore their lifetime is independent of their size, instead going like $\tau \sim \mathrm{e}^{\Delta/\epsilon}$.

Let us now assume that a symmetry excitation is trapped by a trap potential [111] and has a fixed total energy $|C_1|\epsilon$. If the trapped symmetry excitation has $|C_1| \ll \Delta/\epsilon$, it can no longer decay since the trap potential prevents it from contracting to a point. Such a symmetry excitation is an exact quantum many-body scar (QMBS) state [112]. If the trapped symmetry excitation has $|C_1| > \Delta/\epsilon$, it will still decay with a finite lifetime $\tau \sim \mathrm{e}^{\Delta/\epsilon}$. However, it will have a long lifetime when $\epsilon$ is small, making large symmetry excitation loops approximate QMBS states. The larger the trap, the better the QMBS state with a given energy, so infinite-sized loop excitations are exact QMBS states. Furthermore, any small perturbation of the trap would still lead to an approximate QMBS state, but with lifetime $\tau \sim \mathrm{e}^{\Delta/|\epsilon+\delta\epsilon|}$ where $\delta\epsilon$ is the strength of perturbation. Thus the existence of the emergent 1-form symmetry at $E < \Delta$ implies that there exists a large potential trap leading to exact QMBS states at $E < \Delta$ and approximate QMBS states at $E > \Delta$.

Both of these scenarios apply to $p$-form symmetries with $p > 1$ under a straightforward generalization. In the latter, the trap potential can trap $p > 1$ dimensional symmetry excitations which lead to QMBS states. However, it can also trap topologically ordered states which can lead to QMBS states. So, to be precise, we say that an exact emergent $p$-form symmetry implies the existence of $p$-dimensional trap potentials leading to QMBS states besides those corresponding to topologically ordered states.

## VI. FINITE TEMPERATURE EFFECTS

Here we discuss how our results are modified at finite temperature $T$. When $T \neq 0$, the imaginary time direction becomes $S^1$ with radius $1/T$. Thus $S^1$ is a small dimension of space-time when the linear system size $L \gg 1/T$, and space-time can be dimensionally reduced from $(d+1)$-dimensional $M_d \times S^1$ to $d$-dimensional $M_d$ in the thermodynamic limit.

Let us warm up by first considering how the SSB critical dimension for higher-form symmetries changes.[12] At $T = 0$, a $\mathbb{Z}_N^{(p)}$ ($\mathrm{U}(1)^{(p)}$) symmetry can spontaneously break when $d+1 > p+1$ ($d+1 > p+2$) [3,31]. Using dimensional reduction, we can understand the finite temperature case by replacing $d+1$ with $d$. Thus, at $T \neq 0$, a $\mathbb{Z}_N^{(p)}$ ($\mathrm{U}(1)^{(p)}$) symmetry can spontaneously break when $d > p+1$ ($d > p+2$).

So, are emergent higher-form symmetries still exact when $T \neq 0$? At finite temperature, $p$-form symmetry charged operators can act on nontrivial $p$-cycles winding around the compactified imaginary time direction. When dimensionally reducing space-time, this direction becomes negligible, and any such $p$-cycles become $(p-1)$-cycles. So, it is as if $p$-form symmetries acts like $(p-1)$-form symmetries. Since emergent 0-form symmetries are not exact, emergent 1-form symmetries are no longer exact at finite temperature. However, emergent $p$-form symmetries with $p > 1$ are still exact at $T \neq 0$.

This can be seen without having to dimensionally reduce. Indeed, at $T \neq 0$, 1-cycles winding around imaginary time are nontrivial 1-cycles of finite length even in the thermodynamic limit. Assuming the UV theory is local, the low-energy effective Euclidean action includes terms suppressed by $e^{-1/T}$ with the emergent 1-form symmetry's charged operators [36]. Hence, emergent 1-form symmetries at $T \neq 0$ are approximate, not exact. For $p$-form symmetries with $p > 1$, there are no finite sized nontrivial $p$-cycles at $T \neq 0$. Indeed, while a $p$-cycle could wind around the finite-sized imaginary time direction, it must also wind around $(p-1)$ spatial directions, which are not finite-sized in the thermodynamic limit. Therefore emergent $p$-form symmetries with $p > 1$ are still exact at $T \neq 0$.

Let us now apply the above discussion to known examples.

The $p$-form toric code model [see Eq. (B2)] lies in the SSB phase of an anomalous $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ symmetry, where $0 < p < d$. The $e$ excitations are $(p-1)$ branes while the $m$ excitations are $(d-p-1)$ branes. Whether or not its topological order is robust at finite temperature depends on if this symmetry remains spontaneously broken at $T \neq 0$. This requires $d > p+1$ for $\mathbb{Z}_N^{(p)}$ and $d > d-p+1$ for $\mathbb{Z}_N^{(d-p)}$. Therefore, for $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ to spontaneously break at $T \neq 0$, $p$ and $d$ must satisfy

$$2 \leqslant p \leqslant d-2. \tag{60}$$

This is never satisfied for $d = 2$ or 3, recovering that the toric code in $d = 2$ and 3 at finite temperature does not have topological order [113–116]. In the $d = 3$ case, the exact symmetry is $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(2)}$, and there is no topological order at finite temperature since $\mathbb{Z}_N^{(2)}$ cannot spontaneously break. However, the $\mathbb{Z}_N^{(1)}$ still can, giving rise to the "classical topological order" discussed in Ref. [114].

[12]We thank Carolyn Zhang for helpful discussions about this

$d = 4$ is the smallest spatial dimension for which Eq. (60) is satisfied, which recovers that the 2-form toric code's topological order is robust at finite temperature [113,116]. In this case, $p = 2$, so both $e$ and $m$ excitations are loops.

Weakly perturbing the $p$-form toric code explicitly breaks its $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ symmetry. When $T \neq 0$, its can only emergent exactly if $p > 1$ and $d-p > 1$, which is precisely Eq. (60). The "classical topological order" when $d = 3$ is not robust to these perturbations at $T \neq 0$ since the emergent $\mathbb{Z}_N^{(1)}$ symmetry will be approximate.

Next, consider U(1) quantum spin liquids, a class of spin liquid phases whose effective description is the deconfined phase of pure U(1) gauge theory [117]. The prototypical example in $d = 3$ is quantum spin ice [118–121]. More generally, a $p$-form U(1) quantum spin liquid's effective IR theory is $p$-form Maxwell theory, which lies in the SSB phase of a $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ symmetry. If such an SSB phase is robust at finite temperature, $d > p+2$ and $d > d-p-1+2$ for $\mathrm{U}(1)^{(p)}$ and $\mathrm{U}(1)^{(d-p-1)}$, respectively, to spontaneously break. Therefore, for $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ to spontaneously break at $T \neq 0$, $p$ and $d$ must satisfy

$$2 \leqslant p \leqslant d-3. \tag{61}$$

This is never satisfied for $d = 2, 3$, or 4. $d = 5$ is the smallest spatial dimension where a U(1) quantum spin liquid phase is stable at finite temperature. The low-energy description of such a phase is $(5+1)D$ U(1) 2-form Maxwell theory, where both electric and magnetic excitations are loops.

When $d = 4$ and $p = 1$, the emergent symmetry is $\mathrm{U}(1)^{(1)} \times \mathrm{U}(1)^{(2)}$. While the $\mathrm{U}(1)^{(2)}$ symmetry cannot spontaneously break at $T \neq 0$, the $\mathrm{U}(1)^{(1)}$ symmetry can, giving rise to a "classical U(1) topological order." However, this emergent $\mathrm{U}(1)^{(1)}$ symmetry is not exact. Indeed, the emergent higher-form U(1) symmetries are only exact at $T \neq 0$ when Eq. (61) is satisfied.

## VII. CONCLUSION AND DISCUSSION

In this paper, we have investigated the robustness of emergent higher-form symmetries from a UV perspective, considering bosonic lattice Hamiltonian models. In Sec. II, we showed how emergent higher-form symmetries in lattice models are generally exact symmetries and not approximate symmetries. To emphasize this robustness, we referred to emergent higher-form symmetries as *exact emergent symmetries*. This means that lattice models without exact higher-form symmetries can have emergent higher-form symmetries whose effects at low energy are the same as if they were exact symmetries. Therefore emergent higher-form symmetries can exactly characterize phases of systems without exact higher-form symmetries. We considered three examples of this in Sec. III, discussed the general physical consequences in Sec. V, and discussed finite temperature effects in Sec. VI.

As discussed in Sec. II, when higher-form symmetries are emergent, their symmetry and charged operators are "fattened" [50]. The exact expression of these operators depends on the UV parameters, and finding such a closed form requires an exact expression for the local unitary $U_{\mathrm{LU}}$. Not only is this highly nontrivial, but likely analytically intractable for generic models. That said, a promising approach to find exact

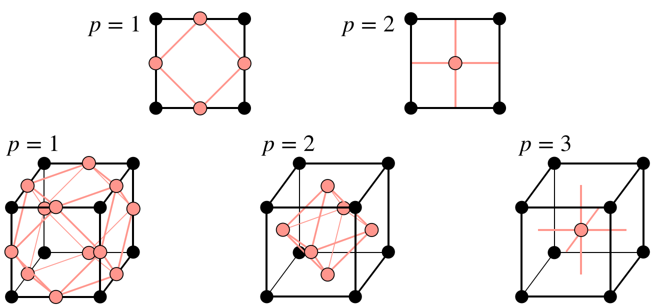


FIG. 15. The $p$-cells of the $d$-dimensional cubic lattice are equivalently the 0-cells of another lattice. Shown here are examples of this equivalent lattice embedded in the conventional unit cell of the cubic lattice, drawn in pink and black, respectively, for (first row) $d = 2$ and (second row) $d = 3$.

expressions is using numerical schemes. This was done for untwisted and twisted $\mathbb{Z}_2$ lattice gauge theory in Ref. [54], which developed a novel unbiased numerical optimization scheme to systemically find the dressed symmetry operators. It would be interesting to extend these machine-learning approaches to other models.

An important follow-up to our paper is an in-depth study of the boundary between regions I and II in Fig. 5. One possibility is that a boundary phase transition separates the two regions, as in Refs. [46,47]. What about a bulk point of view? Recall from Sec. II that emergent higher-form symmetries have an associated energy and length scale. For instance, the emergent symmetry is destroyed if the energy scale vanishes, which is precisely what happens when going from region III to I in Fig. 5. Perhaps when going from region II to I, it is the length scale that blows up. This would prevent the symmetry and charged operators from being well-defined, destroying the emergent symmetry. It would be interesting to further investigate this possibility.

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## APPENDIX A: DISCRETE DIFFERENTIAL GEOMETRY FOR HYPERCUBIC LATTICES

This Appendix reviews relevant parts of discrete differential geometry (in a nonrigorous fashion) used throughout the main text.[13] Consider a hypercubic lattice in $d$-dimensional space, denoted by $M_d$. While a Bravais lattice is a collection of lattice sites $\boldsymbol{x} \in \mathbb{Z}^d$, it is useful to view it as also formed by higher-dimensional objects, like links, plaquettes, cubes, etc.. We call a $p$-dimensional object a $p$-cell, with $0 \leqslant p \leqslant d$. So, a 0-cell is a lattice site, a 1-cell is a link, a 2-cell is a plaquette, etc. The $p$-cells of the $d$-dimensional cubic lattice are equivalently viewed as the 0-cells of some other lattice in $d$ dimensions, as demonstrated for $d = 2$ and 3 in Fig. 15.

[13]We adopt the notation and conventions used in Ref. [122].

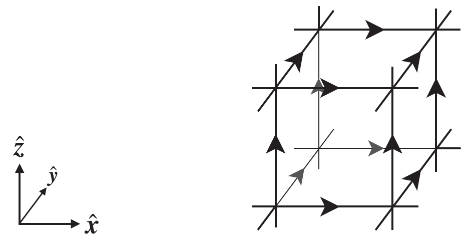


FIG. 16. Example of the branching structure used for a chunk of the cubic lattice in three-dimensional space.

$p$-cells do not add additional structures to the lattice but are just a useful way of organizing the lattice sites. Indeed, denoting a $p$-cell associated with site $\boldsymbol{x}$ as $c_p(\boldsymbol{x})_{\mu_1\mu_2\cdots\mu_p}$, where $\mu_1 < \mu_2 < \cdots < \mu_p$ and $\mu_i \in \{1, 2, \cdots, d\}$, a $p$-cell of the cubic lattice is the set of $2^p$ lattice sites

$$
\begin{aligned}
c_p(\boldsymbol{x})_{\mu_1\mu_2\cdots\mu_p} = \{\boldsymbol{x}\} &\cup \{\boldsymbol{x} + \hat{\boldsymbol{\mu}}_i \,|\, 1 \leqslant i \leqslant p\} \\
&\cup \{\boldsymbol{x} + \hat{\boldsymbol{\mu}}_i + \hat{\boldsymbol{\mu}}_j \,|\, 1 \leqslant i < j \leqslant p\} \\
&\cup \cdots \cup \{\boldsymbol{x} + \hat{\boldsymbol{\mu}}_1 + \ldots + \hat{\boldsymbol{\mu}}_p\}, \qquad \text{(A1)}
\end{aligned}
$$

where $\hat{\boldsymbol{\mu}}_i$ is the unit vector in the $\mu_i$-direction. It is often convenient to drop the requirement that the indices are ordered (i.e., $\mu_1 < \mu_2 < \cdots < \mu_p$) and instead let $c_p(\boldsymbol{x})_{\mu_1\mu_2\cdots\mu_p}$ obey $c_p(\boldsymbol{x})_{\cdots\mu_1\mu_2\cdots} = -c_p(\boldsymbol{x})_{\cdots\mu_2\mu_1\cdots}$.

Introducing the concept of $p$-cells is convenient since "sewing" $p$-cells together gives a natural way to form $p$-dimensional subspaces of the lattice. Furthermore, these subspaces can be given an orientation by defining an orientation structure to the lattice. A nice local scheme for the lattice orientation is a branching structure, where the orientation on each 1-cell is chosen such that a collection of 1-cells cannot form an oriented closed loop. A canonical orientation on all other $p$-cells then follows from the branching structure. We use the branching structure where each 1-cell $c_1(\boldsymbol{x})_\mu$ has an arrow pointing in the $\hat{\boldsymbol{\mu}}$ direction (see Fig. 16). However, it is important to note that the choice of lattice orientation is a formal convention, and choosing different branching structures does not affect the physics.[14]

A $p$-cell can be related to $(p-1)$ cells using the boundary operator $\partial$. The boundary operator acting on a $p$-cell—$\partial c_p$—is the *oriented* sum of $(p-1)$-cells on the boundary of $c_p$. For our branching structure, it is

$$
\begin{aligned}
\partial c_p(\boldsymbol{x})_{\mu_1\cdots\mu_p} = \sum_{k=1}^{p} (-1)^{k+1} \big[ &c_{p-1}(\boldsymbol{x} + \hat{\boldsymbol{\mu}}_k)_{\mu_1\cdots\mathring{\mu}_k\cdots\mu_p} \\
&- c_{p-1}(\boldsymbol{x})_{\mu_1\cdots\mathring{\mu}_k\cdots\mu_p} \big], \qquad \text{(A2)}
\end{aligned}
$$

[14]Reference [123] conjectures that observables are branching structure independent only if the continuum effective field theory is framing anomaly free.

where the notation $\overset{o}{\mu}_k$ indicates that the $\mu_k$ index is omitted. From its definition, the boundary operator satisfies $\partial^2 c_p = 0$ for any $p$-cell. Furthermore, as there are no $(-1)$-cells, the boundary operator acting on a 0-cell is defined to be zero.

On the other hand, a $p$-cell can be related to $(p+1)$-cells using the coboundary operator $\delta$. The coboundary operator acting on a $p$-cell—$\delta c_p$—is an *oriented* sum of all $(p+1)$-cells whose boundary includes $c_p$. For our branching structure we use, it is

$$\delta c_p(\boldsymbol{x})_{\mu_1\cdots\mu_p} = \sum_\nu c_{p+1}(\boldsymbol{x})_{\nu\mu_1\ldots\mu_p} - c_{p+1}(\boldsymbol{x}-\hat{\boldsymbol{\nu}})_{\nu\mu_1\ldots\mu_p}. \tag{A3}$$

From its definition, the coboundary operator satisfies $\delta^2 c_p = 0$ for any $p$-cell. Furthermore, as there are no $(d+1)$-cells, the coboundary operator acting on a $d$-cell is defined to be zero.

Lastly, the lattice has an associated dual lattice. The dual lattice has its lattice sites centered at the $d$-cells of the direct lattice. For the cubic lattice, one way to relate a dual lattice site $\hat{\boldsymbol{x}}$ to a direct lattice site $\boldsymbol{x}$ is by $\hat{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\hat{\boldsymbol{r}}$ with $\hat{\boldsymbol{r}} = \sum_i \hat{\boldsymbol{\mu}}_i$.

Each $p$-cell $c_p$ on the direct lattice corresponds to a $(d-p)$-cell $\hat{c}_{d-p}$ on the dual lattice. Mapping between them is done using the dual operator $*$. A $p$-cell $c_p(\boldsymbol{x})_{\mu_1\cdots\mu_p}$ (with canonical ordering $\mu_1 < \cdots < \mu_p$) and a $(d-p)$-cell of the dual lattice $\hat{c}_{d-p}(\hat{\boldsymbol{x}})_{\mu_1\cdots\mu_{d-p}}$ $(\mu_1 < \cdots < \mu_{d-p})$ are related by

$$\begin{aligned} * c_p(\boldsymbol{x})_{\mu_1\cdots\mu_p} = \epsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_d} \\ \times \hat{c}_{d-p}(\hat{\boldsymbol{x}} - \hat{\boldsymbol{\mu}}_{p+1} - \ldots - \hat{\boldsymbol{\mu}}_d)_{\mu_{p+1}\cdots\mu_d}, \end{aligned} \tag{A4}$$

$$\begin{aligned} * \hat{c}_p(\hat{\boldsymbol{x}})_{\mu_1\ldots\mu_p} = \epsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_d} \\ \times c_{d-p}(\boldsymbol{x} + \hat{\boldsymbol{\mu}}_1 + \ldots + \hat{\boldsymbol{\mu}}_p)_{\mu_{p+1}\cdots\mu_d}, \end{aligned} \tag{A5}$$

where summation is *not* implied on the right-hand side. Here $\epsilon$ is the Levi-Civita symbol, which takes into account the lattice's and dual lattice's relative orientations. From the definition of $*$, acting $*$ twice on a $p$-cell of the direct (dual) lattice yields $**c_p = (-1)^{p(d-p)} c_p$ $(**\hat{c}_p = (-1)^{p(d-p)}\hat{c}_p)$. Furthermore, from the definitions of the boundary, coboundary, and dual operators, they are related to one another by

$$\delta c_p = (-1)^{d(p+1)+1} * \partial * c_p, \tag{A6}$$

which, equivalently, is $*\delta c_p = (-1)^p \partial * c_p$.

## APPENDIX B: TQFT OF THE *p*-FORM TORIC CODE GROUND STATES

In Sec. III B 2 of the main text, we found that the ground states of the $\mathbb{Z}_N^{(p)}$ SSB phase satisfy

$$\prod_{c_p \in \delta c_{p-1}} \widetilde{Z}'_{c_p}|\text{vac}\rangle = |\text{vac}\rangle, \qquad \prod_{c_p \in \partial c_{p+1}} \widetilde{X}'_{c_p}|\text{vac}\rangle = |\text{vac}\rangle, \tag{B1}$$

where $\widetilde{X}'$ and $\widetilde{Z}'$ are the $\mathbb{Z}_N$ clock operators dressed by unitaries. We note that these ground states are also the ground states of the $p$-form toric code Hamiltonian

$$H_{p\text{TC}} = -\sum_{c_{p-1}} \prod_{c_p \in \delta c_{p-1}} \widetilde{Z}'_{c_p} - \sum_{c_{p+1}} \prod_{c_p \in \partial c_{p+1}} \widetilde{X}'_{c_p} + \text{H.c.} \tag{B2}$$

In this section, we relate the lattice description of the ground states to an equivalent topological quantum field theory description. Doing so demonstrates the connection between exact emergent higher-form symmetries in lattice models and exact higher-form symmetries in Lagrangian quantum field theories, where higher-form symmetries are most commonly studied.

To develop a field theory description of these ground states, we take inspiration from Ref. [124] and parametrize the dressed clock operators in the $\mathbb{Z}_N^{(p)}$ SSB phase by

$$\widetilde{X}'_{c_p} = \exp\big[\mathrm{i}\widetilde{\Theta}'_{c_p}\big], \qquad \widetilde{Z}'_{c_p} = \exp\big[\mathrm{i}(*\widetilde{\Phi}')_{c_p}\big]. \tag{B3}$$

Note that in order for $(\widetilde{X}'_{c_p})^N = (\widetilde{Z}'_{c_p})^N = \mathbf{1}$, it must be that the eigenvalues of $\widetilde{\Theta}'_{c_p}$ and $(*\widetilde{\Phi}')_{c_p}$ satisfy $\widetilde{\Theta}'_{c_p} \in 2\pi\mathbb{Z}/N$, and $(*\widetilde{\Phi}')_{c_p} \in 2\pi\mathbb{Z}/N$. Furthermore, in order for the clock operators algebra Eq. (16) to be satisfied, $\widetilde{\Theta}'_{c_p}$ and $(*\widetilde{\Phi}')_{c_p}$ must obey the commutation relation $[\widetilde{\Theta}'_{c_p}, (*\widetilde{\Phi}')_{\widetilde{c}_p}] = \frac{2\pi \mathrm{i}}{N}\delta_{c_p,\widetilde{c}_p}$. In terms of $\widetilde{\Theta}'_{c_p}$ and $(*\widetilde{\Phi}')_{c_p}$, the constraints Eq. (B1) defining the IR are

$$\frac{N}{2\pi}\delta(*\widetilde{\Phi}')_{c_{p-1}} = \frac{N}{2\pi}(\mathrm{d}\widetilde{\Theta}')_{c_{p+1}} = 0. \tag{B4}$$

The lattice Heisenberg operators $\widetilde{\Theta}'_{c_p}(t)$ and $(*\widetilde{\Phi}')_{c_p}(t)$ are related to their continuum counterparts $\widetilde{\Theta}'(t,\boldsymbol{x})$ and $*\widetilde{\Phi}'(t,\boldsymbol{x})$ by

$$\widetilde{\Theta}'_{c_p} = \int_{c_p} \widetilde{\Theta}', \; (*\widetilde{\Phi}')_{c_p} = \int_{c_p} *\widetilde{\Phi}', \tag{B5}$$

where $\int_{c_p}$ denotes spatial integration over the $p$-cell $c_p$.

For simplicity, we will work locally and treat the continuum quantum fields as differential forms in space ($\widetilde{\Theta}'$ is a $p$-form while $\widetilde{\Phi}'$ is a $(d-p)$-form) taking values in $[-\pi,\pi)$, ignoring that the holonomies of $\widetilde{\Theta}'$ and $*\widetilde{\Phi}'$ are restricted to values in $2\pi\mathbb{Z}/N$. In the continuum limit, the lattice operators simply become their continuum versions, so the constraint Eq. (B4) in the continuum limit becomes

$$\frac{N}{2\pi}\mathrm{d}^\dagger * \widetilde{\Phi}' = \frac{N}{2\pi}\mathrm{d}\widetilde{\Theta}' = 0. \tag{B6}$$

Here, $\mathrm{d}^\dagger$ is the adjoint of d, which is $\mathrm{d}^\dagger \equiv (-1)^{d(p+1)+1} * \mathrm{d} *$ when acting on a $p$-form.

The lattice Hamiltonian in the IR is just the ground state energy. Setting this to zero, in the continuum $H_{\text{IR}}^{(\text{III})} = 0$. The continuum Lagrangian is thus

$$\mathcal{L}_{\text{IR}}^{(\text{III})} = \frac{N}{2\pi\, p!}(*\widetilde{\Phi}')_{i_1\cdots i_p}\partial_t \widetilde{\Theta}'_{i_1\cdots i_p}, \tag{B7}$$

which enforces the equal-time commutation relation

$$\frac{\big[\widetilde{\Theta}'_{i_1\cdots i_p}(\boldsymbol{x}), (*\widetilde{\Phi}')_{j_1\cdots j_p}(\boldsymbol{y})\big]}{p!} = \frac{2\pi \mathrm{i}}{N}\delta^{i_1}_{[j_1}\cdots\delta^{i_p}_{j_p]}\delta^d(\boldsymbol{x}-\boldsymbol{y}). \tag{B8}$$

The low-energy path integral only integrates over field configurations satisfying Eq. (B6), and is

$$\mathcal{Z}_{\rm IR}^{\rm (III)} = \int \mathcal{D}[\widetilde{\Theta}']\mathcal{D}[\widetilde{\Phi}']\,\delta\left(\frac{N}{2\pi}{\rm d}^\dagger * \widetilde{\Phi}'\right) \times \delta\left(\frac{N}{2\pi}{\rm d}\widetilde{\Theta}'\right) {\rm e}^{{\rm i}\int {\rm d}t{\rm d}^d\boldsymbol{x}\mathcal{L}_{\rm IR}^{\rm (III)}}. \tag{B9}$$

Let us now massage this path integral into a more familiar form. We can rewrite both functional delta functions by integrating in new fields acting as Lagrange multipliers and modifying the action. The first delta function is rewritten using a $(p-1)$-form Lagrange multiplier $\lambda$ as

$$\delta\left(\frac{N}{2\pi}{\rm d}^\dagger * \widetilde{\Phi}'\right) = \int \mathcal{D}\lambda\, {\rm e}^{\frac{{\rm i}N}{2\pi}\int \frac{\lambda_{i_1\cdots i_{p-1}}({\rm d}^\dagger * \widetilde{\Phi}')_{i_1\cdots i_{p-1}}}{(p-1)!}}, \tag{B10}$$

and using a $(p+1)$-form Lagrange multiplier $*\eta$, the second delta function is

$$\delta\left(\frac{N}{2\pi}{\rm d}\widetilde{\Theta}'\right) = \int \mathcal{D}\eta\, {\rm e}^{\frac{{\rm i}N}{2\pi}\int \frac{(*\eta)_{i_1\cdots i_{p+1}}({\rm d}\widetilde{\Theta}')_{i_1\cdots i_{p+1}}}{(p+1)!}}. \tag{B11}$$

Plugging these expressions into the Eq. (B9) and the components of ${\rm d}\widetilde{\Theta}'$ and ${\rm d}^\dagger * \widetilde{\Phi}'$ and simplifying, the path integral becomes

$$\begin{aligned}\mathcal{Z}_{\rm IR}^{\rm (III)} &= \int \mathcal{D}[\widetilde{\Theta}']\mathcal{D}[\widetilde{\Phi}']\mathcal{D}[\lambda]\mathcal{D}[\eta]\,{\rm e}^{{\rm i}\int {\rm d}t{\rm d}^d\boldsymbol{x}\mathcal{L}_{\rm IR}^{\rm (III)}},\\ \mathcal{L}_{\rm IR}^{\rm (III)} &= \frac{N}{2\pi\, p!}\big((*\widetilde{\Phi}')_{i_1\cdots i_p}\big[\partial_t\widetilde{\Theta}'_{i_1\cdots i_p} + p\,\partial_{[i_1}\lambda_{i_2\cdots i_p]}\big]\\ &\quad + (*\eta)_{i_1 i_2\cdots i_{p+1}}\partial_{[i_1}\widetilde{\Theta}'_{i_2\cdots i_{p+1}]}\big).\end{aligned} \tag{B12}$$

Let's now introduce the $p$-form $a$ and $(d-p)$-form $b$ in space-time whose components are

$$\begin{aligned}a_{i_1\cdots i_p} &= \widetilde{\Theta}'_{i_1\cdots i_p}, \quad a_{0i_2\cdots i_p} = -\lambda_{i_2\cdots i_p},\\ b_{i_1\cdots i_{d-p}} &= (-1)^{d-p}\widetilde{\Phi}'_{i_1\cdots i_{d-p}}, \quad b_{0i_2\cdots i_{d-p}} = -\eta_{i_2\cdots i_{d-p}},\end{aligned}$$

where $b$ satisfies

$$(*\eta)_{i_1\cdots i_{p+1}} = -(*b)_{i_1\cdots i_{p+1}}, \quad (*\widetilde{\Phi}')_{i_1\cdots i_p} = (*b)_{0i_1\cdots i_p}.$$

Using these, the path integral becomes

$$\begin{aligned}\mathcal{Z}_{\rm IR}^{\rm (III)} &= \int \mathcal{D}[a]\mathcal{D}[b]\,{\rm e}^{{\rm i}\int {\rm d}t{\rm d}^d\boldsymbol{x}\mathcal{L}_{\rm IR}^{\rm (III)}},\\ \mathcal{L}_{\rm IR}^{\rm (III)} &= \frac{N}{2\pi\, p!}\big((*b)_{0i_1\cdots i_p}\big[\partial_t a_{i_1\cdots i_p} + (-1)^p p\,\partial_{[i_1}a_{i_2\cdots i_p]0}\big]\\ &\quad - (*b)_{i_1 i_2\cdots i_{p+1}}\partial_{[i_1}a_{i_2\cdots i_{p+1}]}\big).\end{aligned} \tag{B13}$$

The term in square brackets can be rewritten as $(p+1)\partial_{[0}a_{i_1\cdots i_p]}$. Furthermore, working in flat space-time, $X$ is equipped with Minkowski metric $(-,+,\cdots+)$. Using it and summing over space-time indices $\mu$, $\mathcal{L}_{\rm IR}^{\rm (III)}$ can be rewritten as

$$\mathcal{L}_{\rm IR}^{\rm (III)} = -\frac{N}{2\pi}\left(\frac{(*b)^{\mu_1\mu_2\cdots\mu_{p+1}}\partial_{[\mu_1}a_{\mu_2\cdots i_{p+1}]}}{p!}\right). \tag{B14}$$

Lasting, using differential forms notation, we arrive at our final expression for the path integral

$$\mathcal{Z}_{\rm IR}^{\rm (III)} = \int \mathcal{D}[a]\mathcal{D}[b]\,{\rm e}^{{\rm i}\int \frac{N}{2\pi}b\wedge{\rm d}a}. \tag{B15}$$

As anticipated, the low-energy effective field theory, which describes the ground states of the $\mathbb{Z}_N^{(p)}$ symmetry broken phase, is $p$-form $\mathbb{Z}_N$ gauge theory. This is arguably the simplest field theory with an anomalous $\mathbb{Z}_N^{(p)}\times\mathbb{Z}_N^{(d-p)}$ symmetry [89].

### 1. Review of *p*-form BF theory

In the remainder of this section, we will review $p$-form BF theory, focusing on its symmetries and anomalies, working in $D=d+1$ dimensional space-time. From canonical quantization, the fields $a$ and $b$ satisfy the equal-time commutation relations

$$[a_{\mu_1\cdots\mu_p}(\boldsymbol{x}), b_{\mu_{p+1}\cdots\mu_d}(\boldsymbol{y})] = \frac{2\pi{\rm i}}{N}\epsilon_{0\mu_1\cdots\mu_d}\delta^d(\boldsymbol{x}-\boldsymbol{y}). \tag{B16}$$

#### a. $\mathbb{Z}_N$ *p*-form gauge theory in the continuum

Let us first review how $p$-form BF theory can be obtained by condensing charge-$N$ gauge charges in $p$-form Maxwell theory [125–127]. $p$-form Maxwell theory is reviewed in Appendix C 1. We will always assume that $p>0$. For the reader who would like to jump straight to $p$-form BF theory action, they should skip to Eq. (B24).

We modify $p$-form Maxwell theory Eq. (C15) by introducing the dynamical $(p-1)$-form bosonic field $H$ and the gauge redundancy

$$a\to a+{\rm d}\chi, \quad H\to H+N\chi, \tag{B17}$$

where $N\in\mathbb{Z}$. A gauge-invariant globally defined quantity in terms of only $H$ is $F_H={\rm d}H+\omega_H$, where $\omega_H\in 2\pi H^p(X;\mathbb{Z})$. $F_H$ satisfies a Bianchi identity $\frac{1}{2\pi}*{\rm d}F_H=0$ and its periods are quantized as $\oint F_H\in 2\pi\mathbb{Z}$.

With the additional degrees of freedom provided by $H$, we introduce the gauge invariant Wilson operator

$$W_{a,H}(O) = {\rm e}^{{\rm i}\int_O Na-{\rm d}H}, \tag{B18}$$

where $O$ is an open $p$-submanifold. Physically $W_{a,H}(O)$ is an operator that creates a charge excitation on $\partial O$, but one carrying $N$-units of $a$-charge. Minimally coupling $F_H$ to $a$, the partition function is

$$\mathcal{Z} = \int \mathcal{D}[a]\mathcal{D}[H]\,e^{-\int_X \frac{1}{2g^2}|F_a|^2+\frac{v^2}{2}|F_H-Na|^2}. \tag{B19}$$

The Lagrangian density now includes the term $\mathcal{L}\supset a\wedge *Nv^2F_H$ and a mass term $\mathcal{L}\supset\frac{N^2v^2}{2}|a|^2$. Indeed, the new term added to $p$-form Maxwell theory is essentially a Higgs term with $H$ the phase of the Higgs field and $v\in\mathbb{R}$ the vev of the Higgs field. The gauge redundancy described by Eq. (B17) is a $\mathbb{Z}_N$ gauge redundancy, reflecting how the initial U(1) gauge redundancy has been Higgsed down to a $\mathbb{Z}_N$ gauge redundancy.

As discussed in Appendix Sec. C 1 b, these types of theories have a generalized "particle-vortex" like duality called Abelian duality. For instance, since the action's dependency on $H$ is entirely in the form of $F_H$, we can dualize $H \to \hat{H}$ using the same method shown in Sec. C 1 b. Indeed, dualizing $H$ to the $(D-p-1)$-form $\hat{H}$ satisfying $\oint F_{\hat{H}} \in 2\pi\mathbb{Z}$ and $\mathrm{d}F_{\hat{H}} = 0$ (where $F_{\hat{H}} = \mathrm{d}\hat{H} + \omega_{\hat{H}}$), the Euclidean Lagrangian becomes

$$\mathcal{L} = \frac{|F_a|^2}{2g^2} + \frac{|F_{\hat{H}}|^2}{8\pi^2 v^2} - \frac{\mathrm{i}N}{2\pi} a \wedge F_{\hat{H}}. \tag{B20}$$

The Euclidean path integral now integrates over the dynamical fields $a$ and $\hat{H}$ and sums over $\omega_a \in 2\pi H^{p+1}(X;\mathbb{Z})$ and $\omega_{\hat{H}} \in 2\pi H^{D-p}(X;\mathbb{Z})$. We note that without changing the action amplitude, the Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{|F_a|^2}{2g^2} + \frac{|F_{\hat{H}}|^2}{8\pi^2 v^2} - \frac{\mathrm{i}N}{2\pi} \hat{H} \wedge F_a. \tag{B21}$$

Locally, we have just integrated by parts in the $BF$ term. However, keeping track of the globally nontrivial parts of $F_{\hat{H}}$ and $F_a$ makes showing this difficult (it is most naturally seen using Deligne-Beilinson cohomology [128]).

Utilizing Abelian duality, we have found two representations for the theory: the $(a, H)$ representation Eq. (B19) and the $(a, \hat{H})$ representation Eqs. (B20) and (B21), the latter being dual only locally.

In representation $(a, \hat{H})$, the deep IR is governed by $p$-form BF theory. Here, the deep IR refers to energies below the gap of $a$ and $\hat{H}$, which have a gap through topological mass generation. Indeed, to find their energy gaps, first note that in the $(a, \hat{H})$ representation, the Lorentzian action is

$$S = \int_X \left( -\frac{|F_a|^2}{2g^2} - \frac{|F_{\hat{H}}|^2}{8\pi^2 v^2} + \frac{N}{2\pi} a \wedge F_{\hat{H}} \right). \tag{B22}$$

Since this theory is Gaussian, we can show that the $a \wedge F_{\hat{H}}$ term causes all excitations to be gapped using the equations of motion. Minimizing the action and using that $\mathrm{d}^\dagger * F_{b,\hat{H}} = 0$, we find that the classical equations of motion are

$$\begin{aligned} (\delta + N^2 g^2 v^2) * F_{\hat{H}} &= 0, \\ (\delta + N^2 g^2 v^2) * F_a &= 0. \end{aligned} \tag{B23}$$

where $\delta = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$ is the Hodge Laplacian. Therefore we see that the $p$-form $* F_{\hat{H}}$ and the $(D-p-1)$-form $* F_a$ both have an energy gap $Ngv$.

To go below the energy gap into the deep IR, we take the limit $g \to \infty$ and $v \to \infty$. In this limit, the Euclidean path integral becomes

$$\mathcal{Z}_{\mathrm{BF}} = \int \mathcal{D}[a]\mathcal{D}[\hat{H}]\, e^{\frac{\mathrm{i}N}{2\pi} \int_X a \wedge F_{\hat{H}}}. \tag{B24}$$

This is $p$-form $BF$ theory, and it is in terms of the $p$-form bosonic field $a$, which is the U(1) gauge field we started with and $\hat{H}$, which is the Abelian dual of the Higgs field phase. Taking the deep IR limit using the Lagrangian density in this representation written as Eq. (B21), the Lagrangian in the topological limit is equivalent to $\mathcal{L}_{\mathrm{BF}} = -\frac{\mathrm{i}N}{2\pi} \hat{H} \wedge F_a$.

Plugging in $F_{\hat{H}} = \mathrm{d}\hat{H} + \omega_{\hat{H}}$ into Eq. (B24), the path integral becomes

$$\mathcal{Z}_{\mathrm{BF}} = \int \mathcal{D}[a]\mathcal{D}[\hat{H}] \sum_{\frac{\omega_{\hat{H}}}{2\pi} \in H^{D-p}(X;\mathbb{Z})} \mathrm{e}^{\frac{\mathrm{i}N}{2\pi} \int (a \wedge \mathrm{d}\hat{H} + a \wedge \omega_{\hat{H}})} \tag{B25}$$

Integrating by parts on the first term and using Poincaré duality on the second term, we can rewrite this as

$$\mathcal{Z}_{\mathrm{BF}} = \int \mathcal{D}[a]\mathcal{D}[\hat{H}] \sum_{\omega \in H_p(X;\mathbb{Z})} \mathrm{e}^{\frac{\mathrm{i}N}{2\pi} \int_X \hat{H} \wedge \mathrm{d}a + \mathrm{i}N \int_\omega a}. \tag{B26}$$

Integrating over $\hat{H}$ and summing over $\omega$, the path integral becomes

$$\mathcal{Z}_{\mathrm{BF}} = \int \mathcal{D}[a]\, \delta(\mathrm{d}a)\, \delta\left( \oint a \in \frac{2\pi\mathbb{Z}}{N} \right). \tag{B27}$$

Notice that if we would have instead started with the Lagrangian density written as $\mathcal{L}_{\mathrm{BF}} = -\frac{\mathrm{i}N}{2\pi} \hat{H} \wedge F_a$, upon integrating out $a$ and $\omega_a$ we would get

$$\mathcal{Z}_{\mathrm{BF}} = \int \mathcal{D}[\hat{H}]\, \delta(\mathrm{d}\hat{H})\, \delta\left( \oint \hat{H} \in \frac{2\pi\mathbb{Z}}{N} \right). \tag{B28}$$

Having massaged $p$-form BF theory into Eq. (B27), we find that in correlations functions, the U(1) gauge fields are closed forms and have quantized holonomies.[15] Therefore the Wilson operators

$$W_a[C_p] = \mathrm{e}^{\mathrm{i} \oint_{C_p} a}, \quad W_{\hat{H}}[C_{d-p}] = \mathrm{e}^{\mathrm{i} \oint_{C_{d-p}} \hat{H}}. \tag{B30}$$

satisfy $\langle (W_a)^N \rangle = \langle (W_{\hat{H}})^N \rangle = 1$, and are 1 when $C_p \in B_p(X)$ (i.e., there exists an $O_{p+1}$ such that $C_p = \partial O_{p+1}$). The latter property implies that these Wilson operators are topological. At a fixed time slice, in the deep IR, any contractible Wilson operators can condense into the vacuum, but for noncontractible Wilson operators, only $N$ can condense into the vacuum.

The path integral counts the number of nontrivial Wilson operators $W(C)$ which satisfy $W(C)^N = 1$, and thus the number of configurations $a \in H^p(X;\mathbb{Z}_N)$:

$$\mathcal{Z}_{\mathrm{BF}} = \sum_{a \in H^p(X;\mathbb{Z}_N)} 1 = |H^p(X, \mathbb{Z}_N)|. \tag{B31}$$

The number of ground states is given by the partition function evaluated on $X = \mathbb{R} \times M$, where $M$ is a space. Therefore there are $|H^p(M;\mathbb{Z}_N)|$ degenerate ground states.

#### b. Symmetries

Having reviewed the basics of $p$-form BF theory in the previous section, we now turn to identifying the theory's symmetries, showing that there is a $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ symmetry [89].

Let's first consider the symmetries manifest in the $(a, H)$ representation, Eq. (B19). The path integral is invariant under the transformation

$$a \to a + \Gamma, \quad F_H \to F_H + N\Gamma, \tag{B32}$$

[15] This can also be deduced from the equations of motion. Indeed, in the $(a, H)$ representation, Eq. (B19), the $H$ equations of motion in the deep IR are

$$F_H = Na. \tag{B29}$$

Therefore, because of the Bianchi identity $\mathrm{d}F_H = 0$, $a$ is a closed $p$-form. Furthermore, since $F_H$ satisfies $\oint F_H = 2\pi\mathbb{Z}$, the holonomies of $a$ are quantized as $\oint a = 2\pi\mathbb{Z}/N$.

with $d\Gamma = 0$. Since $F_H$ satisfies $\oint F_H \in 2\pi\mathbb{Z}$, in order to shift $F_H \to F_H + N\Gamma$ we require that $\oint \Gamma \in 2\pi\mathbb{Z}/N$. When $\Gamma = d\omega$, the transformation Eq. (B32) becomes the gauge transformation Eq. (B17). Therefore the $\Gamma$ that correspond to physical transformations are $\frac{N}{2\pi}\Gamma \in H^p(X;\mathbb{Z})$.

The quantization condition of the periods of $\Gamma$ has a significant consequence. Indeed, note that the Wilson operator $W_a$ is charged under this symmetry. Because of the quantization condition $\oint \Gamma \in 2\pi\mathbb{Z}/N$, it transforms as

$$W_a[C_p] \to e^{i\oint\Gamma} W_a[C_p], \qquad e^{i\oint\Gamma} \in \mathbb{Z}_N. \tag{B33}$$

Since the charged operators are $p$-dimensional and transform by an element of $\mathbb{Z}_N$, this is a $\mathbb{Z}_N^{(p)}$ symmetry.

It's tempting to think that an additional symmetry may be associated with the $H$ field. Indeed, Eq. (B19) is invariant under the transformation $H \to H + \omega$ for $d\omega = 0$. However, there are no physical observables that transform under this. Indeed, while the Wilson operator $\exp[i\oint H]$ picks up a phase, it is not a physical operator since it is not invariant under the gauge redundancy Eq. (B17).

The $\mathbb{Z}_N^{(p)}$ symmetry can also be seen in the $(a, \hat{H})$ representation, when the Lagrangian is described by Eq. (B20). Indeed, under the symmetry transformation, the action amplitude transforms as $\exp[-S] \to \exp[-S]\exp[-\delta S]$, where

$$e^{-\delta S} = e^{-\frac{iN}{2\pi}\int \Gamma\wedge F_{\hat{H}}}, \tag{B34}$$

with $\frac{N}{2\pi}\Gamma \in H^p(X;\mathbb{Z})$. Plugging in $F_{\hat{H}} = d\hat{H} + \omega_{\hat{H}}$, the phase factor $\exp[-\delta S]$ becomes

$$\exp[-\delta S] = e^{-2\pi i\int \frac{N\Gamma}{2\pi}\wedge\frac{\omega_{\hat{H}}}{2\pi}},$$

where we used integration by parts and that $d\Gamma = 0$. Recall that $\frac{N}{2\pi}\Gamma \in H^p(X;\mathbb{Z})$ and $\frac{\omega_{\hat{H}}}{2\pi} \in H^{D-p}(X;\mathbb{Z})$. Then, since the wedge product preserves integral de Rham cohomology classes, $\frac{N\Gamma}{2\pi}\wedge\frac{\omega_{\hat{H}}}{2\pi} \in H^D(X;\mathbb{Z})$, so $\exp[-\delta S] = 1$.

The Lagrangian in the $(a, \hat{H})$ representation can also be written as Eq. (B21) without changing the partition function. In this form, following the same argument used to show that there is a $\mathbb{Z}_N^{(p)}$ symmetry, we find there is also $\mathbb{Z}_N^{(d-p)}$ symmetry. Indeed, the action amplitude is invariant under $\hat{H} \to \hat{H} + \hat{\Gamma}$, where $\frac{N}{2\pi}\hat{\Gamma} \in H^{d-p}(X;\mathbb{Z})$. The charged operator of this $\mathbb{Z}_N^{(d-p)}$ symmetry is the Wilson operator $W_{\hat{H}} = \exp[i\oint \hat{H}]$, which transforms as

$$W_{\hat{H}}(C) \to e^{i\oint\hat{\Gamma}} W_{\hat{H}}(C), \quad e^{i\oint\hat{\Gamma}} \in \mathbb{Z}_N. \tag{B35}$$

The symmetry operator of the $\mathbb{Z}_N^{(p)}$ symmetry is just $W_{\hat{H}}$ and can be written as

$$U(\Sigma) = e^{i\oint_\Sigma \hat{H}} = \exp^{i\oint_{M_d}\hat{H}\wedge\Gamma}, \tag{B36}$$

where $\Gamma$ is the Poincaré dual of the $p$-cycle $\Sigma$ with respect to space $M_d$. Indeed, using the equal-time commutation relation (B16), which form Eq. (B20) is

$$[a_{\mu_1\cdots\mu_p}(\boldsymbol{x}), \hat{H}_{\mu_{p+1}\cdots\mu_d}(\boldsymbol{y})] = \frac{2\pi i}{N}\epsilon_{0\mu_1\cdots\mu_d}\delta^d(\boldsymbol{x}-\boldsymbol{y}), \tag{B37}$$

we have that

$$U(\Sigma)W_a(C)U^\dagger(\Sigma) = e^{\frac{2\pi i}{N}\int_C\Gamma}W_a(C), \quad = e^{\frac{2\pi i}{N}\#(\Sigma,C)}W_a(C). \tag{B38}$$

Similarly, the symmetry operator of the $\mathbb{Z}_N^{(d-p)}$ symmetry is

$$\hat{U}(\hat{\Sigma}) = e^{i\oint_{\hat{\Sigma}} a}, \tag{B39}$$

which is just $W_a$.

#### c. *Mixed 't Hooft anomaly and anomaly inflow*

In the last section, we reviewed how $p$-form BF theory has $\mathbb{Z}_N^{(p)}$ and $\mathbb{Z}_N^{(d-p)}$ symmetries. However, these symmetries are not independent of one another: the symmetry operator of one symmetry is a charged operator of the other symmetry, thus satisfying the Heisenberg algebra. This is a manifestation of the fact that the $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p)}$ symmetry is anomalous. In this section, we will turn on a background gauge field for these symmetries to learn more about this mixed 't Hooft anomaly.

Let's first turn on a background gauge field for the $\mathbb{Z}_N^{(p)}$ symmetry. We introduce the background gauge field $\mathcal{A} \in \frac{2\pi}{N}H^{p+1}(X;\mathbb{Z})$ and the gauge redundancy

$$a \to a + \beta, \quad \mathcal{A} \to \mathcal{A} + d\beta. \tag{B40}$$

Minimally coupling $\mathcal{A}$, the $p$-form BF theory path integral becomes

$$\mathcal{Z}[\mathcal{A}] = \int \mathcal{D}[a]\mathcal{D}[\hat{H}]\, e^{\frac{iN}{2\pi}\int_X(a\wedge d\hat{H} + (-1)^p\mathcal{A}\wedge\hat{H})}. \tag{B41}$$

Since $\mathcal{Z}[\mathcal{A}] = \mathcal{Z}[\mathcal{A} + d\beta]$, the $\mathbb{Z}_N^{(p)}$ symmetry is anomaly free.

Second, we next turn off $\mathcal{A}$ and turn on a background gauge field for the $\mathbb{Z}_N^{(d-p)}$ symmetry, which introduces the background gauge field $\hat{\mathcal{A}} \in \frac{2\pi}{N}H^{d-p+1}(X;\mathbb{Z})$ and the gauge redundancy

$$\hat{H} \to \hat{H} + \zeta, \quad \hat{\mathcal{A}} \to \hat{\mathcal{A}} + d\zeta. \tag{B42}$$

Minimally coupling $\hat{\mathcal{A}}$, the path integral becomes

$$\mathcal{Z}[\hat{\mathcal{A}}] = \int \mathcal{D}[a]\mathcal{D}[\hat{H}]\, e^{\frac{iN}{2\pi}\int_X a\wedge(F_{\hat{H}} - \hat{\mathcal{A}})}. \tag{B43}$$

Since $\mathcal{Z}[\hat{\mathcal{A}}] = \mathcal{Z}[\hat{\mathcal{A}} + d\zeta]$, the $\mathbb{Z}_N^{(d-p)}$ symmetry is anomaly free.

Now let us turn on both background gauge fields. Coupling them into the action as above yields

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] = \int \mathcal{D}[a]\mathcal{D}[\hat{H}]e^{\frac{iN}{2\pi}\int_X(a\wedge(F_{\hat{H}} - \hat{\mathcal{A}}) + (-1)^p\mathcal{A}\wedge\hat{H})}. \tag{B44}$$

While this is invariant under Eq. (B40), under the gauge transformation Eq. (B42) it transforms as

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}} + d\zeta] \to e^{(-1)^p\frac{iN}{2\pi}\int_X\mathcal{A}\wedge\zeta}\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}]. \tag{B45}$$

In fact, no local counterterms can be added such that the path integral is invariant under both gauge transformations. It always gets multiplied by a phase. Thus we see an obstruction to coupling a background gauge field of both symmetries and hence a mixed 't Hooft anomaly.

A 't Hooft anomaly can be classified by an SPT in one higher dimension whose boundary realizes the symmetry anomalous. Let's now extend the background fields to one higher dimension and have $X$ be the boundary of the new space-time $Y$. The path integral governing all of $Y$ is

$$\mathcal{Z}_Y[\mathcal{A}, \hat{\mathcal{A}}, Y] = \mathcal{Z}_{\text{SPT}}[\mathcal{A}, \hat{\mathcal{A}}, Y]\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}, \partial Y = X], \tag{B46}$$

where the SPT path integral $\mathcal{Z}_{\rm SPT}$ is defined such that it cancels out the phase in Eq. (B45) such that $\mathcal{Z}_Y$ is gauge invariant.

Noting that this phase can be written as

$$e^{(-1)^p \frac{iN}{2\pi}\int_X \mathcal{A}\wedge\zeta} = e^{-\frac{iN}{2\pi}\int_Y \mathcal{A}\wedge d\zeta}, \tag{B47}$$

let's consider the SPT

$$\mathcal{Z}_{\rm SPT}[\mathcal{A},\hat{\mathcal{A}}] = e^{\frac{iN}{2\pi}\int_Y \mathcal{A}\wedge\hat{\mathcal{A}}}. \tag{B48}$$

Under the gauge transformations Eq. (B42), $\mathcal{Z}_{\rm SPT}$ transforms as

$$\mathcal{Z}_{\rm SPT}[\mathcal{A},\hat{\mathcal{A}}+\zeta] = e^{\frac{iN}{2\pi}\int_Y \mathcal{A}\wedge d\zeta}\mathcal{Z}_{\rm SPT}[\mathcal{A},\hat{\mathcal{A}}]. \tag{B49}$$

This phase picked up is the inverse of the phase picked up by $\mathcal{Z}$ in Eq (B45), and thus $\mathcal{Z}_Y$ is indeed gauge invariant. The mixed anomaly between the $\mathbb{Z}_N^{(p)}$ and $\mathbb{Z}_N^{(d-p)}$ symmetry is then said to be classified by the SPT Eq. (B48).

## APPENDIX C: CONTINUUM OF U(1) $p$-GAUGE THEORY

In this Appendix, we take the continuum limit of Eq. (55) in Sec. III C 2 of the main text. Doing so also demonstrates the connection between exact emergent higher-form symmetries in lattice models and exact higher-form symmetries in Lagrangian quantum field theories, where higher-form symmetries are most commonly studied.

The lattice Heisenberg operators $L^z_{c_p}(t)$ and $\Theta_{c_p}(t)$ in the IR are dressed by two local unitary operators $U^{(1)}_{\rm LU}$ and $U^{(2)}_{\rm LU}$ and denoted as $\widetilde{L}'^z_{c_p}(t)$ and $\widetilde{\Theta}'_{c_p}(t)$. Furthermore, in the mid-IR we defined the variable $\omega_{c_{p+1}} \equiv -2\pi\lfloor({\rm d}\widetilde{\Theta})_{c_{p+1}}/(2\pi)\rceil$, which in the IR was dressed by $U^{(2)}_{\rm LU}$ and denoted as $\omega'_{c_{p+1}}(t)$. Therefore the three elementary operators in the IR are $\widetilde{L}'^z_{c_p}(t)$, $\widetilde{\Theta}'_{c_p}(t)$, and $\omega'_{c_{p+1}}(t)$. We relate these lattice operators to their continuum counterparts $\widetilde{L}'^z(t,\boldsymbol{x})$, $\widetilde{\Theta}'(t,\boldsymbol{x})$, and $\omega'(t,\boldsymbol{x})$ by

$$\widetilde{L}'^z_{c_p} = \int_{c_p}\widetilde{L}'^z,\ \widetilde{\Theta}'_{c_p} = \int_{c_p}\widetilde{\Theta}',\ \omega'_{c_{p+1}} = \int_{c_{p+1}}\omega', \tag{C1}$$

where, for instance, $\int_{c_p}$ denotes spatial integral over the $p$-cell $c_p$. The continuum quantum fields are *globally* differential forms in space $M$ ($\widetilde{L}'^z$ and $\widetilde{\Theta}'$ are $p$-forms while $\omega'$ is a $(p+1)$-form), mapping from space-time $X$ to $\mathbb{R}$. In the continuum, the lattice differential operators become their continuum versions. For instance, the lattice operator $F'_{c_{p+1}} = ({\rm d}\widetilde{\Theta}')_{c_{p+1}} + (\omega')_{c_{p+1}}$ becomes $F'_{c_{p+1}} = \int_{c_{p+1}} F'$, where $F' = {\rm d}\widetilde{\Theta}' + \omega'$. So, taking the continuum limit of Eq. (55), the deep IR continuum Hamiltonian is

$$H_{\text{deep IR}} = \int d^d\boldsymbol{x}\left(\frac{\kappa U}{2}\frac{|\widetilde{L}'^z_{i_1\cdots i_p}|^2}{p!} + \frac{U}{2}\frac{|F'_{i_1\cdots i_{p+1}}|^2}{(p+1)!}\right), \tag{C2}$$

where, for instance, $|\widetilde{L}'^z_{i_1\cdots i_p}|^2 \equiv \sum_{i_1,\cdots,i_p=1}^d(\widetilde{L}'^z_{i_1\cdots i_p})^2$.

To write down the path integral, we can find the Lorentzian action and then perform a functional integral over field configurations obeying the following constraints.

(1) Since the IR does not include dressed charge excitations, $\widetilde{L}'^z$ must satisfy $\widetilde{\rho}\,' = 0$. The expression for $\widetilde{\rho}\,'_{c_{p-1}}$ in Eq. (41) can be rewritten using discrete exterior calculus notation as $\widetilde{\rho}\,'_{c_{p-1}} \sim (*{\rm d}*\widetilde{L}'^z)_{c_{p-1}}$. So, $\widetilde{\rho}\,' = 0$ in the continuum limit is the Gauss law

$$\partial_j\widetilde{L}'^z_{ji_1\cdots i_{p-1}} = 0. \tag{C3}$$

Despite there being no dressed charges in the IR, $\widetilde{L}'^z$ can still be sourced along nontrivial $p$-cycles. Because $\widetilde{L}'^z_{c_p}\in\mathbb{Z}$ on the lattice, the flux of $\widetilde{L}'^z$ in the continuum is quantized

$$\oint_{C_{d-p}} *\widetilde{L}'^z \in \mathbb{Z}, \tag{C4}$$

where $C_{d-p}$ is a nontrivial $(d-p)$-cycle in space: $C_{d-p}\in H_{d-p}(M;\mathbb{Z})$.

(2) Since the IR does not include dressed topological defects, $\widetilde{\Theta}'$ and $\omega'$ must satisfy $\hat{\rho}' = 0$. The expression for $(*\hat{\rho}')_{c_{p+2}}$ of Eq. (54) in the continuum limit becomes $\hat{\rho}' = \frac{1}{2\pi}*{\rm d}F'$, and $\hat{\rho}' = 0$ becomes the Bianchi identity

$$\frac{1}{2\pi}*{\rm d}F' = 0. \tag{C5}$$

Despite there being no dressed topological defects in the IR, $*F'$ can still be sourced along nontrivial $(d-p)$-cycles. Indeed, because $\omega'_{c_p}\in 2\pi\mathbb{Z}$ on the lattice, the flux of $*F'$, in the continuum is quantized

$$\oint_{C_{p+1}} F' \in 2\pi\mathbb{Z}, \tag{C6}$$

where $C_{p+1}$ is a nontrivial $(p+1)$-cycle in space. Plugging $F' = {\rm d}\widetilde{\Theta}' + \omega'$ into Eqs. (C5) and (C6), the $\widetilde{\Theta}'$ vanishes and constraints $\omega'$ as

$$\frac{\omega'}{2\pi}\in H^{p+1}(M;\mathbb{Z}), \tag{C7}$$

the $(p+1)$th de Rham cohomology group with integral periods.

Enforcing the three constraints Eqs. (C3), (C4), and (C7) by hand, the path integral in Lorentzian signature is

$$\mathcal{Z}_{\text{deep IR}} = \int\mathcal{D}[\widetilde{\Theta}']\,\mathcal{D}\widetilde{L}'^z\sum_{\omega'\in 2\pi H^{p+1}(M;\mathbb{Z})}\delta\big(\partial_{i_1}\widetilde{L}'^z_{i_1\cdots i_p}\big)$$
$$\times\,\delta\left(\oint *\widetilde{L}'^z\in\mathbb{Z}\right)e^{i\int_X {\rm d}t\,{\rm d}^d\boldsymbol{x}\,\mathcal{L}_{\text{deep IR}}},$$
$$\mathcal{L}_{\text{deep IR}} = \frac{\widetilde{L}'^z_{i_1\cdots i_p}\partial_t\widetilde{\Theta}'_{i_1\cdots i_p}}{p!} - \left(\frac{\kappa U}{2}\frac{|\widetilde{L}'^z_{i_1\cdots i_p}|^2}{p!} + \frac{U}{2}\frac{|F'_{i_1\cdots i_{p+1}}|^2}{(p+1)!}\right). \tag{C8}$$

The first term in $\mathcal{L}_{\text{deep IR}}$ enforces the equal-time commutation relation

$$\frac{[\widetilde{\Theta}'_{i_1\cdots i_p}(\boldsymbol{x}),\widetilde{L}'^z_{j_1\cdots j_p}(\boldsymbol{y})]}{p!} = i\delta^{i_1}_{[j_1}\cdots\delta^{i_p}_{j_p]}\delta^d(\boldsymbol{x}-\boldsymbol{y}), \tag{C9}$$

and the second term in parenthesis is Hamiltonian density from Eq. (C2)

This expression of the path integral is correct, but let us rewrite this phase space path integral as a coordinate space path integral to get it into a more familiar form. We first rewrite the delta functions by integrating in new fields and modifying the action. The delta function enforcing the $\widetilde{L}'^z$ quantization condition can be represented as

$$\delta\left(\oint *\widetilde{L}'^z \in \mathbb{Z}\right) = \sum_{\eta\in 2\pi H^p(M;\mathbb{Z})} \mathrm{e}^{\mathrm{i}\int_X \mathrm{d}t\mathrm{d}^d\boldsymbol{x} \frac{\widetilde{L}'^z_{i_1\cdots i_p}\eta_{i_1\cdots i_p}}{p!}}, \quad \text{(C10)}$$

and the delta function enforcing Gauss law can be rewritten using the $(p-1)$-form Lagrange multiplier $\lambda$ as

$$\delta\big(\partial_{i_1}\widetilde{L}'^z_{i_1\cdots i_p}\big) = \int \mathcal{D}\lambda \mathrm{e}^{\mathrm{i}\int_X \lambda_{i_2\cdots i_p}\partial_{i_1}\widetilde{L}'^z_{i_1 i_2\cdots i_p}/(p-1)!}. \quad \text{(C11)}$$

Plugging these in, the path integral takes the cumbersome form

$$\mathcal{Z}_{\text{deep IR}} = \int \mathcal{D}[\widetilde{\Theta}']\,\mathcal{D}\widetilde{L}'^z\,\mathcal{D}\lambda \sum_{\substack{\omega'\in 2\pi H^{p+1}(M;\mathbb{Z})\\ \eta\in 2\pi H^p(M;\mathbb{Z})}} \mathrm{e}^{\mathrm{i}\int_X \mathrm{d}t\mathrm{d}^d\boldsymbol{x}\,\mathcal{L}_{\text{deep IR}}},$$

$$\mathcal{L}_{\text{deep IR}} = \frac{\lambda_{i_2\cdots i_p}\partial_{i_1}\widetilde{L}'^z_{i_1 i_2\cdots i_p}}{(p-1)!} + \frac{\widetilde{L}'^z_{i_1\cdots i_p}\eta_{i_1\cdots i_p}}{p!} + \frac{\widetilde{L}'^z_{i_1\cdots i_p}\partial_t\widetilde{\Theta}'_{i_1\cdots i_p}}{p!} - \frac{\kappa U|\widetilde{L}'^z_{i_1\cdots i_p}|^2}{2p!} - \frac{U|F'_{i_1\cdots i_{p+1}}|^2}{2(p+1)!}. \quad \text{(C12)}$$

It is now straight forward to integrate out the $\widetilde{L}'^z$ field, after which the path integral becomes

$$\mathcal{Z}_{\text{deep IR}} = \int \mathcal{D}[\widetilde{\Theta}']\,\mathcal{D}\lambda \sum_{\substack{\omega'\in 2\pi H^{p+1}(M;\mathbb{Z})\\ \eta\in 2\pi H^p(M;\mathbb{Z})}} \mathrm{e}^{\mathrm{i}\int_X \mathrm{d}t\mathrm{d}^d\boldsymbol{x}\,\mathcal{L}_{\text{deep IR}}},$$

$$\mathcal{L}_{\text{deep IR}} = \frac{|\partial_t\widetilde{\Theta}'_{i_1\cdots i_p} - p\partial_{[i_1}\lambda_{i_2\cdots i_p]} + \eta_{i_1\cdots i_p}|^2}{2g^2 p!} - \frac{|F'_{i_1\cdots i_{p+1}}|^2}{2g^2(p+1)!}, \quad \text{(C13)}$$

where we have also rescaled $t\to t/(U\sqrt{\kappa})$, $\lambda\to U\sqrt{\kappa}\,\lambda$, and $\eta\to U\sqrt{\kappa}\,\eta$ and introduced $g = 1/(\sqrt{U})$.

Having found the coordinate path integral, let's massage it into a canonical form. Namely, we introduce the $p$-form $a$ and $(p+1)$-form $\omega_a$ in space-time whose components are

$$a_{i_1\cdots i_p} = \widetilde{\Theta}'_{i_1\cdots i_p}, \quad a_{0 i_2\cdots i_p} = \lambda_{i_2\cdots i_p}$$
$$(\omega_a)_{i_1\cdots i_{p+1}} = \omega'_{i_1\cdots i_{p+1}}, \quad (\omega_a)_{0 i_1\cdots i_p} = \eta_{i_1\cdots i_p}.$$

Letting $F_a = \mathrm{d}a + \omega_a$, after some simplifying, we then reexpress the path integral as

$$\mathcal{Z}_{\text{deep IR}} = \int \mathcal{D}[a] \sum_{\omega_a\in 2\pi H^{p+1}(X;\mathbb{Z})} \mathrm{e}^{\mathrm{i}\int_X \mathrm{d}t\mathrm{d}^d\boldsymbol{x}\,\mathcal{L}_{\text{deep IR}}},$$

$$\mathcal{L}_{\text{deep IR}} = \frac{1}{2g^2}\left(\frac{\left|(F_a)_{0 i_1\cdots i_p}\right|^2}{p!} - \frac{\left|(F_a)_{i_1\cdots i_{p+1}}\right|^2}{(p+1)!}\right). \quad \text{(C14)}$$

Furthermore, working in flat space-time, $X$ is equipped with Minkowski metric $(-,+,\cdots+)$. Thus, summing over space-time indices $\mu = 0,\ldots,d$, $\mathcal{L}_{\text{deep IR}}$ can be rewritten as

$$\mathcal{L}_{\text{deep IR}} = -\frac{1}{2g^2(p+1)!}(F_a)_{\mu_1\cdots\mu_{p+1}}(F_a)^{\mu_1\cdots\mu_{p+1}}, \quad \text{(C15)}$$

which is exactly $p$-form Maxwell theory, as stated in the main text.

### 1. Review of $p$-form Maxwell theory

In the remainder of this section, we will review $p$-form Maxwell theory, Eq. (C15), focusing on its symmetries and anomalies, working in $D = d+1$ dimensional space-time. In particular, we consider the theory where there is no electric nor magnetic matter, thus $\mathrm{d}{*F_a} = 0$ and $\mathrm{d}F_a = 0$.

The canonical momentum field $\Pi$ is locally a $(D-p-1)$-form associated with a codimension-1 submanifold of space-time, which we choose to be a constant time slice. Then, $\Pi$'s components are defined by varying the action with respect to $\partial_0 a_{\mu_1\cdots\mu_p}$ which yields $\Pi = \frac{1}{g^2}{*F_a}$. Therefore, from canonical quantization, we have the equal-time commutation relation

$$\left[a_{\mu_1\cdots\mu_p}(\boldsymbol{x}), \frac{(*F_a)_{\mu_{p+1}\cdots\mu_d}(\boldsymbol{y})}{g^2}\right] = \mathrm{i}\epsilon_{0\mu_1\cdots\mu_d}\delta^d(\boldsymbol{x}-\boldsymbol{y}). \quad \text{(C16)}$$

#### a. $\mathrm{U}(1)^{(p)}$ symmetry

The action amplitude is only a function of the field strength, and so the path integral is invariant under $a$ being shifted by a closed $p$-form:

$$a \to a + \Gamma, \quad \mathrm{d}\Gamma = 0. \quad \text{(C17)}$$

This is a symmetry because the Wilson operator $W_a(C_p) = \exp[\mathrm{i}\int a]$ transforms nontrivially under Eq. (C17) as

$$W_a(C_p) \to \mathrm{e}^{\mathrm{i}\oint_{C_p}\Gamma} W_a(C_p). \quad \text{(C18)}$$

There are no restrictions on the holonomies of $a$ and $\Gamma$ satisfies $\oint\Gamma\in\mathbb{R}$, so $\exp[\mathrm{i}\oint_{C^p}\Gamma]\in\mathrm{U}(1)$. Therefore Eq. (C18) is the symmetry transformation of a $\mathrm{U}(1)^{(p)}$ symmetry. Of course, for $\Gamma$ that satisfy $\exp[\mathrm{i}\oint_{C_p}\Gamma] = 1$, the transformation $a\to a+\Gamma$ is instead a gauge transformations corresponding to formal redundancies. The physical transformation on $a$ requires that $\Gamma$ be closed but not exact and have periods not in $2\pi\mathbb{Z}$.

To find the $\mathrm{U}(1)^{(p)}$ symmetry transformation operator, let us turn on the $(p+1)$-form background field $\mathcal{A}$ and the gauge redundancy

$$a\to a+\beta, \quad \mathcal{A}\to\mathcal{A}+\mathrm{d}\beta. \quad \text{(C19)}$$

Minimally coupling the background field such that the theory is gauge invariant, the action becomes

$$S[\mathcal{A}] = -\frac{1}{2g^2}\int_X |F_a - \mathcal{A}|^2. \quad \text{(C20)}$$

The conserved Noether current $J$ ($\mathrm{d}^\dagger J = 0$) of the symmetry will minimally couple to $\mathcal{A}$ as $\int\mathcal{A}\wedge{*J}$. We thus find that $J = \frac{1}{g^2}F_a$, and so the charge operator is $Q\equiv\frac{1}{g^2}\int *J$ and the symmetry operator is

$$U_\alpha(\Sigma) = \mathrm{e}^{\mathrm{i}\alpha\oint_\Sigma \frac{*F_a}{g^2}}, \quad \text{(C21)}$$

where $\alpha\in[0,2\pi)$ parametrizes the U(1) transformation. Notice that because $\mathrm{d}{*J} = 0$, $U_\alpha(\Sigma)$ is a topological operator, depending only on the homology class of $\Sigma$.

Let's check that $U_\alpha$ indeed transforms $W_a$ as Eq. (C18). Letting the $p$-form $\hat{\Sigma}$ be the Poincaré dual of $\Sigma$ with respect to space $M$, we rewrite the symmetry operator as

$U_\alpha(\Sigma) = \exp[\frac{\mathrm{i}\alpha}{g^2}\oint_M *F_a\wedge\hat{\Sigma}]$, and using the BakerCampbell-Hausdorff formula, the Wilson operator transforms as

$$U_\alpha(\Sigma)W_a(C)U_\alpha^\dagger(\Sigma) = \mathrm{e}^{\frac{\alpha}{g^2}[\int_M *F_a\wedge\hat{\Sigma},\int_C a]}W_a(C).$$

Using the canonical commutation relations Eq. (C16), this simplifies to

$$U_\alpha(\Sigma)W_a(C)U_\alpha^\dagger(\Sigma) = \mathrm{e}^{\mathrm{i}\alpha\oint_C\hat{\Sigma}}\,W_a(C), \tag{C22}$$

$$= \mathrm{e}^{\mathrm{i}\alpha\,\#(\Sigma,C)}\,W_a(C), \tag{C23}$$

where $\oint_C\hat{\Sigma} = \int_{\Sigma\cap C}1 = \#(\Sigma, C)$ is the intersection number between $\Sigma$ and $C$ in $M$.

### *b. Abelian duality*

There is another symmetry present in $p$-form Maxwell theory, but to make it manifest, we must effectively change the representation of our degrees of freedom using Abelian duality [129] to dualize the field $a$ to the field $\hat{a}$. To do so, we can use the following trick. Instead of integrating over the equivalence classes of $a$ and summing over $\omega_a$, we can instead integrate over $F_a$ since the action only depends on $F_a$. However, in doing so, we have to ensure that we only integrate over $F_a$ satisfying the Bianchi identity

$$\frac{1}{2\pi}*\mathrm{d}F_a = 0, \tag{C24}$$

and which obey the quantization condition

$$\oint_{C_{p+1}} F_a \in 2\pi\mathbb{Z}, \tag{C25}$$

for all $C_{p+1}\in H_{p+1}(X)$. So, with these two constraints in mind, we can change variables and write the Euclidean path integral of $p$-form Maxwell theory as

$$\mathcal{Z} = \int\mathcal{D}F_a\,\delta\bigg(\frac{*\mathrm{d}F_a}{2\pi}\bigg)\delta\bigg(\oint\frac{F_a}{2\pi}\in\mathbb{Z}\bigg)\mathrm{e}^{-\frac{1}{2g^2}\int_X|F_a|^2}. \tag{C26}$$

Let's now rewrite the delta functions by integrating in fields. For the first delta function, introducing the $(D-p-2)$-form $\hat{a}$, we can represent it as

$$\delta\bigg(\frac{*\mathrm{d}F_a}{2\pi}\bigg) = \int\mathcal{D}\hat{a}\,\mathrm{e}^{\frac{\mathrm{i}}{2\pi}\int_X\mathrm{d}\hat{a}\wedge F_a}. \tag{C27}$$

For the second delta function, we can rewrite it as

$$\delta\bigg(\oint\frac{F_a}{2\pi}\in\mathbb{Z}\bigg) = \sum_{\hat{\omega}_{\hat{a}}\in H_{p+1}(X;\mathbb{Z})}\mathrm{e}^{2\pi\mathrm{i}(\oint_{\hat{\omega}_{\hat{a}}}\frac{F_a}{2\pi})},$$

$$= \sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})}\mathrm{e}^{\frac{\mathrm{i}}{2\pi}\int_X\omega_{\hat{a}}\wedge F_a}, \tag{C28}$$

where we first sum over all closed $(p+1)$-submanifold $\hat{\omega}_{\hat{a}}$, and then using Poincaré duality instead sum over the dual $(D-p-1)$-forms $\omega_{\hat{a}}/(2\pi)$ satisfying $\oint\omega_{\hat{a}}\in 2\pi\mathbb{Z}$. Plugging these representations of the delta functions into the path integral Eq. (C26), it becomes

$$\mathcal{Z} = \int\mathcal{D}F_a\mathcal{D}\hat{a}\sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})}\mathrm{e}^{-\int_X\frac{1}{2g^2}|F_a|^2-\frac{\mathrm{i}}{2\pi}F_{\hat{a}}\wedge F_a}, \tag{C29}$$

where $F_{\hat{a}} = \mathrm{d}\hat{a}+\omega_{\hat{a}}$, which satisfies

$$\oint F_{\hat{a}}\in 2\pi\mathbb{Z}, \quad \frac{1}{2\pi}*\mathrm{d}F_{\hat{a}} = 0. \tag{C30}$$

To integrate out $F_a$, we complete the square and introduce $G = F_a - \mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}$ so the path integral becomes

$$\mathcal{Z} = \int\mathcal{D}G\,\mathcal{D}\hat{a}\sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})}\mathrm{e}^{-\int_X\frac{1}{2g^2}|G|^2+\frac{g^2}{8\pi^2}|F_{\hat{a}}|^2}. \tag{C31}$$

Integrating out $G$, the path integral is only in terms of the dual field $\hat{a}$:

$$\mathcal{Z}[X,g] = \int\mathcal{D}[\hat{a}]\sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})}\mathrm{e}^{-\frac{g^2}{8\pi^2}\int_X|F_{\hat{a}}|^2}. \tag{C32}$$

Remarkably, this theory has the same form as what we started with, expect now that initial $p$-form $a$ is a $(D-p-2)$-form $\hat{a}$ and the coupling constant $g$ is now $2\pi/g$. Thus strongly coupling ($g\gg 1$) in the $a$ representation gets mapped to weak coupling in the $\hat{a}$ representation and vice versa.

Having gone through the process of dualizing $a$ to $\hat{a}$, let's now see how operators in terms of $a$ transform under dualizing. We introduce the map $\mathbb{S}$ that takes an operator in the $a$ representation to the $\hat{a}$ representation.

Let's first check to see what the field strength $F_a$ maps to by inserting $F_a$ into the path integral. When completing the square, we did a change of variables $F_a = G+\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}$ under which the insertion becomes

$$\langle F_a\rangle_a = \langle G\rangle_{\hat{a}} + \bigg\langle\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}\bigg\rangle_{\hat{a}}. \tag{C33}$$

We use the notation that $\langle\cdot\rangle_a$ is the vev evaluated in the $a$ representation and $\langle\cdot\rangle_{\hat{a}}$ is the vev evaluated in the $\hat{a}$ representation. Since the $G$ part of the action is Gaussian, $\langle G\rangle = 0$ and so $\langle F_a\rangle = \langle\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}\rangle$. Thus, in Euclidean signature,

$$\mathbb{S}: F_a\to\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}. \tag{C34}$$

In the Lorentzian signature, this is $\mathbb{S}: F_a\to\frac{g^2}{2\pi}*F_{\hat{a}}$. We can dualizing $\hat{a}$ back to $a$ by simply repeating the same steps as before to find

$$\mathbb{S}: F_{\hat{a}}\to\mathrm{i}\frac{2\pi}{g^2}*F_a. \tag{C35}$$

Therefore dualizing $a$ twice gives $\mathbb{S}^2: F_a\to\mathrm{i}^2**F_a$ in Euclidean space-time, or equivalently

$$\mathbb{S}^2: F_a\to(-1)^{D(p+1)+p}F_a. \tag{C36}$$

When both $D$ and $p$ are even, dualizing twice acts as $\mathbb{S}^2: F_a\to F_a$. However, if $D$ or $p$ or both are odd, then $\mathbb{S}^2: F_a\to -F_a$ and thus one must dualize four times to get the identity map: $\mathbb{S}^4: F_a\to F_a$.

Repeating the argument manipulations, we find that $\mathrm{d}^\dagger F_a$ and $*\mathrm{d}F_a$ get mapped to $*\mathrm{d}F_{\hat{a}}$ and $\mathrm{d}^\dagger F_{\hat{a}}$, respectively, and vice versa. Therefore the excitations (topological defects) of $a$ are the topological defects (excitations) of $\hat{a}$. Hence, Abelian duality is a particle-vortex type duality.

We emphasize that the above mappings do not imply that $F_a = \mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}$. Instead, while operators linear in $F_a$ simply have

$F_a$ replaced with $\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}$, more care is required to find the dual representation of operators nonlinear in $F_a$. For instance, $(F_a)^2$ does *not* become $(\mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}})^2$ due to the addition terms pick up when squaring $F_a = G + \mathrm{i}\frac{g^2}{2\pi}*F_{\hat{a}}$.

Next, let us find what the Wilson operator of $a$ gets mapped to. We assume that $W_a$ is supported on a contractible manifold $C = \partial M$. However, there are infinitely many such $M$ whose boundary is $C$. To avoid this ambiguity, we sum over all such $M$ and write

$$W_a[C] = \sum_{M:\partial M=C} \exp\left[\mathrm{i}\int_M F_a\right]. \tag{C37}$$

Next we introduce the $D-p-1$ form $\hat{M}$ dual to $M$, which satisfies $\oint \hat{M} \in 2\pi\mathbb{Z}$ and is related to the Poincaré dual of $C$ by $\hat{C} = \mathrm{d}\hat{M}/(2\pi)$, and rewrite $W_a$ as

$$W_a[C] = \int \mathcal{D}\hat{M}\,\delta(\mathrm{d}\hat{M} - 2\pi\hat{C})\exp\left[\frac{\mathrm{i}}{2\pi}\int_X F_a \wedge \hat{M}\right].$$

Inserting this into the path integral, integrating in $\hat{a}$, and then integrating out $F_a$, we find

$$\langle W_a[C]\rangle = \int \mathcal{D}\hat{M}\mathcal{D}[\hat{a}] \sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})} \delta(\mathrm{d}\hat{M} - 2\pi\hat{C})\,\mathrm{e}^{-\int_X \mathcal{L}},$$

$$\mathcal{L} = \frac{g^2}{8\pi^2}|F_{\hat{a}} - \hat{M}|^2. \tag{C38}$$

Notice how this is the same as the dualized path integral without the Wilson loop insertion but now with the $p+1$ form connection $\hat{M}$ satisfying $\mathrm{d}\hat{M} = 2\pi\hat{C}$. We can get rid of $\hat{M}$ and have the path integral look similar to Eq. (C32) if we let $\hat{a}$ be a singular field not defined on $C$:

$$\mathcal{Z} = \int \mathcal{D}[\hat{a}] \sum_{\omega_{\hat{a}}\in 2\pi H^{D-p-1}(X;\mathbb{Z})} \mathrm{e}^{-\int_{X/C}\frac{g^2}{8\pi^2}|F_{\hat{a}}|^2}. \tag{C39}$$

However, we require that $\int_\Sigma F_{\hat{a}} = 2\pi$ for any submanifold $\Sigma$ with a nonzero intersection number with $C$. Therefore the Wilson loop in the $a$ representation has become a 't Hooft loop in the $\hat{a}$ representation.

#### c. $\mathbf{U(1)^{(d-p-1)}}$ *symmetry*

In the $a$ representation, it appears that the model only has a $\mathrm{U}(1)^{(p)}$ symmetry. However, upon dualizing $a$ to $\hat{a}$, the path integral Eq. (C32) took a similar form in terms of $F_{\hat{a}}$ but with $g$ replaced by $2\pi/g$. Thus we find a new globally defined differential $(d-p-1)$-form $\hat{a}$, which shifting by a closed form leaves the path integral invariant. Following the same process as used in investigating the $\mathrm{U}(1)^{(p)}$ symmetry, this transformation has a physical part associated with a $\mathrm{U}(1)^{(d-p-1)}$ symmetry.

Everything about this $\mathrm{U}(1)^{(d-p-1)}$ symmetry follows in a similar fashion from the $\mathrm{U}(1)^{(p)}$ case. In particular, the symmetry transformation acts on $\hat{a}$ as

$$\hat{a} \to \hat{a} + \hat{\Gamma}, \quad \mathrm{d}\hat{\Gamma} = 0 \tag{C40}$$

and the charged operators are Wilson operators in terms of $\hat{a}$: $W_{\hat{a}}(C) = \exp[\mathrm{i}\oint_C \hat{a}]$. These correspond to the 't Hooft operators in the $a$ representation. The Noether's current associated with this $\mathrm{U}(1)^{(d-p-1)}$ symmetry is $\hat{J} = \frac{g^2}{4\pi^2}F_{\hat{a}}$, and thus the symmetry operator is

$$\hat{U}_{\hat{\alpha}}(\Sigma) = \mathrm{e}^{\mathrm{i}\hat{\alpha}\frac{g^2}{4\pi^2}\oint_\Sigma *F_{\hat{a}}}, \tag{C41}$$

where $\hat{\alpha} \in [0, 2\pi)$ parametrizes the U(1) transformation.

To see what $\hat{U}$ is in the $a$ representation, let's start with the operator $\exp[\mathrm{i}\theta\int_\Sigma F_a]$ and find its image under $\mathbb{S}$. We've already done this calculation while finding the image of the Wilson operator. This time, we simply do not sum over all $\Sigma$. We, therefore, have that (in Lorentzian signature)

$$\mathbb{S}: \mathrm{e}^{\mathrm{i}\theta\oint_\Sigma F_a} \to \mathrm{e}^{\mathrm{i}\theta\oint_\Sigma \frac{g^2}{2\pi}*F_{\hat{a}} - \mathrm{i}\frac{\theta^2 g^2}{2}\int_X|\hat{\Sigma}|^2}. \tag{C42}$$

Setting $\theta = \frac{\hat{\alpha}}{2\pi}$, the $\mathrm{U}(1)^{(d-p-1)}$ symmetry operator in the $a$ representation is

$$\hat{U}_{\hat{\alpha}}(\Sigma) = \mathrm{e}^{\mathrm{i}\hat{\alpha}\oint_\Sigma \frac{F_a}{2\pi} + \mathrm{i}\frac{\hat{\alpha}^2 g^2}{8\pi^2}\oint_X|\hat{\Sigma}|^2} \tag{C43}$$

since under $\mathbb{S}$ it transforms to Eq. (C41). However, note that the term $\int_X|\hat{\Sigma}|^2$ is an overall phase and therefore does not affect the symmetry transformation. So, we can drop this overall phase and treat the $\mathrm{U}(1)^{(d-p-1)}$ symmetry operator in the $a$ representation instead as

$$\hat{U}_{\hat{\alpha}}(\Sigma) = \mathrm{e}^{\mathrm{i}\hat{\alpha}\oint_\Sigma \frac{F_a}{2\pi}}. \tag{C44}$$

From this expression, it is easy to see that the Noether current of the $\mathrm{U}(1)^{(d-p-1)}$ symmetry in the $a$ representation $\hat{J}$ satisfies

$$*\hat{J} = \frac{1}{2\pi}F_a. \tag{C45}$$

The fact that the $\hat{J}$ is conserved reflects the Bianchi identity.

#### d. *Mixed 't Hooft anomaly and anomaly inflow*

Throughout this subsection thus far, we reviewed that $p$-form Maxwell theory has a $\mathrm{U}(1)^{(p)}$ and a $\mathrm{U}(1)^{(d-p-1)}$ symmetry. However, these two symmetries are not fully independent from one another: there is a mixed 't Hooft anomaly preventing us from simultaneously turning on a background gauge field of both symmetries.

Let's first turn on a background field $\mathcal{A}$ of the $\mathrm{U}(1)^{(p)}$ symmetry which includes the gauge redundancy

$$a \to a + \beta, \quad \mathcal{A} \to \mathcal{A} + \mathrm{d}\beta. \tag{C46}$$

Minimally coupling $\mathcal{A}$ to $a$, the path integral becomes

$$\mathcal{Z}[\mathcal{A}] = \int \mathcal{D}[a]\,\mathrm{e}^{-\int_X \frac{1}{2g^2}|F_a - \mathcal{A}|^2}. \tag{C47}$$

This is a gauge invariant theory, so the $\mathrm{U}(1)^{(p)}$ symmetry is anomaly free.

Let's dualize $a$ to $\hat{a}$ to see how $\mathcal{A}$ couples to $\hat{a}$. Repeating the first few steps of Abelian duality reviewed in Sec. C 1 b, the path integral becomes

$$\mathcal{Z}[\mathcal{A}] = \int \mathcal{D}K\mathcal{D}\hat{a}\,\mathrm{e}^{-\int_X \frac{1}{2g^2}|K|^2 - \frac{\mathrm{i}}{2\pi}F_{\hat{a}}\wedge K - \frac{\mathrm{i}}{2\pi}\mathcal{A}\wedge F_{\hat{a}}}, \tag{C48}$$

where we made the change of variables $F_a = K + \mathcal{A}$. Integrating out $K$, this becomes

$$\mathcal{Z}[\mathcal{A}] = \int \mathcal{D}[\hat{a}]\,\mathrm{e}^{-\int_X \frac{g^2}{8\pi^2}|F_{\hat{a}}|^2 - \frac{\mathrm{i}}{2\pi}\mathcal{A}\wedge F_{\hat{a}}}. \tag{C49}$$

Note that because $\mathrm{d}F_{\hat{a}} = 0$ and space-time is closed, the path integral in the $\hat{a}$ representation is still invariant under the gauge transformation Eq. (C46).

Equation (C49) reveals that turning on a $\mathrm{U}(1)^{(p)}$ symmetry background gauge field is equivalent to adding a topological term in the $\hat{a}$ representation. This new term has a noticeable effect. The Noether current for the $\mathrm{U}(1)^{(D-p-2)}$ symmetry, $\hat{J} = \frac{g^2}{4\pi^2} F_{\hat{a}}$, is no longer conserved:

$$\mathrm{d}^\dagger \hat{J} = \frac{1}{2\pi} *\mathrm{d}\mathcal{A}. \tag{C50}$$

This is a manifestation of the mixed 't Hooft anomaly.

Let's now turn off $\mathcal{A}$ and turn on a background gauge field $\hat{\mathcal{A}}$ for the $\mathrm{U}(1)^{(d-p-1)}$ symmetry with the gauge redundancy

$$\hat{a} \to \hat{a} + \hat{\beta}, \quad \hat{\mathcal{A}} \to \hat{\mathcal{A}} + \mathrm{d}\hat{\beta}. \tag{C51}$$

Inspired by how $\mathcal{A}$ coupled to $\hat{a}$ in Eq. (C48), we minimally couple $\hat{\mathcal{A}}$ to $a$ in a similar fashion and consider

$$\mathcal{Z}[\hat{\mathcal{A}}] = \int \mathcal{D}[a]\, \mathrm{e}^{-\int_X \frac{1}{2g^2}|F_a|^2 - \frac{\mathrm{i}}{2\pi}\hat{\mathcal{A}}\wedge F_a}. \tag{C52}$$

Because $F_a$ is closed, shifting $\hat{\mathcal{A}}$ by an exact form does not change the path integral. Thus the path integral is gauge invariant and $\mathrm{U}(1)^{(d-p-1)}$ is anomaly free. To verify this way of coupling $\hat{\mathcal{A}}$ to $a$ is correct, let's dualize $a$ to $\hat{a}$ in Eq. (C52). Doing so, we find

$$\mathcal{Z}[\hat{\mathcal{A}}] = \int \mathcal{D}[\hat{a}]\mathcal{D}[\hat{\mathcal{A}}]\, \mathrm{e}^{-\int_X \frac{g^2}{8\pi^2}|F_{\hat{a}} - \hat{\mathcal{A}}|^2}, \tag{C53}$$

as expected. Returning back to Eq. (C52), due to the new topological term, the $\mathrm{U}(1)^{(p)}$ symmetry Noether current $J = \frac{1}{g^2}F_a$ is no longer conserved:

$$\mathrm{d}^\dagger J = \frac{1}{2\pi} *\mathrm{d}\hat{\mathcal{A}}. \tag{C54}$$

Once again, this is a manifestation of the mixed 't Hooft anomaly.

Let us now turn on both of the background gauge fields $\mathcal{A}$ and $\hat{\mathcal{A}}$. Working in the $a$ representation and using what we just found, the path integral becomes

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] = \int \mathcal{D}[a]\, \mathrm{e}^{-\int_X \frac{1}{2g^2}|F_a - \mathcal{A}|^2 - \frac{\mathrm{i}}{2\pi}\hat{\mathcal{A}}\wedge F_a}. \tag{C55}$$

This path integral is invariant under the gauge transformation Eq. (C51). However, due to the second term in $\mathcal{L}$, this path integral is no longer invariant under the gauge transformation Eq. (C46) and transforms as

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] \to \mathrm{e}^{\frac{\mathrm{i}}{2\pi}\int_X \hat{\mathcal{A}}\wedge \mathrm{d}\beta}\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}]. \tag{C56}$$

No local counter term can remedy this property, and thus the $\mathrm{U}(1)^{(p)} \times \mathrm{U}(1)^{(d-p-1)}$ symmetry is anomalous.

We can make the theory gauge invariant by introducing the $(D+1)$-dimensional space-time $Y$ such that $X = \partial Y$ and extending the background gauge fields $\mathcal{A}$ and $\hat{\mathcal{A}}$ into $Y$. Indeed, notice how then the phase picked up in Eq. (C56) can be rewritten as

$$\int_{X=\partial Y} \hat{\mathcal{A}}\wedge \mathrm{d}\beta = \int_Y \mathrm{d}(\hat{\mathcal{A}}\wedge \mathrm{d}\beta) = \int_Y \mathrm{d}\hat{\mathcal{A}}\wedge \mathrm{d}\beta. \tag{C57}$$

This then motivates the new *gauge invariant* partition function

$$\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}] = \exp\left[-\frac{\mathrm{i}}{2\pi}\int_Y \mathrm{d}\hat{\mathcal{A}}\wedge\mathcal{A}\right]\int \mathcal{D}[a]\, \mathrm{e}^{-\int_{\partial Y}\mathcal{L}},$$
$$\mathcal{L} = \frac{1}{2g^2}|F_a - \mathcal{A}|^2 - \frac{\mathrm{i}}{2\pi}\hat{\mathcal{A}}\wedge F_a. \tag{C58}$$

Indeed, $\mathcal{Z}[\mathcal{A}, \hat{\mathcal{A}}]$ is invariant under the gauge transformations (C56) since the phase picked up from $\int \mathcal{D}[\hat{a}]\, \mathrm{e}^{-\int_{\partial Y}\mathcal{L}}$ cancels with the phase we added.

Thus we see the 't Hooft anomaly through the modern perspective of anomaly inflow. In order to turn on both $\mathcal{A}$ and $\hat{\mathcal{A}}$, we must have the theory resides on the boundary of an SPT, which in this case was [130]

$$\mathcal{Z}_{\mathrm{SPT}}[\mathcal{A}, \hat{\mathcal{A}}] = \mathrm{e}^{-\frac{\mathrm{i}}{2\pi}\int_Y \mathrm{d}\hat{\mathcal{A}}\wedge\mathcal{A}}. \tag{C59}$$

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*Correction:* The previously published Figure 1 was altered during the production process and did not render correctly. It has been replaced with the correct figure.