Editors' Suggestion

Multipartite entanglement in the measurement-induced phase transition of the quantum Ising chain

Alessio Paviglianiti and Alessandro Silva
International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy

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External monitoring of quantum many-body systems can give rise to a measurement-induced phase transition characterized by a change in behavior of the entanglement entropy from an area law to an unbounded growth. In this paper, we show that this transition extends beyond bipartite correlations to multipartite entanglement. Using the quantum Fisher information, we investigate the entanglement dynamics of a continuously monitored quantum Ising chain. Multipartite entanglement exhibits the same phase boundaries observed for the entropy in the postselected no-click trajectory. Instead, quantum jumps give rise to a more complex behavior that still features the transition, but adds the possibility of having a third phase with logarithmic entropy but bounded multipartiteness.

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I. INTRODUCTION

In recent years, entanglement has become a key tool in condensed-matter and statistical physics [1]. Interesting quantum phenomena, such as quantum criticality or topological order, often give rise to peculiar behavior of entanglement too. For this reason, entanglement now plays an important role in understanding and classifying quantum many-body systems. This approach has lead to numerous theoretical insights, such as probing quantum phase transitions [2,3], understanding thermalization [4], and extracting information on topological properties [5–7]. In addition to its theoretical significance, entanglement is a fundamental resource of practical experimental use. For example, entangled states can enhance the precision of phase estimation in quantum metrology [8,9] and are essential for implementing most quantum computing protocols, with relevant applications in quantum cryptography [10,11], optimization [12,13], and simulation [14].

Measurement-induced phase transitions [15–21] are a notable example of entanglement as a diagnostic tool for quantum criticality in a dynamical setting. Monitoring a system throughout its dynamics can alter its entanglement properties. Specifically, the rate of measurements performed on the system drives a transition from an area law phase to an entangling phase, typically with either volume law [22–27] or logarithmic [28–32] entanglement entropy depending on the model. The two phases have also been related to a qualitative change in the dynamical purification of a mixed state [33–36]. Many features, including critical exponents [16,37,38], hints of conformal symmetry at criticality [39–41], and connections to the percolation universality class [16,42–44], suggest that at least some versions of this phenomenon can be traced back to standard second-order phase transitions. Despite all this progress, an exhaustive characterization of the transition is still missing. For instance, it is still unclear whether there exists a local order parameter and what it would look like in that case. In addition, even though some proposals have been put forward [45–48], observing the transition in an experimental implementation remains an extremely challenging task due to the exponential complexity of postselecting quantum trajectories.

So far, measurement-induced transitions have been mainly characterized on the basis of bipartite entanglement. Nevertheless, the structure of multipartite correlations is potentially richer [49–51] and could give new insight for the characterization of the different phases. In this paper, we investigate multipartite entanglement in a continuously monitored quantum Ising chain by means of the quantum Fisher information (QFI), which witnesses many-body correlations [52–55]. We first investigate the no-click limit [26,29,56] to gain theoretical insight on the measurement-induced transition, and we prove that the QFI manifests the same phase diagram as the entropy. In the logarithmic phase, the QFI density features power-law growth $f_Q \sim L^p$, corresponding to growing multipartiteness with system size, whereas it remains bounded in the area phase (see Fig. 1). We then consider the full dynamics with quantum jumps, revealing that, in general, the behavior of the QFI is more complex. We still observe a transition between extended and limited multipartiteness, but we also find a new region with bounded multipartite entanglement yet logarithmic entanglement entropy.

The rest of this paper is organized as follows. In Sec. II, we introduce the quantum Ising chain and the monitoring protocol we consider, as well as the QFI and its relation to multipartite entanglement. Then, Sec. III presents our results relative to the no-click limit, whereas Sec. IV covers the case of the full dynamics involving quantum jumps. Last, we summarize our findings in Sec. V.

II. MODEL, MEASUREMENT PROTOCOL, AND QUANTUM FISHER INFORMATION

Below, we consider measurement-induced phase transitions in a quantum Ising chain in a transverse field,

$$\hat{H}_0 = -J \sum_j \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x - h \sum_j \hat{\sigma}_j^z, \tag{1}$$

with L lattice sites and periodic boundary conditions. Throughout this paper, we set $J = 1$. Within the formalism of positive-operator-valued measures [57–61], we characterize entirely the measurement protocol by assigning suitable Kraus operators $\hat{A}_m$, $m = 1, \dots, M$, satisfying $\sum_m \hat{A}_m^\dagger \hat{A}_m = \hat{\mathbb{1}}$. In detail, given a state $|\psi_t\rangle$, the evolved state $|\psi_{t+dt}\rangle$ is obtained by applying a projector $\hat{A}_m$ to $|\psi_t\rangle$ and restoring the norm to 1. The choice of the Kraus operator is performed randomly with probabilities set by $p_m = \langle \hat{A}_m^\dagger \hat{A}_m \rangle_t$ (where $\langle \hat{O} \rangle_t = \langle \psi_t | \hat{O} | \psi_t \rangle$). In our case, we assume to measure the z component of each spin randomly and independently of all others with a fixed rate γ. Since the full protocol can be broken down into single-site measurements, we use the local Kraus operators

$$\hat{A}_j^{(0)} = (\hat{\mathbb{1}} - \hat{L}_j) + \sqrt{1 - \gamma dt}\, \hat{L}_j, \tag{2a}$$

$$\hat{A}_j^{(1)} = \sqrt{\gamma dt} \hat{L}_j, \tag{2b}$$

where $\hat{L}_j = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\sigma}_j^z)$. $\hat{A}_j^{(0)}$ has a probability $p_j^{(0)} = \mathcal{O}(1)$ and implements an infinitesimal projection towards a local spin state with down z component. In contrast, $\hat{A}_j^{(1)}$ represents a rare but sudden jump to the state with up z component, occurring with $p_j^{(1)} = \mathcal{O}(\gamma dt)$. The interplay of these Kraus operators can induce nontrivial magnetization dynamics [62,63]. Using this generalized measurement protocol, the dynamics of the system is ruled by the stochastic Schrödinger equation [64,65]

$$d|\psi_t\rangle = -i\hat{H} dt |\psi_t\rangle + \sum_j d\xi_{j,t} \left(\frac{\hat{L}_j}{\sqrt{\langle \hat{L}_j \rangle_t}} - 1 \right) |\psi_t\rangle, \tag{3}$$

where

$$\hat{H} = \hat{H}_0 - i\frac{\gamma}{4} \sum_j \left(\hat{\sigma}_j^z - \langle \hat{\sigma}_j^z \rangle_t \right) \tag{4}$$

is a non-Hermitian Hamiltonian [66] describing an effective nonunitary evolution in the absence of jumps, whereas the functions $d\xi_{j,t} = 0, 1$ are increments of independent Poisson processes satisfying $\overline{d\xi_{j,t}} = \gamma dt \langle \hat{L}_j \rangle_t$. For details on the derivation of Eq. (3), we refer the reader to Ref. [29].

This model has already been considered in Ref. [67] from the perspective of the entanglement entropy. The scope of our study is to investigate the QFI in the stationary state of the dynamics generated by Eq. (3). The QFI is a key quantity in the theory of phase estimation [8,68,69], as it determines the best achievable precision in quantum metrology applications. This quantity, accessible through measurements of dynamical susceptibilities [70,71], is sensitive to the number of entangled degrees of freedom in the state and can thus be used to witness entanglement. When evaluated on pure states, the QFI of an observable $\hat{O}$ takes a simple form proportional to its variance, namely,

$$F_Q[\hat{O}] = 4(\langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2). \tag{5}$$

As shown in Refs. [52,53], the QFI sets rigorous bounds to multipartite entanglement when $\hat{O} = \hat{O}[\{\mathbf{n}_j\}] = \frac{1}{2} \sum_j \mathbf{n}_j \cdot \hat{\boldsymbol{\sigma}}_j = \frac{1}{2} \sum_j \sum_{\alpha=x,y,z} n_j^\alpha \hat{\sigma}_j^\alpha$, where $\mathbf{n}_j$ are unit vectors. Substituting this expression, the QFI takes the form

$$F_Q[\hat{O}[\{\mathbf{n}_j\}]] = \sum_{\alpha,\beta=x,y,z} \sum_{i,j} n_i^\alpha C_{i,j}^{\alpha,\beta} n_j^\beta, \tag{6}$$

where $C_{i,j}^{\alpha,\beta} = \langle \hat{\sigma}_i^\alpha \hat{\sigma}_j^\beta \rangle - \langle \hat{\sigma}_i^\alpha \rangle \langle \hat{\sigma}_j^\beta \rangle$ are connected spin-spin correlators. If the density of QFI $f_Q = F_Q/L$ is larger than some divider k of L, then the state is $(k+1)$-partite entangled. The strictest lower bound is obtained by finding the unit vectors $\{\mathbf{n}_j\}_{\text{opt}}$ that maximize F_Q. The optimization problem is equivalent to the search of the ground state of a classical Hamiltonian $H_{\text{cl}} = -F_Q[\hat{O}[\{\mathbf{n}_j\}]]$ with vector spin variables $\mathbf{n}_j$, where the correlation functions play the role of two-body couplings.

III. NO-CLICK LIMIT

Let us start our analysis from the no-click limit, namely, the specific quantum trajectory in which all $d\xi_{j,t}$ are zero at all times and no quantum jump occurs. At long times, the dynamics converges to a stationary state, which coincides with the vacuum state of the non-Hermitian quasiparticles that diagonalize $\hat{H}$ [72]. Even though this trajectory is exponentially unlikely, it can provide information on what can be expected in generic realizations of the full dynamics. For instance, Ref. [73] shows that the no-click limit of our model manifests the entanglement transition from area to logarithmic law. The logarithmic scaling of the entanglement entropy in the stationary state is linked to the absence of a gap in the decay rate of elementary excitations. For $|h| < 1$ and $\gamma < \gamma_c(h) = 4\sqrt{1 - h^2}$, the imaginary part of the quasiparticle spectrum is gapless, and the entanglement entropy follows a logarithmic law. In contrast, it is gapped outside this region, and the entropy obeys an area law.

The two phases also feature a difference in their correlation functions. The correlators $C_{i,j}^{\alpha,\beta}$ decay exponentially with the distance $|j - i|$ in the gapped phase, whereas they have a power-law envelope, modulated by sinelike oscillations, in the gapless phase [72]. This difference impacts the QFI in the two phases: as mentioned, the maximization of the QFI is mapped into the minimization of a classical Hamiltonian H_{cl} in which the correlation functions set the interactions. In the gapped phase, all correlators are exponential, and thus H_{cl} is a short-range Hamiltonian; as a consequence, we have $F_Q^{\max} \sim L$, and $f_Q^{\max}$ is intensive. Instead, in the gapless phase the power-law decay of correlations opens up the possibility of a long-range H_{cl}. If the correlation functions decay slowly enough, one may expect a superextensive scaling of the QFI with the system size, resulting in $f_Q^{\max} \sim L^p$ with $p > 0$. Recalling the connection between the QFI density and multipartite entanglement, this implies that the degree of multipartiteness of entanglement is bounded in the area phase, whereas it diverges as $\sim L^p$ in the logarithmic phase.

We now test numerically this hypothesis. We find the vacuum (steady) state by solving the model using the Jordan-Wigner map [74]. The spin-spin correlators are computed using the methods described in Refs. [75,76], exploiting the Gaussian structure of the state [72]. Finally, the maximization of the QFI is performed with a classical simulated annealing algorithm [77,78]. For each choice of the parameters h and γ, we evaluate the maximal QFI at different system sizes, and we fit the scaling of $f_Q^{\max}$ to extrapolate the exponent p. Figure 1 shows p in the parameter space. Based on whether $p = 0$ or $p > 0$, we distinguish two phases, which overlap very well with the area and logarithmic phases diagnosed

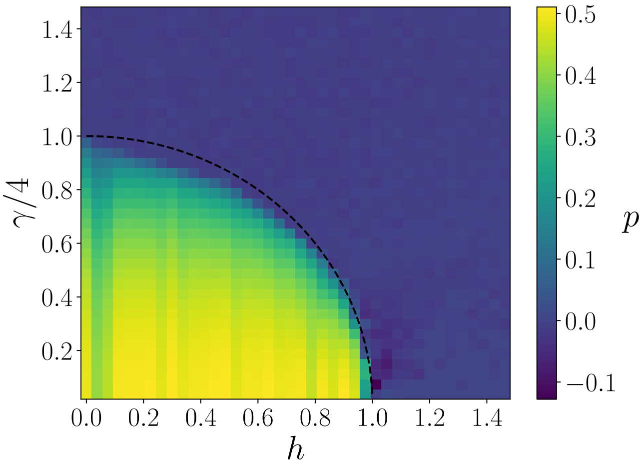

FIG. 1. Exponent p of $f_Q \sim L^p$ as a function of h and γ in the no-click limit. The dashed curve corresponds to the critical line $\gamma_c(h)$ that separates the gapped and gapless phases. The exponent is extrapolated by fitting data for $L = 40$–170.

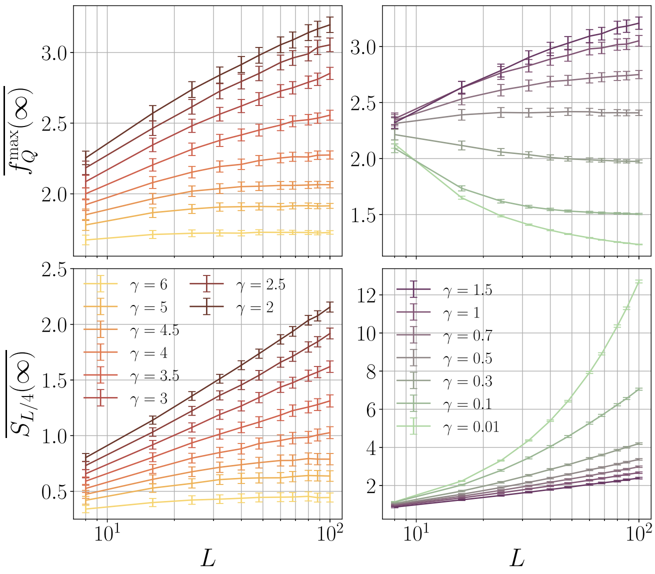

FIG. 2. Disorder-averaged stationary QFI density and entanglement entropy, as functions of L, for $h = 0.2$ and multiple values of γ. Large (left panels) and small (right panels) values of γ are presented separately to help visualization. The stationary values are evaluated as long-time averages.

by the entanglement entropy. This result indicates that the entanglement transition in the no-click limit is witnessed by multipartite entanglement. Our numerical results suggest that p might be a universal function of $\gamma/\gamma_c(h)$ for all values of h [72]. We point out that the effective central charge of the entanglement entropy behaves similarly, being a function of $\gamma/\gamma_c(h)$ only [73].

Surprisingly, despite the translational symmetry of the system, the operator maximizing the QFI is not translationally invariant. In the gapless phase the optimal $\{\mathbf{n}_j\}_{\text{opt}}$ are approximately aligned along the longitudinal direction and alternate between $+\mathbf{x}$ and $-\mathbf{x}$ with a wave vector $k = \pi - k^*$, where k^* is the momentum at which the gap of the quasiparticle decay rate closes. This is understood in terms of correlation functions. We observe numerically that $C_{i,j}^{x,x}$ is the slowest-decaying spin-spin correlator, and it oscillates with a periodicity set precisely by $\pi - k^*$ [72]. This correlation function rules the leading-order behavior of $f_Q^{\max}$ with L, and thus the optimal configuration $\{\mathbf{n}_j\}_{\text{opt}}$ must maximize its contribution in Eq. (6). Assuming the asymptotic ansatz $C_{i,j}^{x,x} \sim \cos[(\pi - k^*)|i-j|]/|i-j|^\lambda$ with $\lambda < 1$, which we find to be a good fit, and considering for simplicity a periodic configuration $n_j^x = \cos[(\pi - k^*)|i-j|]$, we obtain a contribution to the QFI that scales as $L^{2-\lambda}$: this yields a finite $p = 1 - \lambda > 0$.

The operator $\hat{O}[\{\mathbf{n}_j\}_{\text{opt}}]$ that maximizes the QFI can be interpreted as a local order parameter due to its fluctuations, which are superextensive exclusively in the critical gapless phase. This identification holds, for instance, for the quantum Ising chain $\hat{H}_0$, where the order parameter $\sum_j \hat{\sigma}_j^x$ maximizes the QFI, providing $f_Q^{\max} \sim L^{3/4}$ at the critical point [70]. We believe this characterization is reasonable, since we cannot pinpoint any conventional order parameter $\hat{O}$ that changes from $\langle \hat{O} \rangle = 0$ to $\langle \hat{O} \rangle \neq 0$ when crossing the phase boundary.

IV. DYNAMICS WITH QUANTUM JUMPS

The results on the QFI found in the no-click limit extend only partially to the full dynamics produced by Eq. (3). As we show in this section, bipartite and multipartite entanglement do not manifest equivalent behavior and thus provide distinct information and phase boundaries. In particular, we observe a region featuring logarithmic entanglement entropy but intensive f_Q, which possibly corresponds to a new phase.

In our numerics, we start from a product state $|\psi_0\rangle$ with all spins along the positive z direction, and we characterize its dynamics using only the correlation matrices of Jordan-Wigner fermions [72], exploiting the preservation of the Gaussian nature of the state along each quantum trajectory. We compute the maximal QFI using simulated annealing, as in the no-click limit. Since this quantity depends on the previous history of quantum jumps, we repeat the procedure multiple times independently and take a statistical average. To make a comparison, we also evaluate the entanglement entropy, defined as $S_\ell = -\text{Tr}(\hat{\rho}_\ell \ln \hat{\rho}_\ell)$, where $\hat{\rho}_\ell$ is the reduced density matrix associated with a compact subsystem of ℓ spins, using the method described in Ref. [74].

We now illustrate our results for a given $h < 1$, though similar results are obtained qualitatively for other values of h [72]. At large values of γ, the average maximal QFI density saturates to an intensive value at long times. When γ is reduced below $\gamma_c(h)$ ($\gamma_c \approx 4$ for $h = 0.2$), $\overline{f_Q^{\max}(\infty)}$ appears to grow indefinitely with the system size L. This is highlighted in the left panels of Fig. 2, which portray the scaling of the stationary value with L. Our numerics suggest that the crossover of f_Q from an intensive to a size-dependent value occurs at the same γ_c at which S_ℓ transitions from area to logarithmic law, even though it is hard to determine the critical measurement rate precisely. The growth of the QFI density below γ_c is consistent with a power law $\sim L^p$, as in the no-click limit.

When γ is reduced further, we observe a new effect completely at odds with the no-click limit. The steady-state QFI density transitions back to an intensive value, as illustrated in the right panels of Fig. 2. The entanglement entropy shows also a contrasting behavior by developing a volume law at small γ. Our numerics suggest, however, that the entropy crossover takes place at lower γ as compared to the QFI, and it could be interpreted as a finite-size effect occurring at $\gamma \sim 1/L$ [28], where the dynamics is approximately unitary and jumps are rare. These results may then indicate the presence of a third phase at low γ, featuring logarithmic entanglement entropy but only few-partite quantum correlations.

We point out that often states with volume law entanglement entropy have a bounded QFI density, especially in the long-time unitary dynamics following a quantum quench [79]. This intuition might indicate that the intensive scaling of f_Q at small γ is also a finite-size effect. While we cannot rule out this possibility, this seems to be at odds with our numerics, showing that the average QFI density is smaller at larger values of L, where finite-size effects are less relevant.

As in the no-click limit, the behavior of the QFI can be related to the shape of spin-spin correlation functions. Focusing on single quantum trajectories in the long-time regime, where $\overline{f_Q^{\max}(t)}$ has already reached saturation, we define the new distance-dependent correlators

$$\tilde{C}_\ell^{\alpha,\beta} = \frac{1}{L}\sum_i \left|C_{i,i+\ell}^{\alpha,\beta}\right|. \tag{7}$$

Numerically, we observe that all $\tilde{C}_\ell^{\alpha,\beta}$ are exponential at large $\gamma > \gamma_c$, whereas they decay as power laws at smaller γ [72]. This crossover already indicates a qualitative difference between the two regimes. The power-law correlators also suggest that the entangling phase is an extended critical region, compatible with the observation of a logarithmic entanglement entropy.

Given the analogy with the no-click limit, we expect that the average maximal QFI density diverges for $L \to \infty$ if these power laws decay slowly enough, namely, $\tilde{C}_\ell^{\alpha,\beta} \sim |\ell|^{-\lambda_{\alpha,\beta}}$ with $\lambda_{\alpha,\beta} < 1$. We check if typical trajectories with this property exist by simulating M independent realizations, computing their exponents at some long time, and counting the number $M_<$ of them with at least one $\lambda_{\alpha,\beta} < 1$. Notice that the exponents are time dependent, because individual quantum trajectories do not relax to stationary states, and thus we repeat the analysis at multiple times. As shown in Fig. 3, in the region $\gamma \lesssim 4$ we estimate that a finite fraction of the ensemble of all trajectories yields slow-decaying correlators, thus producing the scaling behavior of $\overline{f_Q^{\max}(\infty)}$. At small γ, we observe that $M_</M$ vanishes, meaning that no random realization can support extended multipartiteness. This explains why f_Q returns intensive at low measurement rates, but our limited sample of $M = 100$ trajectories is not sufficient to establish whether or not a sharp transition takes place.

The interpretation of $\hat{O}[\{\mathbf{n}_j\}_{\text{opt}}]$ as the order parameter is less straightforward in the presence of jumps. For each random realization and at each time, the QFI is maximized by a different operator, making it impossible to design a unique, trajectory-independent order parameter. However, this operator may still characterize criticality for each individual trajectory.

Some previous works in the literature [80–83] characterize measurement-induced phase transitions using the fluctuations

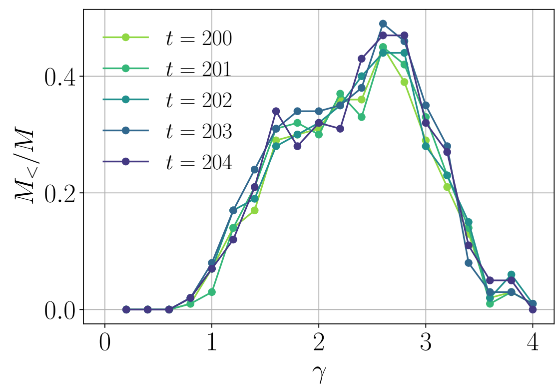

FIG. 3. Fraction $M_</M$ of trajectories with exponents $\lambda_{\alpha,\beta} < 1$, for $M = 100$, $h = 0.2$, and multiple values of γ. Different curves correspond to different times $t = 200$, 201, 202, 203, and 204 (light to dark colors) at which the exponents are measured. Data are for $L = 256$, and the power laws are fitted for $L = 10$–100.

of observables, rather than entanglement. Since our investigation revolves around quantum fluctuations too, we now compare our paper with these studies, clarifying that our results are fundamentally different. First, Refs. [80,81] demonstrate that the fluctuations of the total charge of a subsystem (e.g., half of the chain) manifest a phase transition, shifting from intensive to extensive when tuning the measurement rate. Unfortunately, no information on multipartite entanglement can be gained from this result, as the observable under investigation is not in the required form of $\hat{O}[\{\mathbf{n}_j\}]$, defined below Eq. (5). One may also be interested in the multipartite entanglement content of a subsystem alone, but in this case the QFI takes a more complicated form than Eq. (5) and is no longer directly related to quantum fluctuations. In contrast, Refs. [82,83] do consider fluctuations of globally defined operators. Nevertheless, the models treated in these works are not spin systems as considered in the original Refs. [52,53] that connect the QFI to multipartite entanglement. Still leaving aside this technicality, these papers observe a transition from zero to intensive fluctuations, and both cases correspond to a vanishing QFI density. In contrast, our work features a shift from extensive to superextensive variance, explicitly demonstrating extended multipartiteness of quantum correlations.

V. CONCLUSIONS

In this paper, we investigated the measurement-induced phase transition of a quantum Ising chain from the point of view of multipartite entanglement as witnessed by the QFI. In the postselected trajectory without quantum jumps, the multipartiteness of quantum correlations changes from limited to extended, reproducing the same phase diagram obtained from the entanglement entropy. When quantum jumps are introduced, the entanglement entropy and the QFI manifest distinct behaviors, and we observe a new region with bounded QFI density yet logarithmic entanglement entropy emerges at low γ. Our findings hint at the exciting possibility that this might be a new phase. We stress that, in general, the entanglement entropy and the QFI should not be expected to necessarily behave in the same way, as they probe very different aspects of entanglement: While the former quantifies the strength of bipartite correlations, the latter is a measure of the number of irreducibly entangled degrees of freedom in the system.

The experimental observation of the QFI in monitored systems is unfortunately plagued by the same postselection problem that affects also the entanglement entropy. However, if a new technique to reliably reproduce arbitrary quantum trajectories was discovered, the QFI would then be easily accessible, by measuring either the spin-spin correlators or the fluctuations of operators $\hat{O}$.

We believe our study paves the way to future investigations on the role of multipartite entanglement in measurement-induced phase transitions. A topic of immediate interest is to establish whether or not the region with intensive QFI density at low γ is indeed a stable phase. More broadly, we are still lacking a theoretical understanding of the whole phase diagram. Beyond our model, it will be interesting to study how multipartite entanglement behaves in different instances of the entanglement transition, including circuit models, systems with projective measurements, and, in particular, models with a volume phase.

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