Density of a one-dimensional weakly interacting Bose gas far from an impurity

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We consider an impurity in a one-dimensional weakly interacting Bose gas and analytically calculate the density profile of the Bose gas. Within the mean-field approximation, by increasing the distance from the impurity, the Bose gas density saturates exponentially fast to its mean thermodynamic-limit value at distances beyond the healing length. The effect of quantum fluctuations drastically changes this behavior, leading to a power-law decay of the density deviation from the mean density. At distances longer than the healing length and shorter than a new length scale proportional to the impurity coupling strength, the power-law exponent is 2, while at the longest distances the corresponding exponent becomes 3. The latter crossover does not exist in two special cases. The first one is realized for infinitely strongly coupled impurity; then the density deviation always decays with the exponent 2. The second special case occurs when the new length scale is smaller than the healing length, i.e., at weak impurity coupling; then the density deviation always decays with the exponent 3. The obtained results are exact in the impurity coupling strength and account for the leading order in the interaction between the particles of the Bose gas.

I. INTRODUCTION

Impurities play a pronounced role in one-dimensional quantum liquids and determine their properties. For example, a single impurity severely affects the transport in fermionic liquids [1]. In these systems the impurity also causes Friedel oscillations of the local particle density whose envelope decays following the power law $1/|x|^K$ at long distances [2,3]. Here $|x|$ is the separation from the impurity, and K is the Luttinger liquid parameter.

A related question regarding the behavior of the density in one-dimensional liquids with bosonic particles is less well understood. The goal of this paper is to study this basic problem. Consider a system that consists of a static impurity placed in a weakly interacting Bose gas [4]. In the absence of the impurity, the mean value of the density operator, i.e., the local density, has a constant value. This picture changes once the impurity is present and the density becomes a function of the distance from the impurity. Due to the repulsion from the impurity, the density of the Bose gas will be locally suppressed at the impurity position. Since the particles of the Bose gas interact, the local density depletion will spread over the whole system. This can be visualized as a hole in the Bose gas density created due to the presence of the impurity (see Fig. 1). Our goal is to describe the shape of this hole.

An interacting Bose gas shows superfluid properties. The latter is characterized by a microscopic length scale ξ known as the healing length. It generally denotes the distance where the superfluid density recovers from a local perturbation. For our system, the impurity serves as a localized density perturbation, and one thus expects the density recovery at separations from the impurity of several ξ. This is the correct scenario within the mean-field picture, which will be discussed below. The corresponding nonlinear equation that determines the density has an exact solution. It shows that the

density depletion beyond ξ disappears exponentially fast. In this paper we show that the latter scenario drastically changes beyond the mean-field treatment, i.e., once one accounts for the effect of quantum fluctuations (see Fig. 1). In this case the density depletion decays algebraically with the distance. Formally, $n_0 - n(x) \propto 1/|x|^\eta$, where n_0 and $n(x)$ respectively denote the mean and local densities. The exponent η takes the universal values 2 or 3 depending on whether the distance $|x|$ (which satisfies $|x| \gg \xi$) is shorter or longer than the length
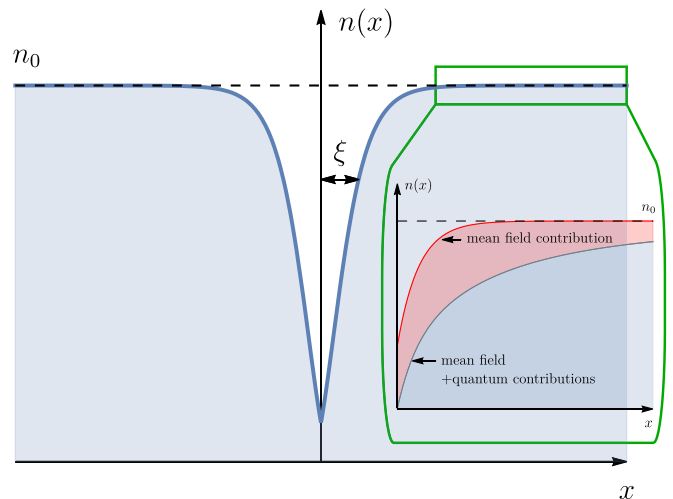

FIG. 1. Plot of the local density $n(x)$ of a weakly interacting Bose gas in the vicinity of the impurity. The inset shows a magnified region of the density at distances from the impurity beyond the healing length ξ. It shows the difference between the density in the mean-field approximation that very quickly approaches the mean density n_0 and the actual density that also contains the contribution arising from quantum fluctuations.

scale $\widetilde{G}\xi$. Here $\widetilde{G}$ denotes the dimensionless impurity strength, which will be defined in the next section. Power-law decay of the density depletion means that the presence of the impurity can be probed in the local density far from the impurity. Note that unlike at distances beyond ξ where the quantum contribution prevails, it is parametrically smaller than the mean-field contribution in the vicinity of the impurity. In the remainder of this paper we derive these results and discuss their further physical implications.

II. THEORETICAL MODEL

We study a one-dimensional system of bosons with short-range repulsion that is locally coupled to an impurity. The corresponding Hamiltonian is given by

$$\hat{H} = \int_{-\infty}^{\infty} dx \left(-\hat{\Psi}^\dagger \frac{\hbar^2 \partial_x^2}{2m} \hat{\Psi} + \frac{g}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right) + G\hat{\Psi}^\dagger(0, t)\hat{\Psi}(0, t). \quad (1)$$

Here m denotes the mass of bosons, $g > 0$ is the repulsion strength, and G denotes the coupling of the system to an impurity at the origin. The bosonic field operators satisfy the standard commutation relations, $[\hat{\Psi}(x, t), \hat{\Psi}^\dagger(y, t)] = \delta(x - y)$ and $[\hat{\Psi}(x, t), \hat{\Psi}(y, t)] = 0$. We study the system in the thermodynamic limit where the particle density n_0 is finite. We are not interested in the impurity dynamics, and thus, the case when it is infinitely heavy is considered. The impurity serves as a source of a local static potential for the system. For repulsive coupling, $G > 0$, the impurity causes the depletion of the Bose gas density at its position in the ground state. Such local perturbation of the density at a single point also affects the local density everywhere in space. The goal of this paper is to calculate how such local depletion will decay as a function of the separation from the impurity. We will calculate the mean boson density

$$n(x) = \langle \hat{\Psi}^\dagger(x, t)\hat{\Psi}(x, t) \rangle. \quad (2)$$

Here the average is with respect to the ground state of the system. Since the impurity is static, we do not expect time dependence in the density profile, and we thus omitted the time coordinate in the left-hand side of Eq. (2).

We treat the Hamiltonian (1) using the Heisenberg equation of motion of the field operator, $i\hbar\partial_t\hat{\Psi}(x, t) = [\hat{\Psi}(x, t), \hat{H}]$. For the purpose of lighter notation, it is convenient to define the dimensionless coordinates X for space and T for time by

$$X = \frac{x}{\xi_\mu}, \quad T = \frac{t\mu}{\hbar}. \quad (3)$$

Here μ is the chemical potential of the interacting Bose gas, and $\xi_\mu = \hbar/\sqrt{m\mu}$. After introducing the dimensionless field operator $\hat{\psi}(X, T)$ and the dimensionless coupling $\widetilde{G}$ by the relations

$$\hat{\Psi}(x, t) = \sqrt{\frac{\mu}{g}}\hat{\psi}(X, T)e^{-iT}, \quad (4)$$

$$\widetilde{G} = \frac{G}{\hbar}\sqrt{\frac{m}{\mu}}, \quad (5)$$

the equation of motion for $\hat{\psi}(X, T)$ takes the form

$$i\partial_T\hat{\psi} = \left[-\frac{\partial_X^2}{2} + \hat{\psi}^\dagger\hat{\psi} - 1 + \widetilde{G}\delta(X) \right]\hat{\psi}, \quad (6)$$

while the nontrivial commutation relation is given by

$$[\hat{\psi}(X, T), \hat{\psi}^\dagger(Y, T)] = \alpha^2\delta(X - Y). \quad (7)$$

Here $\alpha = (\gamma gn_0/\mu)^{1/4}$ is expressed in terms of $\gamma = mg/\hbar^2 n_0$, which denotes the dimensionless parameter of the model (1). Up to this point our consideration is general. Equations (6) and (7) follow exactly from the Hamiltonian (1) and the initial commutation relations.

Let us consider the case of weak interaction corresponding to $\gamma \ll 1$. Then the chemical potential is approximately given by $\mu \approx gn_0$, and thus, $\alpha \approx \gamma^{1/4} \ll 1$. We then seek the solution of Eq. (6) as a series of the form

$$\hat{\psi}(X, T) = \psi_0(X) + \alpha\hat{\psi}_1(X, T) + \alpha^2\hat{\psi}_2(X, T) + \cdots. \quad (8)$$

The function $\psi_0(X)$ can be interpreted as the time-independent single-particle wave function in the absence of quantum fluctuations, whereas $\hat{\psi}_1(X, T)$ and $\hat{\psi}_2(X, T)$ represent, respectively, its first and second quantum corrections.

Substituting Eq. (8) in Eq. (6), we obtain a hierarchy of equations controlled by the different powers of α. At the leading α^0 order we obtain

$$\widehat{\mathcal{L}}_1\psi_0(X) + \widetilde{G}\delta(X)\psi_0(X) = 0, \quad (9)$$

where we have introduced the family of operators

$$\widehat{\mathcal{L}}_j(X) = -\frac{\partial_X^2}{2} + j|\psi_0(X)|^2 - 1. \quad (10)$$

Note that $\widehat{\mathcal{L}}_j(X)$ can be determined only after the nonlinear equation (9) has been solved. Expression (9) is known as the Gross-Pitaevskii equation. At order α we obtain the equation for the first quantum correction of the field operator,

$$[i\partial_T - \widehat{\mathcal{L}}_2(X) - \widetilde{G}\delta(X)]\hat{\psi}_1(X, T) = \psi_0(X)^2\hat{\psi}_1^\dagger(X, T). \quad (11)$$

Equation (11) is linear, but it depends on the solution of the nonlinear problem (9). Finally, at order α^2 we obtain the equation for the second quantum correction,

$$[i\partial_T - \widehat{\mathcal{L}}_2(X) - \widetilde{G}\delta(X)]\hat{\psi}_2(X, T) = \psi_0(X)^2\hat{\psi}_2^\dagger(X, T) + 2\psi_0(X)\hat{\psi}_1^\dagger(X, T)\hat{\psi}_1(X, T) + \psi_0^*(X)\hat{\psi}_1(X, T)^2. \quad (12)$$

III. SOLUTION OF THE EQUATION OF MOTION

A. Mean-field solution

Our goal is to describe the ground state of the system. The corresponding solution of Eq. (9) does not have nodes at any finite value of $\widetilde{G}$. We thus need a strictly positive (or negative) $\psi_0(X)$ that satisfies $\widehat{\mathcal{L}}_1(X)\psi_0(X) = 0$ in the two regions $X < 0$ and $X > 0$, while at the impurity position the boundary conditions

$$\psi_0(0^+) = \psi_0(0^-), \quad (13)$$

$$\psi_0'(0^+) - \psi_0'(0^-) = 2\widetilde{G}\psi_0(0) \quad (14)$$

must be satisfied. For the solution we find [5,6]

$$\psi_0(X) = \tanh(|X| + X_0), \tag{15a}$$

where

$$\tanh X_0 = \frac{2}{\widetilde{G} + \sqrt{4 + \widetilde{G}^2}}. \tag{15b}$$

In the case of a strongly coupled impurity, $\widetilde{G} \gg 1$, we obtain $X_0 \simeq 1/\widetilde{G}$. Therefore, $\psi_0(X) = |\tanh X|$ at $\widetilde{G} \to +\infty$. This is the limiting case where the impurity is so strongly coupled that it completely suppresses the Bose gas density at its position. In the opposite case $\widetilde{G} \ll 1$, the Bose gas density is only slightly perturbed at the impurity position. Indeed, we have $X_0 \simeq \ln(2/\sqrt{\widetilde{G}})$, and thus, $\psi_0(X) = 1 - \widetilde{G}e^{-2|X|}/2 + O(\widetilde{G}^2)$.

B. First quantum correction

Having obtained $\psi_0(X)$ [see Eq. (15)], we are in a position to solve Eq. (11). This is an operator equation. We seek its solution in the form

$$\hat{\psi}_1(X, T) = \sum_k N_k[u_k(X)\hat{b}_k e^{-i\epsilon_k T} - v_k^*(X)\hat{b}_k^\dagger e^{i\epsilon_k T}], \tag{16}$$

where the summation is over real k values. In Eq. (16), N_k is a real normalization factor that is dictated by the commutation relation (7), $\hat{b}_k$ and $\hat{b}_k^\dagger$ are the quasiparticle bosonic operators that satisfy the commutation relations $[\hat{b}_k, \hat{b}_q^\dagger] = \delta_{k,q}$ and $[\hat{b}_k, \hat{b}_q] = 0$, and ϵ_k is the quasiparticle energy for the momentum k. Substituting Eq. (16) in Eq. (11), we obtain a linear system of two coupled equations for the complex functions $u_k(X)$ and $v_k(X)$. They are known as the Bogoliubov–de Gennes equations and take the form

$$[\widehat{\mathcal{L}}_2(X) + \widetilde{G}\delta(X) - \epsilon_k]u_k(X) = \psi_0(X)^2 v_k(X), \tag{17a}$$

$$[\widehat{\mathcal{L}}_2(X) + \widetilde{G}\delta(X) + \epsilon_k]v_k(X) = \psi_0(X)^2 u_k(X). \tag{17b}$$

Here, $\widehat{\mathcal{L}}_j(X)$ operators are defined by Eq. (10) supplemented by $\psi_0(X)$ of Eq. (15). The system (17) can be expressed in

another form. It is given by

$$\widehat{\mathcal{L}}_1(X)[u_k(X) + v_k(X)] = \epsilon_k[u_k(X) - v_k(X)], \tag{18a}$$

$$\widehat{\mathcal{L}}_3(X)[u_k(X) - v_k(X)] = \epsilon_k[u_k(X) + v_k(X)] \tag{18b}$$

and should be considered in the two regions $-\infty < X < 0$ and $0 < X < +\infty$ and supplemented by the boundary conditions

$$u_k(0^+) = u_k(0^-), \tag{19a}$$

$$v_k(0^+) = v_k(0^-), \tag{19b}$$

$$u_k'(0^+) - u_k'(0^-) = 2\widetilde{G}u_k(0), \tag{19c}$$

$$v_k'(0^+) - v_k'(0^-) = 2\widetilde{G}v_k(0). \tag{19d}$$

Therefore, the functions $u_k(X)$ and $v_k(X)$ entering $\hat{\psi}_1(X, T)$ as in Eq. (16) are continuous with a finite jump in its first derivative at the impurity position. Note that in the case of infinite $\widetilde{G}$, the conditions (19) are replaced by $u_k(0) = v_k(0) = 0$. This arises from $\hat{\psi}(0, T) = 0$, which denotes the complete suppression of the boson density at the impurity position.

The solution of the system (18) supplemented by the boundary conditions (19) was achieved in Refs. [6,7]. Here we briefly comment the main steps and give the final results. Combining the two equations of the system (18), we obtain

$$\widehat{\mathcal{L}}_3(X)\widehat{\mathcal{L}}_1(X)[u_k(X) + v_k(X)] = \epsilon_k^2[u_k(X) + v_k(X)]. \tag{20}$$

Once the sum of $u_k(X)$ and $v_k(X)$ is determined by solving Eq. (20), its difference follows from Eq (18a). The fourth-order homogeneous differential equation (20) has four independent particular solutions, which impose the energy dispersion relation of the Bogoliubov form,

$$\epsilon_k = \sqrt{k^2 + k^4/4}. \tag{21}$$

Forming linear combinations of the four particular solutions, one can find the general solution that satisfies the boundary conditions (19). It is given by

$$u_k(X) = \left(1 + \frac{k^2}{2\epsilon_k}\right)[f(X, ik) + r(k)f(X, -ik)] + \left(1 - \frac{q^2}{2\epsilon_k}\right)r_e(k)f(X, q) + \frac{g(X, ik) + r(k)g(X, -ik) + r_e(k)g(X, q)}{\epsilon_k}, \tag{22}$$

$$v_k(X) = \left(1 - \frac{k^2}{2\epsilon_k}\right)[f(X, ik) + r(k)f(X, -ik)] + \left(1 + \frac{q^2}{2\epsilon_k}\right)r_e(k)f(X, q) - \frac{g(X, ik) + r(k)g(X, -ik) + r_e(k)g(X, q)}{\epsilon_k} \tag{23}$$

for $X < 0$ and

$$u_k(X) = \left(1 + \frac{k^2}{2\epsilon_k}\right)t(k)f(X, ik) + \left(1 - \frac{q^2}{2\epsilon_k}\right)r_e(k)f(X, -q) + \frac{t(k)g(X, ik) + r_e(k)g(X, -q)}{\epsilon_k}, \tag{24}$$

$$v_k(X) = \left(1 - \frac{k^2}{2\epsilon_k}\right)t(k)f(X, ik) + \left(1 + \frac{q^2}{2\epsilon_k}\right)r_e(k)f(X, -q) - \frac{t(k)g(X, ik) + r_e(k)g(X, -q)}{\epsilon_k} \tag{25}$$

for $X > 0$. Equations (22)–(25) assume $k > 0$. There we have introduced two auxiliary functions

$$f(x, y) = [y/2 - \text{sgn}(x)\tanh(|x| + X_0)]e^{yx}, \tag{26}$$

$$g(x, y) = \frac{ye^{yx}}{2\cosh^2(|x| + X_0)} \tag{27}$$

and the abbreviations $q = \sqrt{4 + k^2}$, $\eta = \tanh X_0$, and [7]

$$r(k) = \frac{i(1 - \eta^2)k[k^2(2\eta + q) + 4\eta(\eta^2 + 1) + q^3 + 4\eta^2 q + 2\eta q^2]}{(k - iq)[\eta k + i(\eta^2 + 1)][k^2(2\eta + q) + ik(2\eta + q)^2 - 2\eta(2\eta^2 + q^2 + 2\eta q - 2)]}, \qquad (28)$$

$$t(k) = r(k) - \frac{i - \eta k + i\eta^2}{i + \eta k + i\eta^2}, \qquad (29)$$

$$r_e(k) = \frac{4\eta(\eta^2 - 1)k}{(k - iq)[k^2(2\eta + q) + ik(2\eta + q)^2 - 2\eta(2\eta^2 + q^2 + 2\eta q - 2)]}. \qquad (30)$$

We notice the properties $|r(k)|^2 + |t(k)|^2 = 1$, $r^*(k) = r(-k)$, $t^*(k) = t(-k)$, and $r_e^*(k) = r_e(-k)$. Since our problem has a static impurity at the origin, the system is symmetric under the spatial inversion. Therefore, in addition to the solution given by Eqs. (22)–(25) there is another one where the coordinates and the momenta are reversed. It is formally given by

$$u_{-k}(X) = u_k(-X), \quad v_{-k}(X) = v_k(-X), \qquad (31)$$

where $k > 0$. The summation over negative k in $u_k(X)$ and $v_k^*(X)$ in Eq. (16) is thus defined via the relations (31).

Equations (22)–(25) provide the solution for $\hat{\psi}_1(X, T)$ [see Eq. (16)] for arbitrary X. They greatly simplify for $|X| \gg 1$. The expression for $u_k(X)$ then becomes

$$u_k(X) = \left(1 + \frac{k^2}{2\epsilon_k}\right)\left\{\left[\frac{1 - \operatorname{sgn}X}{2}r(k) - \frac{1 + \operatorname{sgn}X}{2}t(k)\right]\right.$$
$$\left.\times \left(1 - \frac{ik}{2}\right)e^{ik|X|} + \frac{1 - \operatorname{sgn}X}{2}\left(1 + \frac{ik}{2}\right)e^{-ik|X|}\right\}, \qquad (32)$$

and the function $v_k(X)$ satisfies

$$v_k(X) = \frac{2\epsilon_k - k^2}{2\epsilon_k + k^2}u_k(X). \qquad (33)$$

Equations (32) and (33) apply for $k > 0$, while for $k < 0$, one should use the prescription (31).

The normalization factor N_k in Eq. (16) is obtained from the requirement [8]

$$N_k N_q \int_{-L/2\xi_\mu}^{L/2\xi_\mu} dX[u_k(X)u_q^*(X) - v_k(X)v_q^*(X)] = \delta_{k,q}. \qquad (34)$$

In the case $L \gg \xi_\mu$ the normalization factor is given by [7]

$$N_k = (\xi_\mu/2L\epsilon_k)^{1/2}. \qquad (35)$$

Indeed, at $L \gg \xi_\mu$ it is sufficient to use $u_k(X)$ and $v_k(X)$ at $|X| \gg 1$ that are given by Eqs. (32) and (33). Substituting them into the condition (34), one obtains Eq. (35).

C. Second quantum correction

The mean-field solution (15) is time independent. Since it enters the quantum contribution to the density as the term $\langle\psi_0(X)\hat{\psi}_2(X, T)\rangle + \langle\psi_0(X)\hat{\psi}_2^\dagger(X, T)\rangle$, for our purpose of evaluating the mean density (2) it is sufficient to consider Eq. (12) averaged with respect to the ground state. Additional

simplification arises as a consequence of the mean-field solution being a real function. For the purpose of the density that is real, we thus need only the real part of the expectation value of the averaged operator. Introducing

$$\psi_2(X) = \operatorname{Re}\langle\hat{\psi}_2(X, T)\rangle, \qquad (36)$$

we find that it satisfies

$$[\widehat{\mathcal{L}}_3(X) + \widetilde{G}\delta(X)]\psi_2(X) = h(X), \qquad (37)$$

where the source term is given by

$$h(X) = -\psi_0(X)\big[2\langle\hat{\psi}_1^\dagger(X, T)\hat{\psi}_1(X, T)\rangle + \operatorname{Re}\langle\hat{\psi}_1^2(X, T)\rangle\big]. \qquad (38)$$

On the left-hand side of Eq. (38) we omitted the time dependence. This is a consequence of the form (16) for $\hat{\psi}_1(X, T)$, yielding the time-independent expression

$$h(X) = \frac{\psi_0(X)}{2}\sum_{k;|k|>\lambda}N_k^2\big[u_k(X)v_k^*(X) + u_k^*(X)v_k(X)$$
$$- 4|v_k(X)|^2\big]. \qquad (39)$$

Note that the summation over real k must be performed carefully, taking into account the condition $|k| > \lambda$, where $\lambda > 0$ is a small cutoff. We have verified that the summand diverges as $1/k$ at $k \to 0$. Another comment concerns the time independence of the source term $h(Y)$, which enabled us to introduce the form of Eq. (36), since $\langle\partial_T\hat{\psi}_2(X, T)\rangle = 0$ in this case [9].

The expression for $\psi_2(X)$ valid at arbitrary X is complicated and is given in the Appendix. Instead of analyzing it, let us find $\psi_2(X)$ at $X \gg 1$ using an alternative method. To do this we notice that $\psi_2(X)$ satisfies the exact relation

$$\frac{d}{dX}\psi_2(X) + 2\tanh(X + X_0)\psi_2(X)$$
$$= 2\cosh^2(X + X_0)\int_X^\infty dY\frac{h(Y)}{\cosh^2(Y + X_0)} \qquad (40)$$

for $X > 0$. Indeed, acting on Eq. (40) with the operator d/dX, we directly show that it reduces to Eq. (37) in the case with $X > 0$ [i.e., to Eq. (37) without the δ-function term]. At $X \gg 1$, the integrand on the right-hand side of Eq. (40) further simplifies as we can use

$$\frac{1}{\cosh^2(Y + X_0)} = 4e^{-2(Y + X_0)} + O(e^{-4Y}). \qquad (41)$$

Equations (32) and (33) enable us to calculate $h(Y)$ of Eq. (39) at $Y \gg 1$, which becomes a sum of exponential functions of Y. The resulting integral in Eq. (40) can then easily be evaluated. Finally, solving the differential equation (40), where on the left-hand side we approximate $\tanh(X + X_0)$ by 1, we find

$$\psi_2(X) = \sum_{k>\lambda} N_k^2 (k\sqrt{4 + k^2} - k^2 - 1)$$
$$\times \left[1 - \frac{r(k)}{2} \frac{k + 2i}{k - 2i} \frac{e^{2ikX}}{k^2 + 1} - \frac{r^*(k)}{2} \frac{k - 2i}{k + 2i} \frac{e^{-2ikX}}{k^2 + 1} \right]$$
$$+ O(e^{-2X}), \tag{42}$$

which is a solution at $X \gg 1$. The term containing the sum on the right-hand side of Eq. (42) arises as a particular solution of the differential equation and does not have the integration constant. On the other hand, the integration constant affects the second term. However, it is exponentially suppressed and thus very small at $X \gg 1$.

IV. THE DENSITY PROFILE OF THE BOSE GAS

The results of the previous section enable us to calculate the density profile of a weakly interacting Bose gas in the presence of an impurity coupled to the system by an arbitrary coupling strength. Using Eq. (4) and the series (8), the local density of the Bose gas (2) can be expressed as

$$n(x) = \frac{\mu}{g} \left\{ |\psi_0(X)|^2 + \alpha^2 [\langle \hat{\psi}_1^\dagger(X, T) \hat{\psi}_1(X, T) \rangle \right.$$
$$\left. + 2\psi_0(X)\psi_2(X)] + O(\alpha^3) \right\}_{X=x/\xi_\mu}. \tag{43}$$

The density expressed as in Eq. (43) depends on the chemical potential of the Bose gas μ that was kept fixed in the equations of motion [see Eq. (6)]. It is more convenient to eliminate μ and instead use the mean density n_0 in $n(x)$. In addition, instead of x/ξ_μ, which depends on μ, we will express the density in terms of x/ξ, where $\xi = \hbar/\sqrt{mgn_0}$ denotes the healing length. We thus need the expression for the chemical potential of a weakly interacting Bose gas that is given by [10] $\mu = gn_0[1 - \sqrt{\gamma}/\pi + O(\gamma)]$. After we express μ in terms of n_0, Eq. (43) takes the form [11]

$$n(x) = n_0[n^{(0)}(X) + \sqrt{\gamma} \, n^{(1)}(X) + O(\gamma)]_{X=x/\xi}. \tag{44}$$

The first term in Eq. (44) is given by

$$n^{(0)}(X) = |\psi_0(X)|^2 = \tanh^2(|X| + X_0) \tag{45}$$

and represents the mean-field contribution. The shift X_0 is related to the impurity coupling by the relation (15b). The second term in Eq. (44) has the form

$$n^{(1)}(X) = \langle \hat{\psi}_1^\dagger(X, T) \hat{\psi}_1(X, T) \rangle + 2\psi_0(X)\psi_2(X)$$
$$- \frac{1}{\pi} |\psi_0(X)|^2 - \frac{X}{2\pi} \frac{d}{dX} |\psi_0(X)|^2. \tag{46}$$

The latter expression with ψ_0 and $\hat{\psi}_1$ given, respectively, by Eqs. (15) and (16) and the expression for $\psi_2(X)$ given in the Appendix define the quantum contribution to the density $n^{(1)}(X)$ at all distances. Taking the average value, Eq. (46)

becomes

$$n^{(1)}(X) = 2\psi_0(X)\psi_2(X) - \frac{1}{\pi} |\psi_0(X)|^2 - \frac{X}{2\pi} \frac{d}{dX} |\psi_0(X)|^2$$
$$+ \sum_{k>\lambda} N_k^2 [|v_k(X)|^2 + |v_k(-X)|^2]. \tag{47}$$

Note that unlike the terms $\langle \hat{\psi}_1^\dagger(X, T) \hat{\psi}_1(X, T) \rangle$ and $2\psi_0(X)\psi_2(X)$ in Eq. (46), which both diverge at small k, thus requiring the small cutoff λ [see Eq. (42)], their sum is finite. Therefore, on the right-hand side of Eq. (47), one can set $\lambda = 0$ once all the terms are grouped under the same sum. This is physically expected since the density is an observable quantity that should not have any divergence in our system.

Analytical evaluation of Eq. (47) for an arbitrary X, which reduces to the integration of elementary functions, is rather complicated. One should first perform the integration over Y in the expression for $\psi_2(X)$ given in the Appendix, followed by the final summation over k (which can be converted to the integral). The actual procedure was previously performed for the special case $\widetilde{G} \to \infty$ in Ref. [12]. There it was shown that $n^{(1)}(X)$ has fundamentally important contributions only at $X \gg 1$ because it is a long-range function. Since the mean-field contribution at $X \gg 1$ rapidly reaches the constant, the leading-order result for the density deviation $n_0 - n(X)$ is controlled by $n^{(1)}(X)$. At $X \lesssim 1$, the quantum contribution $n^{(1)}(X)$, which is multiplied by small $\sqrt{\gamma}$, contains only a small correction to the mean-field result.

Following these remarks, let us evaluate Eq. (47) at $X \gg 1$ considering the case of arbitrary $\widetilde{G}$. Substitution of Eqs. (32)–(42) yields

$$n^{(1)}(X) = - \left[\int_0^\infty \frac{dk}{4\pi} r(k) \frac{k + 2i}{k - 2i} \left(\frac{1 - k^2}{1 + k^2} + \frac{k}{\sqrt{4 + k^2}} \right) \right.$$
$$\left. \times e^{2ikX} + \text{c.c.} \right]. \tag{48}$$

Here c.c. denotes a term that is complex conjugate to the integral. Equation (48) represents the quantum contribution to the mean density at distances $X \gg 1$. Note that the constant term $-1/\pi$ that arises from the second term on the right-hand side of Eq. (47) is canceled by an elementary integral over k. The remaining terms are given by Eq. (48).

The asymptotic series expansion for the integral in Eq. (48) at $X \gg 1$ can be evaluated by the method of partial integration. We found that one part of it, $\int_0^\infty \frac{dk}{4\pi} r(k) \frac{k+2i}{k-2i} \frac{1-k^2}{1+k^2} e^{2ikX} +$ c.c., does not have the power-law expansion. More precisely, we obtained a zero coefficient in front of all powers of $1/X$; we checked numerically and found that the integral is proportional to e^{-2X}, which explains the absence of the asymptotic series. The other part of Eq. (48) contains the square root in the integrand and has power-law contributions. We found

$$\int_0^\infty \frac{dk}{4\pi} r(k) \frac{k + 2i}{k - 2i} \frac{k}{\sqrt{4 + k^2}} e^{2ikX} = \frac{r(0)}{32\pi X^2} + \frac{r(0) + ir'(0)}{32\pi X^3}$$
$$+ \frac{3[5r(0) + 8ir'(0) - 4r''(0)]}{512\pi X^4} + O(X^{-5}). \tag{49}$$

The integral (49) does not diverge at large values of k since $r(k)$ decays sufficiently rapidly in this limit. At $X \gg 1$, it

is thus controlled by the reflection amplitude (28) and its derivatives at $k \to 0$. They are given by

$$r(0) = 0, \qquad (50)$$

$$r'(0) = -\frac{i}{2}\left(\widetilde{G} + \frac{2 + \widetilde{G}^2}{\sqrt{4 + \widetilde{G}^2}} - 1\right), \qquad (51)$$

$$r''(0) = (\widetilde{G} - 1)(\widetilde{G} + \sqrt{4 + \widetilde{G}^2}) - \frac{2}{4 + \widetilde{G}^2} + \frac{5}{2}. \qquad (52)$$

Therefore, for the leading-order result at $X \gg 1$ we obtain

$$n^{(1)}(X) = -\frac{1}{32\pi}\left(\widetilde{G} + \frac{2 + \widetilde{G}^2}{\sqrt{4 + \widetilde{G}^2}} - 1\right)\frac{1}{|X|^3}, \qquad (53)$$

where at the very end we have introduced the absolute value since $n^{(1)}(X)$ is an even function that we have studied so far for positive X. Equation (53) applies for arbitrary $\widetilde{G}$ and should be understood as the first term of the asymptotic expansion of $n^{(1)}(X)$ at $|X| \gg 1$. The mathematical divergence that occurs at $\widetilde{G} \to \infty$ leads to the infinite result, which is unphysical. One should keep in mind that X is assumed to be the largest parameter in Eq. (53), and thus, the operation $\widetilde{G} \to \infty$ is not allowed there. However, $n^{(1)}(X)$ can be easily calculated at very large $\widetilde{G}$. To do so it suffices first to take the limit $\widetilde{G} \to \infty$, which imposes $r(k) = 1$. Substituting this in Eq. (49), we obtain the leading-order result

$$n^{(1)}(X) = -\frac{1}{16\pi X^2}. \qquad (54)$$

A comparison of Eqs. (53) and (54) shows that at $\widetilde{G} \gg 1$, there is a crossover at distances $X \sim \widetilde{G}$. At $1 \ll X \ll \widetilde{G}$, we obtain the result (54), while at $1 \ll \widetilde{G} \ll X$, Eq. (53) applies. Interested readers can easily evaluate the subleading power-law corrections to Eqs. (53) and (54) using Eq. (49). We note that in $n^{(1)}(X)$ we have neglected the terms that decay exponentially since they are always subleading. We also note that the obtained result (54) with the corresponding corrections is in agreement with the result of Ref. [12], where $n^{(1)}(X)$ was calculated analytically for arbitrary X.

The crossover between results (53) and (54) can be described analytically. To do so we need the expression for the reflection amplitude (28) at $k \sim 1/\widetilde{G} \ll 1$, which is given by

$$r(k) = \frac{k}{k + i/\widetilde{G}}. \qquad (55)$$

Equation (55) interpolates between $r(k) = -i\widetilde{G}k$ at $k \ll 1/\widetilde{G}$ and $r(k) = 1$ at $\widetilde{G} \gg 1/k$. Then we can find the result

$$n^{(1)}(X) = -\frac{1}{16\pi}\frac{\partial^2}{\partial X^2}\int_0^\infty dy \frac{e^{-2Xy}}{y + 1/\widetilde{G}}$$

$$= -\frac{1}{16\pi X^2} + \frac{1}{8\pi\widetilde{G}X} + \frac{e^{2X/\widetilde{G}}\text{Ei}(-2X/\widetilde{G})}{4\pi\widetilde{G}^2}. \qquad (56)$$

Here $X > 0$, and $\text{Ei}(x) = -\int_{-x}^\infty dt\, e^{-t}/t$. At $X \gg \widetilde{G}$, Eq. (56) becomes $n^{(1)}(X) = -\widetilde{G}/16\pi X^3$, while in the opposite case $\widetilde{G} \gg X$, it is given by $n^{(1)}(X) = -1/16\pi X^2$. This is in agreement with Eqs. (53) and (54).

V. DISCUSSION

Being perturbed by the presence of an impurity, the local density $n(x)$ of a weakly interacting Bose gas reaches its mean thermodynamic-limit value n_0 at distances $|x| \to \infty$. Within the mean-field approximation, the deviation of the density at large separations from the impurity is an exponentially small function of the distance,

$$n_0 - n(x) = \frac{4\widetilde{G}n_0}{2 + \sqrt{4 + \widetilde{G}^2}}e^{-2|x|/\xi}. \qquad (57)$$

This means that in practice, the local density $n(x)$ saturates very quickly to its mean value, which occurs at distances x of several healing lengths ξ (see Fig. 1). This scenario drastically changes once the effect of quantum fluctuations is taken into account. In this case the density deviation saturates in a much slower fashion, $n_0 - n(x) = -n_0\sqrt{\gamma}\, n^{(1)}(x/\xi)$ at $|x| \gg \xi$. The functional dependence $n^{(1)}(x/\xi)$ is calculated analytically in this paper; see Eqs. (53), (54), and (48) for a more general expression. It has the power-law scaling behavior with the leading-order terms

$$n^{(1)}(x/\xi) \propto \begin{cases} \xi^2/x^2, & \xi \ll |x| \ll \widetilde{G}\xi, \\ \xi^3/|x|^3, & \widetilde{G}\xi \ll |x|, \end{cases} \qquad (58)$$

in the regime of strongly coupled impurity, $\widetilde{G} \gg 1$. In this case the impurity coupling imposes a new length scale in the system, $\widetilde{G}\xi$, which is much longer than the healing length ξ. Between the two lengths there is a wide region where the density deviation shows a decay with the universal, impurity-independent amplitude [see Eq. (54)]. In the limiting case of $\widetilde{G} \to \infty$, we notice the suppression of the regime with inverse cubic scaling. On the contrary, for an impurity weakly coupled to the Bose gas, $\widetilde{G} \ll 1$, only the latter regime with $n^{(1)}(x/\xi) \propto \xi^3/|x|^3$ is realized.

Why should one be interested in studying the deviation of the local density from its mean value beyond the mean-field result? First, this is a fundamental question and should be understood. Second, $n^{(1)}(x/\xi)$ is physically significant because it controls the induced interaction between impurities in the Bose gas, which we will derive now. Consider our original system consisting of a Bose gas with the impurity at the origin that is locally coupled to the gas described by the Hamiltonian $\hat{H}$ of Eq. (1), and let us introduce another impurity at position x. The new system is described by the Hamiltonian

$$\hat{\mathcal{H}} = \hat{H} + G_1\hat{\Psi}^\dagger(x, t)\hat{\Psi}(x, t). \qquad (59)$$

Applying the Hellmann-Feynman theorem, $\langle \partial\hat{\mathcal{H}}/\partial G_1\rangle_{\hat{\mathcal{H}}} = \partial E/\partial G_1$, we obtain the relation

$$\langle \hat{\Psi}^\dagger(x, t)\hat{\Psi}(x, t)\rangle_{\hat{\mathcal{H}}} = \frac{\partial E}{\partial G_1}, \qquad (60)$$

which connects the local density for the system described by the Hamiltonian (59) and the derivative of its ground-state energy E. By $\langle \cdots \rangle_{\hat{\mathcal{H}}}$ we denote the average with respect to the ground state of the Hamiltonian $\hat{\mathcal{H}}$. For the impurity at position x that is weakly coupled to the gas, the left-hand side of Eq. (60) can be understood as the local density [see Eq. (44)] for the system described by the Hamiltonian $\hat{H}$. Indeed, the

effect of the newly added impurity with weak coupling G_1 on the density depletion is negligible. The derivative on the right-hand side of Eq. (60) nontrivially acts only on the part of the ground-state energy that depends on G_1. It involves two terms. One is x independent and represents the impurity binding energy, and the other is the Bose-medium-induced interimpurity interaction $U(x)$. The latter can be extracted by integrating Eq. (60) and is given by $U(x) = G_1\langle n(x)\rangle - G_1\langle n(x \to \infty)\rangle$, where we have chosen the integration constant to obtain the physically expected absence of the interimpurity interaction at infinite separations. Since $\langle n(x \to \infty)\rangle = n_0$, we arrive at the expression

$$U(x) = U^c(x) + U^\varrho(x), \tag{61}$$

where the classical $U^c(x)$ and quantum $U^\varrho(x)$ contributions are given by

$$U^c(x) = G_1 n_0 [n^{(0)}(x/\xi) - 1], \tag{62}$$

$$U^\varrho(x) = G_1 n_0 \sqrt{\gamma}\, n^{(1)}(x/\xi). \tag{63}$$

At short separations, $x \lesssim \xi$, the mean-field term $n^{(0)}(x/\xi)$ in the local density (44) makes dominant the classical contribution (62) in the induced interaction, $U(x) \approx U^c(x)$. However, at larger distances, $x \gg \xi$, the quantum contribution to the density $n^{(1)}(x/\xi)$ prevails, which makes the quantum contribution (63) dominant, $U(x) \approx U^\varrho(x)$.

In several special cases, Eqs. (61)–(63) reduce to the known results. For example, at weak and equal coupling constants, $G_1 = G \ll g/\sqrt{\gamma}$, Eqs. (62) and (63) reduce, respectively, to the results (3) and (4) of Ref. [9]. In the case of induced interaction between an impurity infinitely strongly coupled to the gas, $\widetilde{G} \to \infty$, and another one that is weakly coupled, Eqs. (61)–(63) become the results (22) and (23) of Ref. [13]. On the other hand, the present results (61)–(63) are more general as they describe all the regimes with arbitrary coupling G and weak G_1. We eventually note that the case with equal but arbitrary coupling, $G = G_1$, was studied in Ref. [14].

In this paper, a weakly interacting Bose gas is treated by a method that leads to the quasiparticle excitation spectrum described by the Bogoliubov dispersion (21). This picture is valid at (dimensionful) momenta higher than $\hbar\gamma^{1/4}/\xi$ [15]. Namely, at the lowest momenta, the dispersion (21) and the true quasiparticle dispersion agree only in the leading order; the subleading term of Eq. (21) starts with $|k|^3$, unlike the true dispersion that has a k^2 term. Therefore, at distances much larger than $\xi\gamma^{-1/4}$ the power-law decay of $n^{(1)}(x/\xi)$ [see Eq. (58)] might be affected. At such longest distances, however, the quantum contribution of the induced interaction between the impurities is known to behave as a power law, $U^\varrho(x) = -mv^2\Gamma(G)\Gamma(G_1)\xi^3/32\pi|x|^3$ [16]. Here $v = \pi\hbar n_0/mK$ is the sound velocity, and the dimensionless parameter $\Gamma(G)$ denotes the so-called impurity-phonon scattering amplitude. At weak coupling we have $\Gamma(G) = -G/\hbar v$ [16]. Using Eq. (63), we then find the expression for $n^{(1)}(x/\xi)$ that is in agreement with our result (53) taken at the leading order in $\widetilde{G} \ll 1$. We cannot discuss the case of strong impurity coupling since we do not know $\Gamma(G)$.

The conventional treatment of the low-energy properties of one-dimensional quantum liquids is via the Luttinger liquid theory. The system described by the Hamiltonian (1) can be understood as a quantum liquid, and thus, one may be tempted to use the latter theory. It generically predicts the Friedel oscillations of the particle density around its mean with the envelope function behaving as $|x|^{-K}$ at distances from the impurity longer than $1/n_0$ [17]. Here the Luttinger liquid parameter for weakly interacting bosons is given by $K = \pi/\sqrt{\gamma} \gg 1$. Such rapid saturation of density oscillations is a negligibly small effect in weakly interacting Bose gases. On the contrary, we found that the local density $n(x)$ slowly approaches the mean density n_0 [see, for example, Eqs. (44) and (58)]. This should be contrasted with the Friedel oscillations that exist in the case of a strongly interacting Bose gas. The latter system has an effective description in terms of weakly attractive fermions. In the limit $\widetilde{G}, \gamma \to \infty$, we find the exact result [18]

$$n(x) = n_0 - \frac{\sin(2\pi n_0 x)}{2\pi x}. \tag{64}$$

In the present case we have $K = 1$, and thus, the envelope of the density decay in Eq. (64) behaves according to the generic Luttinger liquid prediction. The disappearance of the oscillations in the density (64) as the value of γ is decreased and the formation of the hole in the density near the impurity drawn in Fig. 1 would be interesting to study.

This paper calls for a study of the density depletion decay in higher-dimensional Bose systems with an impurity. A phase-space argument gives the expectation for the induced interimpurity interaction $U(x) \propto 1/|x|^{2d+1}$ at the longest distances in a d-dimensional system [16], which is in agreement with the detailed calculation of Ref. [19] performed for $d = 3$. Our previous argument based on the Hellmann-Feynman theorem implies the same decay law for the density depletion as the one for $U(x)$. More detailed consideration of this problem is left for future work. Another interesting direction for future research is the study of the density of an interacting system of bosons in one dimension using the techniques of integrable models. In the case $\widetilde{G} \to +\infty$, the system describes interacting bosons in the box potential, which admits an exact solution. We are not, however, aware of any study of the density profile of the latter system at arbitrary interaction.

APPENDIX: SOLUTION OF EQUATION (37)

Equation (37) is a linear second-order differential equation with a singular term at the origin. We consider it at $X \neq 0$. We denote

$$\psi_2(X) = \begin{cases} \psi_2^-(X), & -\infty < X < 0, \\ \psi_2^+(X), & 0 < X < +\infty. \end{cases} \tag{A1}$$

Equation (37) then takes the form

$$\widehat{\mathcal{L}}_3^\pm(X)\psi_2^\pm(X) = h(X), \tag{A2}$$

where

$$\widehat{\mathcal{L}}_3^\pm(X) = -\frac{\partial_X^2}{2} + 3\tanh^2(X \pm X_0) - 1. \tag{A3}$$

The homogeneous equation corresponding to Eq. (A2) is defined by setting $h(X) = 0$. This second-order equation has two solutions that are given by

$$U^{\pm}(X) = \text{sech}^2(X \pm X_0), \tag{A4a}$$

$$V(X)^{\pm} = \text{sech}^2(X \pm X_0)g(X \pm X_0), \tag{A4b}$$

where

$$g(X) = \frac{12X + 8\sinh(2X) + \sinh(4X)}{16}. \tag{A5}$$

Note that $g'(X) = 2\cosh^4 X$.

Using the method of variation of parameters, we can construct the solution of Eq. (A2):

$$\psi_2^-(X) = \left[c_3 - \int_X^0 dY V^-(Y)h(Y) \right] U^-(X)$$
$$+ \left[c_4 + \int_X^0 dY U^-(Y)h(Y) \right] V^-(X), \tag{A6}$$

$$\psi_2^+(X) = \left[c_1 + \int_0^X dY V^+(Y)h(Y) \right] U^+(X)$$
$$+ \left[c_2 - \int_0^X dY U^+(Y)h(Y) \right] V^+(X). \tag{A7}$$

Here c_1, c_2, c_3, and c_4 are the constants that are to be determined from boundary conditions. Noticing that $V^{\pm}(\pm\infty)$ is divergent and requiring a finite value of $\psi_2^{\pm}(\pm\infty)$, we are able to set

$$c_2 = \int_0^{+\infty} dY U^+(Y)h(Y), \tag{A8}$$

$$c_4 = -\int_{-\infty}^0 dY U^-(Y)h(Y). \tag{A9}$$

The remaining two constants are determined by the boundary conditions

$$\psi_2^-(0) = \psi_2^+(0), \tag{A10}$$

$$\left. \left(\frac{d\psi_2^+(X)}{dX} - \frac{d\psi_2^-(X)}{dX} \right) \right|_{X=0} = 2\widetilde{G}\psi_2^+(0). \tag{A11}$$

This leads to

$$c_1 = -g(X_0) \int_0^{+\infty} dY U^+(Y)h(Y)$$
$$+ \frac{g'(X_0)}{2(\widetilde{G} + 2\tanh X_0)} \int_{-\infty}^{+\infty} U(Y)h(Y), \tag{A12}$$

$$c_3 = -g(X_0) \int_{-\infty}^0 dY U^-(Y)h(Y)$$
$$+ \frac{g'(X_0)}{2(\widetilde{G} + 2\tanh X_0)} \int_{-\infty}^{+\infty} U(Y)h(Y). \tag{A13}$$

Here $U(Y) = U^-(Y)$ for $Y < 0$, and $U(Y) = U^+(Y)$ for $Y > 0$.

The latter solution simplifies in our case since $h(Y)$ is an even function, $h(Y) = h(-Y)$, which follows from Eqs. (39) and (31). Then we have two independent constants, e.g., c_1 and c_2, since $c_3 = c_1$ and $c_4 = -c_2$. Moreover, c_1 simplifies and becomes

$$c_1 = \frac{1}{4}\left(\frac{4 + 3\widetilde{G}^2}{\widetilde{G}^2\sqrt{4 + \widetilde{G}^2}} - 3X_0 \right) \int_0^{+\infty} dY U^+(Y)h(Y). \tag{A14}$$

Substituting Eqs. (A14) and (A8) into Eq. (A7) gives $\psi_2(X)$, which is the solution of Eq. (37) at $X > 0$. The solution at $X < 0$ follows from the parity of $\psi_2(X)$.

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[1] C. L. Kane and M. P. A. Fisher, Transmission through barriers and resonant tunneling in an interacting one-dimensional electron gas, Phys. Rev. B **46**, 15233 (1992).

[2] M. Fabrizio and A. O. Gogolin, Interacting one-dimensional electron gas with open boundaries, Phys. Rev. B **51**, 17827 (1995).

[3] R. Egger and H. Grabert, Friedel oscillations for interacting fermions in one dimension, Phys. Rev. Lett. **75**, 3505 (1995).

[4] Note that such a system is rather realistic because it can be realized experimentally [20,21].

[5] Note that Eq. (15) is also the nodeless solution of Eq. (9) in the case with $\tilde{G} < 0$. In this case the solution is nonsingular (normalizable) at any attraction apart from the extreme case $\tilde{G} \to -\infty$.

[6] D. L. Kovrizhin, Exact form of the Bogoliubov excitations in one-dimensional nonlinear Schrödinger equation, Phys. Lett. A **287**, 392 (2001).

[7] A. Petković, Local spectral density of an interacting one-dimensional Bose gas with an impurity, Phys. Rev. A **105**, 043305 (2022).

[8] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, Oxford, 2003).

[9] B. Reichert, Z. Ristivojevic, and A. Petković, The Casimir-like effect in a one-dimensional Bose gas, New J. Phys. **21**, 053024 (2019).

[10] V. N. Popov, Theory of one-dimensional Bose gas with point interaction, Theor. Math. Phys. **30**, 222 (1977).

[11] Below this point the variable X does not have the meaning as in Eq. (3), but rather as x/ξ. For simplicity we decided to keep the same notation, especially since the notion of X is mathematically irrelevant in many expressions apart from its physical importance for Eq. (44).

[12] A. Petković, B. Reichert, and Z. Ristivojevic, Density profile of a semi-infinite one-dimensional Bose gas and bound states of the impurity, Phys. Rev. Res. **2**, 043104 (2020).

[13] B. Reichert, A. Petković, and Z. Ristivojevic, Fluctuation-induced potential for an impurity in a semi-infinite one-dimensional Bose gas, Phys. Rev. B **100**, 235431 (2019).

[14] A. Petković and Z. Ristivojevic, Mediated interaction between polarons in a one-dimensional Bose gas, Phys. Rev. A **105**, L021303 (2022).

[15] M. Pustilnik and K. A. Matveev, Low-energy excitations of a one-dimensional Bose gas with weak contact repulsion, Phys. Rev. B **89**, 100504(R) (2014).

[16] M. Schecter and A. Kamenev, Phonon-mediated Casimir interaction between mobile impurities in one-dimensional quantum liquids, Phys. Rev. Lett. **112**, 155301 (2014).

[17] M. A. Cazalilla, Low-energy properties of a one-dimensional system of interacting bosons with boundaries, Europhys. Lett. **59**, 793 (2002).

[18] D. Sen, The fermionic limit of the δ-function Bose gas: A pseudopotential approach, J. Phys. A **36**, 7517 (2003).

[19] K. Fujii, M. Hongo, and T. Enss, Universal van der Waals force between heavy polarons in superfluids, Phys. Rev. Lett. **129**, 233401 (2022).

[20] J. Catani, G. Lamporesi, D. Naik, M. Gring, M. Inguscio, F. Minardi, A. Kantian, and T. Giamarchi, Quantum dynamics of impurities in a one-dimensional Bose gas, Phys. Rev. A **85**, 023623 (2012).

[21] F. Meinert, M. Knap, E. Kirilov, K. Jag-Lauber, M. B. Zvonarev, E. Demler, and H.-C. Nägerl, Bloch oscillations in the absence of a lattice, Science **356**, 945 (2017).