

Partially topological phase in a quantum loop gas model with tension and pressureJ. Abouie^{1,*} and M. H. Zarei^{2,†}¹*Department of Physics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran*²*Department of Physics, School of Science, Shiraz University, Shiraz 71946-84795, Iran*

(Received 23 May 2023; revised 4 October 2023; accepted 6 October 2023; published 18 October 2023)

Enhancing robustness of topological orders against perturbations is one of the main goals in topological quantum computing. Since the kinetic of excitations is in conflict with the robustness of topological orders, any mechanism that reduces the mobility of excitations will be in favor of robustness. A strategy in this direction is adding frustration to topological systems. In this paper we consider a frustrated toric code on a kagome lattice, and show that although increasing the strength of perturbation reduces the topological order of the system, it cannot destroy it completely. Our frustrated toric code is indeed a quantum loop gas model with string tension and pressure which their competition leads to a partially topological phase (PTP) in which the excitations are restricted to move in particular sublattices. In this phase the ground state is a product of many copies of fluctuating loop states corresponding to quasi-one-dimensional ladders. By defining a nonlocal matrix order parameter and studying the behavior of ground-state global entanglement, we distinguish the PTP from the standard topological phase. The partial mobility of excitations in our system is reminiscent of fracton codes with restricted mobility, and therefore our results propose an alternative way for making such a restriction in three dimensions.

DOI: [10.1103/PhysRevB.108.165133](https://doi.org/10.1103/PhysRevB.108.165133)**I. INTRODUCTION**

One of the mainstreams of research in condensed matter physics and quantum computing is searching for topological phases of matter which are beyond the Landau-Ginzburg symmetry-breaking paradigm [1–6]. In topological states, the ground-state degeneracy is protected by topological properties [7–9], and the excitations are anyonic quasiparticles displaying a fractional statistics intermediate between bosons and fermions [10]. Regarding these interesting features, topological quantum systems are considered for quantum computing [11], due to their robustness against perturbations.

Toric code is an exactly solvable topological quantum system [12–15] which is used as a robust quantum memory [16–20]. The ground state of the toric code is a superposition of closed strings and is called a fluctuating loop state or a quantum loop gas [5]. There are four degenerate ground states for the toric code on torus which are distinguished by different topological classes of loops and are robust against local perturbations [21–24]. However, such a robustness is not unlimited in the sense that increasing perturbations finally lead to a phase transition out of the topological phase, and moving excitations in the system destroy the topological order [25,26].

Regarding the great importance of robustness of the topological quantum codes in their application as quantum memories, providing a mechanism to protect them against perturbations is an important task. An example is a toric code

with random couplings in the presence of a magnetic field where excitations are localized due to the disorder inherent in the code [27–30]. Fracton codes in three dimensions (3D) are another example [31] where mobility of excitations are limited due to quantum glassiness [32]. Introducing frustration to the toric code model in an external magnetic field can also lead to robustness by reducing the mobility of excitations [33]. It has been shown that the interplay between frustration and topological order leads to a rich phase diagram in topological systems [34].

In this paper we consider a frustrated quantum loop gas model with tension and pressure, and show that, in a wide region in the ground-state phase diagram, local perturbations reduce the topological order of the toric code, but cannot destroy it, completely. Our study is based on the mappings between topological quantum codes and classical spin models [35,36] in which topological phase transitions in the quantum models correspond to thermal phase transitions in classical models [37,38]. Recently, it has been demonstrated that the combined effects of string tension and pressure lead to a topological line with infinite robustness against perturbations [39]. Our quantum loop gas model is a frustrated toric code on a kagome lattice. We investigate the ground-state phase diagram of our toric code model by mapping it onto a frustrated Ising model. We show that there is a quantum phase in the ground-state phase diagram with a topological order different from the standard topological phases. In particular, we show that in this phase, dubbed “partially topological phase (PTP),” the ground state is a superposition of a partial set of possible loops. We next describe this phase in terms of the kinetic of excitations, and show that there is a partial restriction on the mobility of excitations in the PTP so that

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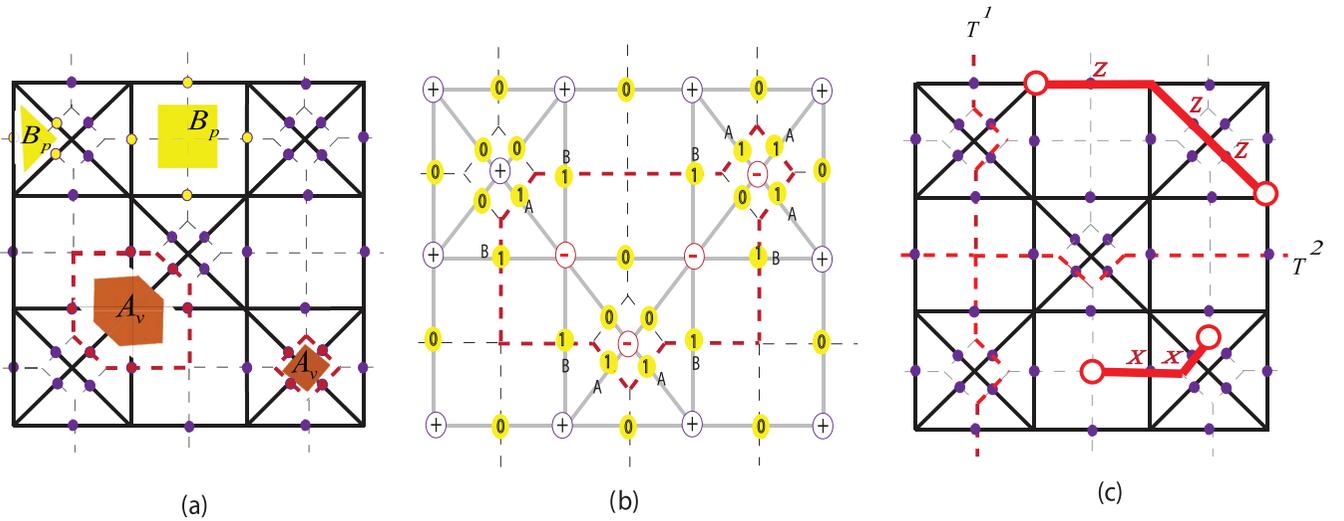


FIG. 1. (a) Toric code on a kagome lattice. Qubits live on edges in the lattice. There are two types of plaquette operator B_p corresponding to the triangular and square plaquettes, and two types of vertex operator which are represented by hexagonal and square loops in the dual lattice. (b) Classical spins in Ising model live on vertices in the kagome lattice. A particular configuration of spins in Ising model can be represented by a loop in the dual lattice where spins inside (outside) the loop are -1 ($+1$). Such a loop configuration is equivalent to a loop state in the toric code where qubits living on the loop are in the state $|1\rangle$ and others are in the state $|0\rangle$. (c) An illustration of two noncontractible loops T^1 (T^2) around the torus which generate other degenerate ground states of the toric code. By applying Z (X) operator on a string in the lattice (dual lattice), excitations are generated at the two end points of the string.

they are forced to move in particular sublattices. In order to characterize such a topological phase, we define a nonlocal matrix order parameter which shows the differences between this PTP and other standard topological phases. We consider a multipartite measure of entanglement which reflects highly entangled nature of topological phases. In particular, we show that global entanglement is able to identify different phases of our frustrated system as well as phase transition points. Importantly, the value of global entanglement in the PTP confirms the semitopological nature of this phase.

The structure of this paper is as follows: In Sec. II, we give a brief review on the toric code as a simple model of a quantum loop gas. In Sec. III, we introduce a simple example of a frustrated toric code model and give a qualitative argument as to why we expect there is a PTP. Next, we give an exactly solvable version of the above frustrated quantum loop gas and present the phase diagram of model. We show that in the PTP, the ground state is a product of many copies of fluctuating loop states corresponding to ladders on the lattice. In Sec. IV, we define a matrix order parameter for characterizing the PTP. We also show that in the PTP, there is a partial restriction on mobility of excitations. Finally, in Sec. V, we investigate the behavior of global entanglement (GE), and show that it can distinguish the PTP from topological and trivial phases.

II. TORIC CODE MODEL AS QUANTUM LOOP GAS

In this section, we provide a review on the important topological properties of the toric code model as a quantum loop gas. The toric code model is defined on an arbitrary oriented two-dimensional lattice. Here, we consider a kagome lattice with qubits living on edges in the lattice. The Hamiltonian is

defined as

$$H_0 = - \sum_p B_p - \sum_v A_v, \quad (1)$$

where the plaquette operator $B_p = \prod_{i \in p} Z_i$ is a tensor product of the Pauli operators Z_i corresponding to the i th qubit around the plaquette p , and the vertex operator $A_v = \prod_{i \in v} X_i$ is a tensor product of the Pauli operators X_i corresponding to the i th qubit incoming to the vertex v [see Fig. 1(a)]. The ground state of H_0 is simply obtained as the operators B_p and A_v are commuted. Consider a product state where all qubits in the lattice are in the state $|0\rangle$, an eigenstate of the operator Z with eigenvalue $+1$. Then, flip the qubits corresponding to a loop configuration in the dual lattice to the state $|1\rangle$ [see Fig. 1(b)]. A loop state is defined as $|\text{loop}\rangle = \prod_{i \in \text{loop}} |1\rangle_i$. Clearly $B_p |\text{loop}\rangle = |\text{loop}\rangle$, meaning that loop states are stabilized states of B_p . On the other hand, each vertex operator also corresponds to a small loop in the dual lattice. Therefore, applying a vertex operator on the state $|\text{loop}\rangle$ generates another loop state corresponding to a different loop configuration. In this regard, the ground state $|G\rangle$ must be a superposition of all loop states, i.e., a loop condensed state. In particular, if one considers each loop configuration as a nonlocal object, the operator A_v plays the role of a kinetic term which leads to loop fluctuations and the ground state is simply a quantum loop gas.

The above quantum loop gas has a fourfold degenerate topological ground state when one considers a periodic boundary condition on a torus [12]. In order to find the above ground states, as shown in Fig. 1(c), consider two noncontractible loops around the torus in two different directions of the lattice, namely, $T^{1(2)}$. Corresponding to these loops, we define two loop operators $T_x^{1(2)} = \prod_{i \in T^{1(2)}} X_i$ which are a product of X operators on all qubits living in $T^{1(2)}$. Other

ground states of the toric code on the torus is generated by applying $T_x^{1(2)}$ to the $|G\rangle$. Regarding the noncontractible nature of the loop operators $T_x^{1(2)}$, it is clear that different ground states belong to the different topological classes of loops. This leads to the important property that the ground state of the toric code is topologically protected against local perturbations.

The above topological protection can be understood by considering excitations of the toric code. Excitations are generated by applying the Pauli operators X or Z on qubits. As shown in Fig. 1(c), consider a string of X (Z) operators with the two end points at two plaquettes (vertices). Since these string operators anticommute with the corresponding plaquette (vertices) operators they generate two excitations on the plaquettes (vertices). Such excitations can be generated by local perturbations added to the initial Hamiltonian. However, they are not able to remove the degeneracy of the ground state because it needs to apply a string of Pauli operators around the torus that is exponentially suppressed as the system size scales.

On the other hand, it is important to note that the above-mentioned topological protection occurs in the presence of weak perturbations. Actually, when the power of perturbations increases, the system undergoes a topological phase transition [21]. For example, suppose that we put the loop gas under a string tension, by adding a Zeeman term to the initial Hamiltonian H_0 in the form of $-h \sum_i Z_i$ with $h > 0$ being an external magnetic field. Due to such a string tension, smaller loops appear that reduce the topological order of the gas. Increasing the string tension removes loop superpositions and destroys the topological order completely at a critical field $h^* = 0.33$.

Topological phase transitions out of topological order in the toric code can be considered in the presence of different forms of perturbations. In particular, one can consider toric code in presence of a nonlinear interaction in the following form:

$$H_0 + \sum_v e^{-\beta h \sum_{i \in v} Z_i}, \quad (2)$$

where β is an additional coupling parameter. For very small values of β , the exponential in (4) reduces to $1 - \beta h$, which is the standard Zeeman term. Such a perturbed toric code is a type of stochastic matrix form Hamiltonian which is exactly solvable [35].

In this regard, it is important to provide a strategy for increasing the robustness of the gas against h , i.e., increase the potency of the system so that the phase transition occurs in larger values of h . Recently, it has been shown that geometrical frustration is an influential factor for topological robustness of the system [33,39]. Following we use such an idea for the toric code on kagome lattice and show that other interesting phenomena emerge due to the interplay of topology and geometrical frustration.

III. QUANTUM LOOP GASES WITH TENSION AND PRESSURE

In order to have a frustrated loop gas, in addition to the string tension, we introduce a string pressure to the system

by adding a magnetic field of opposite direction ($h < 0$) to the toric code Hamiltonian. Although, in contrast with the tension, the pressure leads to the creation of larger loops, but similar to the tension, the pressure also tries to destroy the topological order of the loop gas. Regarding such opposite roles of the string tension and pressure, it is expected their combination results in interesting phenomena. Following we study the ground-state properties of a quantum loop gas model with tension and pressure.

Let us consider a loop gas model with the Hamiltonian of the form

$$H = H_0 - h_A \sum_{i \in A} Z_i - h_B \sum_{i \in B} Z_i, \quad (3)$$

where A and B contain qubits living on diagonal and vertical-horizontal edges in the kagome lattice, respectively. Here, h_A and h_B are external magnetic fields applied to qubits of type A and B , respectively. As shown in Fig. 1(b), each loop configuration in the lattice crosses both types of qubit. In this regard, in the first-order perturbation approximation, the Zeeman terms induce an energy shift of $\Delta \varepsilon = \Delta \varepsilon_A + \Delta \varepsilon_B$ with $\Delta \varepsilon_A = (2L_A - N_A)h_A$ and $\Delta \varepsilon_B = (2L_B - N_B)h_B$, where N_A (N_B) is total number of A (B) qubits and L_A (L_B) is number of A (B) qubits crossed by the corresponding loop configuration. Therefore, the value as well as the sign of h_A and h_B are crucial in determining the lowest-energy state of the Hamiltonian (3). In particular, there are three different possibilities: (1) In the presence of very weak magnetic fields, i.e., when $h_A, h_B \ll 1$, the energy shift is $\Delta \varepsilon \sim 0$ and there is no considerable difference between different loop states, and hence the topological order of H_0 persists. (2) If one of the fields is very strong, i.e., either $h_A \gg 1$ or $h_B \gg 1$, the energy shift becomes either $\Delta \varepsilon_A$ or $\Delta \varepsilon_B$. (3) There is an intermediate regime in which both h_A and h_B are ~ 1 , but have different signs. In this case, the energy shift is $\Delta \varepsilon \sim 2(L_A - L_B) + N_B - N_A$, which can not be simply analyzed for different loop states. In this case our quantum loop gas is actually under both the pressure and tension, simultaneously, and a frustration arises in the ground state of the system. One of the possible scenarios we can consider is the emergence of an interesting phase with a partially topological order. In particular, consider a partial set of loop states Ω including both small and large loop configurations so that the energy shift $\Delta \varepsilon$ is the same for all members of Ω . Therefore, the ground state would be a superposition of all loop states belonging to Ω . Such a partially loop-condensed state is a topological phase, distinct from the topological and trivial phases of toric codes.

A. PTP in an exactly solvable model

The above qualitative argument shows that a frustrated combination of string tension and pressure in quantum loop gas model leads to a PTP. However, in order to realize such a possibility, explicitly, we should find the whole phase diagram of the model. The Hamiltonian in Eq. (3) is not exactly solvable and finding its ground state for arbitrary strengths of h_A and h_B is not straightforward. On the other hand, adding the Zeeman term to the toric code Hamiltonian is not the only way for considering frustration effects of the string tension and pressure. In [39], we have shown that the same effects can

be realized by adding some particular nonlinear interactions to the toric code Hamiltonian. Here, as [39], we consider the toric code on the kagome lattice in presence of a nonlinear interaction of the following form:

$$\sum_v e^{-\beta[h_A \sum_{i \in A} Z_i + h_B \sum_{i \in B} Z_i]}, \quad (4)$$

where β is an additional coupling parameter. For very small values of β , the exponential in (4) reduces to $1 - \beta[h_A \sum_{i \in A} Z_i + h_B \sum_{i \in B} Z_i]$, which is the standard Zeeman term. Such a perturbed toric code is a type of stochastic matrix form Hamiltonian which is exactly solvable [35]. In this regard, the ground state of the toric code in the presence of the above-mentioned term (4) can be exactly calculated as [37,38]

$$|G(\beta)\rangle = \frac{1}{\sqrt{\mathcal{Z}}} e^{\frac{\beta}{2}[h_A \sum_{i \in A} Z_i + h_B \sum_{i \in B} Z_i]} |G_0\rangle, \quad (5)$$

where $|G_0\rangle$ is the ground state of the bare toric code and \mathcal{Z} is a normalization factor. Now, it is important to show how the nonlinear interactions in (4) lead to the string tension or pressure similar to what we discussed for the Hamiltonian (3). In the ground state $|G(\beta)\rangle$ the amplitude of each loop configuration depends on $h_A L_A$ and $h_B L_B$ as $e^{\frac{\beta}{2} \Delta \varepsilon}$. This clearly implies that increasing β decreases the amplitudes in favor of smaller or larger loops depending on the value and the sign of h_A and h_B . These are all things we need for the frustration effect we discussed in the previous section.

B. Mapping onto a frustrated Ising model

Regarding the special form of the ground state in Eq. (5), the normalization factor \mathcal{Z} is equal to the partition function of a two-dimensional Ising model defined on the kagome lattice with classical spins residing at the vertices. The Hamiltonian of such a dual Ising model is given by

$$\mathcal{H} = -h_A \sum_{(\mu, \nu) \in A} S_\mu S_\nu - h_B \sum_{(\mu, \nu) \in B} S_\mu S_\nu, \quad (6)$$

where S_μ is the classical spin resided on the site μ , and the sums run over nearest neighbors in the sublattices A and B . The coupling parameter β in the quantum model corresponds to $1/k_B T$ in the classical Ising model where k_B is the Boltzmann constant and T is temperature. In order to prove the above mapping, we should notice that, similar to the $|G_0\rangle$, $|G(\beta)\rangle$ is also a superposition of all loop states with the difference that each loop state has a particular amplitude. To see this, consider a loop state corresponding to a particular loop configuration which crosses L_A (L_B) numbers of qubits belonging to the edges of A (B). When we apply $e^{-\frac{\beta}{2}[h_A \sum_{i \in A} Z_i + h_B \sum_{i \in B} Z_i]}$ on such a loop state, a term with amplitude $e^{\frac{\beta}{2} \Delta \varepsilon}$ will be generated. This amplitude is equal to the square root of the Boltzmann weight of corresponding spin configuration in the Ising model. To show this, corresponding to a loop configuration we set the classical spins at vertices inside (outside) loops to -1 ($+1$) [see Fig. 1(b)]. The energy of such a spin configuration in the Ising model is equal to $\Delta \varepsilon$. Therefore, $e^{\frac{\beta}{2} \Delta \varepsilon}$ will be the square root of the Boltzmann weight of this configuration. In this regard, the normalization

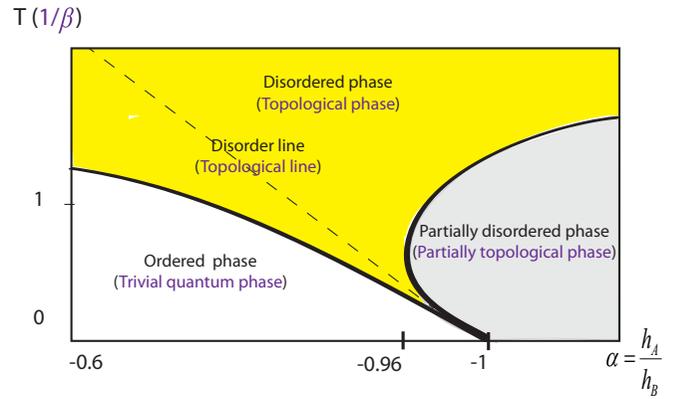


FIG. 2. Phase diagram of the frustrated Ising model (6) on the kagome lattice. There are three different classical phases including disordered, ordered, and partially disordered phases. Corresponding to different classical phases, there are three quantum phases in our frustrated quantum loop gas model. The disordered (ordered) phase in the classical model is equivalent to the topological (trivial) phase, and corresponding to the partially disordered phase there is a PTP. Vertical axis in the classical model is temperature which is mapped onto $1/\beta$ in our quantum loop gas model. Solid lines are phase border and the dashed line is a topological line at which the topological order survives even in the presence of strong perturbations.

factor in Eq. (5) is equal to summation of all Boltzmann weights which is the same as the partition function of the Ising model. We can summarize the above mapping as a mapping from the interaction term $S_\mu S_\nu$ in the Ising model onto the Pauli operator Z_i in the ground state $|G(\beta)\rangle$ where the i th qubit lives on the edge with two end points at μ and ν .

The above mapping can be used for characterizing the phase diagram of our frustrated toric code model. Fortunately, the frustrated Ising model on kagome lattice is well studied [40,41], and as shown in Fig. 2 there are three different phases in the phase diagram of the system. Interestingly, aside from ferromagnetic and paramagnetic phases there is also a partially disordered phase in which the spins on one sublattice are ordered magnetically, whereas on the other one they are free to be $+1$ or -1 . This interesting phase diagram can be characterized by sublattice order parameters. To this end, consider three different sublattices for the kagome lattice: $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ (see Fig. 3). The ensemble averages of spins in these sublattices are given by the magnetizations m_1 , m_2 , and m_3 , respectively. In the disordered phase (ordered phase) the above three order parameters are zero (nonzero). However, in a partially disordered phase the order parameter m_2 is zero while m_1 and m_3 are nonzero. This indicates that in this phase the spins in the sublattice $V^{(2)}$ possess no magnetic order.

C. Ground-state phase diagram

Now, we use the results of the Ising model and present the ground-state phase diagram of our frustrated quantum loop gas model. All phase transitions in the frustrated Ising model correspond to singularities in the thermodynamic functions like heat capacity. Therefore, it is crucial to find the equivalence of such thermodynamic functions in our quantum loop gas model. As we discussed, since Boltzmann weights in the

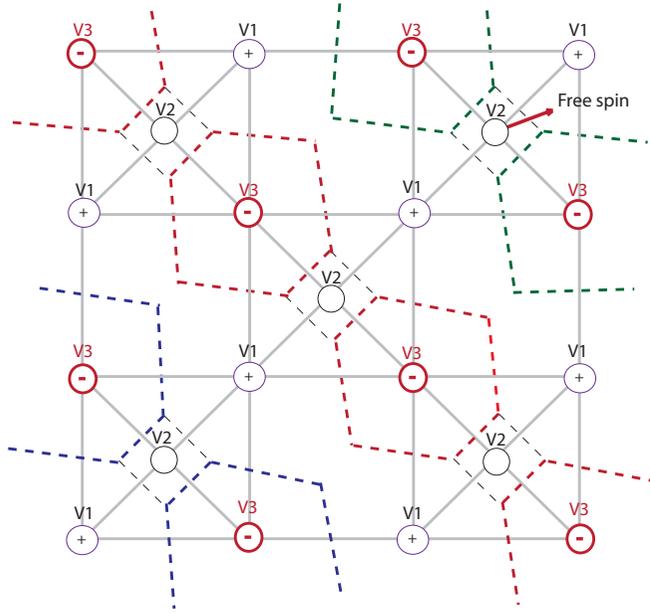


FIG. 3. A loop configuration in the PTP at $\beta = \infty$. The sublattices $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ contain the classical spins which interact with each other with the Ising Hamiltonian (6). At $T = 0$, in the partially disordered phase, the spins in the sublattice $V^{(1)}$ ($V^{(3)}$) are $+1$ (-1), and those in the sublattice $V^{(2)}$ are completely free to be $+1$ or -1 . Such a spin configuration in the Ising model corresponds to particular loop configurations in our quantum loop gas model. According to the sign of the spins in the sublattice $V^{(2)}$, loop configurations are formed in diagonal direction, and the ground state $|G(\beta = \infty)\rangle$ is a product of loop states fluctuating along ladderlike subspaces.

Ising model are encoded in the ground state of the quantum loop gas, one can simply find a mapping between the ground-state fidelity in our quantum model and the heat capacity in the Ising model [38,42]. The ground-state fidelity is defined as $\langle G(\beta)|G(\beta + d\beta)\rangle$ which can be approximated as

$$F = 1 - g_{\beta\beta} d\beta^2, \quad (7)$$

where $g_{\beta\beta} = \frac{\partial^2 F}{\partial \beta^2}$ is the fidelity susceptibility. It has been shown that $g_{\beta\beta}$ is related to the heat capacity of the Ising model as $g_{\beta\beta} = C_v/8\beta^2$. Accordingly, it is clear that any singularity in the heat capacity of the frustrated Ising model, at a critical temperature T^* , corresponds exactly to a singularity in the fidelity susceptibility in the frustrated quantum loop gas, at a critical $\beta^* = 1/k_B T^*$.

Moreover, there is a one-to-one correspondence between classical phases in the frustrated Ising model and quantum phases in our frustrated quantum loop gas (see Fig. 2). Since β in the frustrated quantum loop gas is equivalent to temperature inverse in the Ising model, it is concluded that the paramagnetic (ferromagnetic) phase at high (low) temperatures is equivalent to the topological (trivial) phase at small (large) values of β [38]. There is also a partially disordered phase in the frustrated Ising model which is equivalent to a topological phase different from the standard topological order of the toric code. As seen in Fig. 2, for large values of $1/\beta$, and any value of $\alpha = \frac{h_A}{h_B}$ the system is in the topological phase. By decreasing

$1/\beta$ (increasing the perturbation strength) perturbation tries to destroy the topological order. For $\alpha < 0.96$ the topological order is completely destroyed at a critical point where a topological-trivial phase transition occurs in the system, but for $\alpha > 1$ although the perturbation imposes some changes, the topological order survives at least partially. In this region, by decreasing $1/\beta$ a phase transition to the PTP happens at a critical point. The interval $0.96 < \alpha < 1$ is interesting. In this region interestingly a reentrant phase transition occurs in the system, i.e., by increasing perturbation the system goes to the PTP, and by more increasing perturbation although we expect the topological order to be disappeared completely, but the system prefers to go back to the topological phase. This reentrance phenomenon in our quantum loop gas is due to the frustration and can be seen also in different spin models [43]. The dashed line in the topological phase is a topological line, introduced in Ref. [39], at which the topological order is robust against the nonlinear perturbation of type (4) with arbitrary strength, and no phase transition happens in the system by increasing β . This phenomenon is a combined effect of frustration and the nonlinearity existing in our quantum loop gas model.

IV. CHARACTERIZING THE PTP

According to the mapping between the amplitudes in the ground state of our quantum model and the Boltzmann weights in the Ising model, the ground state of our quantum model in the PTP should be a superposition of loop states corresponding to spin configurations in the partially disordered phase in the Ising model. In order to obtain the allowed loop configurations we look at the sign of the classical spins in the Ising model and draw a loop around the spins -1 . In the partially disordered phase, at $T = 0$, all spins in the sublattice $V^{(1)}$ are $+1$ and therefore they should not be surrounded by any loop, while all spins in $V^{(3)}$ are -1 and they must be surrounded by loops. But the spins in $V^{(2)}$ behave differently. Due to the frustration, they are free to be $+1$ or -1 , and therefore their sign determines their status. In this regard, in the PTP of our frustrated quantum model, in the limit of infinite β , we have a particular set of loop configurations. In this phase, loop configurations are constructed by small loops encompassing the spins -1 in the sublattice $V^{(2)}$ and all spins in $V^{(3)}$. Therefore, the partially topological state is a loop condensed state written as a superposition of particular loop configurations as

$$|G_{\beta=\infty}\rangle = \frac{1}{\sqrt{2^{|V^{(2)}|}}} \prod_{v \in V^{(2)}} (1 + A_v) \prod_{v \in V^{(3)}} A_v |0 \dots 0\rangle, \quad (8)$$

where $|V^{(2)}|$ refers to the number of $V^{(2)}$ -type vertices. Since each A_v corresponds to a small loop around the vertex v , applying $\prod_{v \in V^{(3)}} A_v$ on $|0 \dots 0\rangle$ results in a particular loop state including small loops around vertices belonging to $V^{(3)}$. On the other hand, $\prod_{v \in V^{(2)}} (1 + A_v)$ is also a superposition of different loop configurations generated by loops around vertices belonging to $V^{(2)}$. In this regard, the state $|G_{\infty}\rangle$ is simply a superposition of loop configurations constructed by $V^{(2)}$ - and $V^{(3)}$ -type loops. In particular, notice that since there is no loop around vertices belonging to $V^{(1)}$, the state $|G_{\infty}\rangle$ is

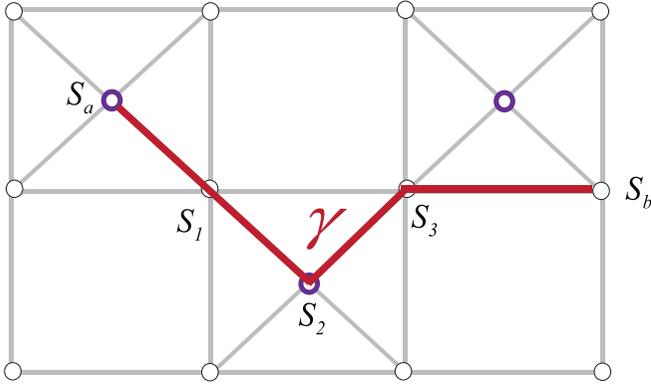


FIG. 4. A schematic illustration of a string γ which starts from a vertex a , passes through different classical spins, and terminates at a vertex b . Corresponding to the string γ there is a string operator in our quantum loop gas model which is equal to the product of Z operators living in the middle of the links that connect neighboring classical spins.

in fact a product of quantum states corresponding to diagonal ladders, as shown in Fig. 3, where each ladder corresponds to a fluctuating loop state constructed by $V^{(2)}$ - and $V^{(3)}$ -type loops. In other words, the ground state at $1/\beta = 0$ is still a loop state, fluctuating along one-dimensional ladders. In order to show that the PTP has topological features, following we define a nonlocal order parameter to distinguish it from the standard topological phases.

A. Nonlocal order parameters

Here, we define a nonlocal order parameter to characterize the PTP, seen in our frustrated quantum loop gas. As mentioned, the connection between the frustrated Ising model on a kagome lattice and our frustrated quantum loop gas is simply obtained by mapping the interaction term $S_\mu S_\nu$ in the classical model onto the Pauli operator Z_i in the quantum model, where the site i is located in the middle of the link $\mu\nu$ [see Fig. 1(a)]. In this mapping the thermal expectation value $\langle S_\mu S_\nu \rangle$ is equivalent to the quantum expectation value $\langle Z_i \rangle = \langle G(\beta) | Z_i | G(\beta) \rangle$. Moreover, we are also able to calculate the spin-spin correlation functions. As shown in Fig. 4, the two spins a and b can be connected to each other by an open string γ . Since for each spin variable we have $S^2 = 1$, we can rewrite $S_a S_b$ as $(S_a S_1)(S_1 S_2)(S_2 S_3) \dots (S_N S_b)$ where S_1, S_2, \dots, S_N are spins that the string γ crosses. Next, according to our quantum-classical mapping, the ensemble average of $(S_a S_1)(S_1 S_2)(S_2 S_3) \dots (S_N S_b)$ is equivalent to the expectation value of Z_i operators on the quantum ground state, i.e.,

$$\langle (S_a S_1)(S_1 S_2)(S_2 S_3) \dots (S_N S_b) \rangle_{\text{en}} = \left\langle \prod_{i \in \gamma} Z_i \right\rangle_{\text{g.s.}} = \Gamma^{ab}, \quad (9)$$

where $i \in \gamma$ refers to all qubits living on the edges that make up the string γ . This means that each spin-spin correlation function in the classical model is equivalent to a string operator in our quantum model where classical spins are living at the two end points of that string. If the distance between spins goes to infinity, the spin-spin correlation in the Ising model

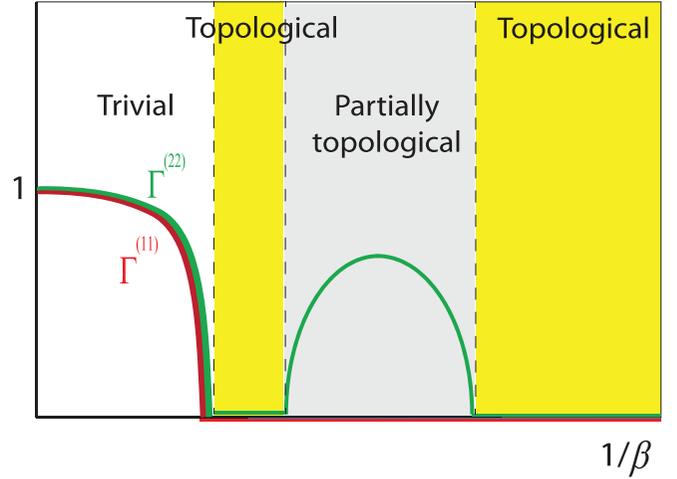


FIG. 5. A schematic illustration of the string order parameters versus $1/\beta$, for $0.96 < |\alpha| < 1$. In the topological phase both the order parameters $\Gamma^{(11)}$ (red) and $\Gamma^{(22)}$ (green) are zero. By decreasing $1/\beta$ the order parameter $\Gamma^{(22)}$ starts to increase at a critical point, and a phase transition from the topological phase to the PTP occurs in the system. In the PTP while $\Gamma^{(11)} = 0$, $\Gamma^{(22)}$ becomes nonzero which indicates the loss of part of topological order. By more decreasing $1/\beta$, $\Gamma^{(22)}$ becomes zero again at a second critical point where a reentrance to the topological phase occurs. Finally, at the third transition point, the systems goes to the trivial phase and both $\Gamma^{(11)}$ and $\Gamma^{(22)}$ become nonzero.

will be simply related to the order parameter as follows:

$$\Gamma^{ab} = \langle S_a S_b \rangle_{\text{en}} = \langle S_a \rangle \langle S_b \rangle = m_a m_b, \quad (10)$$

where m_a and m_b are magnetizations in the Ising model. The two end points of the string γ , i.e., the points a and b , are far apart and their separation distance is the length of the system. So the order parameter Γ^{ab} can be regarded as a nonlocal order parameter for characterizing topological phase transitions in our system. As we discussed, there are three types of vertices in the lattice, $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$, so depending on the location of the two end points of the string γ we can define nine string operators $\Gamma^{(ij)}$ with $i, j = 1, 2, \text{ and } 3$ where $\Gamma^{(ij)}$ is the same as Γ^{ab} so that $a \in V^{(i)}$ and $b \in V^{(j)}$ and the distance between a, b goes to infinity. Since for $a \in V^{(1)}$ and $b \in V^{(3)}$, the magnetizations satisfy $m_a = -m_b$, and also $\Gamma^{(12)} = \Gamma^{(21)} = \sqrt{\Gamma^{(11)}\Gamma^{(22)}}$, we can capture the entire phase diagram of the system by utilizing only the two order parameters: $\Gamma^{(11)}$ and $\Gamma^{(22)}$. In Fig. 5, we have illustrated a schematic plot of $\Gamma^{(11)}$ and $\Gamma^{(22)}$ versus $1/\beta$ for a given α in the narrow interval of $0.966 < |\alpha| < 1$. As seen, for large values of $1/\beta$, when the system is in the topological phase, both these order parameters are zero. By decreasing $1/\beta$, $\Gamma^{(22)}$ starts to increase at a critical point where the system enters to the PTP. In this phase, $\Gamma^{(22)}$ increases by decreasing $1/\beta$ but $\Gamma^{(11)}$ remains zero. By more decreasing $1/\beta$ at a second critical point a reentrance to the topological phase occurs where $\Gamma^{(22)}$ returns to zero again. Finally, at a third critical point, both the order parameters $\Gamma^{(11)}$ and $\Gamma^{(22)}$ become nonzero where a phase transition to the trivial phase occurs.

B. Excitations

For better understanding the nature of our PTP, it is useful to present the ground state of our frustrated quantum loop gas in a basis constructed by the eigenstates of the Pauli operator X . Actually, the perturbation (4) is in terms of Z operators, and therefore examining the behavior of excitations generated by Z operators is worthwhile for understanding the topological features of this phase. To this end, we consider the ground state $|G_{\beta=\infty}\rangle$ which is stabilized by the operators A_v and B_p where $v \in V^{(2)}$ and p runs over all plaquettes. If M is the number of all plaquettes and vertices, the number of vertices in $V^{(2)}$ in addition to the number of all plaquettes is equal to $M - (N_1 + N_3)$, with N_1 and N_3 being the number of vertices belonging to the sublattices $V^{(1)}$ and $V^{(3)}$, respectively. Therefore, for characterizing $|G_{\beta=\infty}\rangle$ as a stabilizer state, we need $N_1 + N_3$ independent stabilizer operators with the eigenstate $|G_{\beta=\infty}\rangle$ and the eigenvalue $+1$. Consider the N_1 (N_3) independent string operators Γ^{ab} whose two end points a, b live in $V^{(1)}$ -type [$V^{(3)}$ -type] vertices. These operators commute with $A_{v \in V^{(2)}}$, due to having an even number of shared qubits and, therefore, they stabilize the state $|G_{\beta=\infty}\rangle$. Consequently, the ground state $|G_{\beta=\infty}\rangle$ can be expressed in terms of the above complete set of stabilizers as

$$\begin{aligned} & |G_{\beta=\infty}\rangle \\ &= \prod_p (1 + B_p) \prod_{a,b \in V^{(1)}} [1 + \Gamma^{ab}] \prod_{a,b \in V^{(3)}} [1 + \Gamma^{ab}] |+\dots+\rangle, \end{aligned} \quad (11)$$

where the products $\prod_{a,b \in V^{(1)}}$ and $\prod_{a,b \in V^{(3)}}$ run over N_1 and N_3 independent string operators, respectively, and $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is the eigenstate of the Pauli operator X with eigenvalue $+1$. Now, we compare the state $|G_{\beta=\infty}\rangle$ with the ground state of the pure toric code $|G_0\rangle = \prod_p (1 + B_p) |+\dots+\rangle$. Since B_p is a loop operator in the kagome lattice, and $Z|+\rangle = |-\rangle$, $|G_0\rangle$ is a superposition of all loop configurations constructed by $|-\rangle$'s in a sea of $|+\rangle$'s. However, the state $|G_{\beta=\infty}\rangle$ in (11) has an important difference from $|G_0\rangle$; that is, in addition to the loop operator B_p there are also open strings Γ^{ab} with $a, b \in V^{(1,3)}$. Therefore, the ground state in the PTP is a superposition of all configurations formed by loops and open strings with end points living in the vertices belonging to $V^{(1)}$ and $V^{(3)}$.

It is interesting to interpret this result from the viewpoint of moving excitations. Since the perturbation term in our quantum loop gas is constructed of Z operators, it leads to open strings that try to remove the topological order of the ground state. However, our result shows that due to the frustration there are three types of excitations in the PTP: excitations of types 1, 2, and 3 living in the $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ sublattices, respectively. In the PTP, the system is robust against excitations of type 2, but the excitations of types 1 and 3 can move in the corresponding sublattices, and reduce the topological order of the system. This restriction on the mobility of excitations is an important property of the PTP that can be compared with other topological systems like fracton codes on 3D lattices. Fracton excitations are restricted to move in certain subspaces due to quantum glassiness [32], and as a

result a topological overprotection occurs in fracton codes. In our quantum loop gas model, frustration creates string tension and pressure which lead to such an overprotection. Frustration does not freeze excitations, but limits their mobility in ladder-like subspaces.

V. GLOBAL ENTANGLEMENT

For characterizing topological phases, a powerful method is to study the behavior of ground-state entanglement. Actually, the nonlocal nature of topological orders is reflected in the long-range entanglement of the topological states [44–55]. Aside from topological entanglement entropy [44,45,56], which is the most important measure for characterizing topological orders, multipartite entanglement measures are also useful to reveal highly entangled nature of topological phases [57–60]. In this regard, although the string order parameters Γ_{11} and Γ_{22} can completely describe the orders of the ground state of the system in different phases, to explore more properties it is useful to study the behavior of the ground-state entanglement. In the following we study the behavior of the global entanglement and its derivative to explore more differences of the TP and PTP phases.

Global entanglement is defined as $GE = 2[1 - \frac{1}{N} \sum_i \text{tr}(\rho_i^2)]$, where ρ_i refers to the reduced density matrix corresponding to the i th qubit. It has been shown that the global entanglement in the toric code model in the presence of the nonlinear perturbation (4) is equivalent to the internal energy in the Ising model [59]. Actually, the linear entropy for an edge qubit i in the perturbed toric code model is equal to the interaction term $S_\mu S_\nu$ in the classical Ising model, where μ and ν are two classical spins resided on the two end points of the edge i . Therefore, the global entanglement in our frustrated quantum loop gas is given by

$$GE = 1 - \frac{1}{2} (\langle S_\mu S_\nu \rangle_A^2 + \langle S_\mu S_\nu \rangle_B^2), \quad (12)$$

where $\langle S_\mu S_\nu \rangle_{A(B)}$ refers to a link energy in A (B). Using the free energy of the frustrated Ising model (see Ref. [40]), one can readily obtain these expectation values and consequently the GE of our quantum loop gas. In Fig. 6(a) we have plotted the GE versus $1/\beta$ for multiple α . According to the phase diagram of our frustrated quantum loop gas, shown in Fig. 2, in the narrow interval of $0.96 < \alpha < 1$ where the reentrance phenomenon can occur, the system experiences different phase transitions by increasing β . Interestingly, the different phase transitions are completely observed in the behavior of the GE. As seen, for large enough $1/\beta$ where the system is deeply in the topological phase, the GE has its maximum value, i.e., $GE = 1$. Far from the reentrance interval, for $\alpha > 1$ ($\alpha < 0.96$), by decreasing $1/\beta$ GE decreases monotonically, falls down to 0.5(0) at the topological–partially topological (topological-trivial) phase transition point, and sticks to 0.5(0) in the partially topological (trivial) phase. In the reentrance interval (see $\alpha = 0.92, 0.94$) GE behaves oscillatory, it is 1 in the topological phase, tries to touch 0.5 in the PTP, and becomes 0 in the trivial phase. The transition points are locations that the GE curve concavity changes. These changes are reflected as cusps and divergence in the GE derivative, as clearly seen in Fig. 6(b).

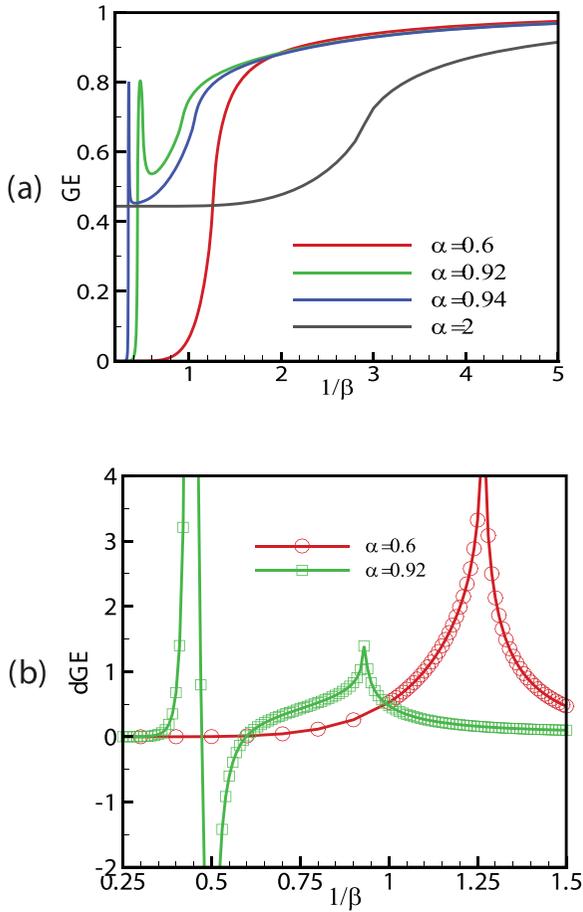


FIG. 6. (a) Global entanglement versus $1/\beta$, for different values of α . For $|\alpha| < 0.96$ there is a phase transition from trivial to topological phase, and the GE changes from zero in the trivial phase to 1 in the topological phase. For $0.96 < |\alpha| < 1$ three phase transitions happen by increasing $1/\beta$. At the first transition point, GE changes from zero in the trivial phase to near 1 in the topological phase, at the second transition point GE changes from 1 to near $\frac{1}{2}$ in the PTP, and finally at the third transition point GE changes from $\frac{1}{2}$ to 1 in the topological phase. (b) Derivative of GE versus $1/\beta$. The cusps and infinite picks reveal exactly the location of phase transitions.

VI. SUMMARY AND OUTLOOK

It is known that the kinetic of excitations in a topological phase is strongly reduced due to frustration. Here, we examined the ground-state phase diagram of a frustrated quantum loop gas model, and demonstrated that there is a PTP in which the kinetic of excitations is partially reduced in the sense that there is a restriction for mobility of excitations living in some sublattices. In this phase the ground state is a tensor product of quasi-one-dimensional fluctuating loop states, there are different types of excitations living in three different sublattices, and excitations belonging to a particular sublattice can move freely. Frustration is a key factor to realizing this phase where string tension and pressure lead to the PTP. This phase can in principle be seen in frustrated toric code in the presence of Zeeman fields, but we considered an exactly solvable model which contains beyond the Zeeman terms. To obtain the ground-state phase diagram, we mapped our frustrated quantum loop gas model onto a classical Ising model. To characterize this phase, we defined two nonlocal order parameters and also considered global entanglement, and showed that these quantities can well distinguish the PTP from other phases of the system.

Finally, it is worth mentioning that topological excitations with restricted mobility are also seen in three-dimensional fracton codes due to quantum glassiness. In this regard, our results propose that similar features can also be realized in two dimensions due to frustration. Moreover, randomness can also lead to the localization of excitations, so it is interesting to study the combined effects of geometrical frustration and randomness in toric codes, and check the possibility of having topological excitations with restricted mobility in certain one-dimensional sublattices similar to lineons in fracton codes.

ACKNOWLEDGMENTS

J.A. is grateful to the Institute for Advanced Studies in Basic Sciences for financial support through research Grant No. G2021IASBS12654.

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